Nondegeneracy of Blowup for Semilinear Heat Equations

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Nondegeneracy of Blowup for Semilinear Heat Equations

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1. Introduction.

This paper is concerned with the blowup of solutions of the semilinear heat equation

$$
(1.1)\qquad \begin{aligned} u_t - \Delta u - |u|^{p-1}u = 0 \quad &\text{in } D \times [0,T) \\ u = 0 \quad &\text{on } \partial D \times [0,T), \end{aligned}
$$

where $D \subset \mathbf{R}^n$, u is scalar–valued, and $p > 1$. Other, related equations and systems are considered as well, for example $u_t - \Delta u = |u|^{p-1}u + |u|^q$ with $q < p$. Following the usual conventions, we understand the *blowup time* T to be the maximal existence time of the L^∞ solution u, and we call $a \in \bar{D}$ a *blowup point* if u is not locally bounded near (a, T). A lot of attention has recently been focussed on the number of blowup points, their possible locations and the local behavior of the blowing–up solution, e.g. [4, 8, 10–13, 15, 16, 18, 19, 29, 34]. Our goal here is to address these issues by making use of the tools developed in [23, 24]. A preliminary report on this work was presented in [25], and related ideas will be found in [22]. Stated informally, our principal results are

(1.2) *A local lower bound on the blowup rate:* if $|u(x,t)| \le \epsilon(T-t)^{-1/(p-1)}$ for x near a and ϵ sufficiently small then a cannot be a blowup point;

(1.3) *Nondegeneracy of blowup limits:* one can tell whether or not $a \in \bar{D}$ is a blowup point by examining the asymptotic behavior of u in a backward spacetime parabola based at (a, T); and

(1.4) *A criterion for excluding blowup:* for any $a \in D$ there is a condition on the initial data, involving the smallness of a certain weighted energy, which assures that a is not a blowup point.

(Our practice throughout this introduction will be to present our results informally, referring to the statements of the theorems for the appropriate hypotheses on p, D, etc.)

The local bound on the growth rate (1.2) plays a key role in our analysis of (1.3) and (1.4), but in addition it seems a matter of independent interest. Only the growth of the nonlinear term is relevant; the result is actually proved in Theorem 2.1 for local solutions of the differential inequality $|v_t - \Delta v| \le K(1 + |v|^p)$. The global analogue of (1.2) is the assertion that

$$
(1.5)\qquad \limsup_{t \to T} \; (T-t)^{1/(p-1)} \|u\|_{L^\infty(D)} \ge \epsilon > 0;
$$

it is easily established by modifying our argument for the local version (see Remark 2.4). However, we note that in the global case u actually satisfies the stronger estimate

$$(1.5)' \qquad \liminf_{t\to T} \ (T-t)^{1/(p-1)}\|u\|_{L^\infty(D)} \geq \epsilon > 0.$$

The latter is a consequence of the local–in–time existence theory with L^∞ initial data, c.f. [2, 21, 33] (where corresponding estimates are presented for L^p norms, $p < \infty$).

Our nondegeneracy result (1.3) complements the analysis of [23,24] concerning the asymptotic behavior of u near blowup. There we considered, among other topics, the solution of (1.1) on a convex domain $D \subset \mathbf{R}^n$ with C^2 boundary, supposing that $p < (n+2)/(n-2)$ or $n \leq 2$; we showed that for any $a \in D$ the *blowup limit*

$$(1.6) \qquad \lim_{t\to T} (T-t)^{1/(p-1)} u(a + y(T-t)^{\frac{1}{2}}, t)$$

exists uniformly on compact sets $|y| \leq C$, is independent of y, and equals 0 or $\pm(p-1)^{-1/(p-1)}$. A similar conclusion was reached in [4] for positive, radial solutions in a ball, with $u_r \leq 0$ and $u_t \geq 0$, even when $p \geq (n+2)/(n-2)$. Here we shall prove that

(1.7) *if a is a blowup point then the blowup limit (1.6) cannot be zero.*

(This is Theorem 4.2.) Since the converse of (1.7) is trivial, it yields a complete characterization of the behavior of u in spacetime parabolas based at blowup points, at least for subcritical p:

(1.8) *$a \in D$ is a blowup point if and only if the blowup limit (1.6) equals $\pm(p-1)^{-1/(p-1)}$.*

(See Corollary 4.3(i).) Taking $y = 0$ in (1.6), we also see that if the solution is not locally bounded near a, then it must actually blow up <u>at</u> a:

(1.9) *$a \in D$ is a blowup point if and only if $|u(a,t)| \to \infty$ as $t \to T$.*

(See Corollary 4.3(ii).) In one space dimension the last assertion can also be deduced from [12].

Since $u = 0$ at ∂D, one might expect that the blowup limit (1.6) should be 0 if $a \in \partial D$. We shall prove that this is so, provided that D is strictly star–shaped about a and $p < (n+2)/(n-2)$ or $n \le 2$. It follows as an application of (1.7) that

(1.10) *for $a \in \partial D$, if a and p are as above then a is not a blowup point.*

(See Theorem 5.3.) This generalizes a result from [15], which asserts that blowup does not occur at ∂D if D is a bounded, convex domain and $u \ge 0$. (However, that result applies even when p is supercritical, i.e. $p \ge (n+2)/(n-2)$.)

Several authors have proved results for one–dimensional solutions of (1.1) which control the number of blowup points in terms of the number of extrema of the initial data [8, 10, 12, 13, 15, 16]. Our criterion for excluding blowup, (1.4), addresses instead the question of whether u blows up at a *particular* point $a \in D$. It involves this weighted "energy" of the initial data u_0:

$$(1.11) \quad \mathcal{E}[u_0] = T^{\frac{2}{p-1} - \frac{n}{2}} \int_D \left(\frac{T}{2} |\nabla u_0|^2 - \frac{T}{p+1} |u_0|^{p+1} + \frac{1}{2(p-1)} |u_0|^2 \right) e^{-|x-a|^2/4T} \, dx.$$

We shall show that

(1.12) *if $\mathcal{E}_a[u_0]$ is sufficiently small then a is not a blowup point.*

(This is Corollary 3.6.) Because of the exponentially decaying weight under the integral in (1.11), $\mathcal{E}_a[u_0]$ will be small if a is far away from the region where $|u_0|$ or $|\nabla u_0|$ is large. Thus (1.12) allows one to localize the blowup set, for example to see that blowup can occur only at points that are not too far from the support of u_0. In particular, we show that

(1.13) *for the Cauchy problem with initial data in $H^1(\mathbf{R}^n)$, the blowup set is compact.*

(See Theorem 5.1.) Such a result was previously known only in one space dimension, for positive inital data decreasing monotonically to zero as $|x| \to \infty$ and satisfying $u_0 \le C|x|^{-2/(p-1)}$ [18,19]. We also apply (1.12) to show the existence of solutions whose blowup set has a particular form. Among our results are

(1.14) *on a one dimensional interval, there exist positive initial data that blow up at exactly two points; and*

(1.15) *in dimensions $n \ge 2$, there exist positive, radially symmetric initial data for which the blowup set is an $n-1$ dimensional sphere.*

(See Corollary 5.7.)

Since we rely primarily on the methods of [24], our analysis requires star–shaped D and subcritical p, *i.e.*

$$p < (n+2)/(n-2) \quad \text{or} \quad n \leq 2. \tag{1.16}$$

Many of our theorems have local versions requiring no star–shapedness condition on D and no particular boundary condition, and which hold even for supercritical p, provided that u satisfies an estimate of the form

$$||u||_{L^\infty(D)}(t) \leq C(T-t)^{-1/(p-1)}. \tag{1.17}$$

Such results are presented in [22]. Unfortunately, (1.17) has so far been proved only for convex D, and only when

$$u|_{\partial D} = 0, \quad \text{and} \quad p < (3n+8)/(3n-4) \quad \text{or} \quad n = 1; \tag{1.18}$$

$$u|_{\partial D} = 0, \quad u \geq 0, \quad \text{and} \quad p < (n+2)/(n-2) \quad \text{or} \quad n \leq 2; \quad \text{or} \tag{1.19}$$

$$u \geq 0, \quad u_t \geq 0, \quad \text{and} \quad u|_{\partial D} = 0 \quad \text{or} \quad \frac{\partial u}{\partial \nu}|_{\partial D} = \alpha u \quad \text{with} \quad \alpha \leq 0; \tag{1.20}$$

(see Sections 3A and 3B of [24], and [15]). On the other hand, no counterexample to (1.17) is known, even for supercritical p.

We turn now to a discussion of our methods. The proof of the local lower bound on the blowup rate, (1.2), uses nothing more than the semigroup representation formula. But the rest of our analysis makes use of "similarity variables", the change of both dependent and independent variables defined for any $a \in \bar{D}$ by

$$w_a(y,s) = (T-t)^{1/(p-1)}u(x,t), \quad y = (x-a)(T-t)^{-\frac{1}{2}}, \quad s = -ln(T-t). \tag{1.21}$$

We observed in [23,24] that $w = w_a$ satisfies

$$w_s - \frac{1}{\rho}\nabla \cdot (\rho \nabla w) + \frac{1}{p-1}w - |w|^{p-1}w = 0 \tag{1.22}$$

with $\rho(y) = \exp(-|y|^2/4)$, and that it has the "energy"

$$E[w] = \int \rho(y)[\frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)}|w|^2 - \frac{1}{p+1}|w|^{p+1}]dy \tag{1.23}$$

as a Liapunov functional provided that D is star–shaped about a. It is important to note that $w = w_a(y,s)$ exists globally in s; using this, we showed in [23,24] that

$$E[w](s) > 0 \textit{ for all } s. \tag{1.24}$$

The basis of our criterion for excluding blowup is the assertion that if the energy ever gets small enough then it must decrease to zero. In other words, there is a $\sigma > 0$ such that

$$\textit{if } E[w](s_1) < \sigma \textit{ for some } s_1 \textit{ then } w(y,s) \to 0 \textit{ and } E[w](s) \to 0 \textit{ as } s \to \infty. \tag{1.25}$$

(See Theorem 3.5 and the proof of Theorem 4.2.) This is not unexpected, since the second variation of E at 0 is positive, so that $w \equiv 0$ should be in some sense a local minimum. There is a subtle aspect to (1.25), however: we assert it not for an arbitrary solution of (1.22), but rather for one that exists globally in s! The statement is false without this extra hypothesis: if, for example, $E[w] = 0$ and $w_s \not\equiv 0$ at the initial time, then $w(y,s)$ will cease to exist at some finite value of s.

Stated in terms of w_a, our nondegeneracy result (1.3) says

$$\textit{if } w_a(y,s) \to 0 \textit{ as } s \to \infty \textit{ then } a \textit{ is not a blowup point.} \tag{1.26}$$

(This is Theorem 4.2.) To prove (1.26), we first establish that if $w_a(y,s) \to 0$ then also $E[w_a](s) \to 0$. Since w_a depends continuously on a, it follows easily that $E[w_b](s)$ stays uniformly small for b near a and s large. An improvement of (1.25) then shows that $w_b(0,s)$ is uniformly small as $s \to \infty$. Restated in terms of u rather than w, the conclusion of this argument is that for any $\epsilon > 0$ there exists $r > 0$ such that

$$\limsup_{T \to t} \sup_{|b-a|<r} (T-t)^{\frac{1}{p-1}} |u(b,t)| \le \epsilon. \tag{1.27}$$

But then, by our local lower bound on the blowup rate (1.2), a cannot be a blowup point.

This presentation of our results has focussed exclusively on the scalar equation (1.1). Actually, much of the analysis can be extended to more general nonlinearities and also to certain systems. Two such classes of extensions are presented in Section 6, involving nonlinearities of the form $|u|^{p-1}u + h(u)$ with h growing slower than $|u|^p$ near infinity, and systems of the form $u_t - \Delta u - \partial G/\partial u = 0$ with $G \geq 0$ positively homogeneous of degree $p+1$.

The bibliography of [24] includes many references to recent work on the blowup of solutions of (1.1) and similar equations. We take this opportunity to note some other, related articles not cited there and not mentioned elsewhere in this introduction: [3], [4] and [28] study the blowup of solutions of $u_t - \Delta u = e^u$; [14] proves single–point blowup for certain semilinear systems; [17] gives an algebraic formula for a nontrivial, self–similar, blowing–up solution of (1.1) when $p = 2$ and $6 < n < 16$ (which we suppose must agree with one of the solutions constructed in [32]); [20] presents evidence that the blowup behaves like an "approximate self–similar solution" (a certain function of y^2/s, in terms of our similarity variables); finally, [5] presents a numerical method for calculating blowing–up solutions, based on rescaling and mesh refinement, and shows that the computed profile is consistent with the conjectured "approximately self–similar" behavior.

2. A Local Lower Bound on the Blowup Rate.

The essential goal of this section is to show that a solution of (1.1) must be large—specifically, of order $(T-t)^{-1/(p-1)}$—near any blowup point. In fact, this property is by no means restricted to (1.1); it holds for any parabolic system whose nonlinearity has p^{th} power growth. For the purposes of this section we shall therefore consider not (1.1) but rather the ***parabolic differential inequality***

$$|v_t - \Delta v| \leq K(1 + |v|^p), \quad 1 < p < \infty, \quad K > 0, \tag{2.1}$$

in which v may be scalar or vector–valued. By a solution of (2.1) on a cylinder $Q = B_r(a) \times [t_0, t_1) \subset \mathbf{R}^n \times \mathbf{R}$, we mean a continuous function v defined on Q for which v_t and Δv are continuous and (2.1) is satisfied pointwise. We say that v ***blows up*** at (a, t_1) if it is not locally bounded nearby, *i.e.* if there is a sequence $\{(x_m, \tau_m)\} \subset Q$ with $(x_m, \tau_m) \to (a, t_1)$ as $m \to \infty$ such that $|v(x_m, \tau_m)| \to \infty$. Our main result is

THEOREM 2.1. *There is a constant $\epsilon > 0$, depending only on K, p and n, with the following property: if v solves (2.1) on $Q_r = B_r(a) \times [t_1 - r^2, t_1)$ for some $a \in \mathbf{R}^n$, $t_1 \in \mathbf{R}$, and $0 < r \leq 1$, and if*

$$|v(x,t)| \leq \epsilon(t_1 - t)^{-1/(p-1)} \text{ for all } (x,t) \in Q_r, \tag{2.2}$$

then v does not blow up at (a, t_1).

The proof uses two elementary lemmas, the first of which estimates certain integrals:

LEMMA 2.2. *For $0 < \alpha < 1$, $\theta > 0$, and $0 < h < 1$, the integral*

$$I(h) = \int_h^1 (s-h)^{-\alpha} s^{-\theta} ds$$

satisfies

$$\text{(i)} \qquad I(h) \leq \left(\frac{1}{1-\alpha} + \frac{1}{\alpha+\theta-1}\right) h^{1-\alpha-\theta} \quad \text{if } \alpha+\theta > 1$$

$$\text{(ii)} \qquad I(h) \leq \frac{1}{1-\alpha} + |\log h| \quad \text{if } \alpha+\theta = 1$$

$$\text{(iii)} \qquad I(h) \leq \frac{1}{1-\alpha-\theta} \quad \text{if } \alpha+\theta < 1.$$

PROOF: If $\alpha + \theta < 1$ then

$$I(h) \leq \int_h^1 (s-h)^{-\alpha-\theta} ds \leq \frac{1}{1-\alpha-\theta},$$

so (iii) is proved. For the other cases we split the integral into two parts,

$$I(h) = \left(\int_h^{2h\wedge 1} + \int_{2h\wedge 1}^1\right) (s-h)^{-\alpha} s^{-\theta} ds = I_1(h) + I_2(h),$$

where $2h \wedge 1 = \min(2h, 1)$. Changing variables to $\sigma = s - h$, we obtain

$$I_1(h) \leq \int_0^h \sigma^{-\alpha}(\sigma+h)^{-\theta} d\sigma \leq h^{-\theta} \int_0^h \sigma^{-\alpha} d\sigma = \frac{1}{1-\alpha} h^{1-\alpha-\theta}.$$

If $2h \geq 1$ then $I_2 = 0$; otherwise

$$I_2(h) = \int_{2h}^1 (s-h)^{-\alpha} s^{-\theta} ds \leq \int_{2h}^1 (s-h)^{-\alpha-\theta} ds.$$

The last integral is bounded by $(\alpha+\theta-1)^{-1} h^{1-\alpha-\theta}$ if $\alpha+\theta > 1$, and by $|\log h|$ if $\alpha+\theta = 1$, yielding (i) and (ii) respectively.

The second lemma is a version of Gronwall's inequality:

LEMMA 2.3. *If $y(t)$, $r(t)$ and $q(t)$ are continuous functions defined on $[t_0, t_1]$ such that*

$$y(t) \le y_0 + \int_{t_0}^{t} y(s)r(s)ds + \int_{t_0}^{t} q(s)ds, \quad t_0 \le t \le t_1, \tag{2.3}$$

then

$$y(t) \le \left[\exp \int_{t_0}^{t} r(\tau)d\tau\right] \left[y_0 + \int_{t_0}^{t} q(\tau)\exp\left(-\int_{t_0}^{\tau} r(\sigma)d\sigma\right) d\tau\right]. \tag{2.4}$$

PROOF: Let $z(t)$ denote the right hand side of (2.3). Clearly $z(t_0) = y_0$, and differentiation of (2.3) gives $\dot{z} \le rz + q$. Integrating this differential inequality, we see that z is bounded by the right hand side of (2.4). The desired conclusion follows, since $y \le z$.

We now begin the proof of Theorem 2.1. Translating coordinates in space and time, we may suppose that $a = 0$ and $t_1 = 0$. By scaling, it is sufficient to consider the case $r = 1$: indeed, if v satisfies the hypotheses of the theorem with $r < 1$, then $v_r(x,t) = r^{2/(p-1)}v(rx, r^2t)$ satisfies them with $r = 1$ (using the same values for ϵ and K), and clearly v_r blows up at $(0,0)$ if and only if v does. So we henceforth suppose that $r = 1$.

The first step is derive an integral equation for a localization of v. Let ϕ be a smooth function supported on B_1 such that $\phi \equiv 1$ on $B_{\frac{1}{2}}$ and $0 \le \phi \le 1$, and consider $w = \phi v$. Setting $f = v_t - \Delta v$, we have $w_t - \Delta w = \phi f + g$ with

$$-g = 2\nabla\phi \cdot \nabla v + v\Delta\phi = 2\nabla \cdot (v\nabla\phi) - v\Delta\phi.$$

The semigroup representation formula for w gives

$$w(t) = e^{(t+1)\Delta}w(-1) + \int_{-1}^{t} e^{(t-s)\Delta}\phi f(s)ds + \int_{-1}^{t} e^{(t-s)\Delta}g(s)ds \tag{2.5}$$

for $-1 < t < 0$, where $e^{t\Delta}$ is the semigroup associated with the heat equation in $\mathbf{R}^n$, i.e.

$$(e^{t\Delta}h)(x) = (4\pi t)^{-n/2}\int_{\mathbf{R}^n} e^{-|x-y|^2/4t}h(y)dy. \tag{2.6}$$

Since

$$\|e^{t\Delta}h\|_\infty \le \|h\|_\infty \tag{2.7a}$$

and

$$\|e^{t\Delta}\nabla h\|_\infty \le Ct^{-\frac{1}{2}}\|h\|_\infty, \tag{2.7b}$$

the last term of (2.5) is dominated by

$$C\int_{-1}^{t}[(t-s)^{-\frac{1}{2}}\|v\nabla\phi\|_\infty + \|v\Delta\phi\|_\infty]ds \le C'\int_{-1}^{t}(t-s)^{-\frac{1}{2}}\|v\|_\infty\, ds,$$

in which the constants depend only on n. Using the hypothesis (2.1), the middle term in (2.5) is dominated by

$$\begin{aligned}\int_{-1}^{t}\|\phi f(s)\|_\infty\, ds &\le K\int_{-1}^{t}\|\phi(1+|v|^p)\|_\infty\, ds\\ &\le K + K\int_{-1}^{t}\|w\|_\infty \cdot \|v\|_\infty^{p-1} ds.\end{aligned}$$

We thus obtain from (2.5) that

$$\begin{aligned}\|w\|_\infty(t) \le \|w\|_\infty(-1) &+ K\int_{-1}^{t}\|v\|_\infty^{p-1}\|w\|_\infty\, ds\\ &+ C\int_{-1}^{t}(t-s)^{-\frac{1}{2}}\|v\|_\infty\, ds + K\end{aligned} \tag{2.8}$$

for every t, $-1 \le t < 0$.

We now make use of (2.2); the smallness conditions on ϵ will emerge in the course of the argument. From (2.8) and (2.2) it follows that

$$\begin{aligned}\|w(t)\|_\infty &\le \epsilon + K\epsilon^{p-1}\int_{-1}^{t}(-s)^{-1}\|w\|_\infty\, ds\\ &+ C\epsilon\int_{-1}^{t}(t-s)^{-\frac{1}{2}}(-s)^{-1/(p-1)}ds + K.\end{aligned} \tag{2.9}$$

Since the third term will be controlled using Lemma 2.2, there are three slightly different cases to consider. The first is when $1/(p-1) < \frac{1}{2}$, *i.e.* $p > 3$. Then part (iii) of Lemma 2.2 applies (with $h = -t$, changing variables to $\sigma = -s$), and (2.9) yields

$$\|w(t)\|_\infty \le (C(n,p)\epsilon + K) + K\epsilon^{p-1}\int_{-1}^{t}(-s)^{-1}\|w\|_\infty\, ds.$$

Therefore, by Gronwall's inequality (Lemma 2.3),

$$\|w\|_\infty(t) \le (C(n,p)\epsilon + K)\cdot(-t)^{-K\epsilon^{p-1}}. \tag{2.10}$$

Choosing ϵ so small that

$$C(n,p)\epsilon \le K \quad \text{and} \quad K\epsilon^{p-1} \le \frac{1}{2(p-1)},$$

we conclude from (2.10) that

$$|v(x,t)| \le 2K\cdot(-t)^{-1/2(p-1)} \text{ for } |x| < \frac{1}{2}, \quad -1 \le t < 0. \tag{2.11}$$

One can see that (2.11) prevents blowup by using parabolic regularity theory (see [27], or Lemma 3.3 of [24]), but a more elementary alternative is to repeat the preceding argument using (2.11) in place of (2.2). If $\tilde{\phi}$ is a smooth function supported on $B_{\frac{1}{2}}$ with $\tilde{\phi} \equiv 1$ on $B_{\frac{1}{4}}$ and $0 \le \tilde{\phi} \le 1$, then $\tilde{w} = \tilde{\phi}v$ satisfies

$$\|\tilde{w}(t)\|_\infty \le CK + K\cdot(2K)^{p-1}\int_{-1}^{t}(-s)^{-\frac{1}{2}}\|\tilde{w}\|_\infty\,ds + CK\int_{-1}^{t}(t-s)^{-\frac{1}{2}}(-t)^{-1/2(p-1)}ds$$

instead of (2.9). By Lemmas 2.2 and 2.3, $\tilde{w}$ is uniformly bounded up to $t=0$, so v does not blow up at $(0,0)$.

The case $p=3$ is almost the same. Using part (ii) of Lemma 2.2, (2.9) yields

$$\|w(t)\|_\infty \le K + C(n,p)\epsilon(1+|ln(-t)|) + K\epsilon^{p-1}\int_{-1}^{t}(-s)^{-1}\|w\|_\infty\,ds,$$

whence by Lemma 2.3

$$\|w(t)\|_\infty \le (-t)^{-K\epsilon^{p-1}}(K + C\epsilon + C\epsilon/K\epsilon^{p-1}), \quad p=3. \tag{2.12}$$

If ϵ is chosen so that $K\epsilon^{p-1} \le \frac{1}{2(p-1)}$, then (2.12) yields

$$\|v(x,t)\|_\infty \le C(K,n,p)\cdot(-t)^{-1/2(p-1)} \text{ for } |x| \le \frac{1}{2}, \quad -1 \le t < 0.$$

This is the same as (2.11) except for the value of the constant, and it follows as before that v does not blow up at (0,0).

The last case, $p < 3$, is only slightly different. Using (2.9) and part (i) of Lemma 2.2 we obtain

$$\|w(t)\|_\infty \le K + C(n,p)\epsilon(-t)^{\frac{1}{2}-\frac{1}{p-1}} + K\epsilon^{p-1}\int_{-1}^{t}(-s)^{-1}\|w\|_\infty\, ds,$$

whence by Lemma 2.3

$$\begin{aligned}\|w(t)\|_\infty &\le (-t)^{-K\epsilon^{p-1}}\left\{K + C\epsilon\int_{-1}^{t}(-s)^{-\frac{1}{2}-\frac{1}{p-1}+K\epsilon^{p-1}}\, ds\right\}\\ &\le C(n,p,K)\cdot(-t)^{\frac{1}{2}-\frac{1}{p-1}},\end{aligned}$$

provided that ϵ is chosen small enough. It follows that

(2.13) $$|v(x,t)| \le C(n,p,K)(-t)^{\frac{1}{2}-\frac{1}{p-1}} \quad \text{for } |x| \le \frac{1}{2}, \quad -1 \le t < 0.$$

From (2.13) one can get a uniform bound on v either by invoking parabolic regularity or by iterating the argument finitely many times; the details of the latter procedure can safely be left to the reader.

REMARK 2.4: Essentially the same argument can be used to prove the global estimate (1.5) for the solution u of (1.1). It suffices to take $\phi \equiv 1$ and to replace $e^{t\Delta}$ by the semigroup associated to the heat equation with a Dirichlet boundary condition. The analogue of (2.7b) is not needed when $\nabla\phi \equiv 0$; that of (2.7a) is a consequence of the maximum principle.

We shall also need a version of Theorem 2.1 "at the boundary":

THEOREM 2.5. *There is a constant $\epsilon > 0$, depending only K, p and n, with the following property: suppose that v solves (2.1) in $\tilde{Q}_r = (B_r(a)\cap D)\times[t_1 - r^2, t_1)$ with $v|_{\partial D\cap B_r(a)} = 0$, where D is any $C^{2,\alpha}$ domain in $\mathbf{R}^n$, $a \in \partial D$, and $0 < r \le 1$. If moreover*

(2.14) $$|v(x,t)| \le \epsilon(t_1 - t)^{-1/(p-1)} \text{ for all } (x,t) \in \tilde{Q}_r,$$

then v does not blow up at (a, t_1).

PROOF: By translation, it suffices to consider $a = 0$ and $t_1 = 0$. For $r = 1$ and $\partial D \cap B_1$ sufficiently close to a hyperplane, the proof proceeds by the argument presented above—using, of course, the semigroup associated to the Laplacian with a Dirichlet boundary condition. The representation formula (2.6) is replaced by

$$(e^{t\Delta}h)(x) = \int G(x, y; t)h(y)dy,$$

where G is the associated Green's function. Our argument requires only the estimates (2.7a) and (2.7b); the former is a consequence of the maximum principle, while the latter follows from the symmetry of G and the fact that

$$|\partial_x G| \leq C_1 t^{-(n+1)/2} \exp(-C_2|x-y|^2/t),$$

see e.g. [27, Theorem 16.3, pg. 413]. The general case follows easily by scaling: indeed, for any $\rho < r$ the function $v_\rho(x,t) = \rho^{2/(p-1)}v(\rho x, \rho^2 t)$ is defined on $(B_1(0) \cap \rho^{-1}D) \times [-1, 0)$. It satisfies both (2.1) and (2.14), and $\rho^{-1}\partial D \cap B_1(0)$ converges to a hyperplane as $\rho \to 0$. Thus, taking ρ to be sufficiently small, we conclude that 0 is not a blowup point.

3. Small Energy Implies No Blowup.

We turn now to the specific equation (1.1), with u scalar–valued and $p > 1$. Our goal is to show that if the weighted energy in similarity variables is small, then the center of scaling is not a blowup point. This will be proved for star–shaped D and subcritical p. There are three main steps to the argument: the first uses energy relations to control certain integral norms of w; the second uses interpolation and parabolic regularization to show that w is uniformly small; and the third applies Theorem 2.1 to rule out the possibility of blowup.

We begin with some definitions and conventions, following [23, 24]. Throughout, D is a domain in $\mathbf{R}^n$, possibly unbounded, with $C^{2,\alpha}$ boundary. The function u is a classical solution of (1.1), with

$$u, \nabla u, \nabla^2 u, \text{ and } u_t \text{ bounded and continuous on } \bar{D} \times [0, \tau) \text{ for every } \tau < T,$$

and T is the blowup time:

$$\sup_{x \in D} |u(x,t)| \to \infty \text{ as } t \to T.$$

If $D \neq \mathbf{R}^n$ then (1.1) imposes a Dirichlet condition at ∂D; if D is unbounded then (3.1) includes the condition that u stays bounded as $|x| \to \infty$. A regularity hypothesis on the initial data is implicit in (3.1):

$$u(x,0) = u_0(x) \in C^2(\bar{D}); \tag{3.2}$$

this is just a matter of technical convenience, since more general initial data will be regularized instantaneously.

For any $a \in \bar{D}$, writing the solution in "similarity variables about (a,T)" means considering the function $w_a(y,s)$, defined by (1.21), which solves (1.22) on the space–time domain

$$W_a = \{(y,s) : s > s_0, \quad e^{-\frac{s}{2}} y + a \in D\}, \quad s_0 = -ln\ T. \tag{3.3}$$

The slice of W_a at time s will be denoted by $\Omega_a(s)$:

$$\Omega_a(s) = e^{\frac{s}{2}}(D - a). \tag{3.4}$$

We shall refer to $E[w_a]$, defined by (1.23), as its "energy", and we shall often suppress the subscript a, writing w for w_a, $\Omega(s)$ for $\Omega_a(s)$, etc.

The following *a priori* estimates for w are more or less the same as those in Proposition 2.2 of [24], except that their dependence on the initial energy is more explicit.

PROPOSITION 3.1. *Let $a \in \bar{D}$, and suppose that D is star–shaped with respect to a. Let $E_0 = E[w_a](s_0)$ denote the initial energy of w_a, and assume that $E_0 \leq 1$. Then $w = w_a$ satisfies*

$$\iint_W |w_s|^2 \rho\, dy\, ds \leq E_0, \tag{3.5}$$

and, for every $s \geq s_0$,

$$\int_{\Omega(s)} |w|^2 \rho\, dy \leq C(n,p) E_0^{\frac{1}{p}} \tag{3.6}$$

(3.7) $$\int_s^{s+1}\left(\int_{\Omega(\tau)}|w|^{p+1}\rho dy\right)^2 d\tau \le C(n,p)E_0^{(p+1)/p}$$

(3.8) $$\int_s^{s+1}\left(\int_{\Omega(\tau)}(|\nabla w|^2+|w|^2)\rho dy\right)^2 d\tau \le C(n,p)E_0^{(p+1)/p}.$$

PROOF: The first assertion is identical to (2.20) of [24]. For the second one we recall (2.24) of [24]: if $g(s) = (\int_{\Omega(s)}|w|^2\rho dy)^{\frac{1}{2}}$, then

$$g\dot{g} + 2E[w](s) \ge Cg^{p+1},$$

with $C = C(n,p) > 0$. Since $E[w]$ is decreasing (this is (2.23) of [24]), it follows that

(3.9) $$Cg^{p+1} \le g\dot{g} + 2E_0.$$

A technical estimate, which we present as Lemma 3.2 below, shows that (3.5) and (3.9) imply (3.6). For the third assertion we recall (2.28) of [24]:

$$\left(\int_{\Omega(s)}|w|^{p+1}\rho dy\right)^2 \le C(p)\left\{g^2(s)\int_{\Omega(s)}|w_s|^2\rho dy + E_0^2\right\}.$$

Integration in s yields (3.7), making use of (3.5) and (3.6), and noting that $E_0^2 \le E_0^{p+1/p}$ since $p > 1$ and $E_0 \le 1$. To get the last result we begin from the definition of $E[w]$, (1.23):

$$\frac{1}{2}\int_{\Omega(s)}\left(|\nabla w|^2 + \frac{1}{(p-1)}|w|^2\right)\rho dy = E[w](s) + \frac{1}{p+1}\int_{\Omega(s)}|w|^{p+1}\rho dy.$$

Since $E[w](s) \le E_0$, it follows that

$$\int_{\Omega(s)}(|\nabla w|^2+|w|^2)\rho dy \le C(p)\left[E_0 + \int_{\Omega(s)}|w|^{p+1}\rho dy\right].$$

Squaring both sides, integrating in s, and making use of (3.7), we easily obtain (3.8).

In proving (3.6), we made use of the following:

LEMMA 3.2. *Let $g(s)$ be a nonnegative H^1 function defined for $s \geq s_0$. Suppose that $\dot{g} = dg/ds$ satisfies*

$$\text{(3.10a)} \qquad c_1 g^{p+1} \leq g\dot{g} + c_2 A$$

$$\text{(3.10b)} \qquad \int_{s_0}^{\infty} \dot{g}^2 ds \leq A$$

for some positive constants c_1, c_2 and A with $A \leq 1$. Then there is a constant C, depending only on c_1, c_2, and p, such that

$$g(s) \leq CA^{1/2p} \text{ for all } s \geq s_0.$$

PROOF: From (3.10a) we see that for almost every s

$$\text{either } \; g(s) \leq A^{1/2p} \quad \text{or } \; c_1 g^p \leq \dot{g} + c_2 A^{1-1/2p}.$$

Since $A \leq 1$ and $p > 1$, it follows using (3.10b) that

$$\text{(3.11)} \qquad \int_s^{s+1} g^{2p}(\tau) d\tau \leq A + 2c_1^{-2} \int_s^{s+1} \left(\dot{g}^2 + c_2^2 A^{2-1/p}\right) d\tau \leq c_3 A,$$

with c_3 depending only on c_1 and c_2. Applying the interpolation inequality

$$\sup_{s \leq \tau \leq s+1} g(\tau) \leq C(p) \left[\int_s^{s+1} (\dot{g}^2 + g^2)\, d\tau\right]^{\frac{\theta}{2}} \left[\int_s^{s+1} g^{2p}\, d\tau\right]^{\frac{1-\theta}{2p}},$$

which holds for $\theta = 1/(p+1)$, we conclude from (3.10b), (3.11), and Hölder's inequality that

$$g(s) \leq C(A^{\frac{1}{2}} + A^{1/2p})^{\theta} A^{(1-\theta)/2p} \leq CA^{1/2p},$$

as desired.

If the initial energy of w is small, then Proposition 3.1 shows that w is small in certain integral norms. When p is subcritical, it follows that w is uniformly small:

PROPOSITION 3.3. *Suppose that w solves (1.22) on $B_R \times (0,1)$, with $p < (n+2)/(n-2)$ or $n \leq 2$. For any $\eta > 0$, there exists $\delta = \delta(R, n, p, \eta) > 0$ such that*

$$\text{(3.12)} \qquad \int_0^1 \int_{B_R} (|\nabla w|^2 + |w_s|^2)\, dy ds + \sup_{0<s<1} \int_{B_R} |w|^2 dy \leq \delta$$

implies

$$|w| \leq \eta \text{ uniformly on } B_{R/4} \times \left(\frac{3}{4}, 1\right).$$

PROOF: Proposition 5.2 of [24] shows that if $\delta \leq \delta_1(R,n,p)$ then w is bounded (though not necessarily small) on a subcylinder:

$$|w| \leq M \text{ on } B_{R/2} \times \left(\frac{1}{2}, 1\right), \tag{3.13}$$

with $M = M(R,n,p)$. Rewriting (1.22), the equation for w is

$$w_s - \Delta w + \frac{1}{2} y \cdot \nabla w + \frac{1}{p-1} w - |w|^{p-1} w = 0. \tag{3.14}$$

Differentiating with respect to y_i, we see that $\partial w / \partial y_i$ solves a parabolic equation with bounded coefficients in $B_{R/2} \times (\frac{1}{2}, 1)$. Therefore, by parabolic regularity (see Lemma 3.3 of [24] or Theorem 7.1, pg. 181 of [27]),

$$|\nabla w| \leq M' \text{ on } B_{R/4} \times \left(\frac{3}{4}, 1\right) \tag{3.15}$$

with $M' = M'(R,n,p)$. Now, the L^2 norm of w is small uniformly in time, by (3.12). This fact can be combined with (3.15) by using the interpolation inequality

$$||f||_{C^\alpha(B)} \leq C \left\{ \left(\int_B |\nabla f|^q \right)^{\theta/q} \left(\int_B f^2 \right)^{(1-\theta)/2} + \left(\int_B f^2 \right)^{\frac{1}{2}} \right\}, \tag{3.16}$$

which holds for $q > n$ and $0 < \alpha < 1 - (n/q)$ when $\theta \in (0,1)$ is chosen so that $-\alpha = (n-q)\theta/q + n(1-\theta)/2$, see e.g. [30]. Applying this to $f = w(\cdot, s)$ on $B = B_{R/4}$ we conclude that

$$|w| \leq C \left\{ (M')^\theta \delta^{(1-\theta)/2} + \delta^{\frac{1}{2}} \right\} \text{ in } B_{R/4} \times (\frac{3}{4}, 1).$$

Since this bound tends to zero with δ, the proof is complete.

Proposition 3.3 is sufficient for studying blowup at interior points, e.g. for Theorems 3.6 and 4.2 with $a \notin \partial D$. When $a \in \partial D$, however, the analysis requires versions at the boundary of the preceding result, and also of Proposition 5.2 in [24]. This is mainly of interest for showing that blowup does not occur at the boundary (Theorem 5.3). No fundamentally new idea is required, but there are some complications due to the presence of the boundary and the fact that the domain of w_a is not a cylinder. Readers who wish to focus on the interior results are advised to skip ahead to the proof of Theorem 3.5.

PROPOSITION 3.4. *Suppose that w solves (1.22) with $p < (n+2)/(n-2)$ or $n \leq 2$, in a spacetime domain of the form*

$$V = \left\{ (y,s) : \quad -1 < s < 1, \quad e^{-s/2} y \in E \right\}, \tag{3.17}$$

for some $C^{2,\alpha}$ domain $E \subset \mathbf{R}^n$ containing 0, and that

$$w = 0 \quad \text{on } \{(y,s) : \quad e^{-s/2} y \in \partial E\}. \tag{3.18}$$

Fixing R, $0 < R < 1$, assume that $\partial E \cap B_R$ is sufficiently close to a hyperplane in the C^2 norm. We identify w with its extension by 0 to all of $B_R \times (0,1)$. For any $K > 0$ there are δ, $M > 0$ and $0 < \theta < 1$ such that

$$\int_{-1}^{1} \int_{B_R} |w_s|^2 dy ds + \sup_{-1<s<1} \int_{B_R} |w|^2 dy < K \tag{3.19}$$

and

$$\int_{-1}^{1} \int_{B_R} |\nabla w|^2 dy ds < \delta \tag{3.20}$$

imply

$$|w(y,s)| \leq M \qquad \text{for } (y,s) \in B_{\theta R} \times (-\theta, \theta). \tag{3.21}$$

The value of θ depends only on R; the constants δ and M depend on K but not on the form of E. Finally, the value of M tends to zero as K tends to zero with R and δ held fixed, and this convergence is uniform with respect to E.

PROOF: Except for the last assertion, this is the boundary analogue of Proposition 5.2 in [24]. We may suppose that

$$\text{dist } (0, \partial E) \leq \frac{1}{4} e^{-\frac{1}{2}} R, \tag{3.22}$$

since otherwise the desired conclusion follows from the interior result.

The first step is to change coordinates in space–time so as to make V a cylinder: we set

$$x = y e^{-s/2}, \quad t = 1 - e^{-s}, \quad \bar{w}(x,t) = w(y,s).$$

This transformation takes V to the cylinder $E \times (t_0, t_1)$ with $t_0 = 1 - e$, $t_1 = 1 - e^{-1}$, and it maps $y = 0$, $s = 0$ to $x = 0$, $t = 0$. The image of $V \cap (B_R \times (-1, 1))$, where (3.19) and (3.20) apply, contains the cylinder $B_{R_1} \times (t_0, t_1)$ with $R_1 = Re^{-\frac{1}{2}}$. We may assume that $\delta \leq K$; then (3.19) and (3.20) yield

$$\int_{t_0}^{t_1} \int_{B_{R_1}} |\bar{w}_t|^2 dx dt + \sup_{t_0 < t < t_1} \int_{B_{R_1}} |\bar{w}|^2 dx \leq C_1 K \tag{3.23}$$

$$\int_{t_0}^{t_1} \int_{B_{R_1}} |\nabla \bar{w}|^2 dx dt \leq C_1 \delta. \tag{3.24}$$

The differential equation satisfied by $\bar{w}$ is obtained by changing variables in (1.22):

$$\bar{w}_t - \Delta \bar{w} = \frac{1}{1-t} \left(|\bar{w}|^{p-1} \bar{w} - \frac{1}{p-1} \bar{w} \right); \tag{3.25}$$

notice that $\bar{w} = 0$ at ∂E. We have assumed that ∂E is close to a hyperplane; it intersects $B_{\frac{1}{4}R_1}$ by (3.22), so it divides B_{R_1} into two roughly equal parts. We shall use the notation

$$D_r = B_r \cap E, \quad \frac{1}{4} R_1 < r < R_1.$$

The second step is to establish an analogue of Lemma 5.3 in [24]: if v solves

$$\begin{aligned} v_t - \Delta v &= 0 \quad \text{in } D_r \times (\alpha, \beta) \\ v &= g \quad \text{on } \partial D_r \times (\alpha, \beta) \\ v &= 0 \quad \text{at } D_r \times \{\alpha\} \end{aligned} \tag{3.26}$$

with g vanishing near $t = \alpha$, then we assert that

$$\sup_{\alpha \leq \tau \leq \beta} ||v||^2_{H^1(D_r)}(\tau) \leq C \int_\alpha^\beta \left(||g||^2_{H^1(D_r)} + ||g_t||^2_{L^2(\partial D_r)} \right) dt. \tag{3.27}$$

The constant in (3.27) is uniform in E, α, β, and r so long as $\beta - \alpha$ and r stay bounded away from 0 and ∞. The main difference between Lemma 5.3 of [24] and (3.27) is that ∂D_r has a corner, i.e. D_r is not a C^1 domain. (The condition that g vanish near $t = \alpha$ is also necessary for Lemma 5.3 of [24], although it is not explicitly mentioned there.) The key estimate required for proving (3.27) is

$$\int_\alpha^\beta \int_{\partial D_r} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma d\tau \leq C \left(\int_\alpha^\beta ||g||^2_{H^1(\partial D_r)} d\tau + \int_{\partial D_r} ||g||^2_{H^{\frac{1}{2}}(\alpha, \beta)} d\sigma \right), \tag{3.28}$$

where $\partial/\partial\nu$ denotes the normal derivative, $d\sigma$ represents surface area, and

$$||f||_{H^{\frac{1}{2}}(\alpha,\beta)} = \int_\alpha^\beta \int_\alpha^\beta \frac{|f(\tau_1) - f(\tau_2)|^2}{|\tau_1 - \tau_2|^2} d\tau_1 d\tau_2.$$

According to a recent result of Brown, [7, Theorem 6.1], (3.28) holds even for Lipschitz domains. Since the right hand side of (3.28) is dominated by

$$C' \int_\alpha^\beta \left(||g||^2_{H^1(\partial D_r)} + ||g_t||^2_{L^2(\partial D_r)} \right) d\tau$$

with C' depending only on $\beta - \alpha$, (3.27) follows from (3.28) by the same argument as was used in [24]. A standard scaling argument shows that the constant in (3.27) can be chosen to be uniform over r and $\beta - \alpha$ as long as they stay bounded away from 0 and ∞. To see that the constant is uniform with respect to E, we recall that ∂E is close to a hyperplane in the C^2 norm. Therefore the corners of ∂D_r are not too sharp, and we have control over the Lipschitz character of ∂D_r. It follows (see the passage after Remark 2.3 of [7]) that the constant in (3.28) is uniform over E, hence so is the one in (3.27).

The third step is to estimate the solution of the linear problem

$$\begin{aligned} U_t - \Delta U &= 0 \quad \text{in } D_r \times (0,\infty) \\ U &= 0 \quad \text{on } \partial D_r \times (0,\infty) \\ U &= g \quad \text{at } t = 0; \end{aligned} \tag{3.29}$$

as usual, we write $U(\cdot,t) = e^{t\Delta}g$. The estimates we desire are

$$||\nabla e^{t\Delta} g||_{L^2(D_r)} \le C t^{-\frac{1}{2}} ||g||_{L^2(D_r)}; \tag{3.30}$$

$$||\nabla e^{t\Delta} g||_{L^2(D_r)} \le ||\nabla g||_{L^2(D_r)} \quad \text{if } g \in H_0^1(D_r); \tag{3.31}$$

and

$$||e^{t\Delta} g||_{L^q(D_r)} \le C t^{-\theta} ||g||_{L^m(D_r)}, \tag{3.32}$$

with θ determined by

$$2\theta = n(\frac{1}{m} - \frac{1}{q}) \ge 0$$

and with a constant C that depends only on m, q, r, and the spatial dimension n. These results are well-known for domains with smooth boundaries; our claim is that they hold also for domains with corners such as D_r. Consider first (3.32). It suffices to prove the result when $g \geq 0$. By the maximum principle, $U \leq \tilde{U}$ where $\tilde{U}$ solves the Cauchy problem

$$\tilde{U}_t - \Delta\tilde{U} = 0 \quad \text{in } \mathbf{R}^n \times (0,\infty)$$

$$\tilde{U}(x,0) = \begin{cases} g(x) & \text{if } x \in D_r \\ 0 & \text{otherwise.} \end{cases}$$

The estimate

$$||\tilde{U}||_{L^q(\mathbf{R}^n)}(t) \leq Ct^{-\theta}||g||_{L^m(\mathbf{R}^n)}$$

is standard, and this implies (3.32). To prove (3.30) and (3.31) we shall use the spectral decomposition of the Laplacian. Recall that $A = -\Delta$ with the Dirichlet boundary condition is defined using a bilinear form

$$a(u,v) = \int_{D_r} (\nabla u, \nabla v)dx \quad \text{for } u,v \in V = H_0^1(D_r).$$

The space V is dense in $X = L^2(D_r)$, and the bilinear form a is symmetric, bounded, and positive on V. We therefore conclude that there is a positive, self-adjoint operator A on X such that

$$a(u,v) = (u, Av) \quad \text{for } v \in D(A), u \in V.$$

We have moreover

$$a(u,v) = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v) \quad \text{for } u,v \in V = D(A^{\frac{1}{2}}), \tag{3.33}$$

where $(\cdot,\cdot)$ is the standard inner product on $L^2(D_r)$; see for example [31, Theorem 2.2.3]. By the spectral decomposition

$$A^\alpha = \int_0^\infty \lambda^\alpha dE(\lambda), \quad \alpha > 0,$$

we observe that

$$e^{t\Delta}g = e^{-tA}g = \int_0^\infty e^{-\lambda t}dE(\lambda)g.$$

This formula and (3.33) yield

$$||\nabla e^{t\Delta}g||^2_{L^2(D_r)} = ||\int_0^\infty \lambda^{\frac{1}{2}} e^{-\lambda t} dE(\lambda) g||^2_{L^2(D_r)}$$
$$\leq t^{-1} \cdot \left(\max_{\tau \geq 0} \tau^{-1} e^{-2\tau}\right) \cdot ||g||^2_{L^2(D_r)},$$

which proves (3.30). Similarly, if $g \in H^1_0(D_r)$ then

$$\begin{aligned} ||\nabla e^{t\Delta}g||_{L^2(D_r)} &= ||A^{\frac{1}{2}} e^{-tA} g||_{L^2(D_r)} \\ &= ||e^{-tA} A^{\frac{1}{2}} g||_{L^2(D_r)} \\ &\leq ||A^{\frac{1}{2}} g||_{L^2(D_r)} = ||\nabla g||_{L^2(D_r)}, \end{aligned}$$

which proves (3.31).

The fourth step is to control the growth of

$$||\overline{w}||^2_{1,r}(t) = \int_{D_r} (|\overline{w}|^2 + |\nabla \overline{w}|^2) dx \tag{3.33}$$

for a suitable choice of r. From this point on the proof is parallel to that in pp. 29–30 of [24]. However, the presentation in [24] is flawed by an error in handling the boundary and initial data; so we present this step in detail to correct the mistake in [24]. We may assume that $\delta \leq K$ in (3.20). By (3.23) and (3.24) there exists r, $\frac{1}{2}R_1 < r < R_1$, such that

$$\int_{t_0}^{t_1} \left(||\overline{w}||^2_{H^1(\partial D_r)} + ||\overline{w}_t||^2_{L^2(\partial D_r)}\right) dt \leq C(R)K. \tag{3.34}$$

Let $t^* \in [\frac{1}{2}t_0, \frac{1}{2}t_1]$ and (to match the notation of [24]) suppose that $||\overline{w}||_{1,r}(t^*) \leq c_6 < \infty$. We claim that there exist $0 < \epsilon_1 < \frac{1}{2}t_1$ and c_6' (depending on n, p, R, K, and c_6) such that

$$||\overline{w}||_{1,r}(t) \leq c_6' \quad \text{for } t \in [t^*, t^* + \epsilon_1). \tag{3.35}$$

Indeed, let v solve

$$\begin{aligned} v_t - \Delta v = 0 &\quad \text{in } D_r \times (t_0, t_1) \\ v = \phi \overline{w} &\quad \text{on } \partial D_r \times (t_0, t_1) \\ v = 0 &\quad \text{at } t = t_0, \end{aligned}$$

where $\phi(t) \geq 0$ is a smooth function on (t_0, t_1) with $\phi \equiv 1$ on $(\frac{1}{2}t_0, t_1)$ and $\phi \equiv 0$ on $(t_0, \frac{3}{4}t_0)$. By (3.27) and (3.34) we have

$$\sup_{t_0 \leq \tau \leq t_1} ||v||_{1,r}^2(\tau) \leq c_7^2 \tag{3.36}$$

for a suitable choice of the constant c_7, depending on R and K. We may rewrite (3.25) in the integral form

$$\overline{w}(t) = v(t) + e^{(t-t^*)\Delta}\tilde{w}(t^*) + \int_{t^*}^{t} e^{(t-\tau)\Delta} f(\tau) d\tau$$

for $t \geq t^* \geq \frac{1}{2}t_0$, with

$$\tilde{w}(t^*) = \overline{w}(t^*) - v(t^*)$$
$$f(\tau) = \frac{1}{1-\tau}(|\overline{w}|^{p-1}\overline{w} - \frac{1}{p-1}\overline{w}),$$

using the notation $\overline{w}(\tau) = \overline{w}(\cdot, \tau)$. Using (3.30)–(3.32), (3.36), and the fact that $\tilde{w}(t^*) \in H_0^1(D_r)$, one verifies that

$$F(\tau) = \sup_{t^* \leq t \leq \tau} ||\overline{w}(t)||_{1,r}$$

satisfies

$$F(\tau) \leq c_7 + (c_6 + c_7) + c_8\left((\tau - t^*)^{1-\theta} F(\tau)^p + (\tau - t^*)F(\tau)\right) \tag{3.37}$$

with $0 < \theta < 1$. Setting $c_6' = 2(c_6 + 2c_7)$, we choose $\epsilon_1 < \frac{1}{2}t_1$ so that

$$c_8(\epsilon_1^{1-\theta}(c_6')^p + \epsilon_1 c_6') < c_6 + 2c_7.$$

Then (3.37) implies that $F(\tau) \leq c_6'$ whenever $t^* \leq \tau \leq t^* + \epsilon_1$, and this establishes (3.35).

The fifth step is to establish a uniform estimate

$$||\overline{w}||_{1,r}(t) \leq C, \quad , \frac{1}{2}t_0 \leq t \leq \frac{1}{2}t_1, \tag{3.38}$$

provided that δ is small enough in (3.24). We shall apply step four with $c_6 = 1$. By (3.35) it will suffice to show that for any $t^* \in (\frac{1}{2}t_0, \frac{1}{2}t_1)$,

$$\inf_{t^* - \epsilon_1 < t < t^*} ||\overline{w}||_{H^1(D_{R_1})}^2 \leq 1, \tag{3.39}$$

with ϵ_1 as in step four. But since $\overline{w} = 0$ on $\partial E \cap B_{R_1}$, Poincaré's inequality gives

$$||\overline{w}||^2_{H^1(D_{R_1})} \leq C_2 \int_{D_{R_1}} |\nabla \overline{w}|^2 dx; \tag{3.40}$$

the constant is independent of E, because it is uniform over all domains which are bi–Lipschitzian images of half–balls, see e.g. [26]. A combination of (3.24) and (3.40) gives

$$\int_{t^*-\epsilon_1}^{t^*} ||\overline{w}||_{H^1(D_{r_1})}(t)dt \leq C_1 C_2 \delta,$$

and when δ is small enough this yields (3.39).

The sixth step is the passage from the uniform H^1 estimate (3.38) to the desired L^∞ estimate (3.21). Here we use the hypothesis that $p < (n+2)/(n-2)$ or $n \leq 2$. For such p, (3.38) implies

$$|\overline{w}(x,t)| \leq C \quad \text{for } x \in D_{\frac{1}{4}R_1}, \quad \frac{1}{4}t_0 < t < \frac{1}{4}t_1 \tag{3.41}$$

by parabolic regularity, e.g. using Lemma 3.4 of [24] and Sobolev's inequality. Since the transformation from (x,t) to (y,s) is continuous, (3.41) yields (3.21) for a suitable choice of $\theta = \theta(R)$.

The seventh and last step is to verify the final assertion, that $M \to 0$ as $K \to 0$. The argument is parallel to that used for Proposition 3.3: parabolic regularity in (x,t) coordinates gives a uniform bound for $|\nabla \overline{w}|$, and interpolation via (3.16) leads to the desired result. (The regularity theorem used for this step requires ∂E to be $C^{2,\alpha}$; the earlier steps work even if it is only C^2.)

We are ready to prove the main result of this section—that if the initial energy is small then the center of scaling is not a blowup point. For any $a \in \bar{D}$, we shall say that D is ***strictly star–shaped about*** a if it is star–shaped with respect to every point of $\bar{D}$ in a neighborhood of a.

THEOREM 3.5. *For $p < (n+2)/(n-2)$ or $n \leq 2$, there is a constant $\sigma = \sigma(n,p)$ with the following property: if D is strictly star–shaped about $a \in \bar{D}$ and $E[w_a](s_0) < \sigma$, then a is not a blowup point of u. The value of σ depends only on n and p, not on D or a.*

PROOF: We consider first the case of an interior point: fix $a \in D$, and suppose that $E[w_a](s_0) < \sigma \le 1$. Since the weighted energy depends continuously on a (see Lemma 2.3 of [24]), there is a neighborhood N of a such that

$$E[w_b](s_0) < \sigma \text{ for all } b \in N, \tag{3.42a}$$

$$\text{dist } (N, \partial D) > 0, \tag{3.42b}$$

$$D \text{ is star–shaped about each point of } N. \tag{3.42c}$$

By (3.42b), there exists $s_1 = s_1(N) > 0$ such that the domain of $w_b(\cdot, s)$ contains the unit ball $B = B_1(0)$ for $s \ge s_1$, and for every $b \in N$. Applying Proposition 3.1, we have

$$\int_s^{s+1} \int_B \left(|w_{b,s}|^2 + |\nabla w_b|^2\right) dy d\tau + \sup_{s \le \tau \le s+1} \int_B |w_b|^2 dy \le C(n,p)\sigma^{\frac{1}{p}} \tag{3.43}$$

for every $s \ge s_1$; note that while s_1 depends on the particular solution under consideration, the constant $C(n,p)$ depends *only* on n and p, not on w or b. By Proposition 3.3, (3.43) with σ small enough—say $\sigma < \sigma_1(n,p,\eta)$—yields

$$|w_b(y,s)| \le \eta \text{ when } b \in N, |y| < \frac{1}{4}, s \ge s_1 + \frac{3}{4}, \tag{3.44}$$

for any $\eta > 0$. Taking $y = 0$ in (3.44) and rewriting the result as a statement about u, we have

$$|u(b,t)| \le \eta(T-t)^{-1/(p-1)} \text{ for } b \in N, t_1 < t < T, \tag{3.45}$$

with $t_1 = T - \exp(-s_1 - \frac{3}{4})$. Now Theorem 2.1 provides a choice of η—say, $\eta = \eta_1(n,p)$—for which (3.45) rules out blowup at any point of N. Thus the assertion of the theorem holds for $a \in D$ provided that $\sigma \le \sigma_1(n,p,\eta_1(n,p))$.

The proof for $a \in \partial D$ is similar: we choose a neighborhood N of a in D such that

$$E[w_b](s_0) < \sigma \text{ for all } b \in N, \tag{3.46a}$$

$$D \text{ is star–shaped about each point of } N. \tag{3.46b}$$

The hypothesis of Proposition 3.4 includes a condition on the curvature of the boundary; it will be satisfied by $E = e^{s/2}(D-b)$ for any $b \in N$, provided that s is sufficiently large—say, $s > s_1(N)$. Now we argue as above, using Proposition 3.4 in addition to Proposition 3.3, and Theorem 2.5 instead of 2.1, to obtain the desired conclusion.

When expressed in the original variables, the smallness of $E[w_a]$ is a condition involving the blowup time and the initial data:

COROLLARY 3.6. *Assume that $p < (n+2)/(n-2)$ or $n \le 2$, and suppose that D is strictly star-shaped about $a \in \bar{D}$. Let u solve (1.1) with initial data $u_0(x)$ and blowup time $T < \infty$. If*

$$\mathcal{E}_a[u_0] = T^{\frac{2}{p-1}-\frac{n}{2}} \int_D \left(\frac{T}{2}|\nabla u_0|^2 - \frac{T}{p+1}|u_0|^{p+1} + \frac{1}{2(p-1)}|u_0|^2 \right) e^{-|x-a|^2/4T}\, dx \tag{3.47}$$

satisfies $\mathcal{E}_a[u_0] < \sigma$ (with σ as in Theorem 3.5) then u does not blow up at a.

PROOF: One easily verifies that $\mathcal{E}_a[u_0] = E[w_a](s_0)$.

REMARK 3.7: If u satisfies the growth rate bound (1.17), then the assertion of Theorem 3.5 holds even for σ equal to

$$\sigma_* = \frac{1}{2}(p-1)^{-2/(p-1)}(p+1)^{-1}(4\pi)^{n/2},$$

the value of $E[w]$ when $w \equiv (p-1)^{-1/(p-1)}$. Indeed, we know that $E[w_a]$ decreases as $s \to \infty$, and that $w_a(y,s) \to w_\infty$ with $w_\infty = \pm(p-1)^{-1/(p-1)}$ or 0, uniformly on compact sets. If u satisfies (1.17) then w_a is uniformly bounded, and the dominated convergence theorem yields $E[w_a](s) \to E[w_\infty]$. In particular, if $E[w](s_0) < \sigma_*$ then w_∞ can only be 0, whence it follows by Theorem 4.2 that a is not a blowup point. Notice that σ_* is the largest possible value for which the assertion of Theorem 3.5 can hold, since the preceding argument shows that $E[w_a](s) \to \sigma_*$ if a *is* a blowup point.

4. Nondegeneracy of Blowup Limits.

The goal of this section is to show that if u blows up at $a \in \bar{D}$, then the blowup limit (1.6) cannot be zero. This is equivalent to showing that if $w_a(y,s) \to 0$ as $s \to \infty$ then a is not a blowup point. In view of Theorem 3.5, the main point is to prove that $E[w_a](s) \to 0$. This will be achieved by applying the following weighted Hardy–type inequality:

LEMMA 4.1. *There is a constant $C(n)$ such that for any H^1_{loc} function f on $\mathbf{R}^n$,*

$$\int_{|y|>1} \rho f^2 dy \le C(n) \left\{ \int_{|y|>1} \rho |y|^2 |\nabla f|^2 dy + \left(\int_{|y|=1} f\, d\sigma \right)^2 \right\}, \tag{4.1}$$

where $\rho(y) = \exp(-|y|^2/4)$ and $d\sigma$ represents surface area.

PROOF: First, we assert that if $f = f(r)$ is radial then

$$\int_1^\infty f^2\theta dr \le C(n)\left\{\int_1^\infty \dot f^2\theta dr + f^2(1)\right\} \tag{4.2}$$

with $\theta = \rho(r)r^{n-1}$ and $\dot f = df/dr$. This is a known result, see e.g. [7], but we prove it from scratch for the sake of completeness. Since $f(r) = \int_1^r \dot f(s)ds + f(1)$, the left side of (4.2) is bounded by

$$2\int_1^\infty \left(\int_1^r \dot f(s)ds\right)^2 \theta(r)dr + 2f^2(1)\int_1^\infty \theta(r)dr. \tag{4.3}$$

By Fubini's theorem and Hölder's inequality, the first term in (4.3) is bounded by

$$2\int_1^\infty \left(\int_1^r \dot f^2(s)ds\right) r\theta(r)dr \le 2\int_1^\infty \left(\int_s^\infty \theta(r)rdr\right) \dot f^2(s)ds. \tag{4.4}$$

We claim that for $s \ge 1$,

$$\int_s^\infty \theta(r)rdr = \int_s^\infty r^n\rho(r)dr \le C(n)s^{n-1}\rho(s). \tag{4.5}$$

This is easily established inductively when n is odd, since

$$\int_s^\infty r\rho(r)dr = -2\int_s^\infty \dot\rho(r)dr = 2\rho(s),$$

$$\int_s^\infty r^n\rho dr = -2\int_s^\infty r^{n-1}\dot\rho dr = 2s^{n-1}\rho(s) + 2(n-1)\int_s^\infty r^{n-2}\rho dr;$$

and it follows for even n, too, since

$$\int_s^\infty r^n\rho dr \le s^{-1}\int_s^\infty r^{n+1}\rho dr.$$

Substituting (4.4) and (4.5) into (4.3), we obtain (4.2).

To deduce the general case from the radial one, we write $f = (f - \bar f(r)) + \bar f(r)$, where $\bar f(r)$ is the spherical average of f:

$$\bar f(r) = \frac{1}{|\partial B(r)|}\int_{|y|=r} f d\sigma.$$

Poincaré's inequality on the unit sphere asserts that

$$\int_{|y|=1} |f-\bar f|^2 d\sigma \le C\int_{|y|=1} |\nabla_T f|^2 d\sigma,$$

where ∇_T denotes the tangential gradient; scaling this inequality to a sphere of radius r yields

$$\int_{|y|=r} |f-\bar{f}|^2 d\sigma \leq Cr^2 \int_{|y|=r} |\nabla_T f|^2 d\sigma.$$

Since $f^2 \leq 2(f-\bar{f})^2 + 2\bar{f}^2$ and $|\nabla_T f|^2 \leq |\nabla f|^2$, we have

$$\int_{|y|=r} f^2 d\sigma \leq 2Cr^2 \int_{|y|=r} |\nabla f|^2 d\sigma + 2\int_{|y|=r} \bar{f}^2 d\sigma;$$

multiplying by ρ and integrating over $(1,\infty)$, this becomes

$$\int_{|y|>1} \rho f^2 dy \leq 2C \int_{|y|>1} |y|^2 \rho |\nabla f|^2 dy + \int_{|y|>1} \rho \bar{f}^2 dy. \tag{4.6}$$

The last term is estimated by (4.2):

$$\int_{|y|>1} \rho \bar{f}^2 dy \leq C \int_{|y|>1} \rho |\partial f/\partial r|^2 dy + C\bar{f}^2(1), \tag{4.7}$$

making use of the fact that

$$|d\bar{f}/dr| \leq \frac{1}{|\partial B(r)|} \int_{|y|=r} |\partial f/\partial r| d\sigma.$$

Combining (4.6) and (4.7), we conclude that

$$\int_{|y|>1} \rho f^2 dy \leq C\left\{ \int_{|y|>1} |y|^2 \rho |\nabla w|^2 dy + \bar{f}^2(1) \right\},$$

which is the same as (4.1).

We are ready to prove the nondegeneracy of blowup limits.

THEOREM 4.2. *For $p < (n+2)/(n-2)$ or $n \leq 2$, assume that D is strictly star-shaped about $a \in \bar{D}$. If*

$$\limsup_{\substack{t\to T \\ |y|<C}} (T-t)^{1/(p-1)} |u(a + y\sqrt{T-t}, t)| = 0. \tag{4.8}$$

for each $C > 0$, then a is not a blowup point.

PROOF: The first (and main) task is to prove that $w = w_a$ satisfies $E[w](s) \to 0$ as $s \to \infty$. We shall make use of the estimate

$$\iint_W (|w_s|^2 + |\nabla w|^2)(1+|y|^2)\rho dy ds < \infty, \tag{4.9}$$

which was established as Proposition 4.3 of [24] for subcritical p. Since

$$2E[w](s) \leq \int_{\Omega(s)} (|\nabla w|^2 + \frac{1}{p-1}|w|^2)\rho dy$$

by (3.7), and $E[w](s) > 0$ and $\frac{d}{ds}E[w](s) \leq 0$ by Proposition 2.1 and (2.25) of [24], it will suffice to show that

$$\liminf_{s\to\infty} \int_{\Omega(s)} (|\nabla w|^2 + \frac{1}{p-1}|w|^2)\rho dy = 0. \tag{4.10}$$

Our hypothesis (4.8) is equivalent to the statement that

$$w(y,s) \to 0 \text{ uniformly for } |y| \leq C \tag{4.11}$$

for every $C > 0$. By parabolic regularity theory, it follows from (4.9) and (4.11) that

$$|\nabla w(y,s)| \to 0 \quad \text{as } s \to \infty, \quad \text{uniformly for } |y| \leq C. \tag{4.12}$$

(For $a \in D$, the proof of (4.12) is similar to that of Proposition 3.3; for $a \in \partial D$ it naturally uses boundary regularity, c.f. Proposition 3.4.) By the dominated convergence theorem, (4.11) and (4.12) yield

$$\lim_{s\to\infty} \int_{\Omega(s)\cap\{|y|\leq C\}} (|\nabla w|^2 + |w|^2)\, \rho dy = 0 \tag{4.13}$$

for any $C > 0$. Since $w = 0$ on $\partial\Omega(s)$, its extension by zero is in $H^1_{loc}(\mathbf{R}^n)$ for any given s, and Lemma 4.1 yields

$$\int_{|y|>1} \rho|w|^2 \leq C\left\{\int_{|y|>1} \rho|y|^2|\nabla w|^2 dy + \left(\int_{|y|=1} w d\sigma\right)^2\right\}.$$

The integral over $|y| = 1$ tends to zero as $s \to \infty$, by (4.11), and so

$$\liminf_{s\to\infty} \int_{|y|>1} (|w|^2 + |\nabla w|^2)\rho \leq C \liminf_{s\to\infty} \int_{|y|>1} \rho|y|^2|\nabla w|^2 dy = 0, \tag{4.14}$$

by (4.9). Taken together, (4.13) and (4.14) yield (4.10), completing the proof that $E[w](s) \to 0$ as $s \to \infty$.

Now we apply Theorem 3.5. Let σ be as in that result, and choose s_1 for which $E[w](s_1) < \sigma$. If $s_1 = -ln(T - t_1)$, then s_1 is the "initial time in similarity variables" of $u_1(x,t) = u(x, t - t_1)$, which blows up at time $T - t_1$. By Theorem 3.5, u_1 does not blow up at a, and so neither does u.

The possible values of the blowup limit (4.8) were classified in Theorem 5.1 of [24], at least if a is an interior point. (We shall address the case $a \in \partial D$ in Section 5.) If D is star–shaped about $a \in D$ and $p < (n+2)/(n-2)$ or $n \le 2$ then the limit (4.8) necessarily exists, is independent of y, and equals 0 or $\pm(p-1)^{-1/(p-1)}$. By combining this fact with Theorem 4.2 we obtain several apparently stronger results:

COROLLARY 4.3. *For $p < (n+2)/(n-2)$ or $n \le 2$, and assuming that D is strictly star–shaped about $a \in D$,*

(i) *a is a blowup point if and only if the blowup limit (4.8) equals $\pm(p-1)^{-1/(p-1)}$;*

(ii) *if a is a blowup point then*

$$\lim_{t \to T} (T-t)^{1/(p-1)} |u(a,t)| = (p-1)^{-1/(p-1)}.$$

(iii) *if, for some $y \in \mathbf{R}^n$,*

$$\liminf_{t \to T} (T-t)^{1/(p-1)} |u(a + y\sqrt{T-t}, t)| < (p-1)^{-1/(p-1)},$$

then a is not a blowup point.

5. Applications to the Form of the Blowup Set.

We continue to consider solutions u of (1.1). The *blowup set* is the closed set

$$\Sigma(u) = \{ a \in \bar{D} : u \text{ is not locally bounded near } (a,T) \}, \tag{5.1}$$

where T is the time of blowup. We shall apply the results of Sections 3 and 4 to establish some general properties of $\Sigma(u)$, and to give examples in which it has a specified form.

Our first application is to show that for the Cauchy problem (i.e. when $D = \mathbf{R}^n$) $\Sigma(u)$ is compact. This was proved in [18, 19] for the one dimensional case, with nonnegative initial data u_0 that decrease monotonically to 0 as $|x| \to \infty$ and satisfy $u_0(x) \le C|x|^{-2/(p-1)}$. Our method is totally different; it is closer to that of [9], where a similar assertion is proved for solutions of the Navier–Stokes equations.

THEOREM 5.1. *Let u solve the Cauchy problem*

$$u_t - \Delta u - |u|^{p-1}u = 0 \quad in\ \mathbf{R}^n \times (0,T)$$

$$u(x,0) = u_0(x) \tag{5.2}$$

with $u_0 \in C^2(\mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n} (|\nabla u_0|^2 + |u_0|^2)dx < \infty, \tag{5.3}$$

and with $p < (n+2)/(n-2)$ or $n \leq 2$. Then the blowup set $\Sigma(u)$ is compact.

PROOF: Since $\Sigma(u)$ is closed as a matter of definition, we need only show that it is bounded. The equation is invariant under the scaling

$$u_\lambda(x,t) = \lambda^{2/(p-1)}u(\lambda x, \lambda^2 t),$$

so we may assume without loss of generality that $T = 1$. By Corollary 3.6, there is a $\sigma > 0$ such that

$$\mathcal{E}_a[u_0] = \int_{\mathbf{R}^n} \left(\frac{1}{2}|\nabla u_0|^2 + \frac{1}{2(p-1)}|u_0|^2 - \frac{1}{p+1}|u_0|^{p+1}\right) e^{-|x-a|^2/4}dx \geq \sigma$$

wherever $a \in \Sigma(u)$. But it is easy to see from (5.3) that $\mathcal{E}_a[u_0] \to 0$ as $|a| \to \infty$, and in particular that for some $R > 0$

$$\mathcal{E}_a[u_0] < \sigma \quad \text{whenever } |a| > R.$$

It follows that $\Sigma(u)$ is contained in the ball of radius R.

REMARK 5.2: We require $u_0 \in C^2(\mathbf{R}^n)$ in Theorem 5.1 only to be consistent with (3.1). The same result actually holds for general H^1 initial data, since u is regularized instantaneously: it suffices to apply Theorem 5.1 to $u_1(x,t) = u(x, t+t_1)$, whose initial data $u_1(x,0) = u(x,t_1)$ are C^2 if $t_1 > 0$.

Our second application is to rule out blowup at the boundary. This was done in [15] for any $p > 1$, if D is bounded and convex and $u \geq 0$. Besides allowing u to change sign, our result is more local: it requires only that D be strictly star–shaped about the boundary point in question. However, we must assume that p is subcritical.

THEOREM 5.3. *Let u solve (1.1) in a (possibly unbounded) domain D with $C^{2,\alpha}$ boundary, and let D be strictly star–shaped about $a \in \partial D$. Then a is not a blowup point, provided that $p < (n+2)/(n-2)$ or $n \leq 2$.*

PROOF: As usual, we work with $w = w_a$; note that its domain $\Omega(s)$ converges to a halfspace as $s \to \infty$ since $a \in \partial D$, and $w \doteq 0$ on $\partial\Omega(s)$ since $u = 0$ on ∂D. Arguing as for Theorem 5.1 of [24] (but using Proposition 3.4 of this article in addition to Proposition 5.2 of [24]), we see that $|\nabla w| \to 0$ uniformly on bounded subsets of $\bar{\Omega}(s)$ as $s \to \infty$. It follows, making use of the boundary condition, that $w(y, s) \to 0$ as $s \to \infty$, uniformly for $|y| > C$. This is equivalent to (4.8), so Theorem 4.2 assures that a is not a blowup point.

REMARK 5.4: If D is star–shaped about every point of $\bar{D}$ (for example if D is convex), then Theorem 5.3 shows that $\Sigma(u)$ omits a neighborhood of ∂D. If D is unbounded but the initial data u_0 satisfy (5.3), then one may argue as for Theorem 5.1 to see that $\Sigma(u)$ is bounded. Thus, for such domains (e.g. a halfspace) and for H^1 initial data, $\Sigma(u)$ is compact in D. Here as usual p must be subcritical: $p < (n+2)/(n-2)$ or $n \leq 2$.

Our remaining applications have to do with the form of the blowup set. Recently much attention has been focussed on bounding the number of blowup points. For example, in the case of a positive solution on an interval, it turns out that the number of blowup points is no greater than the number of local maxima of the initial data [12]; for one or two local maxima this was proved earlier in [8, 13, 15, 16]. In higher dimensions, certain radial solutions are know to blow up just at the origin [15, 29, 33], and single–point blowup has also been established for some non–radial cases [11]. Our focus here is rather different: we consider not the number of blowup points, but instead their location. Specifically, for initial data consisting of "high mountains" or "deep valleys" (where $|u_0|$ is large) separated by "broad plains" (where $|u_0|$ is small), we shall show that blowup does not occur in the middle of the plains. The proof naturally makes use of Corollary 3.6, which eliminates the possibility of blowup at a if the weighted energy $\mathcal{E}_a[u_0]$ is sufficiently small. Since the blowup time $T = T[u_0]$ enters in the definition of $\mathcal{E}_a$, it is important to control T in terms of u_0. For simplicity we consider only the case when D is a ball.

LEMMA 5.5. *Let u solve (1.1) with domain $D = B_R$, a ball of radius R. Assume that the initial data $u_0(x) = u(x,0)$ satisfy*

$$I[u_0] \equiv \frac{1}{2}\int_{B_R} |\nabla u_0|^2 dx - \frac{1}{p+1}\int_{B_R} |u_0|^{p+1} dx < 0. \tag{5.4}$$

Then u blows up in finite time, and its blowup time T is estimated above by

$$T \le C(n,p)\cdot R^{n\gamma}|I[u_0]|^{-\gamma}, \quad \gamma = \frac{p-1}{p+1}, \tag{5.5}$$

where $C(n,p)$ is a constant depending only on n and p.

PROOF: The fact that $I[u_0] < \infty$ implies finite time blowup was shown in [2], and our proof of (5.5) follows the ideas of that paper. Consider first the case $R = 1$. Multiplying the equation by u and integrating yields

$$\frac{1}{2}\frac{dy}{dt} = -2I[u] + \frac{p-1}{p+1}\int_{B_1} |u|^{p+1} dx,$$

with

$$y(t) = \int_{B_1} |u|^2 dx.$$

Since $I[u]$ is monotonically decreasing, an application of Hölder's inequality gives

$$\frac{dy}{dt} \ge -4I[u_0] + cy^{(p+1)/2}.$$

This differential inequality forces finite time blowup if $I[u_0] = I_0 < 0$, and the blowup time is estimated above by

$$T \le \int_0^\infty \frac{dy}{4|I_0| + cy^{(p+1)/2}} = C(n,p)\,|I_0|^{-\gamma} \tag{5.6}$$

with $\gamma = (p-1)/(p+1)$. This proves (5.1) when $R = 1$.

The general case $R > 0$ follows easily by scaling: if u solves (1.1) on B_R with initial "energy" $I[u_0] = I_0$ and blows up at time T, then $\tilde{u}(x,t) = R^{2/(p-1)}u(Rx, R^2t)$ solves the same equation on B_1 with initial "energy" $\tilde{I}_0 = R^{-n+(2/\gamma)}I_0$, and it blows up at time $\tilde{T} = R^{-2}T$. By (5.6), $\tilde{T} \le C(n,p)|\tilde{I}_0|^{-\gamma}$, and rewriting this as a relation between T and I_0 yields (5.5).

Our first examples are radial ones, with initial data vanishing near 0 but large (in absolute value) near $|x| = R$: fixing a C^2 function $\psi(r)$ supported on $(0,1)$ such that

$$\frac{1}{2}\int_0^1 |\psi_r|^2 dr - \frac{1}{p+1}\int_0^1 |\psi|^{p+1} dr < 0, \tag{5.7}$$

we consider initial data

$$u_0^R(x) = \psi(|x| - R + 1) \tag{5.8}$$

on B_R. Notice that it is easy to find functions ψ satisfying (5.7); indeed, for any C^2 $\phi \not\equiv 0$ supported on $(0,1)$, $\psi = \lambda\phi$ will do for λ sufficiently large.

PROPOSITION 5.6. *Let u solve (1.1) on $B_R \subset \mathbf{R}^n$ with initial data u_0^R, and assume that $1 < p < (n+2)/(n-2)$ or $n \leq 2$. If in addition R is sufficiently large, then 0 is not a blowup point, and the blowup set $\Sigma(u)$ contains an $n-1$ dimensional sphere.*

PROOF: By Corollary 3.6, we need only show that

$$\mathcal{E}[u_0^R] = T^{\frac{2}{p-1}-\frac{n}{2}} \int_{B_R} \left(\frac{T}{2}|\nabla u_0^R|^2 - \frac{T}{p+1}|u_0^R|^{p+1} + \frac{1}{2(p+1)}|u_0^R|^2\right) e^{-|x|^2/4T} dx \tag{5.9}$$

satisfies

$$\limsup_{R\to\infty} \mathcal{E}[u_0^R] \leq 0; \tag{5.10}$$

here $T = T(R)$ is the blowup time of the solution with initial data u_0^R. Substitution of (5.8) into (5.4) gives

$$I[u_0^R] = \omega \int_0^1 \left(\frac{1}{2}|\psi_\sigma|^2 - \frac{1}{p+1}|\psi(\sigma)|^{p+1}\right)(R+\sigma-1)^{n-1} d\sigma,$$

where ω is the area of the $n-1$ dimensional unit sphere. Therefore $I(u_0^R)$ is a polynomial of degree $n-1$ in R, and the coefficient of R^{n-1} is negative, by (5.7). In particular $I(u_0^R) < 0$ for sufficiently large R, so that u does indeed blow up, and (5.5) yields

$$T(R) \leq c_1 R^{(p-1)/(p+1)} \quad \text{for } R \geq R_0, \tag{5.11}$$

with c_1 a constant depending on ψ, p, and n, but not on R. We need a lower bound to go along with (5.11). This can be obtained from the estimate

$$T(R) \geq C\|u_0^R\|_{L^r(B_R)}^{-1/\theta}, \quad \theta = \frac{1}{2}\left(\frac{2}{p-1} - \frac{n}{r}\right) > 0, \quad r > 1, \tag{5.12}$$

which is a consequence of the local–in–time existence theory, see [2, 21, 33]. Notice that (5.12) is scale–invariant, so the constant does not depend on R; choosing, for example, $r = pn$, it yields

$$T(R) \geq c_2 R^{\mu_1} \tag{5.13}$$

with $c_2 = c_2(n, p, \psi) > 0$ and $\mu_1 = \mu_1(n, p) \leq 0$.

We turn to the task of estimating $\mathcal{E}[u_0^R]$. Since $|u_0^R|$ and $|\nabla u_0^R|$ are uniformly bounded, (5.9) yields

$$\mathcal{E}[u_0^R] \leq c_3 T^{\frac{2}{p-1} - \frac{n}{2}}(T+1) \int_{R-1<|x|<R} e^{-|x|^2/4T}\, dx,$$

with c_3 independent of R, and $T = T(R)$. By (5.11),

$$|x|^2/4T \geq c_4 R^{1+2/(p+1)} \tag{5.15}$$

for $R - 1 < |x| < R$ and R sufficiently large, with $c_4 > 0$. A combination of (5.11), (5.13), (5.14), and (5.15) yields

$$\mathcal{E}[u_0^R] \leq c_5 R^{\mu_2} \exp(-c_4 R^{\nu})$$

with $\nu = 1 + \frac{2}{p+1} > 0$, for a suitable value of μ_2. This clearly implies (5.10). Thus 0 is not a blowup point if R is sufficiently large. It follows in particular that $\Sigma(u)$ contains an $n-1$ dimensional sphere, since it is nonempty and radially symmetric.

This construction can be combined with the results in [8, 10, 12, 13] to give examples for which the form of the blowup set is known exactly:

Corollary 5.7. *On any one–dimensional interval there are positive initial data for which $\Sigma(u)$ consists of two distinct points; similarly, on any ball in $\mathbf{R}^n$, if $p < (n+2)/(n-2)$ or $n \leq 2$, then there are positive initial data for which $\Sigma(u)$ is exactly an $n-1$ dimensional sphere.*

PROOF: If ψ is nonnegative and has only one local maximum, then according to [8, 12, 13] the solution on an interval with initial data u_0^R blows up at most at two points. Therefore $\Sigma(u)$ consists of exactly two points whenever 0 is not a blowup point. This gives the desired example on $(-R, +R)$ when R is large, and scaling leads easily to an example on any interval. The second assertion is proved similarly using a theorem from [10], according to which $\Sigma(u)$ must be either $\{0\}$ or an $n-1$ dimensional sphere. (It is assumed in [10] that the initial data are not identically constant in any subinterval of $(0, R)$, so we should use not u_0^R but a perturbation of it. The proof of Proposition 5.6 still rules out blowup at 0 if the perturbation is sufficiently small.)

The method used for Proposition 5.6 is capable of treating many other cases; in particular, radial symmetry is not at all required. We restrict ourselves here to one more class of applications, this time without symmetry. Fixing a pair of C^2 functions ψ_1, ψ_2 supported on $B_1 \subset \mathbf{R}^n$ such that

$$\frac{1}{2}\int_{B_1} |\nabla\psi_j|^2 dx - \frac{1}{p+1}\int_{B_1} |\psi_j|^{p+1} dx < 0, \quad j = 1, 2,$$

we consider initial data

$$v_0^R(x) = \psi_1(x_1 + R - 1, x_2, \ldots, x_n) + \psi_2(-x_1 - R + 1, x_2, \ldots, x_n).$$

PROPOSITION 5.8. *Let v solve (1.1) on $B_R \subset \mathbf{R}^n$ with initial data v_0^R, and with $p < (n+2)/(n-2)$ or $n \leq 2$. If in addition R is sufficiently large then v does not blow up on the hyperplane $x_1 = 0$.*

PROOF: If $R > 2$ then $I[v_0^R]$ is independent of R, so (5.5) gives

$$T(R) \leq c_1 R^{n(p-1)/(p+1)}$$

for the blowup time of v. A lower bound

$$T(R) \geq c_2$$

follows similarly from (5.12) with $r = np$. From (3.35) we have

$$\mathcal{E}_a[v_0^R] \leq c_3 T^{\frac{2}{p-1}-\frac{n}{2}}(T+1) \int_{\mathrm{supp}\, v_0^R} e^{-|x-a|^2/4T}\, dx.$$

Now, if a lies in the hyperplane $x_1 = 0$ and $x \in$ supp v_0^R, then for large R

$$|x-a|^2/4T \leq c_4 R^\nu$$

with $c_4 > 0$ and $\nu = 2 - n(p-1)/(p+1) > 0$. So

$$\mathcal{E}_a[v_0^R] \leq c_5 R^\mu \exp(-c_4 R^\nu)$$

for some $\mu \in \mathbf{R}$. In particular $\mathcal{E}_a[v_0^R] \to 0$ as $R \to \infty$. Therefore, by Corollary 3.6, the blowup set omits the hyperplane $x_1 = 0$ when R is sufficiently large.

We note that if $\psi_1(x_1, x_2, \dots, x_n) = \psi(-x_1, x_2, \dots, x_n)$ then the solution v is symmetric under reflection, so $\Sigma(v)$ must contain at least two points.

6. Generalizations.

The principal results of the preceding sections extend to equations with more general nonlinear terms, provided that $|u|^{p-1}u$ is the "principal" nonlinearity. Moreover, most of our results extend to an appropriate class of systems, namely those with gradient structure and a similar scale invariance. We indicate here the necessary changes in the arguments.

6A. More general nonlinearities.

Consider, instead of (1.1), the equation

$$\begin{aligned} u_t - \Delta u - |u|^{p-1}u &= h(u) \quad \text{in } D \times [0,T) \\ u &= 0 \quad \text{on } \partial D \times [0,T) \end{aligned} \tag{6.1}$$

where $h : \mathbf{R} \to \mathbf{R}$ is C^1 and satisfies

$$|h(\xi)| \leq b(1 + |\xi|^q), \quad b > 0, \quad 1 < q < p. \tag{6.2}$$

As we observed in [24], the analogue for (6.1) of $E[w]$ is

$$J[w](s) = E[w](s) - e^{-\beta(p+1)s} \int_{\Omega(s)} H(e^{\beta s} w(y,s))\rho dy,$$

where $\beta = \frac{1}{p-1}$, E is defined by (1.23), and $H(\xi) = \int_0^\xi h$. In particular, if D is star-shaped about $a \in \bar{D}$ then $J[w](s)$ remains bounded above and below as $s \to \infty$ (this is (6.16) of [24]), and it satisfies

$$\text{(6.3)} \qquad \frac{1}{2}\frac{d}{ds}\int_{\Omega(s)} |w|^2 \rho dy \geq -2J[w](s) + \frac{1}{2}\frac{p-1}{p+1}\int_{\Omega(s)} |w|^{p+1}\rho dy - \mu e^{-\delta s},$$

$$\text{(6.4)} \qquad \frac{d}{ds}J[w](s) \leq -\frac{1}{2}\int_{\Omega(s)} |w_s|^2 \rho dy + \mu e^{-\delta s},$$

where $\delta = (p-q)/(p-1)$, and μ is a positive constant depending on p, $s_0 = -lnT$, and on the constants b and q in (6.2) (see (6.12) and (6.15) of [24]). Armed with these facts, one can argue in Section 3 of [22] to obtain this substitute for Proposition 3.1:

PROPOSITION 3.1′. *Let u solve (6.1) with $p > 1$, and suppose that D is star-shaped about $a \in \bar{D}$. Then $\lim_{s\to\infty} J[w](s)$ exists and is nonnegative. Moreover, for every $\epsilon > 0$ there exists $s^* = s^*(\epsilon, p, q, b, n, s_0) > 0$ with the following property: if, for some $s_1 \geq s_*$, $J[w_a](s_1) \leq \epsilon$, then*

$$\text{(6.5)} \qquad \int_{s_1}^{\infty}\int_{\Omega(s)} |w_s|^2 \rho dy dt \leq 3\epsilon$$

$$\text{(6.6)} \qquad \int_{\Omega(s)} |w|^2 \rho dy \leq C\epsilon^{1/p}, \quad s \geq s_1$$

$$\text{(6.7)} \qquad \int_s^{s+1}\left(\int_{\Omega(\tau)} |w|^{p+1}\rho dy\right)^2 \leq C\epsilon^{(p+1)/p}, \quad s \geq s_1$$

$$\text{(6.8)} \qquad \int_s^{s+1}\left(\int_{\Omega(\tau)} [|\nabla w|^2 + |w|^2]\rho dy\right)^2 dt \leq C\epsilon^{(p+1)/p}, \quad s \geq s_1.$$

The constant C depends only on p and n.

Since u solves (6.1) instead of (1.1), w solves

$$w_s - \nabla w + \frac{1}{2} y \cdot \nabla w + \beta w - |w|^{p-1} w - e^{-\beta p s} h(e^{\beta s} w) = 0, \quad \beta = \frac{1}{p-1}, \tag{6.9}$$

instead of (1.22). The assertions of Propositions 3.3 and 3.4 remain valid for (6.9), with essentially the same proofs; however, the constant δ may now depend on h, since the proofs involve differentiating the equation. Theorem 3.5 must be altered so that it refers only to large s:

THEOREM 3.5′. *For solutions of (6.1) with $p < (n+2)/(n-2)$ or $n \leq 2$, there are constants σ and $s_{\#}$ with the following property: if D is strictly star-shaped about $a \in \bar{D}$, and if $J[w_a](s) < \sigma$ for some $s \geq s_{\#}$, then a is not a blowup point. The constants σ and $s_{\#}$ are independent of a and D, and they depend on the solution u only through $s_0 = -lnT$.*

The proof is fundamentally as before.

The analysis of Section 4 extends directly: the key point is that (4.9) remains true for solutions of (6.1) (this is Proposition 6.2 of [24]). In particular, Theorem 4.2 and Corollary 4.3 remain valid for solutions of (6.1).

The analogues of Theorem 5.1 and 5.3 hold, too. The proof of the latter is essentially the same, while that of the former is based on the observation that $J[w_a](s_{\#}) \to 0$ as $|a| \to \infty$, with $s_{\#}$ as in Theorem 3.5′.

6B. Systems.

Little is changed if (1.1) is replaced by the system

$$\begin{aligned} u_t^j - \Delta u^j - g^j(u) &= 0 \quad \text{in } D \times [0,T) \\ u &= 0 \quad \text{on } \partial D \times [0,T), \end{aligned} \tag{6.10}$$

provided that $g : \mathbf{R}^m \to \mathbf{R}^m$ has the form

$$g^j(\xi) = \frac{1}{p+1} \frac{\partial G}{\partial \xi j} \tag{6.11}$$

for some C^2 function G satisfying

$$G(\lambda \xi) = \lambda^{p+1} G(\xi), \quad \lambda > 0 \tag{6.12}$$

(6.13) $$G(\xi) \geq c|\xi|^{p+1}, \quad c > 0.$$

Clearly u satisfies (2.1), so the local lower bound on the blowup rate established in Section 2 applies to (6.10). The analysis of Section 3 was based on certain "energy relations" for the functional $E[w]$, defined by (1.23). the corresponding functional for the study of (6.10) is obtained by replacing $|u|^{p+1}$ with $G(u)$. It satisfies the *same* energy relations as in the scalar case, and so all the results of Section 3 extend directly to solutions of (6.10). The only change is that certain constants which used to depend only on n and p (e.g. σ in Theorem 3.5) now depend on G as well. The results in Sections 4 and 5 extend directly, except for Corollary 4.4, which is based on the classification of possible blowup limits. Its analogue is

COROLLARY 4.4″. *Let u solve (6.10) with $p < (n+2)/(n-2)$ or $n \leq 2$, and assume that the domain D is strictly star–shaped with respect to $a \in \bar{D}$. Let k_1 (respectively, k_2) be the smallest (largest) modulus of a nonzero solution of $\xi^j = (p-1)g^j(\xi)$ in $\mathbf{R}^m$. Then*

(i) *a is blowup point if and only if*

(6.14) $$\limsup_{t \to T} (T-t)^{1/(p-1)} |u(a + y\sqrt{T-t}, t)| \geq k_1$$

for some $y \in \mathbf{R}^n$; moreover,

(ii) *if a is a blowup point then*

(6.15) $$\begin{aligned} k_1 &\leq \varliminf_{t \to T} (T-t)^{\frac{1}{(p-1)}} \left| u\left(a + y\sqrt{T-t}, t\right) \right| \\ &\leq \varlimsup_{t \to T} (T-t)^{\frac{1}{(p-1)}} \left| u\left(a + y\sqrt{T-t}, t\right) \right| \leq k_2. \end{aligned}$$

PROOF: We begin with (i). Since $k_1 > 0$, if (6.14) holds then a is clearly a blowup point. For the converse, suppose that (6.14) fails; in other words, using similarity variables based at (a, T), suppose that

(6.16) $$\limsup_{s \to \infty} |w(y, s)| < k_1$$

for each y. Let $\{s_l\}$ be any sequence tending to infinity, and consider $w_l(y,s) = w(y, s+s_l)$. Arguing as for Theorem 5.1 of [24], we see that a subsequence of $\{w_l\}$ converges uniformly on compact sets to a constant solution of $\xi = (p-1)g(\xi)$. By (6.16) and the definition of k_1, the only possible limit is zero. It follows that $w(y,s) \to 0$ as $s \to \infty$, uniformly on compact sets $|y| \leq C$; hence by (the extension of) Theorem 4.4, a is not a blowup point.

The proof of (ii) is similar: with $\{w_l\}$ as above, we must show that if *any* subsequence has limit zero then *every* subsequence tends to zero. Suppose to the contrary that $w_l(y,s) = w(y, s+s_l) \to 0$ but $\tilde{w} = w(y, s+\tilde{s}_l) \to \xi \neq 0$. By the definition of k_1, $|\xi| \geq k_1 > 0$. Evaluating w_l and $\tilde{w}_l$ at $(0,0)$, we see that $|w(0,s_l)| \to 0$ and $|w(0,\tilde{s}_l)| \to |\xi|$. Since w is continuous, it follows that there is a sequence $\tau_l \to \infty$ with $|w(0,\tau_l)| \to \frac{1}{2}k_1$. But by the last paragraph, a subsequence of $\{w(0,\tau_l)\}$ converges to a constant solution of $\xi = (p-1)g(\xi)$. This is a contradiction, so (6.15) is proved.

We note that for $g^j(u) = |u|^{p-1}u^j$, (6.15) says that

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} |u|(a + y\sqrt{T-t}, t) = (p-1)^{-1/(p-1)},$$

just as in the scalar case.

Bibliography

[1] Ball, J., *Remarks on blowup and nonexistence theorems for nonlinear evolution equations*, Quart. J. Math. Oxford 28, 1977, pp. 473–486.

[2] Baras, P., *Nonunicité des solutions d'une equation d'evolution nonlineaire*, Ann. Fac. Sci. Toulouse 5, 1983, pp. 287–302.

[3] Bebernes, J., Bressan, A., and Eberly, D., *A description of blowup for the solid fuel ignition model*, Indiana Univ. Math. J. 36, 1987, pp. 295–305.

[4] Bebernes, J., and Eberly, D., *A description of self-similar blowup for dimension* $n \geq 3$, Ann. Inst. H. Poincaré Analyse Nonlinéaire 5, 1988, pp. 1–22.

[5] Berger, M., and Kohn, R., *A rescaling algorithm for the numerical calculation of blowing up solutions*, Comm. Pure Appl. Math., to appear.

[6] Bradley, J.S., *Hardy inequalities with mixed norms*, Canad. Math. Bull. 21, 1978, pp. 405–408.

[7] Brown, R., *The method of layer potentials for the heat equation in Lipschitz cylinders*, Amer. J. Math., to appear.

[8] Caffarelli, L., and Friedman, A., *Blowup of solutions of nonlinear heat equations*, J. Math. Anal. Appl. 129, 1988, pp. 409–419.

[9] Caffarelli, L., Kohn, R., and Nirenberg, L., *Partial regularity of suitable weak solutions of the Navier–Stokes equations*, Comm. Pure Appl. Math. 35, 1982, pp. 771–831.

[10] Chen, Y.–G., *On blowup solutions of semilinear parabolic equations; analytical and numerical studies*, Thesis, Tokyo University, Dec. 1987.

[11] Chen, Y.–G., and Suzuki, T., *Single-point blowup for semilinear equations in some non-radial domains*, Proc. Japan Acad. 64, Ser. A., 1988, pp. 57–60.

[12] Chen, X.–Y., and Matano, H., *Convergence, asymptotic periodicity, and finite-point blowup in one-dimensional semilinear heat equations*, J. Differential Equations, to appear.

[13] Friedman, A., *Blowup of solutions of nonlinear parabolic equations*, in proc. of micro-program on Nonlinear Diffusion Equations and their Equilibrium States, MSRI, Berkeley, 1986, to appear.

[14] Friedman, A., and Giga, Y., *A single point blowup for solutions of semilinear*

parabolic systems, J. Fac. Sci. Univ. Tokyo, Sect. I, 34, 1987, pp. 65–79.

[15] Friedman, A., and McLeod, B., *Blowup of solutions of semilinear heat equations*, Indiana Univ. Math. J. 34, 1985, pp. 425–447.

[16] Fujita, H., and Chen, Y.-G., *On the set of blowup points and asymptotic behaviors of blowing-up solutions to a semilinear parabolic equation*, Analyse Math. et Appl., to appear.

[17] Galaktionov, V., Kurdyumov, S., and Samarskii, A., *Asymptotic stability of invariant solutions of nonlinear heat-conduction equation with sources*, Differential Equations 20, 1984, pp. 461–476.

[18] Galaktionov, V., and Posashkov, S., *The equation $u_t = u_{xx} + u^\beta$: localization and asymptotic behavior of solutions*, preprint, Keldysh Inst. of Appl. Math., Moscow, 1985, No. 97 (in Russian).

[19] Galaktionov, V., and Posashkov, S., *Application of new comparison theorem in the investigation of unbounded solutions of nonlinear parabolic equations*, Differential Equations 22, 1986, pp. 809–815.

[20] Galaktionov, V., and Posashkov, S., *On some properties of unbounded solutions of semilinear parabolic equations*, preprint, Keldysh Inst. of Applied Math., Moscow, 1987, No. 232 (in Russian).

[21] Giga, Y., *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations 62, 1986, pp. 186–212.

[22] Giga, Y., *A local characterization of blowup points of semilinear heat equations*, in Recent Topics in Nonlinear P.D.E. IV, M. Mimura and T. Nishida, eds., North Holland, to appear.

[23] Giga, Y., and Kohn, R., *Asymptotically self-similar blowup of semilinear heat equations*, Comm. Pure Appl. Math. 38, 1985, pp. 297–319.

[24] Giga, Y., and Kohn, R., *Characterizing blowup using similarity variables*, Indiana Univ. Math. J. 36, 1987, pp. 1–40.

[25] Giga, Y., and Kohn, R., *Removability of blowup points for semilinear heat equations*, in Proc. EQUADIFF 1987, G. Papanicolaou, ed., Marcel Dekker, to appear.

[26] Kohn, R., *New integral estimates for deformations in terms of their nonlinear*

strains, Arch. Rat. Mech. Anal. 78, 1982, pp. 131–172.

[27] Ladyzhenskaya, O., Solonnikov, V., and Ural'ceva, N., *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, Vol. 23, AMS, 1968.

[28] Liu, W., *The blowup rate of solutions of semlinear heat equations*, preprint, Purdue University, 1987.

[29] Meuller, C., and Weissler, F., *Single point blowup for a general semilinear heat equation*, Indiana Univ. Math. J. 34, 1985, pp. 881–913.

[30] Nirenberg, L., *On elliptic partial differential equations*, Ann. Scuola Normale Pisa, Ser. III, 13, 1959, pp. 115–162.

[31] Tanabe, H., *Equations of Evolution*, Pitman Press, 1979.

[32] Troy, W., *The existence of bounded solutions of a semilinear heat equation*, SIAM J. Math. Anal. 18, 1987, pp. 332–336.

[33] Weissler, F., *Existence and nonexistence of global solutions for a semilinear heat equation*, Israel J. Math. 38, 1981, pp. 29–40.

[34] Weissler, F., *Single point blowup for a semilinear initial value problem*, J. Differential Equations 55, 1984, pp. 204–224.