

On dangerous self-tangencies in families of conflict sets

M. van Manen (manen@math.sci.hokudai.ac.jp)
Hokkaido University
Sapporo, Japan

Abstract. In this note we observe an interesting link between an intersection of two Legendrian manifolds in $ST^*\mathbb{R}^2$ - or dangerous self-tangencies, on the one hand and hyperbolic Morse transition of immersed curves on a torus on the other hand. The connection between these two arises naturally in the study of one-parameter families of conflict sets.

1. Legendrian properties of conflict sets.

In this section, we fix some notation and we explain what a conflict set is. We start with a review of some well-known symplectic geometry; see (Arnol'd and Givental', 2001).

The slit cotangent bundle $T_{\setminus 0}^*\mathbb{R}^n$ of \mathbb{R}^n is the cotangent bundle minus its zero section. Coordinates on $T_{\setminus 0}^*\mathbb{R}^n$ will be denoted x for the base ξ for the fibers. The sphere cotangent bundle is $ST^*\mathbb{R}^n = T_{\setminus 0}^*\mathbb{R}^n \cap \{\|\xi\| = 1\}$. On $T_{\setminus 0}^*\mathbb{R}^n$ we have the canonical 1-form $\alpha = \xi dx$. **Conic Lagrangian** manifolds in $T_{\setminus 0}^*\mathbb{R}^n$ are manifolds of dimension n on whose tangent space α vanishes. Let $\rho: \mathbb{R}_{>0} \times T_{\setminus 0}^*\mathbb{R}^n \rightarrow T_{\setminus 0}^*\mathbb{R}^n$ be defined by $\rho(\lambda, x, \xi) = (x, \lambda\xi)$. If for a submanifold $L \subset ST^*\mathbb{R}^2$ the image $\rho(\mathbb{R}_{>0} \times L)$ is conic Lagrangian, then we say that L is **Legendrian**. Legendrian manifolds in the spherical cotangent bundle $ST^*\mathbb{R}^n$ are in 1 – 1 correspondence with conic Lagrangian manifolds in $T_{\setminus 0}^*\mathbb{R}^n$. Hence, we can interchangeably use the terms “Legendrian” and “conic Lagrangian”. The singularities of the projection of Legendrian manifolds L to \mathbb{R}^n , i.e. $L \hookrightarrow ST^*\mathbb{R}^n \rightarrow \mathbb{R}^n$, are called Legendrian singularities. The image of the projection is called a front. When a subset Y of \mathbb{R}^n is the projection of a smooth embedded Legendrian manifold in $ST^*\mathbb{R}^n$, we say that Y is Legendrian. This use of the term is maybe a little misleading, but it is convenient.

A C^∞ function $H: T_{\setminus 0}^*\mathbb{R}^n \rightarrow \mathbb{R}^n$ that satisfies $H(\rho(\lambda, x, \xi)) = \lambda H(x, \xi)$ defines the Hamiltonian vectorfield $\text{vf}(H)$ by $d\alpha(\text{vf}(H), \cdot) = -dH$. For instance, a Riemannian metric $g_{ij}(x)$ gives rise to a Hamiltonian $H(x, \xi) = \sqrt{g^{ij}(x)\xi_i\xi_j}$. We will assume that the Hessian of H^2 with respect to the ξ variables is everywhere positive definite, so that our Hamiltonians correspond to Finsler metrics. Throughout we will as-

sume that the flow of $\text{vf}(H)$ is defined for all $t \in \mathbb{R}$, and that for any two points $p \neq q$ in \mathbb{R}^n , there is a unique integral curve in $H = 1$ of $\text{vf}(H)$ such that the projections to \mathbb{R}^n is a curve from p to q . The distance from p to q is the integral of α along that curve. Thus, a Riemannian metric is assumed to have an infinite injectivity radius everywhere.

Consider a deformation $F = F(x, s)$ of a germ or function $f = F(x_0, s) \in \mathcal{C}^\infty(\mathbb{R}^k)$. The deformation is required to be a non-degenerate phase function, i.e. the level set

$$\Sigma(F) = \{(x, s) \in \mathbb{R}^{n+k} \mid d_s F = 0 \quad F = 0\} \quad (1)$$

is required to be a manifold. If the $(n+k) \times (k+1)$ matrix $K(F)$

$$K(F) = d_{x,s}(F, d_s F)$$

has maximal rank for all $(x, s) \in \Sigma(F)$, then by the implicit function theorem $\Sigma(F)$ is at least locally a manifold. This requisite is called the **rank condition**. When the rank condition holds the image of the manifold $\Sigma(F) \times \mathbb{R}_{>0}$ in $T_{\setminus 0}^* \mathbb{R}^n$ by

$$(x, s, \lambda) \rightarrow (x, \lambda d_x F)$$

is a conic Lagrangian manifold L in $T_{\setminus 0}^* \mathbb{R}^n$.

By γ , γ_1 or γ_2 we will always mean embeddings of the circle S^1 in the plane \mathbb{R}^2 . Important examples of deformations as in the above are squared distance functions $F(x, s) = d^2(\gamma(s), x) - C^2$, for some constant C , associated to Hamiltonians as in the above. They always satisfy the rank condition, and the fronts are the equidistants of γ . Also $G(x_0, x, s) = d^2(\gamma(s), x) - x_0$ is a non-degenerate phase function and the projection to \mathbb{R}^3 of $\Sigma(G)$ is called the **big front**. However, distance functions are not the only source of non-degenerate phase functions.

Suppose we have two embeddings $\gamma_i: S^1 \rightarrow \mathbb{R}^2$, and for each embedding a homogeneous Hamiltonian on \mathbb{R}^2 . We can consider not a wavefront but points x where there exist critical points s_1 and s_2 of two possibly different distance functions $d_1(x, \gamma_1(s_1))$ from x to γ_1 and $d_2(x, \gamma_2(s_2))$ from x to γ_2 with a common critical value. These points x make up the **conflict set**. The conflict set is the locus of intersecting wavefronts. With identical distance functions the closure of the conflict set of a curve and itself is called the **symmetry set**. In that case, if we consider only minima of the distance function, rather than all critical values, we get the **medial axis**.

We will use the abbreviation **CS** for conflict set(s). The CS is specified by the set of equations

$$F_i(x, s_i) = d_i^2(x, \gamma_i(s_i)) \quad F_1 - F_2 = 0, \quad \frac{\partial F_i}{\partial s_i} = 0 \quad i = 1, 2 \quad (2)$$

In the study of CS, a particular kind of deformation of a particular kind of germ occurs. In both the germ and in the deformation, the variables are separated.

In (van Manen, 2003), it was proved that generically - that is, under perturbations of γ_1 and γ_2 - the matrix $K(F_1 - F_2)$ has maximal rank. Hence, generically the CS lifts to a Legendrian manifold in $ST^*\mathbb{R}^2$. These deformations are different from families of distance functions that always satisfy the maximal rank condition.


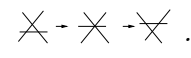

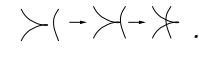


After the generic case come 1-parameter families. In generic 1-parameter families of CS for isolated values of the parameter $K(F_1 - F_2)$ may no longer have maximal rank. What happens in 1-parameter families of CS in the plane is the subject matter of this article.

It will turn out that the behavior of 1-parameter families of CS is quite different from the behavior of regular fronts. In particular dangerous self-tangencies appear in these families. After passing through a dangerous self-tangency the CS breaks in two in the generic case. Thus when the CS is no longer Legendrian for some isolated value of the parameter, there can occur a topological change on the front.

1.1. GENERIC SINGULARITIES OF FAMILIES OF WAVEFRONTS IN THE PLANE.

We recall the standard theory of singularities of wavefronts in \mathbb{R}^2 , see (Arnol'd and Givental', 2001). Generic legendrian singularities are the cusp $x_1^3 - x_2^2 = 0$, a double point $x_1^2 - x_2^2 = 0$ and a smooth curve $x_1 = 0$.

PROPOSITION 1.1. *A 1-parameter family of fronts generically admits, apart from the generic Legendrian singularities, only the following singularities:*

- *4/3-lipschitz smoothness points:*  .
- *triple points T:*  .
- *safe self-tangency, the transition K:*  .
- *the transition II:*  .
- *lips:*  .
- *beaks:*  .

- *elliptic morse transition*: $\emptyset \rightarrow \bullet \rightarrow \bigcirc$.
- *hyperbolic morse transition*: $\succ(-\times-\smile)$.

Only K , T , Π and $4/3$ happen in families of evolving wavefronts from a smooth planar curve. Arnol'd also established a list of singularities of 2-parameter families of fronts, see (Tchernov, 2002). However, in this case, there are no longer normal forms up to diffeomorphism available. For instance, in such a list, there must be quadruple points. The four corresponding lines in the tangent space define a cross-ratio, that is projectively invariant. Hence, in 2-parameter families of fronts in the plane there are moduli.

DEFINITION 1.2. A ***dangerous self-tangency*** of an almost everywhere Legendrian, except at isolated points, front $L \mapsto ST^*\mathbb{R}^2$ is a point $p \in \mathbb{R}^2$ such that L has a self-intersection at some point in $\pi^{-1}(p) \cap L$.

At a beaks point, the lift of the front to $ST^*\mathbb{R}^2$ has a self-intersection: the lift to $ST^*\mathbb{R}^2$ is no longer an embedded manifold. Again, with a Hamiltonian isotopy, such a self-intersection cannot come into existence.

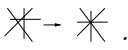

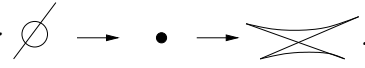
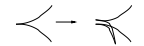
2. Statement of the main result

A 1-parameter family of CS can be obtained in two ways. We can vary the Hamiltonians with a parameter λ , or we can vary the embeddings with a parameter. In both ways we obtain a 1-parameter family of CS.

With the restrictions on the Hamiltonians described in the above we have the following theorem.

PROPOSITION 2.1. *Generically - under small 1-parameter perturbations of the Hamiltonians H_1 and H_2 and/or small 1-parameter perturbations of the embeddings γ_1 and γ_2 - in 1-parameter families of CS in the plane we can expect only the following singularities:*

- A_1^2 , A_2 ,
- $4/3$, T , K , Π , *beaks and lips*,
- *cusps that meet with a different normal vector on the intersection of an evolute and a symmetry set (these are legendrian and this case is called III in (Tchernov, 2002)), \succ .*

- *two symmetry sets that meet: TT* .
- *the nib:* .
- *the moth:* .
- *the slide:* .

The meeting symmetry sets and the meeting cusps happen in two parameter families of wavefronts evolving from surfaces. The other three cases: moth, nib and slide are sections of the A_2A_2 surface, see figure 1. These happen when two caustics meet on the conflict set. Notice that the nib is another example of a dangerous self-tangency.

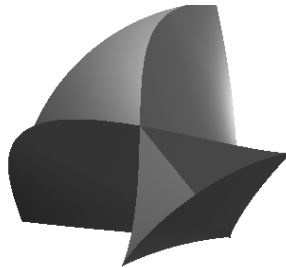


Figure 1. The A_2A_2 surface.

The proof follows from a simple consideration involving big fronts. See the section on intersections and transitions below. Interestingly, the beaks and lips we get do not occur in families of symmetry sets as is proved in an article of Giblin and Bruce, (Bruce and Giblin, 1986). Curiously, in their list the slide case is also absent.

The validity of the normal forms (that are really just pictures) we obtain is up to strata preserving homeomorphism. We refer to (Bruce, 1986) for the technical details. It would be interesting to know whether a stronger equivalence holds, say diffeomorphisms outside of a point.

Theorem 2.1 does not provide any information on the orientations of the different transitions. If we orient the conflict set, then there are, according to lecture 1 in (Arnol'd, 1994), two different transitions T . Similarly, signs can be associated to the TT crossing. It is not clear whether all types occur in transitions of conflict sets.

3. Intersections and transitions

3.1. INTERSECTIONS OF BIG FRONTS

For the classification of singularities of CS we obtained in (van Manen, 2003), the idea was to take the intersection of the big fronts to get a transversal intersection. That is, taking instead of $F_1(x, s_1) = F_2(x, s_2)$ the intersection of $x_0 = F_1(x, s_1)$ and $x_0 = F_2(x, s_2)$ there where $d_{s_1}F_1 = d_{s_2}F_2 = 0$. After that, we project the intersection back to the x -space. This last projection generically does not induce any extra singularity.

Generically, the big front in \mathbb{R}^3 is a Whitney stratified set. For such nice stratifications, we have obvious notions of transversality, and the transversality theorem of Thom also holds for Whitney stratified sets; see (Gibson et al., 1976).

A generic big front is stratified by the singularity type of the momental fronts listed in theorem 1.1. A transversal intersection of two big fronts, is in the generic case, a transversal intersection of stratified sets. That is, each stratum S_1 of one big front πL_1^h lies transverse to another stratum $S_2 \subset \pi L_2^h$. Because the sum of the codimension of the strata that intersect transversally cannot add up to more than the dimension of space-time, there occur only three types of intersections of strata:

- A_1 and A_1 , sum of the codimensions is two, the CS is a regular curve.
- A_2 and A_1 , sum of the codimensions is three, the CS has a cusp.
- $2A_1$ and A_1 , sum of the codimensions is three, the CS has a self-intersection.

In terms of normal forms for big fronts given by the calculations in (Arnol'd, 1976), the construction runs as follows: consider two big fronts coming from curves in \mathbb{R}^2 . At a point $(x_0, x) \in \mathbb{R}^3$ where the singularity of the distance function is not worse than A_2 , the front can locally be put in a normal form where it has a non-degenerate phase function of the form:

$$G_1 \quad : \quad x_0 = s^3 + x_1s + x_2. \quad (3)$$

From this equation we read that the cuspidal edge has equation

$$x_0 - x_2 = 0 \text{ and } x_1 = 0.$$

If G_1 is one of the moving fronts that forms a CS and G_2 is another, then in a generic CS we expect to have a transversal intersection of big fronts. We expect G_2 to look like

$$G_2 \quad : \quad x_0 = -x_2. \quad (4)$$

In this way, all strata of the big front G_2 intersect all strata of the big front G_1 transversally. The resulting equations for the CS are

$$0 = s^3 + x_1 s + 2x_2, \quad (5)$$

which results in a cusp at $x_1 = 0, x_2 = 0$.

3.2. THE A_2A_2 -SURFACE

Similar techniques apply when we consider the singularities of CS of two surfaces in \mathbb{R}^3 .

Here, however, we meet one singularity that we do not meet generically on wavefronts in \mathbb{R}^3 . Namely, cuspidal edges on a big wavefront emanating from a surface in \mathbb{R}^3 have a 2 dimensional edge. Generically, two of these edges can intersect. The projection to \mathbb{R}^3 of this intersection is the A_2A_2 surface, depicted in figure 1.

With these techniques in hand, let us proceed to prove proposition 2.1.

3.3. PROOF OF PROPOSITION 2.1

In the 1-parameter case, there are more intersections of strata to consider than in the case of CS sets without a parameter, and the intersection need no longer be transversal. The codimensions of the strata on the big wavefront can add up to 4 instead of three, and when the sum of the codimensions of the strata in the intersection is smaller than 4, there can be intersections that are not transversal. Let us first consider the cases where the sum of the codimensions is four.

If the sum of the codimensions is four then we need to consider the codimension 3 strata on the big wavefront: 4/3, triple point, K and Π . The codimension 3 strata can intersect with A_1 and with no other stratum because, as said, the codimension can add up to 4 at most. The intersections of strata of codimensions 3 and 1 thus give rise to the singularities of 1-parameter families of evolving wavefronts in 1 parameter families of CS.

There can also be intersections of two codimension 2 strata. There are three different cases: A_1^2 - the symmetry set - meeting the evolute, A_1^2 meeting A_1^2 , and the evolute meeting another evolute.

The symmetry set meeting the symmetry set leads to the quadruple point $A_1^2A_1^2$. The symmetry set meeting the evolute results in a III case.

Quite different is the A_2A_2 case, where two moving wavefronts and their caustics meet at the same instant. Here we must encounter a section of the A_2A_2 - surface which we encounter generically in CS of two surfaces in \mathbb{R}^3 .

Indeed, the parameter λ in the family of CS is, for a generic family, a generic function on the A_2A_2 surface. What happens with the CS is thus the evolution of level sets on A_2A_2 when the value of λ varies. An exploration of possible level sets or sections leads to the moth, nib and slide.

Next, consider the cases where the sum of the codimensions is three.

If the intersection of the strata adds up to three, this means that there can happen the following:

The A_1^2 or symmetry set stratum can touch the A_1 stratum. One gets the transition K .

The A_2 stratum can touch the A_1 stratum: beaks or lips.

If the sum of the codimensions is two the A_1 stratum can touch the A_1 stratum; this would result in Morse transitions. This cannot happen.

LEMMA 3.1. *The Morse transitions do not occur in families of CS.*

Proof. Suppose that the A_1 strata of two different big fronts did become tangent at some point (x_0, x) . Then the directions and the speeds of the flows of the two Hamiltonians coincide. Both these events are non-generic. Hence we need at least two parameters to have them occurring together.

The proof of proposition 2.1 is complete.

3.4. "NORMAL FORM" FOR THE SLIDE

We will pause here to explain in some detail how we can produce normal forms for the slide. Relevant pictures can all be made with the software (Greuel et al., 2001).

Consider the deformation $F = s^3 + x_1s + x_2$. The front in the plane associated to $F = 0$ and $d_sF = 0$ is parameterized by $(-3s^2, 2s^3)$. To get a moving front, we can vary x_0 in $F = s^3 + x_1s + x_2 - x_0$. The caustic of this moving front is $x_2 - x_0 = 0$. If we want to move the whole picture, caustic and front, we can take $F = s^3 + (x_1 - x_0)s + x_2$. The caustic is $x_1 = x_0$ and the front moves as $(-3s^2 + x_0, 2s^3)$.

In order to produce normal forms of moving CS, we can look at two such families:

$$F_1(A_1, A_2, s_1) = s_1^3 + A_1s_1 + A_2 \text{ and } F_2(A_3, A_4, s_2) = s_2^3 + A_3s_2 + A_4,$$

where the A_i are functions of x_1, x_2 and x_0 . To get the CS we need to solve the equations

$$F_1 = F_2 \quad d_{s_1}F_1 = 0 \quad d_{s_2}F_2 = 0 \quad (6)$$

This problem can be handled by elimination and resultants. The surface in the A_1, A_2, A_3, A_4 -space we obtain from (6) is irreducible. Up to some constant factors, its parameterization is

$$(u_1, u_2, u_3) \rightarrow (u_1^2, u_1^3 + u_2^3 + u_3, u_2^2, u_3) \quad (7)$$

Of course, the A_2A_2 surface is a section ($u_3 \equiv \text{constant}$) of the image of (7).

To get a slide one can choose the four functions A_i to be

$$A_1 = x_1 + \lambda \quad A_2 = x_2 \quad A_3 = x_1 - x_2 \quad A_4 = x_1 - x_2$$

3.5. PRE-IMAGES AND CURVES ON A TORUS.

The CS of two manifolds can also be regarded as the image of a subset of $S_1 \times S_1$. This pre-image of the CS was also introduced in the context of affine symmetry sets in (Giblin and Sapiro, 1997).

The CS is locally defined by the projection of

$$\Sigma(F_1 - F_2) \subset M_1 \times M_2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

The **pre-image of the CS** in $M_1 \times M_2$ is the projection

$$\Sigma(F_1 - F_2) \subset M_1 \times M_2 \times \mathbb{R}^2 \rightarrow M_1 \times M_2$$

The next proposition says that singularities of the CS are resolved by the pre-image if the CS is Legendrian.

PROPOSITION 3.2. *The pre-image of the CS is a smooth curve in $M_1 \times M_2$ if the CS is Legendrian. Cusps on the CS, that lie on the caustic of M_1 result in horizontal tangents, while cusps that lie on the caustic of M_2 result in vertical tangents on the pre-image of the CS. The transitions moth, nib, beaks and lips are mapped to Morse transitions, i.e. level sets $s_1^2 \pm s_2^2 = \lambda$, as λ goes from positive to negative on the pre-image.*

Proof. For the smoothness, note that $\Sigma = \Sigma(F_1 - F_2)$ is smooth if the CS is Legendrian. If the projection to the s_1, s_2 -plane is singular, we have

$$\frac{\partial^2 F_1}{\partial s_1^2} = \frac{\partial^2 F_2}{\partial s_2^2} = 0$$

But that implies that the CS can no longer be lifted to an embedded Legendrian curve.

For the statement on the cusps, notice that if the CS has a cusp, then the rank of the matrix

$$\begin{pmatrix} \frac{\partial^2 F_1}{\partial s_1^2} & 0 \\ 0 & \frac{\partial^2 F_2}{\partial s_2^2} \end{pmatrix}$$

is < 2 . Hence one of the derivatives

$$\frac{\partial^2 F_1}{\partial s_1^2} = 0 \text{ or } \frac{\partial^2 F_2}{\partial s_2^2} = 0$$

We suppose that $d_{s_1}^2 F_1 = 0$. This implies that the projection to the s_2 axis is singular. Hence the pre-image of the CS has a vertical tangent.

REMARK 3.3. *Note that the presence of vertical or horizontal tangents does not imply that there are cusps, in figure 3, we can see vertical tangents on the closed component that do not correspond to cusps. Cusps come and go in pairs, it is not clear whether this is true for the other vertical and horizontal tangents as well.*

That the points nib, moth, beaks and lips where the rank condition fails are mapped to Morse transitions on the pre-image is the subject of the following subsection.

3.6. "NORMAL FORMS" FOR MOTH, NIB, BEAKS AND LIPS.

According to the previous paragraphs we have to consider two cases; intersecting cuspidal edges on the big front or a cuspidal edge touching a regular part.

Intersecting cuspidal edges correspond to the nib or moth. The equations are

$$F_1 = s_1^3 + x_1 s_1 + x_2 \quad F_2 = s_2^3 + (x_1 + \lambda) s_2 - x_2 \quad (8)$$

which is the nib transition or,

$$F_1 = s_1^3 + x_1 s_1 + x_2 \quad F_2 = -s_2^3 + (x_1 + \lambda) s_2 - x_2 \quad (9)$$

which is the moth. The nib case leads to an equation $\lambda = 3s_2^2 - 3s_1^2$. If λ flips through zero there is a topological change on the pre-image. On the other hand, in the x plane we see cusps that meet. The moth case leads to $-\lambda = 3s_2^2 + 3s_1^2$, and to a moth in the x -plane as $\lambda < 0$.

The moth transition in the plane is everywhere Legendrian, because also an isolated point has a Legendrian lift to $ST^*\mathbb{R}^2$. The nib is not

Legendrian when $\lambda = 0$, because there is a dangerous self-tangency in that case.

The next case is beaks or lips when we have a cuspidal edge that “touches” a regular big front.

$$F_1 = s_1^3 + x_1 s_2 + x_2, \quad F_2 = s_2^2 + x_2 + \lambda \quad (10)$$

The CS is $F_1 = F_2$, $d_{s_1} F_1 = 0$, $d_{s_2} F_2 = 0$. As long as x_0 is not zero, the front F_2 is regular at $x = 0$. If λ varies from negative to positive, there are two components of the CS that annihilate each other. In practice, something else happens. In the first order approximation, a vanishing takes place but generically there appear afterwards two cusps. Hence the equations (10) are not entirely accurate. This is because at $\lambda = 0$ the cuspidal edge of the big front $x_0 = F_2(x, s_2)$ is entirely contained in the big front $x_0 = F_1(x, s_1)$. The normal forms in this case make the geometry too special.

For a generic situation, the cuspidal edge should only touch the other big front and not have a higher order contact. That beaks and lips arise is explained by the pictures in figure 2. The first one is beaks

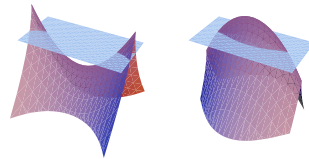


Figure 2. Beaks and lips

and the second one lips. The pictures also illustrate that if the cuspidal edge were straight, we would have the annihilation corresponding to equations (10).

The lips transition in the plane is everywhere Legendrian, because also an isolated point has a Legendrian lift to $ST^*\mathbb{R}^2$. The beaks is not Legendrian when $\lambda = 0$.

Clearly the beaks lead to $s_1^2 - s_2^2 = \lambda$ on the pre-image and the lips to $s_1^2 + s_2^2 = \lambda$. It is remarked that the slide case does not lead to a Morse transition on the pre-image. The interested reader can also verify these assertions using the normal forms and the software (Greuel et al., 2001).

We see that when the CS no longer has a Legendrian lift there is a dangerous self-tangency beaks or nib and that dangerous self-tangencies correspond to hyperbolic Morse perestroikas on the pre-image.

4. An Example

Let us have a look at an image of the transitions of proposition 2.1 in a situation that is not just purely a local normal form.

In the pictures 3 and 4, we take two different parabolas and place them in generic position with respect to intersections of big fronts. We then vary the Hamiltonian that governs the front from the parabola $x_2 - x_1^2$. As a phase function we take

$$F = F_1 - F_2 = (1 + \lambda) \|x - \gamma_1(s_1)\| - (1 - \lambda) \|x - \gamma_2(s_2)\|$$

On the *rhs.* of the pictures 3 and 4, we see the pre-image of the CS. On the *lhs.* we see the CS itself. The transitions are marked with boxes.

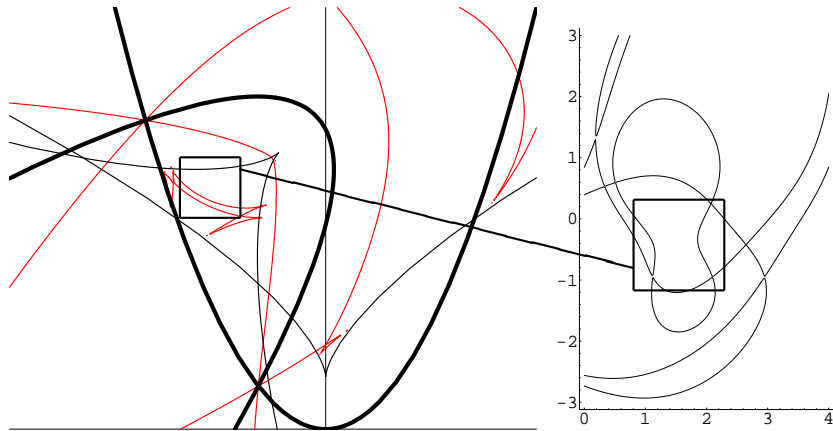


Figure 3. Beaks: two regular parts of the CS become tangential.

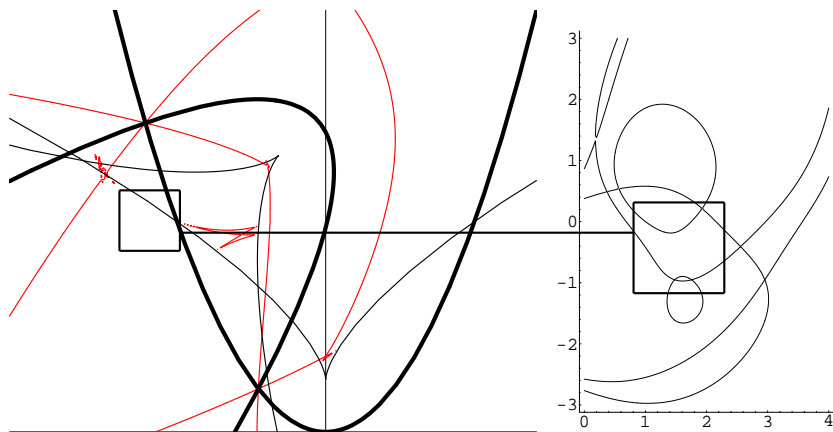


Figure 4. After the two regular parts of the CS have become tangential

5. Acknowledgement

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