# COMPARISON ESTIMATES FOR THE GREEN FUNCTION AND THE MARTIN KERNEL

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ABSTRACT. A comparison estimate for the product of the Green function and the Martin kernel is given in a uniform domain. As its application, we show the equivalence of ordinary thinness and minimal thinness of a set contained in a non-tangential cone. We also give a comparison estimate for the Martin kernels with distinct singularities.

## 1. INTRODUCTION

The purpose of this paper is to estimate the product of the Green function and the Martin kernel by an explicit function. In the unit ball $B$ of $\mathbb{R}^n$ with $n \geq 2$, the formulas of the Green function $G_B(\cdot, 0)$ with pole at the origin and the Martin kernel $K_B(\cdot, \xi)$ with singularity at a boundary point $\xi$ give that

$$|x - \xi|^{2-n} \leq G_B(x, 0) K_B(x, \xi) \leq c_n |x - \xi|^{2-n}$$

for $x = r\xi$ with $2^{-1} < r < 1$, where $c_n$ is a positive constant depending only on the dimension $n$. Such an estimate in a general domain is interesting because, even though the Green function and the Martin kernel are not clear, the product can be controlled by an explicit function. We will consider it in a uniform domain. An open subset $\Omega$ of $\mathbb{R}^n$, where $n \geq 2$, is said to be a uniform domain if there exists a constant $C_0 > 1$ such that each pair of points $x$ and $y$ in $\Omega$ can be connected by a rectifiable curve $\gamma$ in $\Omega$ for which

$$\begin{aligned} &\ell(\gamma) \leq C_0 |x - y|, \\ &\min\{\ell(\gamma(x, z)), \ell(\gamma(z, y))\} \leq C_0 \delta_\Omega(z) \quad \text{for all } z \in \gamma, \end{aligned}$$

(1.1)

where $\ell(\gamma(x, z))$ denotes the length of the subarc $\gamma(x, z)$ of $\gamma$ from $x$ to $z$ and $\delta_\Omega(z)$ stands for the distance from $z$ to the boundary $\partial\Omega$ of $\Omega$. We denote by $G_\Omega$ the Green function for $\Omega$. Let $x_0$ be a reference point in $\Omega$. The Martin kernel of $\Omega$ is defined by

$$K_\Omega(x, y) = \frac{G_\Omega(x, y)}{G_\Omega(x_0, y)} \quad \text{for } (x, y) \in (\Omega \times \Omega) \setminus \{(x_0, x_0)\}.$$

It is known that if $\Omega$ is a uniform domain, then $K_\Omega(x, \cdot)$ can be extended continuously to the boundary (cf. [2, Theorem 3]). The Martin kernel with singularity at $\xi \in \partial\Omega$ is denoted by the same symbol $K_\Omega(\cdot, \xi)$. For $\xi \in \partial\Omega$ and $\alpha > 1$, we write

$$\Gamma_\alpha(\xi) = \{x \in \Omega : |x - \xi| < \alpha \delta_\Omega(x)\},$$

a non-tangential cone at $\xi$. An open ball and a sphere of center $x$ and radius $r$ are denoted by $B(x, r)$ and $S(x, r)$, respectively. Throughout the paper, we use the

symbol $C$ to denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we write $C(a, b, \cdots)$ to denote a constant depending only on $a, b, \cdots$. For two positive functions $f_1$ and $f_2$, we write $f_1 \approx f_2$ if there exists a constant $C > 1$ such that $C^{-1} f_1 \leq f_2 \leq C f_1$. The constant $C$ is called the constant of comparison. Our result in higher dimensions is as follows.

**Theorem 1.1.** *Suppose that $\Omega$ is a uniform domain in $\mathbb{R}^n$ with $n \geq 3$. Let $\xi \in \partial\Omega$ and $\alpha > 1$. Then*

$$(1.2) \qquad G_\Omega(x, x_0) K_\Omega(x, \xi) \approx |x - \xi|^{2-n} \quad \textit{for } x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1}\delta_\Omega(x_0)),$$

*where the constant of comparison depends only on $\alpha$ and $\Omega$.*

This result may be relating to the 3G inequality in bounded subdomains of $\mathbb{R}^n$, where $n \geq 3$: there exists a positive constant $C = C(\Omega)$ such that

$$(1.3) \qquad \frac{G_\Omega(x, y) G_\Omega(x, z)}{G_\Omega(y, z)} \leq C(|x - y|^{2-n} + |x - z|^{2-n}) \quad \text{for } x, y, z \in \Omega.$$

The 3G inequality was first proved in a bounded Lipschitz domain by Cranston, Fabes and Zhao [13] to study the conditional gauge theory for the schrödinger operator. See also Bogdan [8]. Recently, Aikawa and Lundh [4] proved the 3G inequality in a bounded uniformly John domain. Theorem 1.1 may be interpreted as the limiting case of the 3G inequality. Indeed, if we let $z = x_0$ and tend $y$ to $\xi \in \partial\Omega$, then we have

$$K_\Omega(x, \xi) G_\Omega(x, x_0) \leq C(|x - \xi|^{2-n} + |x - x_0|^{2-n}) \leq C|x - \xi|^{2-n}$$

whenever $x \in \Omega \cap B(\xi, 2^{-1}\delta_\Omega(x_0))$. Note that Theorem 1.1 is a local estimate although the 3G inequality is a global one. Theorem 1.1 asserts that the product $G_\Omega(\cdot, x_0) K_\Omega(\cdot, \xi)$ is bounded from below by the function $|\cdot - \xi|^{2-n}$ as well.

The 3G inequality in two dimensions was proved by Bass and Burdzy [7] using probabilistic methods: for any bounded domain in $\mathbb{R}^2$, there exists a positive constant $C = C(\Omega)$ such that

$$\frac{G_\Omega(x, y) G_\Omega(x, z)}{G_\Omega(y, z)} \leq C \left( 1 + \log^+ \frac{1}{|x - y|} + \log^+ \frac{1}{|x - z|} \right) \quad \text{for } x, y, z \in \Omega,$$

where $\log^+ f = \max\{0, \log f\}$. If $\Omega$ is a bounded uniform domain in $\mathbb{R}^2$ and $\xi \in \partial\Omega$, then the same process as above gives that for $x \in \Omega$ sufficiently near $\xi$,

$$K_\Omega(x, \xi) G_\Omega(x, x_0) \leq C \log \frac{1}{|x - \xi|}.$$

In the particular case that $\Omega$ is a unit disc in $\mathbb{R}^2$, this inequality is not sharp as seen in the starting paragraph of this section. But the above inequality is sharp when $\xi$ is an isolated boundary point. Indeed, letting $\delta = 2^{-1} \min\{1, \operatorname{dist}(\xi, \{x_0\} \cup (\partial\Omega \setminus \{\xi\}))\}$, we have for $x \in B(\xi, \delta) \setminus \{\xi\}$ that $G_\Omega(x, x_0) = G_{\Omega \cup \{\xi\}}(x, x_0) \approx G_{\Omega \cup \{\xi\}}(\xi, x_0)$ by the Harnack inequality, and that

$$K_\Omega(x, \xi) = \frac{G_{\Omega \cup \{\xi\}}(x, \xi)}{G_{\Omega \cup \{\xi\}}(x_0, \xi)} \geq \frac{G_{B(\xi, 2\delta)}(x, \xi)}{G_{\Omega \cup \{\xi\}}(x_0, \xi)} \geq \frac{2\delta}{G_{\Omega \cup \{\xi\}}(x_0, \xi)} \log \frac{1}{|x - \xi|}.$$

In order to obtain comparison estimate (1.2) for $n = 2$, we assume the following exterior condition at $\xi \in \partial\Omega$:

(1.4)    There exists a positive constant $C_1$ such that for each $r > 0$ small, there is a point $z_r \in B(\xi, r) \setminus \overline{\Omega}$ so that $B(z_r, C_1 r) \subset \mathbb{R}^n \setminus \overline{\Omega}$.

Obviously, Lipschitz domains and NTA domains (in the sense of Jerison and Kenig [18]) satisfy condition (1.4) at every boundary point. Our result in two dimensions is as follows.

**Theorem 1.2.** *Let $\Omega$ be a uniform domain in $\mathbb{R}^n$ with $n = 2$, and let $\alpha > 1$. Suppose that $\xi \in \partial\Omega$ satisfies condition (1.4). Then*

$$G_\Omega(x, x_0) K_\Omega(x, \xi) \approx 1 \quad \text{for } x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1}\delta_\Omega(x_0)),$$

*where the constant of comparison depends only on $\alpha$, $\Omega$ and $C_1$.*

Our results are also relating to the following. In a domain $\Omega_\phi$ whose boundary is described as the graph of a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\phi(0) = 0$, Burdzy [9, 10], Carroll [11, 12] and Gardiner [14] showed that the convergence of the integrals

$$(1.5) \qquad \int_{\{|x'|<1\}} \frac{\max\{\phi(x'), 0\}}{|x'|^n} dx' \quad \text{and} \quad \int_{\{|x'|<1\}} \frac{\max\{-\phi(x'), 0\}}{|x'|^n} dx',$$

controls the limit of $G_{\Omega_\phi}(te, e)/t$ as $t \to 0$, where $e = (0, \cdots, 0, 1) \in \mathbb{R}^n$. The author [16] obtained a corresponding result for the Martin kernel of $\Omega_\phi$ with singularity at the origin, that is, the convergence of the integrals in (1.5) controls the limit of $t^{n-1} K_{\Omega_\phi}(te, 0)$ as $t \to 0$. Theorems 1.1 and 1.2 are independent of the convergence of the integrals in (1.5), and give a direct connection beween the boundary decay of the Green function and the boundary growth of the Martin kernel.

Theorems 1.1 and 1.2 will be proved simply using the boundary Harnack principle and estimates of the Green function. A certain modification of Theorem 1.1, stated in Section 2, will be enable us to show the equivalence of ordinary thinness and minimal thinness for a set contained in a non-tangential cone. See Section 3. Using Theorems 1.1 and 1.2, we also give in Section 4 comparison estimates for the product of two Martin kernels with distinct singularities.

## 2. Proof of Theorems 1.1 and 1.2

We start by preparing some materials: the boundary Harnack principle in a uniform domain (cf. [2, Theorem 1]) and estimates for the Green function. We say that a property holds quasi-everywhere if it holds apart from a polar set.

**Lemma 2.1.** *Let $\Omega$ be a uniform domain in $\mathbb{R}^n$ with $n \geq 2$. Then there exist constants $r_0 > 0$ and $C_2 > 1$ depending only on $\Omega$ with the following property: Let $\xi \in \partial\Omega$ and $0 < r \leq r_0$. If $h_1$ and $h_2$ are positive and bounded harmonic functions in $\Omega \cap B(\xi, C_2 r)$ vanishing quasi-everywhere on $\partial\Omega \cap B(\xi, C_2 r)$, then*

$$\frac{h_1(y)}{h_2(y)} \approx \frac{h_1(y')}{h_2(y')} \quad \text{for } y, y' \in \Omega \cap B(\xi, r),$$

*where the constant of comparison depends only on $\Omega$.*

A uniform domain can be characterized in terms of the quasi-hyperbolic metric:

$$k_\Omega(x, y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_\Omega(z)},$$

where the infimum is taken over all rectifiable curve $\gamma$ in $\Omega$ connecting $x$ to $y$, and $ds$ stands for the line element on $\gamma$. Gehring and Osgood [15] showed that $\Omega$ is a uniform domain if and only if there exists a positive constant $C$ such that

$$(2.1) \qquad k_\Omega(x, y) \le C \log\left[ \left( \frac{|x - y|}{\delta_\Omega(x)} + 1 \right) \left( \frac{|x - y|}{\delta_\Omega(y)} + 1 \right) \right] + C \quad \text{for } x, y \in \Omega.$$

We say that a finite sequence of balls $\{B(x_j, 2^{-1}\delta_\Omega(x_j))\}_{j=1}^N$ in $\Omega$ is a Harnack chain between $x$ and $y$ if $x_1 = x$, $x_N = y$, and $x_{j+1} \in B(x_j, 2^{-1}\delta_\Omega(x_j))$ for $j = 1, \cdots, N-1$. The number $N$ is called the length of the Harnack chain. We observe in any proper subdomains of $\mathbb{R}^n$ that the shortest length of the Harnack chain between $x$ and $y$ is comparable to $k_\Omega(x, y) + 1$. The following lemma follows from the Harnack inequality.

**Lemma 2.2.** *Let $\Omega$ be a proper subdomain of $\mathbb{R}^n$ with $n \ge 2$. Then there exists a constant $C > 1$ depending only on the dimension $n$ such that if $x, y \in \Omega$, then*

$$\exp(-C(k_\Omega(x, y) + 1)) \le \frac{h(x)}{h(y)} \le \exp(C(k_\Omega(x, y) + 1))$$

*for every positive harmonic function $h$ in $\Omega$.*

To apply Lemma 2.2 to the Green function, we need the following: If $z \in \Omega$, then

$$(2.2) \qquad k_{\Omega \setminus \{z\}}(x, y) \le 3k_\Omega(x, y) + \pi \quad \text{for } x, y \in \Omega \setminus B(z, 2^{-1}\delta_\Omega(z)).$$

The proof of this inequality may be found in [3, Lemma 7.2].

**Lemma 2.3.** *Let $\Omega$ be a uniform domain in $\mathbb{R}^n$ with $n \ge 2$. If $x, y \in \Omega$ satisfy*

$$|x - y| \le C_3 \min\{\delta_\Omega(x), \delta_\Omega(y)\}$$

*for some positive constant $C_3$, then there exists a positive constant $C$ depending only on $C_3$ and $\Omega$ such that*

$$G_\Omega(x, y) \ge C|x - y|^{2-n}.$$

*Proof.* We may assume, without loss of generality, that $\delta_\Omega(x) \le \delta_\Omega(y)$ and $|x - y| \ge 2^{-1}\delta_\Omega(x)$. Take $w \in S(x, 2^{-1}\delta_\Omega(x))$. Then, by assumption,

$$|y - w| \le 2|x - y| \le 4C_3 \min\{\delta_\Omega(w), \delta_\Omega(y)\}.$$

Hence Lemma 2.2, together with (2.1) and (2.2), yields that

$$G_\Omega(x, y) \approx G_\Omega(x, w) \ge G_{B(x, \delta_\Omega(x))}(x, w) \approx \delta_\Omega(x)^{2-n} \ge C|x - y|^{2-n}.$$

Thus the lemma is proved. $\qquad\qquad \square$

In general, if $n \ge 3$, then $G_\Omega(x, y) \le |x - y|^{2-n}$ for $x, y \in \Omega$. But, in two dimensions, such an upper bound does not necessarily hold. This is a reason to assume condition (1.4).

**Lemma 2.4.** *Let $\Omega$ be a proper subdomain of $\mathbb{R}^2$ and let $\alpha > 1$. Suppose that $\xi \in \partial\Omega$ satisfies condition (1.4). Then there exists a positive constant $C$ depending only on $\alpha$ and $C_1$ such that*

$$G_\Omega(x, y) \leq C \quad \text{for } x \in \Gamma_\alpha(\xi) \text{ and } y \in \Omega \setminus B(x, 2^{-1}\delta_\Omega(x)).$$

*Proof.* Let $x \in \Gamma_\alpha(\xi)$ and put $r = |x - \xi|$. By assumption, there is $z_r \in B(\xi, r) \setminus \overline{\Omega}$ such that $B(z_r, C_1 r) \subset \mathbb{R}^2 \setminus \overline{\Omega}$. We now write $y^*$ for the inverse of $y$ with respect to $S(z_r, C_1 r)$. Then we obtain that for $y \in S(x, 2^{-1}\delta_\Omega(x))$,

$$G_\Omega(x, y) \leq G_{\mathbb{R}^2 \setminus \overline{B(z_r, C_1 r)}}(x, y) = \log\left(\frac{|y - z_r|}{C_1 r} \frac{|x - y^*|}{|x - y|}\right) \leq C(\alpha, C_1).$$

Hence the maximum principle yields the lemma. $\qquad\qquad\square$

Substituting for Theorem 1.1, we prove the following modification, which will be used in Section 3.

**Proposition 2.5.** *Suppose that $\Omega$ is a uniform domain in $\mathbb{R}^n$ with $n \geq 3$. Let $\xi \in \partial\Omega$, $\alpha > 1$ and $\kappa \geq 1$. Then for $x \in \Gamma_\alpha(\xi) \cap B(\xi, (2\kappa)^{-1}\delta_\Omega(x_0))$ and $y \in \overline{\Omega} \cap B(\xi, \kappa|x - \xi|)$,*

$$(2.3) \qquad\qquad G_\Omega(x, x_0) K_\Omega(x, y) \approx |x - y|^{2-n},$$

*where the constant of comparison depends only on $\alpha$, $\kappa$ and $\Omega$.*

*Proof.* Let $x \in \Gamma_\alpha(\xi) \cap B(\xi, (2\kappa)^{-1}\delta_\Omega(x_0))$ and $y \in \Omega \cap B(\xi, \kappa|x - \xi|)$. Then $x, y \notin B(x_0, 2^{-1}\delta_\Omega(x_0))$. Let $C_4$ be a constant sufficiently large so that

$$C_4 > \max\left\{5, C_2, \frac{\delta_\Omega(x_0)}{r_0}\right\},$$

where $C_2$ and $r_0$ are the constants in Lemma 2.1. We put $r = C_4^{-1}\delta_\Omega(x)$. Since

$$\delta_\Omega(x) \leq |x - \xi| < \delta_\Omega(x_0) < C_4 r_0,$$

we have $r < r_0$. We consider two cases: $\delta_\Omega(y) < r$ and $\delta_\Omega(y) \geq r$.

*Case 1 :* $\delta_\Omega(y) < r$. Let $y' \in \partial\Omega$ be a point such that $\delta_\Omega(y) = |y - y'|$. Then

$$|x - y'| \geq \delta_\Omega(x) > C_2 r \quad \text{and} \quad |x_0 - y'| \geq \delta_\Omega(x_0) \geq \delta_\Omega(x) > C_2 r.$$

In view of the second inequality of (1.1), we can take a point $y_r$ in $S(y', r) \cap \Omega$ so that $\delta_\Omega(y_r) \geq 2^{-1}C_0 r$. We apply Lemma 2.1 to obtain

$$(2.4) \qquad\qquad K_\Omega(x, y) = \frac{G_\Omega(x, y)}{G_\Omega(x_0, y)} \approx \frac{G_\Omega(x, y_r)}{G_\Omega(x_0, y_r)}.$$

Note that $y_r \notin B(x_0, 2^{-1}\delta_\Omega(x_0))$. Indeed, since $2r < C_4 r = \delta_\Omega(x) \leq \delta_\Omega(x_0)$, we have

$$|x_0 - y_r| \geq \delta_\Omega(x_0) - \delta_\Omega(y_r) \geq \delta_\Omega(x_0) - r \geq \frac{1}{2}\delta_\Omega(x_0).$$

Since $|y - \xi| \leq \kappa|x - \xi| \leq \alpha\kappa\delta_\Omega(x) = \alpha\kappa C_4 r$ and $|y - y_r| \leq 2r$, we have

$$|x - y_r| \leq |x - \xi| + |\xi - y| + |y - y_r| \leq C(\alpha, \kappa, C_4)r.$$

It therefore follows from (2.1), Lemma 2.2 and (2.2) that

$$(2.5) \qquad\qquad G_\Omega(x, x_0) \approx G_\Omega(y_r, x_0),$$

and from Lemma 2.3 that $G_\Omega(x, y_r) \approx |x - y_r|^{2-n}$. Since $|x - y| \geq \delta_\Omega(x) - \delta_\Omega(y) \geq (C_4 - 1)r \geq 4r$ by $\delta_\Omega(y) < r$, we have

$$|x - y_r| \leq |x - y| + |y - y_r| \leq |x - y| + 2r \leq \frac{3}{2}|x - y|$$

and

$$|x - y_r| \geq |x - y| - |y - y_r| \geq |x - y| - 2r \geq \frac{1}{2}|x - y|.$$

Therefore

$$(2.6) \qquad\qquad G_\Omega(x, y_r) \approx |x - y|^{2-n}.$$

Combining (2.4), (2.5) and (2.6), we obtain (2.3) in this case.

*Case 2*: $\delta_\Omega(y) \geq r$. Since $|x - y| \leq C(\alpha, \kappa, C_4)r$, it follows from (2.1), Lemma 2.2, (2.2) and Lemma 2.3 that

$$G_\Omega(x, x_0) \approx G_\Omega(y, x_0) \quad \text{and} \quad G_\Omega(x, y) \approx |x - y|^{2-n},$$

and so (2.3) holds.

Finally, tending $y$ to the boundary, we also obtain (2.3) for $y \in \partial\Omega \cap B(\xi, \kappa|x - \xi|)$. Thus the proposition is proved. $\qquad\square$

Theorem 1.2 (two dimensional case) may be established by repeating the same argument as in the proof of Proposition 2.5 with $\kappa = 1$. Indeed, the lower bound

$$G_\Omega(x, x_0)K_\Omega(x, \xi) \geq C \quad \text{for } x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1}\delta_\Omega(x_0)),$$

holds, since Lemma 2.3 holds for $n \geq 2$. We need to use Lemma 2.4 to obtain the upper bound

$$(2.7) \qquad G_\Omega(x, x_0)K_\Omega(x, \xi) \leq C \quad \text{for } x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1}\delta_\Omega(x_0)),$$

since $G_\Omega(x, y) \leq C$ does not hold in general. For completeness, we give a proof. Let $x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1}\delta_\Omega(x_0))$ and $y \in \Omega \cap B(\xi, r)$, where $r = C_4^{-1}\delta_\Omega(x)$. Note that $\delta_\Omega(y) < r$. Let $y_r$ be a point as in the proof of Proposition 2.5. Since

$$|x - y_r| \geq \delta_\Omega(x) - \delta_\Omega(y_r) \geq \delta_\Omega(x) - r \geq \frac{1}{2}\delta_\Omega(x),$$

we have by Lemma 2.4 that $G_\Omega(x, y_r) \leq C$. Hence this, together with (2.4) and (2.5), yields that $G_\Omega(x, x_0)K_\Omega(x, y) \leq C$. Tending $y$ to $\xi$, we obtain (2.7).

## 3. Equivalence of ordinary thinness and minimal thinness

In this section, we show, as an application of Proposition 2.5, the equivalence of ordinary thinness and minimal thinness for a set contained in a non-tangential cone of a uniform domain in $\mathbb{R}^n$, where $n \geq 3$. Let $E$ be a subset of $\mathbb{R}^n$ and let $\xi \in \mathbb{R}^n$ be a limit point of $E$. We say that $E$ is thin at $\xi$ (in the ordinary sense) if there exists a positive superharmonic function $u$ in $\mathbb{R}^n$ such that $u(\xi) < +\infty$ and $u(x) \to +\infty$ as $x \to \xi$ along $E$. By Wiener's criterion (cf. [6, Theorem 7.7.2]), thinness can be characterized in terms of the regularized reduced function. We denote by $\widehat{R}_1^E$ the regularized reduced function of the constant function 1 relative to $E$ in $\mathbb{R}^n$, and write $E_j = \{x \in E : 2^{-j-1} \leq |x - \xi| \leq 2^{-j}\}$. Then $E$ is thin at $\xi$ if and only if $\sum_{j=1}^{\infty} \widehat{R}_1^{E_j}(\xi) < +\infty$. The original definition of minimal thinness by Naïm [20] is based on the regularized reduced function of the Martin kernel. We define minimal

thinness by the following equivalent condition (cf. [6, Theorem 9.2.7]): Let $E$ be a subset of $\Omega$ and let $\xi$ be a minimal Martin boundary point of $\Omega$, which is a Martin topology limit point of $E$. We say that $E$ is minimally thin at $\xi$ with respect to $\Omega$ if there exists a Green potential $G_\Omega \mu$ in $\Omega$ such that $\int K_\Omega(x, \xi) d\mu(x) < +\infty$ and

$$\lim_{y \to \xi, y \in E} \frac{G_\Omega \mu(y)}{G_\Omega(x_0, y)} = +\infty \quad \text{(in the Martin topology)}.$$

Now, suppose $n \geq 3$ and let $E$ be a set containd in a non-tangential cone at a boundary point $\xi$. In [19], Lelong-Ferrand proved in the half-space that $E$ is thin at $\xi$ if and only if $E$ is minimally thin at $\xi$. The extension of this to a bounded Lipschitz domain was established by Aikawa [1]. Note that when $n = 2$, the equivalence does not hold in general (cf. [17]). We now give the extension to a uniform domain. Note that if $\Omega$ is a bounded uniform domain, then the Martin compactification of $\Omega$ is homeomorphic to the Euclidean closure and all Martin boundary points are minimal (cf. [2, Corollary 3]).

**Theorem 3.1.** *Suppose that $\Omega$ is a bounded uniform domain in $\mathbb{R}^n$, where $n \geq 3$. Let $\xi \in \partial\Omega$ and $\alpha > 1$, and let $E$ be a subset of $\Gamma_\alpha(\xi)$. Then $E$ is thin at $\xi$ if and only if $E$ is minimally thin at $\xi$ with respect to $\Omega$.*

*Remark* 3.2. The boundedness of $\Omega$ is not essential and we may leave it out, since if $D$ is a domain containing $\Omega$ such that $D \cap B(\xi, 1) = \Omega \cap B(\xi, 1)$ and if $E$ is minimally thin at $\xi$ with respect to $\Omega$, then so is with respect to $D$ (cf. [20, Théorème 15]).

Note again that 3G inequality (1.3) in a bounded uniform domain yields that for $x \in \Omega \setminus B(x_0, 2^{-1}\delta_\Omega(x_0))$ and $y \in \overline{\Omega}$,

$$(3.1) \qquad K_\Omega(x, y) G_\Omega(x, x_0) \leq C(|x - y|^{2-n} + |x - x_0|^{2-n}) \leq C|x - y|^{2-n}.$$

Here, in the last inequality, we used $|x - y| \leq 2(\text{diam } \Omega)\delta_\Omega(x_0)^{-1}|x - x_0|$.

*Proof of Theorem 3.1.* We may assume, without loss of generality, that $\xi$ is a limit point of $E$ and $E \subset B(\xi, 6^{-1}\delta_\Omega(x_0))$. We first show the necessity. For $j \geq 1$, we let $E_j = \{x \in E : 2^{-j-1} \leq |x - \xi| \leq 2^{-j}\}$. Since $E$ is thin at $\xi$, there exists a sequence $\{a_j\}$ of positive numbers such that

$$\lim_{j \to +\infty} a_j = +\infty \quad \text{and} \quad \sum_{j=1}^{\infty} a_j \widehat{R}_1^{E_j}(\xi) < +\infty.$$

Let $\mu_j$ be the Riesz measure associated with $\widehat{R}_1^{E_j}$, and let $d\nu_j(x) = G_\Omega(x, x_0) d\mu_j(x)$. Note that the support of $\nu_j$ is contained in $\overline{E}_j$. It follows from Proposition 2.5 with $\kappa = 3$ that for $y \in E_j$,

$$\widehat{R}_1^{E_j}(y) = \int |x - y|^{2-n} d\mu_j(x) \leq C \int K_\Omega(x, y) d\nu_j(x).$$

Since $\widehat{R}_1^{E_j} = 1$ quasi-everywhere on $E_j$, we have

$$\frac{1}{C} \leq \frac{G_\Omega \nu_j(y)}{G_\Omega(x_0, y)} \quad \text{for quasi-every } y \in E_j.$$

Let $u(y) = \sum_{j=1}^{\infty} a_j G_\Omega \nu_j(y)$. Then $u$ is a Green potential in $\Omega$ satisfying

$$\lim_{y \to \xi, y \in E \setminus F} \frac{u(y)}{G_\Omega(x_0, y)} = +\infty,$$

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where $F$ is a polar set. Also, Proposition 2.5 with $y = \xi$ gives

$$\sum_{j=1}^{\infty} a_j \int K_{\Omega}(x, \xi) d\nu_j(x) \leq C \sum_{j=1}^{\infty} a_j \widehat{R}_1^{E_j}(\xi) < +\infty.$$

Hence $E \setminus F$ is minimally thin at $\xi$ with respect to $\Omega$, and so is $E$.

We next show the sufficiency. Since $E$ is minimally thin at $\xi$ with respect to $\Omega$, there exists a Green potential $G_{\Omega}\mu$ such that $\int K_{\Omega}(x, \xi) d\mu(x) < +\infty$ and

$$\lim_{y \to \xi, \, y \in E} \frac{G_{\Omega}\mu(y)}{G_{\Omega}(x_0, y)} = +\infty.$$

Replacing $G_{\Omega}\mu$ by its regularized reduced function relative to $\overline{\Gamma_{\alpha}(\xi) \cap B(\xi, 2^{-1}\delta_{\Omega}(x_0))}$ in $\Omega$ if necessary, we may assume that the support of $\mu$ is in $\overline{\Gamma_{\alpha}(\xi) \cap B(\xi, 2^{-1}\delta_{\Omega}(x_0))}$. Let $d\nu(x) = G_{\Omega}(x, x_0)^{-1} d\mu(x)$. It then follows from (3.1) that

$$\frac{G_{\Omega}\mu(y)}{G_{\Omega}(x_0, y)} = \int K_{\Omega}(x, y) d\mu(x) \leq C \int |x - y|^{2-n} d\nu(x),$$

so that

$$\lim_{y \to \xi, \, y \in E} \int |x - y|^{2-n} d\nu(x) = +\infty.$$

Also, Proposition 2.5 with $y = \xi$ gives

$$\int |x - \xi|^{2-n} d\nu(x) \leq C \int K_{\Omega}(x, \xi) d\mu(x) < +\infty.$$

Hence $E$ is thin at $\xi$. Thus the proof is complete. $\qquad\square$

## 4. Further result

In this section, we give comparison estimates for the product of two Martin kernels with distinct singularities. Estimates will be obtained on a certain curve connecting singularities. We observe that the properties in (1.1) can be extended to the boundary of $\Omega$, that is, if $\Omega$ is a uniform domain, then each pair of points $\xi$ and $\eta$ in $\partial\Omega$ can be connected by a rectifiable curve $\gamma$ such that $\gamma \setminus \{\xi, \eta\} \subset \Omega$ and

(4.1) $$\ell(\gamma) \leq C_5 |\xi - \eta|,$$

(4.2) $$\min\{\ell(\gamma(\xi, z)), \ell(\gamma(z, \eta))\} \leq C_5 \delta_{\Omega}(z) \quad \text{for all } z \in \gamma,$$

where the constant $C_5$ depends only on $C_0$ in (1.1). See [5, Lemma 2.1], in which this was proved for a uniformly John domain but their argument is applicable to our case after replacing the internal metric by the Euclidean metric. For $\xi, \eta \in \partial\Omega$, we denote by $z_{\xi,\eta}$ the middle point of $\gamma$ so that $\ell(\gamma(\xi, z_{\xi,\eta})) = \ell(\gamma(z_{\xi,\eta}, \eta)) = 2^{-1}\ell(\gamma)$.

**Theorem 4.1.** *Let $\Omega$ be a bounded uniform domain in $\mathbb{R}^n$. Let $\xi, \eta \in \partial\Omega$ be distinct points and suppose that $\gamma$ is a curve connecting $\xi$ to $\eta$ for which (4.1) and (4.2) are satisfied. Let $z_{\xi,\eta}$ be the middle point of $\gamma$ and put*

$$g(\xi, \eta) = \max\left\{1, \frac{|\xi - \eta|^{2-n}}{G_{\Omega}(z_{\xi,\eta}, x_0)^2}\right\}.$$

*The following statements hold:*

(i) *If $n \geq 3$, then for $x \in \gamma$,*

$$(4.3) \qquad K_\Omega(x, \xi) K_\Omega(x, \eta) \approx g(\xi, \eta)(|x - \xi|^{2-n} + |x - \eta|^{2-n}),$$

*where the constant of comparison depends only on $\Omega$.*

(ii) *If $n = 2$ and every boundary point of $\Omega$ satisfies condition (1.4), then (4.3) holds for $x \in \gamma$.*

We observe that if $\gamma$ is a curve connecting $\xi$ to $\eta$ with (4.1) and (4.2), then

$$(2C_5)^{-1}|\xi - \eta| \leq C_5^{-1}\ell(\gamma(\xi, z_{\xi,\eta})) \leq \delta_\Omega(z_{\xi,\eta}) \leq \ell(\gamma(\xi, z_{\xi,\eta})) \leq C_5|\xi - \eta|.$$

Note that if $\Omega$ is a bounded $C^{1,1}$-domain, then

$$G_\Omega(x, x_0) \approx \delta_\Omega(x) \quad \text{for } x \in \Omega \setminus B(x_0, 2^{-1}\delta_\Omega(x_0)).$$

Hence, in this case, we obtain the following.

**Corollary 4.2.** *Let $\Omega$ be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ with $n \geq 2$. Let $\xi, \eta \in \partial\Omega$ be distinct points and suppose that $\gamma$ is a curve connecting $\xi$ to $\eta$ for which (4.1) and (4.2) are satisfied. Then for $x \in \gamma$,*

$$K_\Omega(x, \xi)K_\Omega(x, \eta) \approx \frac{1}{|\xi - \eta|^n}(|x - \xi|^{2-n} + |x - \eta|^{2-n}),$$

*where the constant of comparison depends only on $\Omega$.*

Let us give a proof of Theorem 4.1.

*Proof of Theorem 4.1.* We give a proof only when $n \geq 3$. Let $r_0$ and $C_2$ be the constants in Lemma 2.1. We may assume, without loss of generality, that $\delta_\Omega(x_0) \geq 2\max\{C_2, C_5\}r_0$. Let $r = (C_2 + 2)^{-1}|\xi - \eta|$. We consider two cases: $r \leq r_0$ and $r > r_0$.

*Case 1*: $r \leq r_0$. Let $x \in \gamma \cap \overline{B(\xi, r)}$. Then $|x - \eta| > C_2 r$, and so Lemma 2.1 gives

$$(4.4) \qquad K_\Omega(x, \eta) \approx \frac{G_\Omega(x, y_r)}{G_\Omega(x_0, y_r)},$$

where $y_r$ is a point such that $y_r \in \gamma \cap S(\eta, r)$. Since $|y_r - \xi| > C_2 r$, we again apply Lemma 2.1 to obtain

$$(4.5) \qquad \frac{G_\Omega(x, y_r)}{G_\Omega(x, x_0)} \approx \frac{G_\Omega(x_r, y_r)}{G_\Omega(x_r, x_0)},$$

where $x_r$ is a point such that $x_r \in \gamma \cap S(\xi, r)$. Since $x \in \Gamma_{C_5}(\xi) \cap B(\xi, 2^{-1}\delta_\Omega(x_0))$ by (4.2), it follows from (4.4), (4.5) and Theorem 1.1 that

$$(4.6) \qquad \begin{aligned} K_\Omega(x, \eta) &\approx \frac{G_\Omega(x_r, y_r)}{G_\Omega(x_r, x_0)G_\Omega(y_r, x_0)}G_\Omega(x, x_0) \\ &\approx \frac{G_\Omega(x_r, y_r)}{G_\Omega(x_r, x_0)G_\Omega(y_r, x_0)}\frac{|x - \xi|^{2-n}}{K_\Omega(x, \xi)}. \end{aligned}$$

Note from (4.2) that every $\delta_\Omega(x_r), \delta_\Omega(y_r), \delta_\Omega(z_{\xi,\eta})$ is greater than $C_5^{-1}r$ and that

$$|x - x_0| \geq \delta_\Omega(x_0) - |x - \xi| \geq \frac{1}{2}\delta_\Omega(x_0).$$

Since $|x_r - z_{\xi,\eta}|$ and $|y_r - z_{\xi,\eta}|$ are bounded by $\ell(\gamma) \leq C_5|\xi - \eta| = C_5(C_2 + 2)r$, we have from (2.1), Lemma 2.2 and (2.2)

$$(4.7) \qquad G_\Omega(x_r, x_0) \approx G_\Omega(z_{\xi,\eta}, x_0) \approx G_\Omega(y_r, x_0).$$

Also, since $|x_r - y_r| \leq C_5(C_2 + 2)r$ and

$$|x_r - y_r| \geq |\xi - \eta| - |x_r - \xi| - |y_r - \eta| \geq (C_2 + 2)r - r - r = C_2 r,$$

we have by Lemma 2.3 (ane Lemma 2.4 when $n = 2$)

$$(4.8) \qquad G_\Omega(x_r, y_r) \approx |x_r - y_r|^{2-n} \approx r^{2-n} \approx |\xi - \eta|^{2-n}.$$

Combining (4.6), (4.7) and (4.8), we obtain

$$(4.9) \qquad K_\Omega(x, \xi) K_\Omega(x, \eta) \approx \frac{|\xi - \eta|^{2-n}}{G_\Omega(z_{\xi,\eta}, x_0)^2} |x - \xi|^{2-n}$$

whenever $x \in \gamma \cap \overline{B(\xi, r)}$. If $x \in \gamma(\xi, z_{\xi,\eta}) \setminus B(\xi, r)$, then $|x - x_r| \leq C_5(C_2 + 1)r$ and $\delta_\Omega(x) \geq C_5^{-1} r$ by (4.1) and (4.2), and therefore (2.1) and Lemma 2.2 give

$$K_\Omega(x, \xi) K_\Omega(x, \eta) \approx K_\Omega(x_r, \xi) K_\Omega(x_r, \eta).$$

Observe that $|x - \xi| \approx r = |x_r - \xi|$. Hence (4.9) holds for $x \in \gamma(\xi, z_{\xi,\eta})$. Since $|x - \xi|^{2-n} \approx |x - \xi|^{2-n} + |x - \eta|^{2-n}$ for $x \in \gamma(\xi, z_{\xi,\eta})$ and $|\xi - \eta|^{2-n} G_\Omega(z_{\xi,\eta}, x_0)^{-2} \geq C(\Omega) > 0$, we obtain (4.3) for $x \in \gamma(\xi, z_{\xi,\eta})$. Similarly, we can obtain (4.3) for $x \in \gamma(z_{\xi,\eta}, \eta)$. Thus (4.3) holds for all $x \in \gamma$ in this case.

*Case 2 :* $r > r_0$. Let $x \in \gamma \cap \overline{B(\xi, r_0)}$ and let $x_{r_0} \in \gamma \cap S(\xi, r_0)$. Then we observe that

$$K_\Omega(x_{r_0}, \eta) \approx 1 \quad \text{and} \quad G_\Omega(x_{r_0}, x_0) \approx 1,$$

where the constants of comparisons depend on $r_0$, $\delta_\Omega(x_0)$ and the diameter of $\Omega$. Note that $|\xi - \eta| = (C_2 + 2)r > C_2 r_0$. We apply Lemma 2.1 and Theorem 1.1 to obtain

$$K_\Omega(x, \eta) \approx \frac{K_\Omega(x_{r_0}, \eta)}{G_\Omega(x_{r_0}, x_0)} G_\Omega(x, x_0) \approx \frac{|x - \xi|^{2-n}}{K_\Omega(x, \xi)} \approx \frac{|x - \xi|^{2-n} + |x - \eta|^{2-n}}{K_\Omega(x, \xi)}.$$

If $x \in \gamma(\xi, z_{\xi,\eta}) \setminus B(\xi, r_0)$, then $\delta_\Omega(x) \geq C_5^{-1} r_0$ by (4.2), and so

$$K_\Omega(x, \xi) \approx 1 \approx K_\Omega(x, \eta) \quad \text{and} \quad |x - \xi| \approx 1 \approx |x - \eta|,$$

where the constants of comparisons depend on $C_5^{-1} r_0$, $\delta_\Omega(x_0)$ and the diameter of $\Omega$. Noting $|\xi - \eta|^{2-n} G_\Omega(z_{\xi,\eta}, x_0)^{-2} \leq C(\Omega)$, we have (4.3) for all $x \in \gamma(\xi, z_{\xi,\eta})$. Similarly, we can obtain (4.3) for $x \in \gamma(z_{\xi,\eta}, \eta)$. Thus the proof of Theorem 4.1 is complete. $\qquad\square$

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