Some Typical Ideal In a Uniform Algebra

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Some Typical Ideal In a Uniform Algebra

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ABSTRACT: Let H^∞ be a weak-* closed subalgebra of $L^\infty(m)$ on which m is multiplicative. Let I be a weak-* closed linear span of functions in H^∞ that are zero on sets of positive measure. Then I is a weak-* closed ideal of H^∞. In this paper this typical ideal is studied.

§1. Introduction

Throughout this note $(X, \mathcal{A}, m)$ will be a fixed nontrivial probability measure space and A will be a complex subalgebra of $L^\infty = L^\infty(m)$ containing the constants and satisfying the following condition:

$$\int_X fgdm = \int_X fdm \int_X gdm \quad (f, g \in A).$$

The abstract Hardy space $H^p = H^p(m), 0 < p \leq \infty$, determined by A is defined to be the closure of A in $L^p = L^p(m)$, when p is finite and to be the weak-* closure of A in L^∞ when $p = \infty$. The measure m is multiplicative on H^∞ and so determines a point ϕ in the maximal ideal space $M(H^\infty)$ of H^∞. We denote the Gleason part determined by ϕ by $G(\phi)$; i.e. $G(\phi) = \{\psi \in M(H^\infty); \| \phi - \psi \| < 2\}$. The ideal J of H^∞ is called primary if $f, g \in H^\infty$ and $fg \in J$ implies $f \in J$ or $g \in J$. The hull of J consists of all $\psi \in M(H^\infty)$ such that $\psi(f) = 0$ for all $f \in J$.

DEFINITION: $I = I(H^\infty)$ is a weak-* closed linear span of functions on H^∞ that are zero on sets of positive measure. Then I is an ideal.

The questions arises: (1) Is I a primary ideal?, (2) $I = \{f \in H^\infty : \psi(f) = 0$ for all $\psi \in$ *hull* $I\}$?, (3) What is H^∞/I ?, (4) If *hull* $I = \{\phi\}$ then what happens in H^∞?, (5) What relations are there between $G(\phi)$ and hull I?. In this paper mainly we give the answers in case $A + \bar{A}$ is weak-* dense in L^∞, that is, A is a weak-* Dirichlet algebra [19].

For any measurable subset E of X, the function χ_E is the characteristic function of E. If $f \in L^p$, write E_f for the support set of f and write χ_f for the characteristic function of E_f. Suppose $0 < p \leq s \leq \infty$. For any subset $M \subset L^s(m)$, denote by $[M]_p$ the $L^p(m)$-closure of the linear span of M (weak-* closure for $p = \infty$).

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§2. $I = \{0\}$.

If $I = \{0\}$ then it is clear that I is a primary ideal. $M(H^\infty) = \textit{hull } I$ if and only if $I = \{0\}$. In many concrete examples, $I = \{0\}$. If D is a weak-* closed subalgebra of L^∞ which contains H^∞ and is not an invariant subspace under multiplications of functions in $\bar{A}$, then we call D an essential algebra. The following proposition is a version of Helson and Quigley [5].

PROPOSITION 1. *If there exists an essential algebra which contains H^∞ and is maximal among the proper weak-* closed subalgebras of L^∞, then $I = \{0\}$ and H^∞ is an integral domain.*

PROOF: Let B be the essential maximal weak-* closed subalgebra. Let $f \in H^\infty$ and $0 < m(E_f) < 1$. Put

$$D = [\chi_f B]_\infty + (1 - \chi_f)L^\infty,$$

then D is a weak-* closed subalgebra and $B \subsetneq D \subsetneq L^\infty$. For $[f[\chi_f B]_\infty] = [\chi_f f B]_\infty = [fB]_\infty$ and $[fB]_\infty \neq \chi_f L^\infty$ because B is essential.

In general the converse of Proposition 1 is not true. However if A is a weak-* Dirichlet albebra, then Nakazi [12, Theorem 2] showed the converse is true, using a method due to Muhly [11].

EXAMPLE 1: Let Γ be an ordered discrete abelian group and G the compact dual group of Γ. Let Γ_+ be the semigroup in Γ which orders Γ. If A is the uniform algebra on G which is generated by Γ_+ and σ is a Haar measure on G, then A is a weak-* Dirichlet algebra in $L^\infty(\sigma)$. $I = \{0\}$ if and only if Γ_+ is a maximal semigroup of Γ.

§3. $I \subseteq H^\infty_\phi$.

If $I = \{0\}$ then $H^\infty_\phi \supset I$. In general we don't know $H^\infty_\phi \supset I$, equivalently hull $I \ni \phi$.

PROPOSITION 2. *If m is absolutely continuous with respect to some Jensen measure μ of ϕ, then $H^\infty_\phi \supset I$.*

PROOF: If $f \in H^\infty$ and $m(E_f) < 0$, then $\mu(E_f) > 0$ because m is absolutely continuous with respecct to μ. Since μ is a Jensen measure,

$$\int_X log \mid f \mid d\mu \geq log \mid \phi(f) \mid$$

and hence $\phi(f) = 0$. This implies the proposition.

If A is a weak-* Dirichlet algebra then m is a Jensen measure and hence by Proposition 2 $H^\infty_\phi \supset I$. In general we don't know $I \neq H^\infty$ or $I \neq H^\infty_\phi$. Let H_{min} be the intersection of all weak-* closed subalgebras of L^∞ which contains H^∞ properly. We say that H^∞ is ϕ-weak-* maximal if whenever B is a weak-* closed subalgebra of L^∞ such that $B \supset H^\infty$ and ϕ extends multiplicative to B, then $B = H^\infty$ (cf. [4, Theorem 5.5]).

PROPOSITION 3. *If $H_{min} \supsetneq H^\infty$ then I is an ideal of H_{min} and so $I \neq H^\infty$. If moreover H^∞ is ϕ-weak-* maximal and $I \subseteq H^\infty_\phi$ then $I \subsetneq H^\infty_\phi$.*

PROOF: Let $f \in I$ and $0 < m(E_f) < 1$. Put

$$B_f = [\chi_f H^\infty]_\infty + [(1-\chi_f)H^\infty]_\infty$$

then B_f is a weak-* closed superalgebra of H^∞ and hence $B_f \supset H_{min}$. Moreover $B_f[fH^\infty]_\infty \subset [fH^\infty]_\infty \subset I$ and hence $H_{min}[fH^\infty]_\infty \subset I$. Let g be the linear combination of $f_1, \ldots, f_n$ in I with $0 < m(E_j) < 1$ $(j = 1, \ldots, n)$, then $H_{min}[gH^\infty]_\infty \subset I$ and hence $H_{min} I \subset I$. This implies the proposition.

If A is a weak-* Dirichlet algebra then $I \subset H^\infty_\phi$ and hence $I \neq H^\infty$. Nakazi [14, Corollary 5] showed that the converse of Proposition 3 is true. In Example $I \subsetneqq H^\infty_\phi$ if and only if there exists the least semigroup of Γ which contains Γ_+ properly, where ϕ is a complex homomorphism determined by $m = \sigma$.

EXAMPLE 2: . Let $L^\infty(T)$ be the algebra of essentially bounded, measurable functions with respect to Lebesgue measure on the circle T. $H^\infty(T)$ denotes the algebra of functions in $L^\infty(T)$ where Fourier coefficients with negative indices vanish. Let $\phi \in M(H^\infty(T))$ be not in the Shilov boundary of $H^\infty(T)$ and m the representing measure of ϕ, then $H^\infty(T)$ is a weak-* Dirichlet algebra of $L^\infty(m)$. If ϕ is an evalution at a point in the open unit disc, then $I = \{0\}$. If ϕ is not so and $G(\phi) \neq \{\phi\}$ then $\{0\} \neq I \subsetneqq H^\infty_\phi(m)$ by [8, p.492]. If $G(\phi) = \{\phi\}$ then we know nothing about I.

When *hull* $I \ni \phi$, if *hull* $I \neq \{\phi\}$ then $I \subsetneqq H^\infty_\phi$. However we don't know that if $I \subsetneqq H^\infty_\phi$ then *hull* $I \neq \{\phi\}$.

§4. $I = H^\infty_\phi$.

It is easy to construct H^∞ with $I = H^\infty_\phi$. In fact when $I \subsetneqq H^\infty_\phi$ let B be the weak-* closed linear span of 1 and I, then $B_\phi = \{f \in B : \phi(f) = 0\} = I$. However this B is not ϕ-weak-* maximal. Hence $B + \bar{B}$ is not weak-* dense in L^∞. But we have examples in weak-* Dirichlet algebras. In Example 1, there does not exist the least semigroup of Γ which contains Γ_+ properly if and only if $I = H^\infty_\phi$.

EXAMPLE 3: Let $(z_1(t) : t \geq 0), \ldots, (z_\ell(t) : t \geq 0)$ be ℓ independent complex Brownian motions on a complete probability space (Ω, P) such that $P(z_1(0) = \cdots = z_\ell(0) = 0) = 1$. For every $t \geq 0$, $\mathcal{F}(t)$ denotes the σ-field generated by $\{z_j(t) : 0 \leq s \leq t; j = 1, \ldots, m\}$ and the P-null sets, and $\mathcal{F}$ denotes the σ-field generated by $\bigcup_{t\geq 0} \mathcal{F}(t)$. Let us denote by $H^\infty(\Omega)$ the algebra of bounded $(\mathcal{F}(t))$-martingales $(X_t : t \geq 0)$ which admit an Ito integral representation of the form

$$X_t = X_0 + \sum_{j=1}^{\ell} \int_0^t \alpha_j(s) dz_j(s) \quad (t \geq 0)$$

where $\alpha_1, \ldots, \alpha_\ell$ are predictable processes. Then $H^\infty = \{X_\infty : (X_t : t \geq 0) \in H^\infty(\Omega)\}$ is a weak-* Dirichlet algebra on $(\Omega, \mathcal{F}, P)$ [20, Theorem 3.1]. By [1, Corollary 1] and [14, Corollary 5], $I = H^\infty_\phi$.

§5. H^∞/I.

If $H^\infty_\phi = I$ then H^∞/I is a field. We wish to know H^∞/I when $H^\infty_\phi \neq I$.

THEOREM 4. *Let A be a weak-* Dirichlet algebra and $I \subsetneq H^\infty_\phi$.*

(1) There exists a weak- closed subalgebra $\mathcal{H}^\infty$ of H^∞ and we have the direct sum decomposition*

$$H^\infty = \mathcal{H}^\infty \oplus I.$$

(2) I is a primary ideal and hence $\mathcal{H}^\infty$ is an integral domain.

(3) $I = \{f \in H^\infty : \psi(f) = 0 \quad \text{for all} \quad \psi \in hull\ I\}$.

(4) $hull\ I = M(\mathcal{H}^\infty)$ and $H^\infty \mid hull\ I = \mathcal{H}^\infty \mid hull\ I$.

(5) If $\mathcal{L}^\infty$ is the commutative von-Neumann algebra generated by $\mathcal{H}^\infty$ then $\mathcal{H}^\infty$ is a weak- Dirichlet algebra and maximal among the proper weak-* closed subalgebra of $\mathcal{L}^\infty$.*

PROOF: By [14,Corollary 5], $H_{min} \neq H^\infty$ and by [14, Theorem 2], $I = \{f \in H_{min} : \int_X fg\,dm = 0 \quad \text{for all} \quad g \in H_{min}\}$. By [7, Theorem 1.5], $H_{min} = \mathcal{L} \oplus I$ where $\mathcal{L} = H_{min} \cap \bar{H}_{min}$. Put $\mathcal{H}^\infty = H^\infty \cap \mathcal{L}$ then

$$H^\infty = \mathcal{H}^\infty \oplus I$$

and (1) follows. (2) is known in [14, Corollary 2], but we give a simple proof using (1). If $f, g \in H^\infty$ and $fg \in I$ then we can write $f = u + f_0$ and $g = v + g_0$ where $u, v \in \mathcal{H}^\infty$ and $f_0, g_0 \in I$. $fg \in I$ implies $uv \in I \cap \mathcal{H}^\infty$ and so $uv = 0$. Hence u or v belongs to $I \cap \mathcal{H}^\infty$ and $u = 0$ or $v = 0$. This implies (2)

Let Φ be a homomorphism from H^∞ onto $\mathcal{H}^\infty$ with the kernel I, then it is a contraction. The restriction map of elements in $M(H^\infty)$ to $\mathcal{H}^\infty$ is continuous from $M(H^\infty)$ into

$M(\mathcal{H}^\infty)$. If $\phi_0 \in M(\mathcal{H}^\infty)$, put $\phi(f) = \phi_0(\Phi(f))$ for any $f \in H^\infty$ then $\phi \in M(H^\infty)$ and $\phi \mid \mathcal{H}^\infty = \phi_0$. Hence the restriction map is onto and one to one on *hull* I. From this (3) and (4) follows. It is clear that $\mathcal{L}^\infty \subset \mathcal{L}$. By [15, Proposition 7], $\mathcal{L} + I + \bar{I}$ is weak-* dense in L^∞. Since $\mathcal{H}^\infty + \bar{\mathcal{H}}^\infty + I + \bar{I}$ is weak-* dense in L^∞, $\mathcal{H}^\infty + \bar{\mathcal{H}}^\infty$ is weak-* dense in $\mathcal{L}$ and hence $\mathcal{L} = \mathcal{L}^\infty$. This implies that $\mathcal{H}^\infty$ is a weak-* Dirichlet algebra in $\mathcal{L}^\infty$. If D is a weak-* closed subalgebra of $\mathcal{L}^\infty$ which contains $\mathcal{H}^\infty$ properly, then $D \oplus I$ is a weak-* closed superalgebra of H^∞ in H_{min} and $H_{min} = D \oplus I$ by the definition of H_{min}. Hence $D = \mathcal{L}^\infty$ and (5) follows.

When A is a weak-* Dirichlet algebra and $I \subsetneq H^\infty_\phi$ then *hull* $I \supsetneq \{\phi\}$.

§6. Structure of $\mathbf{H}^\infty$ in case $\mathbf{I} = H^\infty_\phi$.

THEOREM 5. *Let A be a weak-* Dirichlet algebra and $I = H^\infty_\phi$.*

(1) There exist a weak- closed subalgebra $\mathcal{H}^\infty$ of H^∞ and a weak-* closed ideal of H^∞ with $J \subsetneq I$, and we have the direct sum decomposition*

$$H^\infty = \mathcal{H}^\infty \oplus J.$$

(2) J is not a primary ideal and hence $\mathcal{H}^\infty$ is not an integral domain.

(3) $J = \{f \in H^\infty; \psi(f) = 0 \quad \text{for all} \quad \psi \in \textit{hull } J\}$.

(4) hull $J = M(\mathcal{H}^\infty)$ and $H^\infty \mid \textit{hull } J = \mathcal{H}^\infty \mid \textit{hull } J$.

(5) If $\mathcal{L}^\infty$ is a commutative von-Neumann algebra generated by $\mathcal{H}^\infty$ then $\mathcal{H}^\infty$ is a weak- Dirichlet algebra and $I(\mathcal{H}^\infty) = \mathcal{H}^\infty_\phi$.*

PROOF: Since $I \neq \{0\}$, by Proposition 1 there exists a weak-* closed superalgebra B of H^∞ with $H^\infty \subsetneq B \subsetneq L^\infty$. Let $J = \{f \in B; \int_X fgdm = 0 \quad \text{for all} \quad g \in B\}$, then by [7, Theorem 1.5] $B = \mathcal{L} \oplus J$ and $\mathcal{L} = B \cap \bar{B} \neq \{1\}$. Let $\mathcal{H}^\infty = H^\infty \cap \mathcal{L}$ then $H^\infty = H^\infty \oplus J$ and (1) follows. By Proposition 3, there exists a weak-* closed superalgebra B_1 of H^∞ with $H^\infty \subsetneq B_1 \subsetneq B$. Let $J_1 = \{f \in B_1 : \int_X fgdm = 0 \quad \text{for all} \quad g \in B_1\}$, then $J_1 \supsetneq J$ and $B_1 \cap \bar{B}_1 \neq \{1\}$. There exists a characteristic function $\chi_{E_0} \in B_1$ such that

$$\chi_E(\chi_{E_0}J_1) \supsetneqq \chi_E(\chi_{E_0}J)$$

for any $\chi_E \in B_1$ with $\chi_E\chi_{E_0} \neq 0$ (see [13, Theorem 1]). As in the proof of [12, Lemma 3], by [12, Lemma 2] there exists a characteristic function $\chi_E \in B_1$ such that $\chi_E\chi_{E_0}J_1 \neq \{0\}$ and $(1-\chi_E)\chi_{E_0}J_1 \neq \{0\}$. If $f \in \chi_E\chi_{E_0}J_1$ and $g \in (1-\chi_E)\chi_{E_0}J_1$ then $f \notin J$ and $g \in J$, and $fg \in J$. This implies (2). (3), (4) and that $\mathcal{H}^\infty$ is a weak-* Dirichlet algebra in $\mathcal{L}^\infty$, can be proved as in the proof of Theorem 4.

We shall show that $I(\mathcal{H}^\infty) = \mathcal{H}_\phi^\infty$. If $\mathcal{H}_\phi^\infty \neq I(\mathcal{H}^\infty)$, by [14, Corollary 5] $\mathcal{H}_{min} \supsetneqq \mathcal{H}^\infty$ and hence $\mathcal{H}_{min} \oplus J \supsetneqq H^\infty$. $\mathcal{H}_{min} \oplus J$ is a minimal weak-* closed subalgebra of L^∞ that contains H^∞ properly. By [17], $\mathcal{H}_{min} \oplus J = H_{min}$ and $H_{min} \neq H^\infty$. While by hypothesis and [14, Corollary 5], $H_{min} = H^\infty$. This contradiction implies $\mathcal{H}_\phi^\infty = I(\mathcal{H}^\infty)$.

§8. Nontrivial Gleason part

We wish to know the relation between $G(\phi)$ and *hull* I.

PROPOSITION 7. *Let m be a Jensen measure of ϕ. If there exists a Jensen measure μ of $\psi \in G(\phi)$ such that $\psi \neq \phi$ and μ is absolutely continuous with respect to m, then $H_\phi^\infty \supsetneqq I$.*

Proof is similar to the proof of Proposition 2.

THEOREM 8. *Let A be a weak-* Dirichlet algebra and $G(\phi) \neq \{\phi\}$.*

(1) $I = \{f \in H^\infty : \psi(f) = 0 \quad$ for all $\quad \psi \in G(\phi)\}$ and hull $I =$ the closure of $G(\phi)$.

(2) $H^\infty = \mathcal{H}^\infty \oplus I$ and $\mathcal{H}^\infty$ is isometrically isomorphic to $H^\infty(T)$ in Example 2.

PROOF: By Theorem 4 $H^\infty = \mathcal{H}^\infty \oplus I$ and $\mathcal{H}^\infty$ is a weak-* Dirichlet algebra. Since $G(\phi) \neq \{\phi\}$ and *hull* $I \supset G(\phi)$ by Proposition 7, the Gleason part of ϕ in $M(\mathcal{H}^\infty)$ is non-trivial. Hence $\mathcal{H}_\phi^\infty = Z\mathcal{H}^\infty$ for some $Z \in \mathcal{H}_\phi^\infty$ with $|\,Z\,| = 1$ (see [9, p.469]). Then $H_\phi^\infty = ZH^\infty$. Let $J = \{f \in H^\infty : \psi(f) = 0 \quad$ for all $\quad \psi \in G(\phi)\}$, then $H^\infty = \mathcal{H} \oplus J$ where $\mathcal{H}$ denotes the weak-* closure of the polynomials in Z, and $ZJ = J$ (see [9, Lemma 5]). Then $\mathcal{H} \subset \mathcal{H}^\infty$ becasue $Z \in \mathcal{H}^\infty$, and $J \supset I$. Let $\mathcal{L}$ be the weak-* closure of the polynomials in Z and $\bar{Z}$, then $\mathcal{L}J \subset J$ and hence $J \subset I$. Thus $\mathcal{H} = \mathcal{H}^\infty$ and $J = I$. It is known that $\mathcal{H}$ is isometrically isomorphic to $H^\infty(T)$ (cf. [21] and [9, Lemma 6]).

Theorem 8 is essentially a famous theorem of Wermer (cf. [21], [9] and [8]). It happens that *hull* $I \neq \{\phi\}$ and $G(\phi) = \{\phi\}$. Hence Theorem 4 is a generalization of Wermer's theorem in case *hull* $I \neq \{\phi\}$.

§8. General weak-* closed ideal

In this section, assuming that A is a weak-* Dirichlet algebra, we wish to consider general weak-* closed ideals in H^∞ using the typical weak-* closed ideal I. Let J be a weak-* closed ideal of H^∞ and put

$$B(J) = \{f \in L^\infty : fJ \subset J\}.$$

It is reasonable to assume $B(J) = H^\infty$ because of the results in [12], [13] and [7]. $| J | = | H^\infty |$ by [16] where $| J | = \{| f |; f \in J\}$. If $[H^\infty_\phi J]_\infty \subsetneq J$ then $J = qH^\infty$ for some unimodular function q in H^∞ [19]. We wish to know about J when $[H^\infty_\phi J]_\infty = J$.

THEOREM 9. *Suppose $I \neq H^\infty_\phi$. Let J be a weak-* closed ideal of H^∞ with $B(J) = H^\infty$, then*

$$J = J_0 \oplus qI$$

where $| J_0 | = | \mathcal{H}^\infty |$ and $q \in H_{min}$ with $| q | = 1$.

PROOF: Set $J_1 = [H_{min}J]_\infty$, then $\chi_E J_1 \supsetneq \chi_E[IJ_1]_\infty$ for any $\chi_E \in H_{min}$ with $\chi_E \neq 0$. For if $\chi_E J_1 = \chi_E[IJ_1]_\infty$ for some $\chi_E \in H_{min}$ with $\chi_E \neq 0$, then

$$\chi_E J \subset \chi_E J_1 = \chi_E[IJ]_\infty \subset J$$

because $IH_{min} \subset I$, and this contradicts $B(J) = H^\infty$. By [7, Theorem 1.5] the measure m is quasi-multiplicative on H_{min} and by [13, Theorem 2] $J_1 = qH_{min}$ for some unimodular function $q \in H_{min}$. Hence $H_{min} \supset \bar{q}J \supset I$ and $\bar{q}J = J_2 \oplus I$ where $\mathcal{L}^\infty \supset J_2$ and $\mathcal{H}^\infty J_2 \subset J_2$. Then by [16] $| J_2 |=| \mathcal{H}^\infty |$ because $\mathcal{H}^\infty$ is a weak-* Dirichlet algebra in $\mathcal{L}^\infty$. Put $J_0 = qJ_2$ then the theorem follows.

THEOREM 10. *Suppose $I = H^\infty_\phi$ and B_α ($\alpha \in \Lambda$) are weak-* closed superalgebras of H^∞ such that $B_\alpha \neq H^\infty$ and $\cap\{B_\alpha : \alpha \in \Lambda\} = H^\infty$. Let J be a weak-* closed ideal of H^∞ with $B(J) = H^\infty$. Then*

$$[\{\cup I_\alpha J : \alpha \in \Lambda\}]_\infty \subseteq J \subseteq \cap\{[B_\alpha J]_\infty : \alpha \in \Lambda\}$$

where $I_\alpha = \{f \in B_\alpha : \int_X fg dm = 0 \quad \text{for all} \quad g \in B_\alpha\}$.

(1) For any $\alpha \in \Lambda$ $[B_\alpha J]_\infty = q_\alpha B_\alpha$ for some unimodular function $q_\alpha \in B_\alpha$ and if $[B_\alpha J]_\infty = q'_\alpha B_\alpha$ then $q_\alpha \bar{q}'_\alpha \in B_\alpha \cap \bar{B}_\alpha$.

(2) If $[\cup I_\alpha J]_\infty \neq J$ then $J = qH^\infty$ for some unimodular function $q \in H^\infty$.

(3) If $J \neq \cap[B_\alpha J]_\infty$ then $J = qH^\infty_\phi$ for some unimodular function $q \in H^\infty$.

(4) If $[\cup I_\alpha J]_\infty = J = \cap[B_\alpha J]_\infty$ then

$$J = [\cup q_\alpha I_\alpha]_\infty = \cap q_\alpha B_\alpha$$

where q_α is a unimodular function in B_α.

PROOF: $[B_\alpha J]_\infty = q_\alpha B_\alpha$ for some unimodular function $q_\alpha \in B_\alpha$ as in the proof of Theorem 9 and it is clear that if $[B_\alpha J]_\infty = q'_\alpha B_\alpha$ then $q_\alpha \bar{q}'_\alpha \in B_\alpha \cap \bar{B}_\alpha$. This implies (1). (2) is clear because $[\cup I_\alpha]_\infty = I = H^\infty_\phi$ [19]. (3) follows by the dual method. (1) implies (4).

EXAMPLE 4: Let A be the algebra of continuous complexvalued functions on the infinite torus T^∞ which are uniform limits of polynomials in $z_1^{\ell_1}, z_2^{\ell_1}, \dots, z_n^{\ell_n}$ where $(\ell_1, \ell_2, \dots, \ell_n, 0, 0, \dots) \in \Gamma$ and Γ the set of $(\ell_1, \ell_2, \dots) \in Z^\infty$ whose first non-zero entry is positive, together with 0. Denote by m the normalized Haar measure on T^∞, then A is the weak-* Dirichlet algebra of $L^\infty(m)$. Let B_n be the weak-* closure of $\bigcup_{i=0}^\infty \bar{z}_n^i H^\infty$, then $B_1 \supsetneq B_2 \supsetneq B_3 \supsetneq \dots H^\infty$ and $\bigcap_{n=1}^\infty B_n = H^\infty$. Let J be a weak-* closed ideal of H^∞ with $B(J) = H^\infty$.

If $[\bigcup_n I_n J]_\infty = J = \bigcap_n [B_n J]_\infty$ then $J = [\bigcup_n q_n I_n]_\infty = \bigcap_n q_n B_n$, $q_n B_n \supset q_{n+1} B_{n+1}$, $q_n I_n \subset q_{n+1} I_{n+1}$ and $I_n = z_n B_n$, where $q_n \in B_n$ and $| q_n | = 1$.

In (4) of Theorem 10 (even if in Example 4) we don't know that there exists a weak-* closed ideal J of H^∞ such that $[\cup I_\alpha J]_\infty = J = \cap [B_\alpha J]_\infty$. In the same method we can prove Theorems 9 and 10 for weak-* closed invariant subspace of L^∞ under multipliations of functions of H^∞. Since H_{min} is an extended weak-* Dirichlet algebra, we apply to this algebra the general theory for extended weak-* Dirichlet algebras [15]. For example if w is positive in L^1 we can calculate $inf\{\int_X | 1 - g |^2 wdm; g \in I\}$.

§10. **An application of I**

In this section we assume that A is a weak-* Dirichlet algebra. In Example 1, Shapiro [18] showed that if Γ_+ is a maximal semigroup of Γ and has not the least positive element, then for $0 < p < 1$ each continuous linear functional on H^p is a constant multiple of the linear functional determined by m. We shall show the same result for Example 1 not assuming that Γ_+ is a maximal semigroup of Γ, and for Example 3.

THEOREM 11. *For $0 < p < 1$ each continuous linear functional on H^p is zero on I.*

PROOF: We may assume $I \neq \{0\}$. By Proposition 1 there exsits a weak-* closed subalgebra B with $H^\infty \subsetneq B \subsetneq L^\infty$. Put $I_B = \{f \in B : \int_X fgdm = 0 \quad \text{for all} \quad g \in B\}$ then $I_B \subset I$. For $I = \{f \in H_{min} : \int_X fgdm = 0 \quad \text{for all} \quad g \in H_{min}\}$. Let ℓ be a continuous linear functional on H^p, put for any fixed $g \in I_B$

$$\tilde{\ell}(u) = \ell(ug) \quad (u \in \mathcal{L}_B).$$

Then $\tilde{\ell}$ is a continuous linear functional on the L^p-closure of $\mathcal{L}_B$ and hence $\tilde{\ell} = 0$ by Day's theorem [3]. This implies $\ell(g) = 0$ for each $g \in I$ and hence $\ell = 0$ on I_B. When $I \neq H^\infty_\phi$, by [14, Corollary 5] $H_{min} \neq H^\infty$. If $B = H_{min}$ then $I = I_B$ and this implies the theorem. When $I = H^\infty_\phi$, by [14, Corollary 5] $H_{min} = H^\infty$. Hence there exist superalgebras B_α of H^∞ such that $\cap_\alpha B_\alpha = H^\infty$ and $B_\alpha \neq H^\infty$. Then $[\cup_\alpha I_{B_\alpha}]_\infty = I$ and hence $\ell = 0$ on I.

COROLLARY 1. *Suppose $I = H^{\infty}_{\phi}$. Then for $0 < p < 1$ each continuous linear functional on H^p is a constant multiple of the linear functional determined by m.*

Muhly [10] showed Shapiro's theorem for ergodic H^{∞}. Corollary 1 does not contain it because $I = \{0\}$ for ergodic H^{∞}. In Example 1, if Γ_+ has not the least positive element, then $I = H^{\infty}_{\phi}$ or $H^{\infty} = \mathcal{H}^{\infty} \oplus I$ where $\mathcal{H}^{\infty}$ is isometrically isomorphic to the Hardy space in case Γ_+ is a maximal semigroup. Hence Corollary 1 and Shapiro's theorem imply that for $0 < p < 1$ each continuous linear functional on H^p is a constant multiple of the linear functional determined by m.

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