Threefolds Homeomorphic to a Hyperquadric in P^4

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Threefolds Homeomorphic to a Hyperquadric in P^4

Dedicated to Professor Masayoshi NAGATA on his 60-th birthday

By Iku NAKAMURA

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§ 0 Introduction. The purpose of this article is to prove

(0.1) Theorem. A compact complex threefold homeomorphic to a nonsingular hyperquadric Q^3 in P^4 is isomorphic to Q^3 if $H^1(X,O_X) = 0$ and if there is a positive integer m such that dim $H^0(X,-mK_X) > 1$.

As its corollaries, we obtain

(0.2) Theorem. A Moishezon threefold homeomorphic to Q^3 is isomorphic to Q^3 if its Kodaira dimension is less than three.

A compact complex threefold is called a Moishezon threefold if it has three algebraically independent meromorphic functions on it.

(0.3) Theorem. An arbitrary complex analytic (global) deformation of Q^3 is isomorphic to Q^3.

We shall prove a stronger theorem (2.1) in arbitrary characteristic and apply this in complex case to derive (0.1). The above theorems in arbitrary dimension have been proved by Brieskorn [2] under the assumption that the manifold is kählerian. See also [3],[9],[11] for related results. When I completed the major parts of the present article, I received a preprint [14] of Peternell, in which he claims that he is able to prove the theorems (0.2) and (0.3) without assuming the condition on Kodaira dimension. See [12,(3.3)].

The main idea of the present article is the same as that of our previous work [12], in which we proved the similar theorems for complex projective space P^3. However there arises a new problem that we have never seen in [12]. See (0.4) below.

Let X be a complex threefold with $H^1(X,O_X) = 0$,$\kappa(X,-K_X) \geq 1$ (see [6]), which is homeomorphic to a nonsingular hyperquadric Q^3. Let L be the generator of Pic X ($\cong$ Z) with L^3 equal to two. Then $K_X = -3L$ by Brieskorn [2], Morrow [11] and [12,(1.1)]. In the same manner as in [12], we see that dim |L| is not less than four.

Let D and D' be an arbitrary pair of distinct members of |L|, ℓ the scheme-theoretic complete intersection $D \cap D'$ of D and D'. Then ℓ is a pure one dimensional connected closed analytic subspace of X containing Bs |L|, the base locus of the linear system |L|. By studying ℓ and ℓ_{red} in detail, we eventually prove that the base locus Bs |L| is empty. Indeed, we are able to verify;

(0.4) Lemma. ℓ_{red} is a connected (possibly reducible) curve whose irreducible components are nonsingular rational curves intersecting transversally and either

(0.4.1) ℓ is an irreducible nonsingular rational curve, or

(0.4.2) ℓ is "a double line" with ℓ_{red} irreducible nonsingular,

(0.4.3) ℓ is "a double line" plus a nonsingular rational curve,

(0.4.4) ℓ is reduced everywhere and is the union of two rational curves ("lines") and a (possibly empty) chain of rational curves connecting the "lines", each component of the chain being algebraically equivalent to zero.

It turns out after completing the proof of (0.1) that the case (0.4.3) is impossible and the chain in (0.4.4) is empty.

It follows from (0.4) that Bs $|L|$ is empty so that the complete intersection $\mathfrak{L} = D \cap D'$ is irreducible nonsingular for a general pair D and D', and that dim $|L|$ is equal to four. Thus we have a bimeromorphic morphism f of X onto a (possibly singular) hyperquadric in P^4 associated with the linear system $|L|$. It follows from Pic X $\cong$ Z and an elementary fact about singular hyperquadrics in P^4 that the image f(X) is nonsingular and that f is an isomorphism of X onto Q^3.

The article is organized as follows. In section one, we recall elementary facts about algebraic two cycles on singular hyperquadrics in P^4. In sections 2-8, we consider a threefold X with a line bundle L such that Pic X $\cong$ ZL, $K_X = -3L$, L^3 is positive, $\kappa(X,L) \geq 1$ (see [7]). In section 2, we prove the vanishing of certain cohomology groups. We also prove $L^3 \geq 2$ and $h^0(X,L) \geq 5$.

In section 3, first we state without proof five lemmas (3.2)-(3.6) which are detailed forms of (0.4) and then by assuming these, prove that X is isomorphic to Q^3. In sections 4-8, we study a scheme-theoretic complete intersection $\mathfrak{L} = D \cap D'$ to prove the lemmas (3.2)-(3.6).

In section 9, we first give a slight improvement of a theorem in [12] and complete the proofs of (0.1) by applying the results in sections 2-8.

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List of notations

$\mathbf{Z}$	integers or the infinite cyclic group
$\mathbf{C}$	complex numbers
X	a nonsingular threefold
$\kappa(X,L)$	L-dimension of X, L being a line bundle on X [7]
$\mathrm{Bs}\ \lvert L\rvert$	the set of base points of the linear system $\lvert L\rvert$
$H^q(X,F)$	the q-th cohomology group of X with coefficients in a coherent sheaf F
$h^q(X,F)$	$\dim_{\mathbf{C}} H^q(X,F)$
$\chi(X,F)$	$\sum_{q\in \mathbf{Z}} (-1)^q h^q(X,F)$
O_X, O_X^*	the sheaf of germs over X of holomorphic (resp. nonvanishing holomorphic) functions
I_C, I_ℓ	the ideal sheaf in O_X defining C, resp. ℓ
Ω_X^p	the sheaf of germs over X of holomorphic p-forms
K_X	the canonical line bundle of X
$[D]$	the line bundle associated with a Cartier divisor D
b_q	the q-th Betti number (of X)
c_q	the q-th Chern class (of X)
$c_1(E)$	the first Chern class of a vector bundle E

cl(C)	the homology class of an irreducible curve C
Q^3, Q^3_ν	hyperquadrics in P^4, see (1.1)

§ 1 Hyperquadrics in P^4

(1.1) We recall elementary facts about hyperquadrics in P^4. Let x_i $(0 \leq i \leq 4)$ be the homogeneous coordinate of P^4, $F_\nu = \sum_{i=0}^{\nu+1} x_i^2$, Q^3_ν a hypersurface defined by $F_\nu = 0$. The hypersurface Q^3_ν $(\nu = 1,2,3)$ is irreducible and Q^3 $(:= Q^3_3)$ only is nonsingular.

The hypersurface Q^3_1 contains a conic $q := Q^3_1 \cap \{x_3 = x_4 = 0\}$ and a line $\ell := \{x_0 = x_1 = x_2 = 0\}$. Let U be a sufficiently small open neighborhood of ℓ in Q^3_1. We may assume that $Q^3_1 \backslash U$ (resp. U) is homotopic to q (resp. ℓ) and that ∂U ,the boundary of U,is an S^3-bundle over the conic q. By the Thom-Gysin sequence , we have,

(1.1.1) $$H_n(\partial U, Z) = \begin{cases} Z & n = 0,2,3,5 \\ 0 & n = 1,4 \end{cases}$$

In particular, $H_3(\partial U, Z) \cong H_0(q, Z)$.

Also by the Mayer-Vietoris sequence of $Q^3_1 = (Q^3_1 \backslash U) \cup$ (the closure of U), we have,

(1.1.2) $$H_n(Q^3_1, Z) = \begin{cases} Z & n = 0,2,4,6 \\ 0 & n = 1,3,5 \end{cases}$$

By (1.1.1) and (1.1.2), we have,

(1.1.3) $$H_4(Q^3_1, Z) \cong H_3(\partial U, Z) \cong H_0(q, Z) \cong Z.$$

(1.2) Lemma. <u>There is a Weil divisor on Q^3_1 which is not an integral multiple of a hyperplane section</u> H <u>of Q^3_1 in</u> $H_4(Q^3_1, Z)$.

<u>Proof</u>. Let $a = [a_0, a_1, a_2]$ be a point of the conic q, D_a = the closure of $\{[a_0, a_1, a_2, x_3, x_4] \in P^4 \,;\, x_3, x_4 \in C\}$. Then by (1.1.3), $H = 2D_a$ in $H_4(Q^3_1, Z)$. q.e.d.

(1.3) Lemma. *Let* Q *be a quadric surface* $Q_2^3 \cap \{x_4 = 0\}$ *contained in* Q_2^3. *Then* $H_4(Q_2^3, \mathbf{Z}) \cong H_2(Q, \mathbf{Z})$ $(\cong \mathbf{Z} \oplus \mathbf{Z})$ *and* $H_2(Q, \mathbf{Z})$ *is generated by fibers of two rulings via the isomorphism of* Q *with* $\mathbf{P}^1 \times \mathbf{P}^1$.

Proof. Similar to the above. q.e.d.

(1.4) Remark. In arbitrary characteristic, any singular hyperquadric in $\mathbf{P}^4$ is a cone over a hyperplane section of it, whence it has a Weil divisor which is not (algebraically equivalent to) an integral multiple of a hyperplane section.

§ 2 Lemmas

Our first aim is to prove the following

(2.1) Theorem. Let X be a compact complex threefold or a complete irreducible nonsingular algebraic threefold defined over an algebraically closed field of arbitrary characteristic L a line bundle on X. Assume that $H^1(X,O_X) = 0$, $\text{Pic } X \cong \mathbb{Z}L$, $L^3 > 0$, $K_X = -3L$, $\kappa(X,L) \geq 1$. Then $L^3 = 2$ and X is isomorphic to a nonsingular hyperquadric in P^4.

Compare [2], [8].

Sections 2-8 are devoted to proving (2.1). Throughout sections 2-8, we always assume that X is a compact complex threefold satisfying the conditions in (2.1). Our proof of (2.1) is completed in (3.8) by assuming (0.4), or more precisely, (3.2)-(3.6).

(2.2) Lemma. $H^1(X,O_X) = 0$ and $c_1c_2 \geq 24$, $L^3 \geq 2$, $\chi(X,mL) \geq (m+1)(m+2)(2m+3)/6$.

Proof. We see $h^3(X,O_X) = 0$, $\chi(X,O_X) = 1 + h^2(X,O_X) \geq 1$ and $c_1c_2 = 24\chi(X,O_X) \geq 24$, $\chi(X,mL) = \chi(X,O_X) + m(c_1^2 + c_2)L/12 + m^2c_1L^2/4 + m^3L^3/6$. Assume $L^3 = 1$ to derive a contradiction. Let $c_1c_2 = 24a$, $a \geq 1$. Hence $c_2L = 8a$ by $L^3 = 1$. We also see that $\chi(X,L) = (5+5a)/3$, whence $1 + a = 0 \bmod 3$ and $a \geq 2$. Let $a = 3b + 2$, $b \geq 0$. Then $\chi(X,2L) = 7b + (21/2)$, which is absurd. Consequently $L^3 \geq 2$ and $\chi(X,mL) \geq (m+1)(m+2)(2m+3)/6$ by $c_2L = c_1c_2/3 \geq 8$.

q.e.d.

(2.3) Lemma. $h^0(X,L) \geq 5$.

Proof. The same proof as in [11,(1.5)] works by taking $d = 3$, $\chi(X,L) \geq 5$ instead of $d \geq 4$ and $\chi(X,L) \geq 4$. q.e.d.

(2.4) Lemma. Let D and D' be distinct members of $|L|$, $\ell = D \cap D'$ the scheme-theoretic intersection of D and D'. Then we have,

(2.4.1) $H^q(X,-mL)=0$ for $q=0,1,m>0; q=2,0\leq m\leq 3; q=3,0\leq m\leq 2$,

(2.4.2) $H^q(D,-mL_D)=0$ for $q=0,m>0; q=1,0\leq m\leq 2; q=2,m=0,1$,

(2.4.3) $H^0(\ell,-L_\ell) = 0$, $H^1(\ell,O_\ell) = 0$,

(2.4.4) $H^0(X,O_X) \cong H^0(D,O_D) \cong H^0(\ell,O_\ell) \cong C$,

(2.4.5) $H^3(X,-3L) \cong H^2(D,-2L_D) \cong H^1(\ell,-L_\ell) \cong C$.

Proof. The same as in [11,(1.7)] by using an exact sequence $0 \to O_D(-L) \to O_D \to O_\ell \to 0$ [11,(1.5.1) and (1.6)].

q.e.d.

(2.5) Corollary. $H^2(X,O_X) = 0$ and $\chi(X,O_X) = 1$.

(2.6) Corollary. $Bs\ |L| = Bs\ |L_D| = Bs\ |L_\ell|$.

§ 3 A complete intersection $\ell = D \cap D'$

Let X, L be the same as in section 2.

(3.1) Lemma. <u>Let</u> D <u>and</u> D' <u>be distinct members of the linear system</u> $|L|$, $\ell := D \cap D'$ <u>the complete intersection of</u> D <u>and</u> D'. <u>Let</u> $\ell_{red} = A_1 + \dots + A_s$ <u>be the decomposition of</u> ℓ_{red} <u>into irreducible components. Then</u>

(3.1.1) <u>each</u> A_j <u>is a nonsingular rational curve with</u> $LA_j \leq 2$,

(3.1.2) <u>if there is an irreducible component</u> A_i <u>with</u> $LA_i = 2$, <u>then</u> $LA_j \leq 1$ <u>for</u> $j \neq i$.

<u>Proof</u>. By (2.4.3), $H^1(\ell, O_\ell) = 0$. Hence $H^1(A_j, O_{A_j}) = 0$ for any j, whence A_j is a nonsingular rational curve. In view of (2.4.5), $h^1(\ell, -L_\ell) = 1$, whence $h^1(\ell_{red}, -L_{\ell_{red}}) \leq 1$. Therefore $\sum_{i=1}^{s} h^1(A_i, -L_{A_i}) = \sum_{i=1}^{s} h^0(A_i, O_{A_i}(-2+LA_i)) \leq 1$. The assertions are therefore clear. See [11,(2.3)]. q.e.d.

In the subsequent sections 4-8, we shall prove the following five lemmas;

(3.2) Lemma. <u>Let</u> $\ell = D \cap D'$ <u>be the complete intersection in</u> (3.1). <u>Assume that there is an irreducible component</u> C <u>of</u> ℓ_{red} <u>with</u> $LC \geq 2$. <u>Then</u>

(3.2.1) $LC = 2$ <u>and</u> ℓ <u>is an irreducible nonsingular rational curve, isomorphic to</u> C,

(3.2.2) $I_C/I_C^2 \cong O_C(-2) \oplus O_C(-2)$.

(3.3) Lemma. Let $\ell = D \cap D'$ be the complete intersection in (3.1). Assume that there is an irreducible component C of ℓ_{red} with $LC = 1$ such that ℓ is nonreduced anywhere along C. Let I_ℓ (resp. I_C) be the ideal sheaf of O_X defining ℓ (resp. C). Then $I_\ell + I_C^2/I_C^2 \cong O_C$ or $O_C(-1)$. If $I_\ell + I_C^2/I_C^2 \cong O_C$, then

(3.3.1) ℓ_{red} is an irreducible nonsingular rational curve, isomorphic to C,

(3.3.2) ℓ is "a double line", to be precise, at any point p of C, the ideal sheaf I_ℓ (resp. I_C) is given by;

$$I_\ell = O_{X,p}x + O_{X,p}y^2,$$
$$I_C = O_{X,p}x + O_{X,p}y$$

for suitable local parameters x and y at p,

(3.3.3) $I_C \supset I_\ell \supset I_C^2$, $I_C/I_C^2 \cong O_C \oplus O_C(-1)$, $I_C/I_\ell \cong O_C(-1)$, $I_\ell/I_C^2 \cong O_C$.

(3.4) Lemma. Let $\ell = D \cap D'$ be the complete intersection in (3.1). Assume that there is an irreducible component C of ℓ_{red} with $LC = 1$ such that ℓ is nonreduced anywhere along C. Assume that $I_\ell + I_C^2/I_C^2 \cong O_C(-1)$ and that if ℓ_{red} is reducible, then C meets an irreducible component C' of ℓ_{red} not contained in Bs $|L|$. Then ℓ is a double line plus a nonsingular rational curve C'. To be more precise,

(3.4.1) ℓ_{red} is the union of C and C' with $LC = 1$, $LC' = 0$, the curve C intersecting C' transversally at a unique point p_0,

(3.4.2) the ideal sheaf I_ℓ (resp. $I_C, I_{C'}$) defining ℓ (resp. C, C') is given at p_0 by

$$I_\ell = O_{X,p_0}x + O_{X,p_0}zy^2,$$

$$I_C = O_{X,p_0}x + O_{X,p_0}y,$$

$$I_{C'} = O_{X,p_0}x + O_{X,p_0}z$$

for a local parameter system x, y and z at p_0 and except at p_0, ℓ is a double line along C in the sense of (3.3.2), and reduced along C',

(3.4.3) $I_C/I_C^2 \cong O_C \oplus O_C(-1)$, $I_{C'}/I_{C'}^2 \cong O_{C'}(2) \oplus O_{C'}$.

(3.5) Lemma. Let $\ell = D \cap D'$ be the complete intersection in (3.1). Assume that ℓ is reduced at a point of an irreducible component C_0 of ℓ_{red} with $LC_0 = 1$ and that C_0 intersects an irreducible component C' of ℓ_{red} not contained in Bs $|L|$. Then,

(3.5.1) ℓ is reduced everywhere,

(3.5.2) there exist another irreducible component C_m of ℓ with $LC_m = 1$ and a chain of irreducible components C_j of ℓ with $LC_j = 0$ $(1 \leq j \leq m-1)$ such that ℓ is the union of C_j $(0 \leq j \leq m)$, the pair C_j and C_k $(j < k)$ intersect iff $j = k-1$. If $j = k-1$, then C_{j-1} and C_j intersect at a unique point p_j $(1 \leq j \leq m)$ transversally, to be precise,

$$\hat{O}_{\ell,p_j} \text{ (:= the completion of } O_{\ell,p_j}) \cong \mathbb{C}[[x,y,z]]/(x,yz),$$

for suitable local parameters x, y, z at p_j,

(3.5.3) $$I_C/I_C^2 = \begin{cases} O_C \oplus O_C(-1) & (C = C_0, C_m) \\ O_C(1) \oplus O_C(1) \text{ or } O_C(2) \oplus O_C & (C = C_1, \ldots, C_{m-1}) \end{cases}$$

(3.6) Lemma Let $\ell = D \cap D'$ be the complete intersection in (3.1). Let C be an irreducible component of ℓ_{red} with $LC = 1$. If

$\mathfrak{L}_{red}$ _is reducible, then_ C _intersects an irreducible component_ C' _of_ $\mathfrak{L}_{red}$ _not contained in_ Bs |L|.

From (3.2)-(3.6), we infer the following

(3.7) Lemma. _The linear system_ |L| _is base point free and_ dim |L| = 4, $L^3 = 2$.

Proof by assuming (3.2)-(3.6). In view of (2.3), we are able to choose distinct members D and D' from |L|. Let $\mathfrak{L} = D \cap D'$ be the complete intersection. Let $\mathfrak{L}_{red} = A_1 + \dots + A_s$ be the decomposition into irreducible components. Then $cl(\mathfrak{L}) = n_1 cl(A_1) + \dots + n_s cl(A_s) \in H_2(X,Z)$ for some $n_i > 0$ (see [11,(2.1)]). Since $L^3 = L\mathfrak{L} = n_1 LA_1 + \dots + n_s LA_s$, there is at least a component A_i with $LA_i > 0$. We see that there are only three cases;

Case 1. $\mathfrak{L}_{red}$ contains an irreducible component C with $LC \geq 2$,

Case 2. $\mathfrak{L}_{red}$ contains no irreducible components C' with $LC' \geq 2$, but contains an irreducible component C with LC = 1 along which $\mathfrak{L}$ is nonreduced anywhere,

Case 3. $\mathfrak{L}_{red}$ contains no irreducible components C' with $LC' \geq 2$, but contains an irreducible component C_0 with $LC_0 = 1$ such that $\mathfrak{L}$ is reduced at a point of C_0.

Case 1. By (3.2), $\mathfrak{L}$ is isomorphic to C. By (2.6), Bs |L| = Bs $|L_{\mathfrak{L}}|$. Since $L^3 = L\mathfrak{L} = LC = 2$, we have $L_{\mathfrak{L}} = O_{\mathfrak{L}}(2)$, so that $|L_{\mathfrak{L}}|$ is base point free. Consequently |L| is base point free and $h^0(X,L) = 2 + h^0(\mathfrak{L},L_{\mathfrak{L}}) = 5$.

$\underline{\text{Case}}$ 2. First we assume that ℓ_{red} is irreducible. By (3.3) and (3.4), ℓ_{red} is isomorphic to C and $I_C/I_\ell \cong O_C(-1)$. Hence we have an exact sequence,

$$0 \to (I_C/I_\ell)\otimes L \to O_\ell(L) \to O_C(L) \to 0,$$

whence $0 \to O_C \to O_\ell(L) \to O_C(1) \to 0$ is exact. It follows that

$$0 \to H^0(C,O_C) \to H^0(\ell,L_\ell) \to H^0(C,O_C(1))$$
$$\to H^1(C,O_C) \to H^1(\ell,L_\ell) \to H^1(C,O_C(1)) \to 0$$

is exact. Hence $|L|$ is base point free. Moreover $h^0(X,L) = 2 + h^0(\ell,L_\ell) = 5$. The intersection number $L^3 = L\ell = 2$ because $h^0(\ell,sL_\ell) = 2s + 1$. In this case, the proof of (3.7) is complete.

Next we consider the case where ℓ_{red} is reducible. Then by (3.4) and (3.6), ℓ is a double line plus a nonsingular rational curve C',whence $cl(\ell) = 2cl(C) + cl(C')$ and $L\ell = 2$. We define a subsheaf I_2 of I_C by $I_2 = O_C(-1)+I_C^2$ via the isomophism $I_C/I_C^2 \cong O_C\oplus O_C(-1)$. Let $p = C \cap C'$. We note that with the notations in (3.4), $I_{2,p}$ (:= the stalk of I_2 at p) $= O_{X,p}x + O_{X,p}y^2$. Then we have exact sequences;

$$0 \to O_\ell(L) \to O_{C'}\oplus(O_X/I_2)(L) \to C^2(\cong O_X/I_{C'}+I_2) \to 0,$$
$$0 \to O_C(1) \to (O_X/I_2)(L) \to O_C(1) \to 0$$

because $I_C/I_2 \cong O_C$. We see that a subspace $H^0(O_{C'}) \oplus H^0((I_C/I_2)(L))$ of $H^0(O_{C'}) \oplus H^0((O_X/I_2)(L))$ is mapped onto $O_X/I_{C'}+I_2$ by the natural homomorphism. Therefore $h^0(X,L) = h^0(\ell,L_\ell) + 2 = 5$, Bs $|L|$ = Bs $|L_\ell| = \phi$. This completes the proof of (3.7) in Case 2.

Case 3. By (3.5) and (3.6), ℓ is reduced everywhere and $\ell = C_0 + \dots + C_m$ with $LC_0 = LC_m = 1$, $LC_j = 0$ $(1 \leq j \leq m-1)$. Then $L^3 = L\ell = L(C_0 + \dots + C_m) = 2$. Consider an exact sequence,

$$0 \to O_\ell(L) \to O_{C_0}(1) \oplus O_{C_1} \oplus \dots \oplus O_{C_{m-1}} \oplus O_{C_m}(1) \to C^m \to 0.$$

It follows from this that $h^0(X,L) = 2 + h^0(\ell, L_\ell) = 5$, and that $|L|$ is base point free.

Thus we complete the proof of (3.7). q.e.d.

(3.8) Completion of the proof of (2.1) by assuming (3.2)-(3.6). Let X be a compact complex threefold with a line bundle L satisfying the conditions in (2.1). By (3.7), we have a bimeromorphic morphism of X onto a hyperquadric in P^4. The image f(X) endowed with reduced structure is one of Q^3_ν $(\nu = 1,2,3)$. We note Pic X = ZL = Z[f^*H], where H is a hyperplane section of f(X) and [f^*H] is the line bundle associated with f^*H. If we are given a Weil divisor (an analytic two cycle) E of f(X), then f^*E is a Cartier divisor of X and $E = f_*(f^*E)$ because f is bimeromorphic. Since [f^*E] is an integral multiple of L, any Weil divisor of f(X) is homologically (algebraically) equivalent to an integral multiple of H [3, Theorem 1.4]. Hence $f(X) \neq Q^3_1$, Q^3_2 in view of (1.2) and (1.3). We note that over an algebraically closed field of arbitrary characteristic, any singular hyperquadric in P^4 has a Weil divisor which is not an integral multiple of a hyperplane section. Consequently $f(X) = Q^3$. Since f_* is an isomorphism of Pic X onto Pic Q^3 (= Z[H]), the exceptional set (det(Jac f)) of f is empty (see [11,(2.8)]). Therefore f is an isomorphism of X onto Q^3. q.e.d.

Before closing this section, we prepare three lemmas for sections 4-8.

(3.9) Lemma. Let $\ell = D \cap D'$ be the complete intersection in (3.1), C an irreducible component of ℓ_{red}, I_C the ideal sheaf of O_X defining C, $c_1(I_C/I_C^2) = s \in H^2(C,\mathbb{Z})$ ($\cong \mathbb{Z}$). Then $\chi(X, O_X/I_C^n) = n(n+1)(sn-s+3)/6$, $s = -3LC+2$.

Proof. The first assrtion is clear from Riemann-Roch for $C = P^1$. Next consider an exact sequence,

$$0 \to I_C/I_C^2 \to \Omega_X^1 \otimes O_C \to \Omega_C^1 \to 0.$$

Then we have $s = c_1(I_C/I_C^2) = K_X C + 2 = -3LC + 2$. q.e.d.

(3.10) Lemma. Let ℓ and C be the same as in (3.9). Let ϕ : $(I_\ell/I_\ell{}^2)\otimes O_C \to I_C/I_C^2$ be the natural homomorphism induced from the inclusion of I_ℓ into I_C. Then ϕ is injective everywhere on C iff ℓ is reduced at a point of C.

Proof. We note that $(I_\ell/I_\ell^2)\otimes O_C \cong O_C(-L)\oplus O_C(-L)$ is locally free, hence torsion free. Therefore the following conditions are equivalent to each other;

(3.10.1) ϕ is injective everywhere,

(3.10.2) ϕ is injective at a point q of C,

(3.10.3) $\mathrm{Coker}(\phi) = 0$ at a point p of C,

(3.10.4) $I_\ell + I_C^2 = I_C$ at a point p of C,

(3.10.5) $I_\ell = I_C$ at a point p of C.

Thus the assertion is clear. q.e.d.

(3.11) Lemma. *Let* I *and* I' $(\neq O_X)$ *be ideal sheaves of* O_X. *Suppose that* $I \subset I'$ *and* $h^1(O_X/I) = 0$, $\dim \operatorname{supp}(O_X/I) \leq 1$. *Then* $h^1(O_X/I') = 0$ *and* $\chi(O_X/I') \geq 1$.

Clear.

§ 4 Proof of (3.2)

We apply a method of Mori [9, pp. 167-170].

Assume that C is an irreducible component of ℓ_{red} with LC $\geqq 2$. Then by (3.1.1), we have LC = 2. Then $(I_\ell/I_\ell^2)\otimes O_C \cong O_C(-2)\oplus O_C(-2)$. Since $C = P^1$, by a theorem of Grothendieck, we express $I_C/I_C^2 = O_C(a)\oplus O_C(b)$, $a \geqq b$. By (3.9), $a+b = -4$.

(4.1) Lemma. $I_\ell \not\subset I_C^2$

Proof. Suppose $I_\ell \subset I_C^2$. Hence $h^1(O_X/I_C^2) = 0$ by (2.4.3). Hence $\chi(O_X/I_C^2) \geqq 1$. However by (3.9), $\chi(O_X/I_C^2) = s+3 = -1$ because $s = -4$. This is a contradiction. q.e.d.

In view of (4.1), we have a nontrivial natural homomorphism $\phi : (I_\ell/I_\ell^2)\otimes O_C \rightarrow I_C/I_C^2$. We shall prove

(4.2) Lemma. ϕ is injective.

Proof. Suppose not. Then both Ker ϕ and Im ϕ are torsion free sheaves of rank one, hence locally O_C-free. By a theorem of Grothendieck, we express Ker $\phi = O_C(c)$, Im $\phi = O_C(d)$ for some $c, d \in Z$. Then we have an exact sequence,

$$0 \rightarrow O_C(c) \rightarrow O_C(-2)\oplus O_C(-2) \rightarrow O_C(d) \rightarrow 0.$$

Hence $c + d = -4$, $b \leqq c \leqq -2 \leqq d \leqq a$. Now we shall prove $b = d$ (hence $a = b = c = d = -2$). Assume $b < d$ to derive a contradiction. Then since $Hom_{O_C}(O_C(d), O_C(b)) = 0$, the sheaf $O_C(d)$ is contained in a direct summand $O_C(a)$ of I_C/I_C^2. Here we note that if $b < d$, then $b < a$ so that the subsheaf $O_C(a)$ in the splitting of I_C/I_C^2 is uniquely determined in I_C/I_C^2 . Define a subsheaf I of I_C by $I = O_C(a) + I_C^2$. Then we see readily that $I_C \supset I \supset I_\mathfrak{L}$, $I/I_C^2 \cong O_C(a)$, $I_C/I \cong O_C(b)$. By $H^1(O_\mathfrak{L}) = H^1(O_X/I_\mathfrak{L}) = 0$, we have,

$$1 \leqq \chi(O_X/I) = \chi(O_X/I_C) + \chi(I_C/I) = 2 + b,$$

whence $b \geqq -1$. This contradicts $b \leqq c \leqq -2$. Hence $a = b = c = d = -2$. Next we let $J = \mathrm{Im}\,\phi + I_C^2 = I_\mathfrak{L} + I_C^2$. Then $J \supset I_\mathfrak{L}$, $I_C/J \cong O_C(-2)$. Therefore

$$1 \leqq \chi(O_X/J) = \chi(O_X/I_C) + \chi(O_C(-2)) = 0,$$

which is a contradiction. q.e.d.

(4.3) <u>Completion of the proof of</u> (3.2). By (4.2), we have the exact sequence,

$$0 \rightarrow (I_\mathfrak{L}/I_\mathfrak{L}^2)\otimes O_C \rightarrow I_C/I_C^2$$

Therefore $-2 \leqq a$, $-2 \leqq b$, whence $a = b = -2$. Hence $(I_\mathfrak{L}/I_\mathfrak{L}^2)\otimes O_C \cong I_C/I_C^2$. Let p be a point of X, $I_{\mathfrak{L},p}$ (resp. $I_{C,p}$) be the stalk of $I_\mathfrak{L}$ (resp. I_C) at p. Then $I_{\mathfrak{L},p} + I_{C,p}^2 = I_{C,p}$ for any point p of C, whence $I_{\mathfrak{L},p} = I_{C,p}$. This shows that $\mathfrak{L}$ is isomorphic to C anywhere on C. Since $\mathfrak{L}_{red}$ is connected by (2.4.4), $\mathfrak{L}$ is isomorphic to C.

This completes the proof of (3.2). q.e.d.

§ 5 Proof of (3.3)

(5.1) Lemma. <u>Assume that</u> C <u>is an irreducible component of</u> ℓ_{red} <u>with</u> $LC = 1$. <u>Then</u> $I_C/I_C^2 \cong O_C \oplus O_C(-1)$.

<u>Proof</u>. Since $C = \mathbf{P}^1$, we express $I_C/I_C^2 = O_C(a)\oplus O_C(b)$, $a \geq b$. Then by (3.9), $a + b = c_1(I_C/I_C^2) = -3LC + 2 = -1$, $a \geq 0$, $b \leq -1$. We shall show $a = 0$, $b = -1$. We assume $b \leq -2$ to derive a contradiction. Consider the natural homomorphism ϕ : $(I_\ell/I_\ell^2)\otimes O_C \to I_C/I_C^2$. When $b \leq -2$, Im ϕ is contained in $O_C(a) = O_C(a)\oplus\{0\}$. Let $I = O_C(a)+I_C^2$. Then $I_C \supset I \supset I_\ell$ and $I_C/I \cong O_C(b)$. Hence by (3.11),

$$1 \leq \chi(O_X/I) = \chi(O_X/I_C) + \chi(I_C/I) = 2 + b.$$

This is a contradiction. Hence $a = 0$, $b = -1$. q.e.d.

In what follows, we assume that C is an irreducible component of ℓ_{red} with $LC = 1$ along which ℓ is nonreduced anywhere.

(5.2) Lemma. $I_\ell \not\subset I_C^2$.

<u>Proof</u>. Assume $I_\ell \subset I_C^2$. Let $I = I_C I_\ell$. Then $I_\ell \supset I \supset I_\ell^2$, $I_C^3 \supset I$ and $I_\ell/I \cong (I_\ell/I_\ell^2)\otimes O_C \cong O_C(-1)\oplus O_C(-1)$. Therefore by (2.4), $h^0(O_X/I) = 1$, $h^1(O_X/I) = 0$. Consider the natural inclusion

$$\iota : I + I_C^4/I_C^4 \to I_C^3/I_C^4 \cong O_C\oplus O_C(-1)\oplus O_C(-2)\oplus O_C(-3).$$

Then Im ι is contained in $O_C\oplus O_C(-1)\oplus O_C(-2)$ because the following natural homomorphism is surjective,

$$(I_C/I_C^2)\otimes(I_\ell/I_\ell^2)\otimes O_C \to I_C I_\ell+I_C^4/I_C^4 = I+I_C^4/I_C^4 \quad (\subset I_C^3/I_C^4)$$

and $(I_C/I_C^2)\otimes(I_\ell/I_\ell^2)\otimes O_C \cong O_C(-1)\oplus O_C(-1)\oplus O_C(-2)\oplus O_C(-2)$.

Let $I' = O_C \oplus O_C(-1) \oplus O_C(-2) + I_C^4$. Then $I_C^3 \supset I' \supset I$, $I_C^3/I' \cong O_C(-3)$. By $h^1(O_X/I) = 0$ and (3.11), we have,

$$1 \leq \chi(O_X/I') = \chi(O_X/I_C^3) + \chi(O_C(-3)) = 0$$

which is a contradiction. Hence $I_\ell \not\subset I_C^2$. q.e.d.

(5.3) <u>Completion of the proof of</u> (3.3). We consider the natural homomorphism $\phi : (I_\ell/I_\ell^2) \otimes O_C \rightarrow I_C/I_C^2$. By (5.2), Im ϕ is not zero. Since Im ϕ ($\cong I_\ell + I_C^2/I_C^2$) is a subsheaf of a torsion free sheaf I_C/I_C^2, it is locally O_C-free. Since ℓ is nonreduced along C, Im ϕ is of rank one by (3.10). Here we may set Im $\phi = O_C(c)$ for some $c \in \mathbf{Z}$. Then $c = 0$ or -1 because $(I_\ell/I_\ell^2) \otimes O_C \cong O_C(-1) \oplus O_C(-1)$. In view of our assumption in (3.3), Im $\phi \cong O_C$, Ker $\phi \cong O_C(-2)$. Let $E = \text{Coker } \phi \cong O_C(-1)$, $F = \text{Im } \phi \cong O_C$. Then we may view $I_C/I_C^2 = E \oplus F$ because $H^1(C, E^\vee \otimes F) = 0$, $E^\vee$ being the dual of E. So we consider again the homomorphism ϕ as,

$$\phi : (I_\ell/I_\ell^2) \otimes O_C \rightarrow F \subset E \oplus F = I_C/I_C^2.$$

Let p be an arbitrary point of C. Then there are two generators x ,y of $I_{C,p}$, and two generators f, g of $I_{\ell,p}$ such that $\phi(f) = x$, $\phi(g) = 0$, x mod $I_{C,p}^2$ (resp. y mod $I_{C,p}^2$) generates F (resp. E). Since f = x mod $I_{C,p}^2$, it is easy to see that f and y generates $I_{C,p}$ over $O_{X,p}$, so that we may take f instead of x. Then by deleting an $O_{X,p}$-multiple of x from g, we may assume $g = \beta y^m$ for some $\beta \in O_{X,p}$ and $m > 0$, the restriction of β to C being not identically zero. Thus we obtain local parameters x and $y \in I_{C,p}$ and $\beta \in O_{X,p}$, $m > 0$ such that

(5.3.1) $\quad I_{C,p} = O_{X,p}x + O_{X,p}y$

$$I_{\ell,p} = O_{X,p}x + O_{X,p}\beta y^m$$

where the restriction β_C of β to C is not identically zero. The integer m is uniquely determined by the point p, but it is independent of the choice of $p \in C$. We note that $m \geq 2$ because $g \in I_{C,p}^2$.

Let $\Phi = \{U_j\}$ be a sufficiently fine covering of an open neighborhood of C by Stein (or affine) open sets U_j. Then by (5.3.1), we have $x_j \in \Gamma(U_j, I_\ell)$, $y_j \in \Gamma(U_j, I_C)$, $\beta_j \in \Gamma(U_j, O_X)$ such that

(5.3.2) $\Gamma(U_j, I_C) = \Gamma(U_j, O_X)x_j + \Gamma(U_j, O_X)y_j$

$\Gamma(U_j, I_\ell) = \Gamma(U_j, O_X)x_j + \Gamma(U_j, O_X)\beta_j y_j^m$.

Since $I_C/I_C^2 = F \oplus E \cong O_C \oplus O_C(-1)$, we may assume that

(5.3.3) $x_j = x_k \bmod I_C^2$, $y_j = \ell_{jk} y_k \bmod I_C^2$,

where ℓ_{jk} stands for the one cocycle $L_C = O_C(1) \in H^1(C, O_C^*)$. Note that the second equation in (5.3.3) does make sense.

Hence $D_j \beta_j y_j^m$ and $D_k \beta_k y_k^m \in \operatorname{Ker} \phi$ ($\cong O_C(-2)$) are identified iff (we may assume that)

(5.3.4) $(D_{j|C}) = \ell_{jk}^{-2}(D_{k|C})$.

This shows that

(5.3.5) $(\beta_{j|C}) = \ell_{jk}^{2-m}(\beta_{k|C})$.

In particular, $\beta_C := \{\beta_{j|C};\ U_j \in \Phi\}$ is a nontrivial element of $H^0(C, O_C(2-m))$. This is possible only when $m = 2$ and β_C is a nonzero constant. Consequently ℓ_{red} is nonsingular anywhere on C, and it is isomorphic to C because it is connected by (2.4.4). Moreover ℓ is "a double line" in the sense that at any point p of C, there exist local parameters x, $y \in I_{C,p}$ such that

$$I_{C,p} = O_{X,p}x + O_{X,p}y,$$
$$I_{\ell,p} = O_{X,p}x + O_{X,p}y^2.$$

This completes the proof of (3.3). q.e.d.

§ 6 Proof of (3.4)

Let C be an irreducible component of ℓ_{red} with LC = 1, along which ℓ is nonreduced anywhere.

By (5.3), there are two possibilities Im $\phi \cong O_C$ or $O_C(-1)$. The case Im $\phi \cong O_C$ was discussed in section 5. In this section, we shall discuss the case Im $\phi \cong O_C(-1)$. We note that via the isomorphism $I_C/I_C^2 \cong O_C \oplus O_C(-1)$, the subsheaf $O_C = O_C \oplus \{0\}$ of I_C/I_C^2 is uniquely determined. First we prove

(6.1) Lemma. Assume Im $\phi \cong O_C(-1)$. Then Im ϕ is not contained in O_C $(= O_C \oplus \{0\}) \subset I_C/I_C^2$.

Proof. Assume Im $\phi = O_C(-1) \subset O_C$ to derive a contradiction. Let p be an arbitrary point of C. Then there are two generators x,y of $I_{C,p}$ and two generators f,g of $I_{\ell,p}$ such that x mod $I_{C,p}^2$ (resp. y mod $I_{C,p}^2$) generates $O_C \oplus \{0\}$ (resp. $\{0\} \oplus O_C(-1)$) in $I_{C,p}/I_{C,p}^2$, and $\phi(f)$ generates Im ϕ, $\phi(g) = 0$, or equivalently g $\in I_{\ell,p} \cap I_{C,p}^2$. Since Im ϕ is contained in $O_C \oplus \{0\}$, f = αx mod $I_{C,p}^2$ for some $\alpha \in O_{X,p}$. Thus we obtain,

(6.1.1) $$I_{C,p} = O_{X,p}x + O_{X,p}y,$$
$$I_{\ell,p} = O_{X,p}f + O_{X,p}g,$$
$$f = \alpha x \bmod I_{C,p}^2$$

where $\alpha \in O_{X,p}$, $g \in I_{\ell,p} \cap I_{C,p}^2$.

Let $\Phi = \{U_j\}$ be a sufficiently fine covering of an open neighborhood of C by Stein (or affine) open sets U_j. Then by (6.1.1), we have $x_j, y_j \in \Gamma(U_j, I_C)$, $f_j \in \Gamma(U_j, I_\ell)$, $g_j \in \Gamma(U_j, I_\ell \cap I_C^2)$, $\alpha_j \in \Gamma(U_j, O_X)$ such that

$$(6.1.2)\quad \begin{aligned} &\Gamma(U_j,I_C) = \Gamma(U_j,O_X)x_j + \Gamma(U_j,O_X)y_j \\ &\Gamma(U_j,I_\ell) = \Gamma(U_j,O_X)f_j + \Gamma(U_j,O_X)g_j \\ &f_j = \alpha_j x_j \mod I_C^2 , \end{aligned}$$

Moreover by the choice of the generators, we may assume

$$(6.1.3)\quad \begin{aligned} &f_j = \ell_{jk}f_k \mod I_C I_\ell , \quad g_j = \ell_{jk}g_k \mod I_C I_\ell , \\ &x_j = x_k \mod I_C^2 , \quad y_j = \ell_{jk}y_k \mod I_C^2 . \end{aligned}$$

where ℓ_{jk} is the one cocycle $L_C = O_C(1) \in H^1(C,O_C^*)$.

Then one checks that $\alpha_C := \{\alpha_{j|C}\}$ is a nontrivial element of $H^0(C,O_C(1))$. Hence α_C has a single zero at a point p_0 of C and it vanishes nowhere else.

If $\alpha_{|C}$ is nonvanishing at p in (6.1.1), then f and y generates $I_{C,p}$, so that we may take f instead of x and can normalize g as βy^m for some $\beta \in O_{X,p}$ and for some $m \geq 2$ so that the restriction of β to C is not identically zero . The integer m is independent of the choice of p.

If $\alpha_{|C}$ has a single zero at p in (6.1.1), then $z := \alpha$ forms a regular sequence at p together with the parameters x and y. Since $f = zx \mod I_{C,p}^2$ in (6.1.1), we may assume, by a suitable coordinate change, that $I_{C,p} = O_{X,p}x + O_{X,p}y$, $f = zx$ or $zx - y^s$ for some $s \geq 2$.

Therefore by taking a suitable refinement of Φ if necessary, we may assume that

(6.1.4) U_j contains p_0 iff $j = 0$,

$$(6.1.5)\quad \begin{aligned} &\Gamma(U_j,I_C) = \Gamma(U_j,O_X)x_j + \Gamma(U_j,O_X)y_j , \\ &\Gamma(U_j,I_\ell) = \Gamma(U_j,O_X)x_j + \Gamma(U_j,O_X)g_j , \end{aligned}$$

for $j \neq 0$, $g_j = \beta_j y_j^m$, $\beta_j \in \Gamma(U_j,O_X)$, $m \geq 2$, $\beta_{j|C}$ being not identically zero, (6.1.6) $\quad \Gamma(U_0, I_C) = \Gamma(U_0, O_X)x_0 + \Gamma(U_0, O_X)y_0$

$$\Gamma(U_0, I_\ell) = \Gamma(U_0, O_X)f_0 + \Gamma(U_0, O_X)g_0$$

$$f_0 = z_0x_0 \quad \text{or } z_0x_0 - y_0^s \quad (s \geq 2)$$

where x_0, y_0 and z_0 form a regular sequence everywhere on U_0, $g_0 \in I_C^2$, and moreover

(6.1.7) $\quad \beta_j$ ($j \neq 0$) (resp. z_0) vanishes nowhere on U_{ij} (: $= U_i \cap U_j$, $i \neq j$) (resp. on U_{0j}).

Now we define β_0 as follows. Let $x = x_0$, $y = y_0$, $z = z_0$.

<u>Case</u> 0. Assume $f_0 = zx$. Then the second generator g_0 of I_ℓ is normalized (mod f_0) into $g_0 = A_n(x,y)x + B_n(y,z)y^n$ for some $n \geq 2$, $A_n \in \Gamma(U_0, I_C)$, $B_n \in \Gamma(U_0, O_X)$, B_n being not identically zero on C. At a general point q of C sufficiently close to p, $I_{C,q}$ (resp. $I_{\ell,q}$) is generated by x and y (resp. x and y^m) by (6.1.7) because β does not vanish at q. It follows that $n \geq m$. We now define

$$\beta_0 = B_n(y,z)y^{n-m}.$$

Next we consider the case $f = zx - y^s$, $s \geq 2$.

<u>Case</u> 1. Assume $s > m$. We can choose f_j such that $f_j = x_j$ for $j \neq 0$ and $f_j = \ell_{jk}f_k$ mod I_CI_ℓ for any j,k. We see

$$x = (f_0 + y^s)/z = (\ell_{0j}/z)x_j \mod I_CI_\ell .$$

$$y = c_{0j}x_j + d_{0j}y_j$$

for some c_{0j} and d_{0j} such that $c_{0j|C} = 0$, $d_{0j|C} = \ell_{0j}$. On the other hand,

$$g_0 = A_2(x,y)x + B_2(y,z)y^2$$

for some $A_2 \in \Gamma(U_0, I_C)$, $B_2 \in \Gamma(U_0, O_X)$. Since $\Gamma(U_{0j}, I_CI_\ell)$ is generated by x_j^2, x_jy_j, y_j^{m+1}, we have,

$$g_0 = B_2(y,z)y^2 \qquad \mod I_CI_\ell$$

$$= B_2(d_{0j}y_j, z)(d_{0j}y_j)^2 \quad \mathrm{mod}\ (x_j^2, x_jy_j, y_j^{m+1})$$

$$= 0 \quad \mathrm{mod}\ (x_j^2, x_jy_j, y_j^m)$$

by (6.1.5), whence $B_2(d_{0j}y_j, z)$ is divisible by $(d_{0j}y_j)^{m-2}$. Hence $B_2(y,z)$ is divisible by y^{m-2}. So we define

$$\beta_0 = B_2(y,z)y^{-m+2}.$$

<u>Case</u> 2. Assume $s \leqq m$. Let $g_0 = \sum_{\substack{\nu+\mu\geqq 1\\ \nu,\mu\geqq 0}} A_{\nu\mu}(z)x^\nu y^\mu$. Then on U_{0j}, we see $g_0 = \sum_{\nu+\mu\geqq 1} A_{\nu\mu}(z)y^{\nu s+\mu}/z^\nu \ \mathrm{mod}\ I_C I_\Omega$. By (6.1.7), we have $\sum_{\nu s+\mu=k} A_{\nu\mu}/z^\nu = 0$ for $k < m$, and we define

$$\beta_0 = \sum_{\substack{\nu s+\mu=m\\ \nu,\mu\geqq 0}} A_{\nu\mu}/z^\nu.$$

By these definitions, we have,

$$\beta_0 y_0^m = g_0 \quad \mathrm{mod}\ I_C I_\Omega \quad \text{on } U_{0j}$$

$$= \ell_{0j} g_j \quad \mathrm{mod}\ I_C I_\Omega \quad \text{by (6.1.3)}$$

$$= \ell_{0j}\beta_j y_j^m \quad \mathrm{mod}\ I_C I_\Omega \quad \text{by (6.1.5)}$$

$$\beta_i y_i^m = \ell_{ij}\beta_j y_j^m \quad \mathrm{mod}\ I_C I_\Omega \quad (i,j \neq 0).$$

Therefore we have

$$(\beta_{i|C}) = \ell_{ij}^{1-m}(\beta_{j|C}) \quad \text{for any } i,j.$$

In Case 0 and Case 1, β_0 is holomorphic on U_0. In Case 2, $\beta_{0|C}$ is holomorphic except at p_0 and meromorphic at p_0, and it has a pole at p_0 of order at most $\nu_{max} := \max\{\nu;\ \nu s + \mu = m,\ A_{\nu\mu} \neq 0\}$. Clearly $\nu_{max} \leqq m-2$ when $s < m$. Hence $\beta_C := \{\beta_{j|C};\ U_j \in \Phi\}$ is a nontrivial element of $H^0(C, \mathcal{O}_C(1-m+\nu_{max})) = \{0\}$, which is a contradiction. Thus Im ϕ is not contained in $\mathcal{O}_C$.

q.e.d.

(6.2) <u>Completion of the proof of</u> (3.4). Let $E = \mathrm{Coker}\,\phi$, $F = \mathrm{Im}\,\phi$. Then by (6.1), $E \cong O_C$, $F \cong O_C(-1)$ and we may view $I_C/I_C^2 = E \oplus F$ because $H^1(C, E^\vee \otimes F) = 0$. So we consider again the homomorphism ϕ as,

$$\phi \ : \ (I_\ell/I_\ell^2)\otimes O_C \ \to \ F \ \subset E \oplus F = I_C/I_C^2.$$

Let p be an arbitrary point of C. Then there are two generators x, y of $I_{C,p}$, and two generators f, g of $I_{\ell,p}$ such that $\phi(f) = x$, $\phi(g) = 0$, and that $x \bmod I_{C,p}^2$ (resp. $y \bmod I_{C,p}^2$) generates F (resp. E). Since $f = x \bmod I_{C,p}^2$, we may take f instead of x. Then in the same manner as in (5.3), by taking a sufficiently fine covering $\Phi = \{U_j\}$ of an open neighborhood of C by Stein open sets U_j, we have $x_j \in \Gamma(U_j, I_\ell)$, $y_j \in \Gamma(U_j, I_C)$, $\beta_j \in \Gamma(U_j, O_X)$ such that x_j and y_j (resp. x_j and $\beta_j y_j^m$) generate $\Gamma(U_j, I_C)$ (resp. $\Gamma(U_j, I_\ell)$).

Since $I_C/I_C^2 = F \oplus E \cong O_C(-1)\oplus O_C$, we may assume

(6.2.1) $\quad x_j = \ell_{jk}x_k \bmod I_C^2, \quad y_j = y_k \bmod I_C^2$

where ℓ_{jk} stands for the one cocycle $L_C \in H^1(C, O_C^*)$. Since Ker ϕ is isomorphic to $O_C(-1)$, and it is generated locally by $\beta_j y_j^m$, we see that

(6.2.2) $\quad (\beta_{j|C}) = \ell_{jk}(\beta_{k|C})$.

In particular, $\beta_C := \{\beta_{j|C};\ U_j \in \Phi\}$ is a nontrivial element of $H^0(C, O_C(1))$ by (6.2.2). Consequently β_C has a single zero at a unique point $p_0 \in C$ and it vanishes nowhere else. Then $z := \beta_0$ forms a regular sequence at p_0 with the parameters x and y.

The curve C intersects a unique irreducible component C' of ℓ_{red} at p_0, but nowhere else. In particular, ℓ_{red} is reducible. Let $I_{C'}$ be the ideal sheaf of O_X defining C'. Then $I_{C',p_0} = O_{X,p_0}x + O_{X,p_0}z$. By the assumption in (3.4), we have $\delta := LC' \geqq 0$. Let $I_{C'}/I_{C'}^2 \cong O_{C'}(a) \oplus O_{C'}(b)$, $a \geqq b$, and $\phi_{C'}$: $(I_{\mathfrak{Q}}/I_{\mathfrak{Q}}^2) \otimes O_{C'} \to I_{C'}/I_{C'}^2$ the natural homomorphism. Then since $\mathfrak{Q}$ is reduced generically along C', we have by (3.9) $a + b = -3\delta+2$, and $\dim(\mathrm{Coker}\ \phi_{C'}) = a + b + 2\delta = -\delta + 2$. Since $\beta_0 = z$ is a local parameter, $\mathrm{Coker}\ \phi_{C',p_0} \cong \mathbf{C}[y]/(y^m)$, whence $m \leqq -\delta + 2$. Hence $m = 2$, $\delta = 0$, $\mathrm{Coker}\ \phi_{C'} = \mathrm{Coker}\ \phi_{C',p_0} = \mathbf{C}[y]/(y^2)$. By $(I_{\mathfrak{Q}}/I_{\mathfrak{Q}}^2) \otimes O_{C'} \cong O_{C'} \oplus O_{C'}$ and the form of $\mathrm{Coker}\ \phi_{C',p_0}$, we have $a = 2$, $b = 0$. Moreover this shows that C' meets no irreducible component other than C.

Thus the proof of (3.4) is complete. q.e.d.

§ 7 Proof of (3.5)

Assume that there is an irreducible component C_0 of $\mathfrak{L}_{red}$ with $LC_0 = 1$ such that $\mathfrak{L}$ is reduced at a point of C_0. Assume moreover that C_0 intersects an irreducible component C_1 of $\mathfrak{L}_{red}$ not contained in Bs $|L|$.

(7.1) Lemma. Let $C = C_0$. We have,

(7.1.1) C intersects the unique irreducible component C' of $\mathfrak{L}_{red} - C$ (:= the closure of $\mathfrak{L}_{red} \setminus C$) at a unique point p transversally, to be more precise, we can choose local parameters x, y and z at p such that

$$I_{C,p} = O_{X,p}x + O_{X,p}y,$$

$$I_{\mathfrak{L},p} = O_{X,p}x + O_{X,p}zy,$$

(7.1.2) $\mathfrak{L}$ is reduced everywhere on C, and reduced generically along C'.

Proof. In view of (5.1), $I_C/I_C^2 \cong O_C \oplus O_C(-1)$. We consider the natural homomorphism $\phi : (I_{\mathfrak{L}}/I_{\mathfrak{L}}^2) \otimes O_C (\cong O_C(-1) \oplus O_C(-1)) \rightarrow I_C/I_C^2$. By (3.10), ϕ is injective and Coker $\phi \cong O_C/O_C(-1) \cong \mathbb{C}$. Let p be Supp Coker ϕ. The in the same manner as in (5.3), we can find local parameters x, y, w and a germ $\beta \in O_{X,p}$ such that

$$I_{C,p} = O_{X,p}x + O_{X,p}y,$$

$$I_{\mathfrak{L},p} = O_{X,p}x + O_{X,p}\beta(y,w)y^m$$

where $\beta(0,w)$ is not identically zero and $m \geqq 1$. Since ϕ is injective, we have $m = 1$. Moreover we see that $\beta(0,w)$ has a single zero at p. Hence $\beta(y,w)$ forms a parameter system at p with x and y. So (7.1.1) is clear by setting $z = \beta(y,w)$. (7.1.2) is clear from (7.1.1). q.e.d.

(7.2) Completion of the proof of (3.5). By the assumption, the irreducible component C_0 of $\mathfrak{L}_{red}$ with $LC_0 = 1$ intersects an irreducible component C_1 of $\mathfrak{L}_{red}$ which is not contained in Bs $|L|$. Then $LC_1 = 0$ or 1. Assume $LC_1 = 0$. Let $I_{C_1}/I_{C_1}^2 = O_{C_1}(a) \oplus O_{C_1}(b)$, $a \geqq b$. By (7.1.2), $\mathfrak{L}$ is reduced generically along C_1, whence the natural homomorphism $\phi_1 : (I_{\mathfrak{L}}/I_{\mathfrak{L}}^2) \otimes O_{C_1} \to I_{C_1}/I_{C_1}^2$ is injective. Hence $a \geqq 0$, $b \geqq 0$. Moreover $a + b = -3LC_1 + 2 = 2$. Therefore dim Coker $\phi_1 = 2$, $(a,b) = (1,1)$ or $(2,0)$. Since ϕ_1 is not surjective at $p_0 := C_0 \cap C_1$, there is a unique point p_1 of C_1, $p_1 \neq p_0$ such that ϕ_1 is not surjective at p_1. By the same argument as in (7.1), we can choose local parameters x,y,z at p_1 such that

$$I_{C_1,p_1} = O_{X,p_1}x + O_{X,p_1}y,$$

$$I_{\mathfrak{L},p_1} = O_{X,p_1}x + O_{X,p_1}zy.$$

Consequently there is the third component C_2 of $\mathfrak{L}_{red}$ intersecting C_1. Then C_2 is not contained in Bs $|L|$. Otherwise, C_1 is contained in Bs $|L|$ because $LC_1 = 0$. Hence $LC_2 = 0$ or 1. If $LC_2 = 0$, then we repeat the same argument as above and after a finite repetition of these steps, we eventually obtain C_m and a chain of rational curves $C_1, \ldots, C_{m-1}$ of $\mathfrak{L}_{red}$ such that $LC_j = 0$ $(1 \leqq j \leqq m-1)$ and $LC_m = 1$, and the pair C_j and C_k $(j < k)$ intersect at a unique point p_j transversally iff $j = k-1$. By the same argument as above no C_j $(0 \leqq j \leqq m)$ is contained in Bs $|L|$. Moreover by (5.1) and (7.1), $I_{C_m}/I_{C_m}^2 = O_{C_m} \oplus O_{C_m}(-1)$ and C_m intersects C_{m-1} only. By (2.4.4), $\mathfrak{L}$ is connected so that it is the union of $C_0, \ldots, C_m$. Hence $\mathfrak{L}$ is reduced everywhere.

Thus the proof of (3.5) is complete.

§ 8 Proof of (3.6)

(8.1) Lemma. <u>Let</u> C <u>be an arbitrary irreducible component of</u> ℓ_{red} <u>with</u> LC = 1. <u>Then we have,</u>

(8.1.1) $I_C/I_C^2 \cong O_C \oplus O_C(-1)$,

(8.1.2) C <u>intersects</u> $\ell_{red} - C$ (:= <u>the closure of</u> $\ell_{red}\setminus C$) <u>at a unique point</u> p <u>transversally, to be more precise, we can choose local parameters</u> x,y <u>and</u> z <u>at</u> p <u>such that</u>

$$I_{C,p} = O_{X,p}x + O_{X,p}y,$$
$$I_{\ell,p} = O_{X,p}x + O_{X,p}zy^m ,$$

for some $m \geq 1$, <u>we call</u> m <u>the multiplicity of</u> C <u>in</u> ℓ

(8.1.3) C <u>is not contained in</u> Bs |L|.

<u>Proof</u>. The assertion (8.1.1) follows from (5.1). If ℓ is reduced at a point of C, then (8.1.2) follows in the same manner as in (7.1). Next we consider the case where ℓ is nonreduced along C. Consider the natural homomorphism $\phi : (I_\ell/I_\ell^2)\otimes O_C \rightarrow I_C/I_C^2$. Then by (3.3) and (6.1), Im $\phi = O_C(-1)$ and Im ϕ is not contained in $O_C\oplus\{0\}$. Let E = Coker ϕ, F = Im ϕ. Then we may view I_C/I_C^2 = E $\oplus$ F and consider the homomorphism ϕ as,

$$\phi : (I_\ell/I_\ell^2)\otimes O_C \rightarrow F \subset E \oplus F = I_C/I_C^2.$$

Then we are able to choose an open covering $\Phi = \{U_j\}$ of an open neighborhood of C and x_j,y_j and β_j satisfying (5.3.2). Here we may assume that

$$x_j = \ell_{jk}x_k \bmod I_C^2,\ y_j = y_k \bmod I_C^2,\ \beta_{j|C} = \ell_{jk}\beta_{k|C}$$

and that x_j (resp. y_j) generates F (resp. E). Hence $\beta_C = \{\beta_{j|C};U_j \in \Phi\}$ is a nontrivial element of $H^0(C,O_C(1))$, whence β_C has a single zero at a unique point p_0 of C. Then $x = x_j, y = y_j$

and $z = \beta_j$ at p_0 form a regular parameter system at p_0. This completes the proof of (8.1.2).

Now we are able to construct a partial "normalization" of ℓ by using the expression (8.1.2) of I_C and I_ℓ as follows;

With the notations in (8.1.2), we define an ideal subsheaf $I_{\ell'}$ of O_X by;

$$I_{\ell',p} = I_{\ell,p} \qquad (p \in X \setminus C)$$
$$I_{\ell',p} = O_{X,p}x + O_{X,p}z \qquad (p = p_0)$$
$$I_{\ell',p} = O_{X,p} \qquad (p \in C \setminus \{p_0\})$$

where $I_{\ell',p}$ is the stalk of $I_{\ell'}$ at p. Let ℓ' be an analytic subspace of X with $\ell'_{red} = (\ell_{red} \setminus C) \cup \{p_0\}$, $O_{\ell'} = O_X/I_{\ell'}$, and $I_k = I_C^k + I_\ell$ $(1 \leqq k \leqq m)$. Then we have exact sequences;

(8.1.4) $\quad 0 \to O_\ell \to O_X/I_{\ell'} \oplus O_X/I_m \to O_X/I_m+I_{\ell'} \to 0,$

(8.1.5) $\quad 0 \to I_{k-1}\cap I_{\ell'}/I_k\cap I_{\ell'} \to I_k+I_{\ell'}/I_k \to I_{k-1}+I_{\ell'}/I_{k-1} \to 0,$

(8.1.6) $\quad 0 \to I_k+I_{\ell'}/I_k \to O_X/I_k \to O_X/I_k+I_{\ell'} \to 0$

We note that $O_X/I_k+I_{\ell'} = C[y]/(y^k), I_{k-1}/I_k \cong O_C, I_{k-1}\cap I_{\ell'}/I_k\cap I_{\ell'} = O_C(-p_0)$. Let $V_k = H^0((I_k+I_{\ell'}/I_k)\otimes O_X(L))$, $\eta_{i,j}$ the natural homomorphism of V_i into V_j for $i > j$. From (8.1.4)-(8.1.6) tensored by $O_X(L)$, we infer long exact sequences,

(8.1.7) $\quad 0 \to H^0(O_\ell(L)) \to H^0((O_X/I_{\ell'})(L))\oplus H^0((O_X/I_m)(L)) \to C^m$

(8.1.8) $\quad 0 \to H^0(O_C) \to V_k \to V_{k-1} \to 0$

(8.1.9) $\quad 0 \to V_k \to H^0((O_X/I_k)(L)) \to O_X/I_k+I_{\ell'} \ (\cong C^k)$

Then by (8.1.9), $V_k = \mathrm{Ker}(H^0((O_X/I_k)(L)) \to O_X/I_k+I_{\ell'})$, whereas V_m is a subspace of $H^0(O_\ell(L))$ by (8.1.7). By (8.1.8), $\eta_{k,k-1}$ is surjective and $\dim V_k = \dim V_{k-1} + 1$, whence $\eta_{m,1} = \eta_{m,m-1}\eta_{m-1,m-2}\cdots\eta_{2,1} : V_m \to V_1$ is surjective. Since $V_1 = H^0(C,L_C-p_0)$(:= elements of $H^0(C,L_C)$ vanishing at p_0) is a nontrivial subspace of $H^0(C, L_C)$, $C \setminus \{p_0\}$ is disjoint from Bs $|L_\ell|$ (= Bs $|L|$ by (2.6)). This completes the proof of (8.1.3).

q.e.d.

(8.2) Corollary. $\dim V_k = k$, $h^0((O_X/I_k)(L)) = 2k$.

Proof. By the above proof, $\dim V_k = \dim V_1 + k - 1$. From the exact sequence

$$0 \to (I_{k-1}/I_k)(L) \to (O_X/I_k)(L) \to (O_X/I_{k-1})(L) \to 0$$

and $(I_{k-1}/I_k)(L) \cong O_C(1)$, we infer $h^0((O_X/I_k)(L)) = h^0((O_X/I_{k-1})(L)) + 2$, whence the second assertion. q.e.d.

(8.3) Proof of (3.6) — Start. Assume that there is an irreducible component C of ℓ_{red} with LC = 1 such that C intersects an irreducible component C' of ℓ_{red} not contained in Bs $|L|$. Then by (3.2)-(3.5) and (3.7), Bs $|L|$ is empty so that for any $\ell' = D'' \cap D''$, D'', $D'' \in |L|$, any irreducible component C" of ℓ'_{red} with LC" = 1 intersects a component C" of ℓ'_{red} not contained in Bs $|L|$. Therefore it remains to consider the case where for any $\ell = D \cap D'$, $D, D' \in |L|$, any irreducible component C of ℓ_{red} with LC = 1 intersects a component of ℓ_{red} contained in Bs $|L|$. Then C is not contained in Bs $|L|$ and there is a unique irreducible component C' of ℓ_{red} intersecting C by (8.1). In what follows, we assume this to derive a contradiction in (8.10).

First we shall prove,

(8.4) Lemma. Let C_j $(1 \leq j \leq s)$ be all the irreducible components of ℓ_{red} with $LC_j = 1$, B_j the unique irreducible component of $\mathfrak{L}_{red}$ contained in Bs $|L|$ that C_j intersects. By choosing a general pair D and D', $B_1 = B_2 = \ldots = B_s$.

Proof. We apply a variant of the argument in [11,(2.6)]. Assume the contrary. Then we can choose a one parameter family D'_t $(t \in P^1)$ and a Zariski dense open subset U of P^1 with the following properties;

(8.4.1) $\mathfrak{L}_{t,red} = C_{1,t} + \ldots + C_{s(t),t} + B_1 + B_2 + \ldots$, $t \in U$, $B_j \subset$ Bs $|L|$, $B_1 \neq B_2$ where $\mathfrak{L}_t = D \cap D'_t$,

(8.4.2) $LC_{j,t} = 1$, $(1 \leq j \leq s(t))$,

(8.4.3) $C_{1,t}$ (resp. $C_{2,t}$) intersects B_1 (resp. B_2) for any $t \in U$.

Let d (resp. d'_t) be the equation defining D (resp. D'_t), and define an analytic subset Z of $X \times P^1$ by $Z = \{(x,t) \in X \times P^1;\ d(x) = d'_t(x) = 0\}$. Let p_j be the j-th projection of $X \times P^1$, Z_j all the irreducible components of Z_{red}, $g_j : Y_j \to Z_j$ the normalization of Z_j, $Y_j \overset{\pi_j}{\to} U_j \overset{h_j}{\to} P^1$ the Stein factorization of $p_2 g_j$ $(1 \leq j \leq s)$. We may assume that $C_{j,t_j} = p_1 g_j(\pi_j^{-1}(u_j))$, $t_j = h_j(u_j)$ for some $u_j \in U_j$ $(j = 1,2)$. By (8.1), $p_1 g_j(\pi_j^{-1}(v))$ is irreducible nonsingular and intersects B_j only at one point when v moves in a Zariski dense open subset V_j of U_j. Then C_{1,t_1} and C_{2,t_2} intersect nowhere for general $u_1 \in V_1, u_2 \in V_2$. In fact, if $C_{1,t_1} \cap C_{2,t_2} = \{p,\ldots\} \neq \phi$ and if $p \neq C_{1,t_1} \cap B_1$, then D'_{t_2} contains C_{1,t_1} by $D'_{t_2} C_{1,t_1} = LC_{1,t_1} = 1$. Since D'_t is chosen general, this contradicts that C_{1,t_1} is not contained in Bs $|L|$. Therefore we may assume that $C_{1,t_1} \cap C_{2,t_2} = C_{1,t_1} \cap B_1$

$= C_{2,t_2} \cap B_2$. This shows that $C_{1,t}$ intersects $\mathfrak{L}_{t,red} - C_{1,t}$ at the intersection $B_1 \cap B_2$ ($\neq \phi$) for general t. However this is impossible by (8.1.2). This proves that C_{1,t_1} and C_{2,t_2} intersect nowhere for general $u_1 \in V_1, u_2 \in V_2$. Hence the intersection of $p_1 g_1(Y_1)$ and $p_1 g_2(Y_2)$ is at most one dimensional. However $D = p_1 g_1(Y_1) = p_1 g_2(Y_2)$ because D is irreducible. This is a contradiction. q.e.d.

By (8.4), all B_j are the same, say, $B_j = B$ for any j, by choosing a sufficiently general pair D ,D' and $\mathfrak{L} = D \cap D'$. Let $n = -LB$, and let $\phi_B : (I_{\mathfrak{L}}/I_{\mathfrak{L}}^2)\otimes O_B \to I_B/I_B^2$ be the natural homomorphism. One sees $n \geq 0$ in view of (3.2) and (8.1).

(8.5) Lemma. Let $\mathfrak{L} = D \cap D'$ for D, D' sufficiently general. Let m_j be the multiplicity of C_j in $\mathfrak{L}$, $m := m_1+\ldots+m_s$. Then $m = n + 2$,$n \geq 0$.

Proof. By (8.1.2), $\mathfrak{L}$ is reduced generically along B, so that ϕ_B is injective by (3.10). Hence Coker ϕ_B is finite. One sees that $\dim \operatorname{Coker} \phi_B = c_1(I_B/I_B^2) - c_1((I_{\mathfrak{L}}/I_{\mathfrak{L}}^2)\otimes O_B) = -LB + 2 = n + 2$, $\dim \operatorname{Coker} \phi_{B,p_j} = m_j$ at any intersection point $p_j := C_j \cap B$ by (8.1.2). Since p_j's are all distinct by (8.1.2), we have $m \leq n + 2$. Since $\mathfrak{L}$ is of multiplicity m_j generically along C_j and it is reduced generically along B, $cl(\mathfrak{L})$ is equal to $m_1 cl(C_1) + \ldots + m_s cl(C_s) + cl(B) + cl(B')$ for some effective one cycle B' such that Supp $B' \subset$ Bs $|L|$, Supp $B' \not\supset B$. Then LB'

$\leqq 0$. In fact, there is no irreducible component B" of B'_{red} with LB" $\geqq 1$ by (3.2) and (8.1). Therefore we have by (2.2),

$$2 \leqq L^3 = L\ell \leqq L(m_1C_1 + \ldots + m_sC_s + B) = m - n.$$

This proves $m \geqq n + 2$, hence $m = n + 2$, $L^3 = 2$. q.e.d.

(8.6) Corollary. <u>Let</u> $\ell = D \cap D'$ <u>for</u> D, D' <u>sufficiently general. The curve</u> B <u>intersects</u> C_j $(1 \leqq j \leqq s)$ <u>only</u>, ℓ <u>is reduced everywhere along</u> $B \setminus \bigcup_j (B \cap C_j)$, $\ell = m_1C_1 + \ldots + m_sC_s + B$, Bs $|L|$ = Bs $|L_\ell|$ = B.

(8.7) Lemma. <u>Let</u> D" <u>and</u> D" <u>be arbitrary members of</u> $|L|$, D" $\neq$ D", $\ell' = D'' \cap D''$. <u>Then</u> $\ell' = m'_1C'_1 + \ldots + m'_{s'}C'_{s'} + B$ <u>for some</u> m'_j <u>and</u> s' <u>where</u> $LC'_j = 1$, $m' := m'_1 + \ldots + m'_{s'} = n + 2$, <u>the structures of</u> ℓ', C'_j <u>and</u> B <u>at</u> $C'_j \cap B$ <u>are described in</u> (8.1). .

<u>Proof</u>. The proof of (3.7) shows that Bs$|L| = \phi$ in the cases (3.2)-(3.5). Since Bs $|L|$ = B in our case, any irreducible component C'_j of ℓ'_{red} with $LC'_j = 1$ intersects B. By the above argument, we see $\ell' = m'_1C'_1 + \ldots + m'_{s'}C'_{s'} + B$ for some m'_j and s' where $LC'_j = 1$, $m' := m'_1 + \ldots + m'_{s'} = n + 2$. The rest is clear from (8.1). q.e.d.

(8.8) Lemma. $h^0(O_\ell(L)) = m$, and $n = -LB > 0$.

<u>Proof</u>. Let ℓ" be an analytic subspace of X whose ideal in O_X is defined by $I_{\ell''} = I_\ell + \bigcap_{1 \leqq j \leqq s} (I_{C_j})^{m_j}$ where m_j= multiplicity of C_j in ℓ. We easily see that

$$I_{\ell'',p} = I_{\ell,p} \qquad (p \in \ell_{red} \setminus B),$$

$$O_{X,p}x + O_{X,p}y^{m_j} \quad (p = B\cap C_j)$$
$$O_{X,p} \quad (p \notin \bigcup_{1\leq j\leq s} C_j)$$

Then there is an exact sequence

$$0 \to O_{\ell}(L) \to O_{\ell''}(L)\oplus O_B(L) \to \bigoplus_j C^{m_j} \to 0.$$

Since the support of ℓ'' is the disjoint union of C_j ($1 \leq j \leq s$) we see by (8.1.9) and (8.1) $h^0(O_{\ell''}(L)) = 2(m_1+\ldots+m_s) = 2m$, and that the natural homomorphism $H^0(O_{\ell''}(L)) \to \bigoplus_j C^{m_j}$ is surjective. Since $H^0(O_{\ell}(L))$ is mapped to zero in $H^0(O_B(L))$, we have $h^0(O_{\ell}(L)) = m$, $h^0(O_B(L)) = 0$. It follows from $O_B(L) = O_B(-n)$ that $n > 0$. q.e.d.

Let $h : Y \to X$ be the blowing-up of X with B center, $E = h^{-1}(B)_{red}$, $N = h^*L - [E]$, $\bar{D} = h^*D - E$, $\bar{D}' = h^*D' - E$, $\bar{C}_j$ = the proper transform of C_j ($1 \leq j \leq s$). Then one checks

(8.9) Lemma. For general D and D' in $|L|$, $\bar{C}_j$ is isomorphic to C_j and $(\bar{D} \cap \bar{D}')_{red} = \bigcup_{j=1}^{s} \bar{C}_j$, $\bar{D}\bar{D}' = m_1\bar{C}_1 + \ldots + m_s\bar{C}_s$, $\bar{B}' := E \cap \bar{D}$ is isomorphic to B.

Proof. We note that a general member of $|L|$ is nonsingular along B. Indeed, assume that D and D' are singular at a point p of B, that is, $I_{D,p} \subset m^2_{X,p}$, $I_{D',p} \subset m^2_{X,p}$. Let $\ell = D\cap D'$. Then

$$\begin{aligned} T_{\ell,p} &= \mathrm{Hom}(m_{\ell,p}/m^2_{\ell,p}, C) \\ &= \mathrm{Hom}(m_{X,p}/I_{D,p}+I_{D',p}+m^2_{X,p}, C) \\ &= \mathrm{Hom}(m_{X,p}/m^2_{X,p}, C) \end{aligned}$$

whence $\dim T_{\mathcal{l},p} = 3$. However by (8.1.2), $\dim T_{\mathcal{l},p} \leq 2$, which is absurd. Let $q := B \cap C_j$, $a := m_j$. It suffices to consider the problem near q to prove (8.9). Then by (8.1.2),

$$I_{\mathcal{l},q} = O_{X,q}x + O_{X,q}zy^a,$$
$$I_{B,q} = O_{X,q}x + O_{X,q}z, \quad I_{C_j,q} = O_{X,q}x + O_{X,q}y.$$

We may assume without loss of generality that $I_{D,q} = O_{X,q}x$, $I_{D',q} = O_{X,q}(x+zy^a)$ because general D and D' are nonsigular along B. Now it is easy to check the assertions by a direct computation. q.e.d.

(8.10) <u>Completion of the proof of</u> (3.6). Since $\bar{C}_j$ is a movable part of $\bar{D}\cap\bar{D}'$, Bs $|N|$ consists of at most finitely many points, whence $N\bar{C}_j \geq 0$ for any j. Let $I_B/I_B^2 = O_B(a)\oplus O_B(b)$, $a \geq b$, $c = a-b$. Then by (3.9), $a + b = c + 2b = 3n+2$ and E is a rational ruled surface Σ_c. By (8.9), $n > 0$. Let e (resp. f) be a section (resp. a fiber) of the ruling of Σ_c with $e^2 = c$, $f^2 = 0$, $ef = 1$. Let e_∞ be a section of Σ_c with $e_\infty^2 = -c$, $ee_\infty = 0$. Let $\bar{B}' := E \cap \bar{D}$. We see $[E]_E = -e-bf$, $E\bar{B}' = E^2\bar{D} = E^2(h^*L - E) = -(2n+2)$, $E^3 = c_1(I_B/I_B^2) = 3n + 2$, $N^3 = L^3 + 3h^*LE^2 - E^3 = 2 + 3n - (3n+2) = 0$. Consequently $m_1N\bar{C}_1 + \ldots + m_sN\bar{C}_s = N\bar{D}\bar{D}' = N^3 = 0$, $N\bar{C}_j = 0$, and $|N|$ is base point free.

Let $\bar{B}' = pe + qf \in \mathrm{Pic}\, E$. In view of (8.9), $\bar{B}'$ is isomorphic to B and $p = 1$. Since $[\bar{B}'] = [\bar{D}]_E = (h^*L-[E])_E = e + (b-n)f$, we have $q = b-n$. If $\bar{B}' = e_\infty$, then $q = -c$, $b = 2n+2$, $a-b = -n-2 < 0$. This is a contradiction. Hence $\bar{B}' \neq e_\infty$, so that $q \geq 0$, $b \geq n$. Therefore $h^0(E,N\otimes O_E) = h^0(\Sigma_c,e + (b-n)f) = n + 4$. We have by (8.8) $h^0(Y,N) = h^0(\dot{X},L) = h^0(\mathcal{l},L_{\mathcal{l}}) + 2 = m + 2 = n +$

4, and by (8.1.2) $h^0(Y,N-E) = h^0(X,I_B^2L) = 0$, whence we have a natural isomorphism $H^0(Y,N) \cong H^0(E,N\otimes O_E)$.

Let $g : Y \to P^{n+3}$ $(n > 0)$ be the morphism associated with the linear system $|N|$, g_E the restriction of g to E. Any point y of Y is contained in some $\bar{C}_j$ of some $\bar{D} \cap \bar{D}'$ because $h^0(Y,N) \geq 5$. By $N\bar{C}_j = 0$, $\bar{C}_j$ is mapped to one point. Moreover $E\bar{C}_j = (N+E)\bar{C}_j = h^*L\bar{C}_j = LC_j = 1$, whence $g(y) = g(\bar{C}_j \cap E)$. Hence $g(Y) = g(E)$. We note that the linear system $|N\otimes O_E|$ defines an isomorphism g_E of E into P^{n+3} iff $b > n$. If $b = n$, then g_E is an isomorphism over $E \setminus e_\infty$.

We shall define a morphism ψ of Y onto E as follows. If $b > n$, then we define ψ to be the morphism g. In the general case, take an arbitrary point y of Y. Then choose two general members $\bar{D}$ and $\bar{D}'$ of $|N|$ such that $\bar{D},\bar{D}'$ pass through y. The intersection $\bar{\Omega}_{red} = (\bar{D} \cap \bar{D}')_{red}$ is the disjoint union of $\bar{C}_j$ $(j = 1,\dots,s)$ by (8.9). Hence there is a unique $\bar{C}_j$ passing through y. Since $E\bar{C}_j = 1$, E intersects $\bar{C}_j$ at a unique point y' of E. We define $\psi(y) = y' = E \cap \bar{C}_j$. The point y' is independent of the choice of $\bar{D}$ and $\bar{D}'$. To show this, take $\bar{D}''$ and $\bar{D}''$ of $|N|$ such that $\bar{D}''$ and $\bar{D}''$ pass through y. Let $\bar{\Omega}' = \bar{D}'' \cap \bar{D}''$, $\bar{\Omega}'_{red} = \bar{C}_1+\cdots+\bar{C}_{s'}$. Then there is a unique $\bar{C}'_i$ passing through y. In fact, since $\bar{D}\bar{C}'_i = \bar{D}'\bar{C}'_i = N\bar{C}'_i = 0$, both $\bar{D}$ and $\bar{D}'$ contain $\bar{C}'_i$, whence $\bar{C}'_i$ is also the unique irreducible component of $\bar{\Omega}_{red}$ passing through y. Hence $\bar{C}_j = \bar{C}'_i$. Therefore $\bar{C}_j \cap E = \bar{C}'_i \cap E$. It is easy to check that for any point y of E $(\cong \Sigma_c)$, there exist two members H and H' of $|N\otimes O_E|$ such that $H \cap H'$ is reduced and it contains y. By the isomorphism $H^0(Y,N) \cong H^0(E,N\otimes O_E)$, for a given y of Y, we can choose two members D" and D‴ of |L| such that $\bar{D}'' \cap \bar{D}''' \cap E$ is reduced and $\bar{D}''$,$\bar{D}'''$ pass through y where $\bar{D}'' = h^*D'' - E$, $\bar{D}''' = h^*D''' - E$. Hence $D'' \cap D'''$ has $m_j' = 1$ for any j in (8.7). By using this it is easy to see that ψ is a morphism of Y into (indeed, onto) E. If b = n, the morphism ψ coincides with the morphism g on $Y \setminus g^{-1}(g(e_\infty))$. Note that $g(Y \setminus g^{-1}(g(e_\infty))) \cong E \setminus e_\infty$.

One also sees readily that any fiber of ψ endowed with reduced structure is P^1, one of the irreducible components of $\bar{D} \cap \bar{D}'$ for some $\bar{D}, \bar{D}' \in |N|$. Since both Y and E are smooth, ψ is flat and any general (scheme-theoretic) fiber of ψ is P^1. Any fiber is mutually algebraically equivalent and $\psi^{-1}(x)E = 1$ for general $x \in E$. By the criterion of multiplicity one, we see that a fiber $\psi^{-1}(x)$ is generically reduced for any $x \in E$. Since any fiber is Gorenstein, it is therefore reduced everywhere, whence it is a nonsingular rational curve P^1. Thus the morphism ψ gives a P^1-bundle structure of Y over E with a section E. Therefore there is a rank two vector bundle F on E such that $Y \cong P(F)$ and the following is exact;

$$0 \to O_E \to F \to \det F \to 0.$$

The surface E (= P(det F)) is embedded into Y by viewing det F as a quotient bundle of F as above. Then we have $\det F = -N_{E/Y} = -[E]_E = e + bf$. Since $H^1(E,-e-bf) = 0$, $F \cong \det F \oplus O_E$. Now it is easy to see that $b_2(Y)$ (or Picard number of Y) = 3. The threefold X is obtained from Y by contracting E to a curve B, whence $b_2(X)$ (or Picard number of X) = 2. This is a contradiction.

This completes the proof of (3.6).

(8.11) Remark. The proofs in sections 2-8 work as well in arbitrary characteristic.

§ 9 Proof of (0.1)

(9.1) Theorem. *Let* X *be a compact complex threefold homeomorphic to* P^3 *is isomorphic to* P^3 *if* $H^1(X,O_X) = 0$ *and if* $h^0(X,-mK_X) \geq 2$ *for some positive integer* m.

Proof. Since $H^1(X,O_X) = 0$, we have a natural exact sequence $1 \to H^1(X,O_X^*) \to H^2(X,\mathbb{Z})(\cong \mathbb{Z}) \to H^2(X,O_X)$. Since $H^1(X,O_X^*)$ has a nontrivial element K_X and $H^2(X,O_X)$ is torsion free, $H^1(X,O_X^*)$ is mapped isomorphically onto $H^2(X,\mathbb{Z})$. Let L be the generator of $H^1(X,O_X^*)$ with $L^3 = 1$. Since the second Stiefel Whitney class w_2 (= $c_1(X)$ mod 2) and rational Pontrjagin classes are topological invariants, we see by [5,pp. 207-208]

$$c_1(X) = (2s+4)L, \quad \chi(X,O_X) = (s+1)(s+2)(s+3)/6$$

for an integer s. We see $h^3(X,O_X) = h^0(X,\Omega_X^3) \leq b_3 = 0$, $\chi(X,O_X) = 1 + h^2(X,O_X) \geq 1$. This shows $s \geq 0$, $K_X = -(2s+4)L$, $2s+4 \geq 4$. Thus all the assumptions in [11,(1.1)] are satisfied. Hence X is isomorphic to P^3. q.e.d.

(9.2) *Proof of* (0.1). By the same argument as in (9.1), we see Pic X $\cong H^2(X,\mathbb{Z}) \cong \mathbb{Z}$. Let L be a generator of Pic X with $L^3 = 2$. Then by [1, p.188] or [10, p.321], we have,

$$c_1(X) = (2s+3)L, \quad \chi(X,O_X) = (s+1)(s+2)(2s+3)/6$$

for an integer s . By (1.2), $\chi(X,O_X) \geq 1$, whence $s \geq 0$. If $s > 0$, then X is isomorphic to P^3 by [11,(1.1)] which is a contradiction. Hence $s = 0$ and X is isomorphic to Q^3 by (2.1). q.e.d.

Theorems (0.2) and (0.3) are derived from (0.1). See [11],[12].

Appendix.

As was pointed out by Fujita, the proof of (2.7) in [11] does not work in positive characteristic because it uses Bertini's theorem.

We shall give an alternative proof of [11,(1.1)] in arbitrary characteristic.

(A.1) Theorem. Let X be a complete irreducible nonsingular algebraic threefold defined over an algebraically closed field of arbitrary characteristic. Assume $\mathrm{Pic}\, X = \mathbb{Z}L$, $H^1(X,O_X) = 0$, $K_X = -dL$ $(d \geq 4)$, $L^3 > 0$, $\kappa(X,L) \geq 1$. Then X is isomorphic to P^3.

For the proof of (A.1), in view of [11,(2.8) or (2.9)],it suffices to prove that a complete intersection $\ell = D \cap D'$ for any pair D and $D' \in |L|$ is a nonsingular rational curve.

In view of [11,(2.2)], $d = 4$, $K_X = -4L$. And by [11,(2.3)], there is a unique irreducible component A_1 of ℓ_{red} with $LA_1 = 1$. Here we write $C = A_1$ for simplicity.

First we show,

(A.2) Lemma. $I_\ell \not\subset I_C^2$.

Proof. Let I_ℓ (resp. I_C) be the ideal sheaf in O_X defining ℓ (resp. C). By Grothendieck's theorem, let $I_C/I_C^2 \cong O_C(a)\oplus O_C(b)$, $a \geq b$. We infer $c_1(I_C/I_C^2)+c_1(\Omega_C^1) = c_1(\Omega_X^1\otimes O_C) = K_XC = -4LC = -4$. Hence $a + b = -2$. Hence $b \leq -1$. Assume $I_\ell \subset I_C^2$. We consider the natural homomorphism $\phi : (I_\ell/I_\ell^2)\otimes O_C \to I_C^2/I_C^3$ $(\cong O_C(2a)\oplus O_C(a+b)\oplus O_C(2b))$. Since $(I_\ell/I_\ell^2)\otimes O_C \cong O_C(-1)\oplus O_C(-1)$, $\mathrm{Im}\,\phi$ is contained in $O_C(2a)$. Let $I = O_C(2a)+I_C^3$. Then since by

[11,(1.7.3)] $H^1(O_X/I_\ell) = 0$, we have $\chi(O_X/I) \geqq 1$. It follows that

$$1 \lneqq \chi(O_X/I) = \chi(O_X/I_C) + \chi(I_C/I_C^2) + \chi(I_C^2/I) = 1 + 2b,$$

which is absurd. q.e.d.

(A.3) Lemma. $I_C/I_C^2 \cong O_C(-1)\oplus O_C(-1)$ <u>and the natural homomorphism</u> $\phi : (I_\ell/I_\ell^2)\otimes O_C \to I_C/I_C^2$ <u>is an isomorphism</u>.

<u>Proof</u>. By the proof of (A.2), we note $a + b = -2$, $b \lneqq -1 \lneqq a$. If $a > b$, then Im ϕ is contained in $O_C(a)$. Let $I = O_C(a)+I_C^2$. Then $I_\ell \subset I \subset I_C$, $I_C/I = O_C(b)$. In the same manner as in (A.2), by [11,(1.7.3)] and (3.11), we have

$$1 \leqq \chi(O_X/I) = \chi(O_X/I_C) + \chi(I_C/I) = 2 + b.$$

This is a contradiction. Hence $a = b = -1$. Assume ϕ is not injective. We note that by (A.2), ϕ is a nontrivial homomorphism. Since Im ϕ ($\cong I_\ell+I_C^2/I_C^2$) is a rank one subsheaf of a torsion free sheaf I_C/I_C^2, it is locally O_C-free. Here we may set Im $\phi = O_C(c)$ for some $c \in \mathbb{Z}$. Then $c = -1$ because $(I_\ell/I_\ell^2)\otimes O_C \cong O_C(-1)\oplus O_C(-1)$. Let E = Coker $\phi \cong O_C(-1)$, F = Im $\phi \cong O_C(-1)$. Then we may view $I_C/I_C^2 = E \oplus F$ because $H^1(C,E^\vee\otimes F) = 0$, $E^\vee$ being the dual of E. So we consider again the homomorphism ϕ as,

$$\phi : (I_\ell/I_\ell^2)\otimes O_C \to F \subset E \oplus F = I_C/I_C^2.$$

Let p be an arbitrary point of C. Then there are two generators x, y of $I_{C,p}$, and two generators f, g of $I_{\ell,p}$ such that $\phi(f) = x$, $\phi(g) = 0$, $x \bmod I_{C,p}^2$ (resp. $y \bmod I_{C,p}^2$) generates F (resp. E). In the same manner as in (5.3), we obtain local parameters x and $y \in I_{C,p}$ and $\beta \in O_{X,p}$, $m > 0$ such that

(A.3.1) $I_{C,p} = O_{X,p}x + O_{X,p}y$

$$I_{\ell,p} = O_{X,p}x + O_{X,p}\beta y^m$$

where the restriction β_C of β to C is not identically zero. We note that $m \geq 2$ because $g \in I^2_{C,p}$.

Let $\Phi = \{U_j\}$ be a sufficiently fine covering of an open neighborhood of C by Stein open sets U_j. Then by (A.3.1), we have $x_j \in \Gamma(U_j, I_\ell)$, $y_j \in \Gamma(U_j, I_C)$, $\beta_j \in \Gamma(U_j, O_X)$ such that x_j and y_j (resp. x_j and $\beta_j y_j^m$) generate $\Gamma(U_j, I_C)$ (resp. $\Gamma(U_j, I_\ell)$).

Since $(I_\ell/I_\ell^2)\otimes O_C \cong O_C(-1)\oplus O_C(-1)$, we may assume that

(A.3.2) $x_j = \ell_{jk}x_k \mod I_C^2$, $y_j = \ell_{jk}y_k \mod I_C^2$,

where ℓ_{jk} stands for the one cocycle $L_C \in H^1(C, O_C^*)$.

Two elements $D_j\beta_j y_j^m$ and $D_k\beta_k y_k^m \in \operatorname{Ker}\phi$ ($\cong O_C(-1)$) are identified iff (we may assume that)

(A.3.3) $(D_{j|C}) = \ell_{jk}^{-1}(D_{k|C})$.

This shows that

(A.3.4) $(\beta_{j|C}) = \ell_{jk}^{1-m}(\beta_{k|C})$.

In particular, $\beta_C := \{\beta_{j|C};\ U_j \in \Phi\}$ is a nontrivial element of $H^0(C, O_C(1-m))$. This is possible only when $m = 1$ and β_C is a nonzero constant. This contradicts $m \geq 2$. Consequently ϕ is injective whence it is an isomorphism. This completes the proof of (A.3). q.e.d.

By (A.3), $I_{\ell,p} + I^2_{C,p} = I_{C,p}$ whence $I_{\ell,p} = I_{C,p}$ for any point p of C. This shows that ℓ is nonsingular anywhere along C. Since ℓ is connected by [11,(1.7)], ℓ is isomorphic to C. Then it is easy to see that Bs $|L| = \phi$ and that the morphism associated with $|L|$ is an isomorphism of X onto P^3. q.e.d.

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