Timelike hypersurfaces in de Sitter space and Legendrian singularities

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Abstract

We construct a basic framework for the study of extrinsic differential geometry on timelike hypersurfaces from the view point of the theory of Legendrian singularities. As an application, we study the contact of timelike hypersurfaces with flat totally umbilic timelike hypersurfaces in de Sitter space.

1 Introduction

In this paper we present some results of the project constructing the extrinsic differential geometry on submanifolds of pseudo-spheres in Minkowski space (cf., [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]). There are three kinds of pseudo-spheres in Minkowski space (i.e., hyperbolic space, the lightcone and de Sitter space). In the previous papers we consider submanifolds in hyperbolic space or the lightcone. In these cases submanifolds are always spacelike or lightlike. Only in de Sitter space, we have timelike submanifolds. Therefore we only consider timelike hypersurfaces in de Sitter space here. We construct a basic framework for the study of timelike hypersurfaces from the view point of the theory of Legendrian singularities here. Actually almost all results in this paper are analogous to the previous results on spacelike hypersurfaces in hyperbolic space or the lightcone. However, there are no contexts describing extrinsic differential geometry on the timelike hypersurface in de Sitter space from this point of view. Moreover there might be some new applications of such a framework to conformal geometry (discussions with M. C. Romero-Fuster and E. S. Sanabria-Codesal). Detailed descriptions of such applications will be appeared in elsewhere.

In §2 we describe the basic notions on Minkowski space and contact geometry. Especially, the Legendrian duality theorem (Proposition 2.2) between de Sitter spaces is the key to understand the whole story. In §3 we introduce the notion of de Sitter Gauss images and de Sitter curvatures of timelike hypersurfaces in de Sitter space. On of the results is that the totally umbilic timelike hypersurfaces are given as intersections of de Sitter space with timelike

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hyperplanes (i.e., timelike hyperbolic hyperquardrics). In particular the flat totally umbilic timelike hypersurface is the intersection of de Sitter space with a timelike hyperplane through the origin. We call it a *flat timelike hyperquadric.* We can interpret that the de Sitter Gauss image is a wave front set of a natural Legendrian submanifold. In §4 we introduce the notion of de Sitter height functions on timelike hypersurfaces in order to connect the notion of differential geometry in de Sitter space with the theory of Legendrian singularities. As a consequence, we can show that the de Sitter height function of a timelike hypersurface is a generating family of the corresponding Legendrian submanifold (cf., §5). We apply the theory of Legendrian singularities to study the contact of timelike hypersurfaces with flat timelike hyperquadrics in §6. In §7 we consider generic properties of timelike hypersurfaces in the low dimensional case. We give some examples in§8.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 Basic notations and the duality theorem

In this section we prepare basic notions on Minkowski space and contact geometry. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) | x_i \in \mathbb{R},\ i = 0, 1, \dots, n\}$ be an $(n+1)$-dimensional cartesian space. For any vectors $\boldsymbol{x} = (x_0, \dots, x_n)$, $\boldsymbol{y} = (y_0, \dots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *Minkowski* $n+1$*-space* and denoted by $\mathbb{R}^{n+1}_1$. For basic notions and properties of Minkowski space from the view point of Lorentz geometry, see [16].

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1} \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Given a vector $\boldsymbol{n} \in \mathbb{R}^{n+1}_1$ and a real number c, the hyperplane with pseudo normal $\boldsymbol{n}$ is defined by

$$HP(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}.$$

We say that $HP(\boldsymbol{n}, c)$ is a *spacelike* , *timelike* or *lightlike hyperplane* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively.

We have the following three kinds of pseudo-spheres in $\mathbb{R}^{n+1}_1$: *Hyperbolic n-space* is defined by

$$H^n(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 |\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

de Sitter n-space by

$$S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\ \}$$

and the (*open*) *lightcone* by

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\ \}.$$

In this paper we stick to timelike hypersurfaces in de Sitter space S^n_1. Typical such hypersurfaces are given by the intersection of S^n_1 with a hyperplane in $\mathbb{R}^{n+1}_1$:

$$DH(\boldsymbol{n}, c) = HP(\boldsymbol{n}, c) \cap S^n_1.$$

We say that $DH(\boldsymbol{n}, c)$ is a *quadric hypersurface in de Sitter space* (or briefly, *de Sitter hyperquadric*). We also say that $DH(\boldsymbol{n}, c)$ is *elliptic*, *hyperbolic* or *parabolic* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively. However, elliptic and parabolic de Sitter hyperquadrics are always spacelike. For a hyperbolic de Sitter hyperquadric $DH(\boldsymbol{n}, c)$, it is timelike if and only if $\langle \boldsymbol{n}, \boldsymbol{n} \rangle > c$. Hyperbolic de Sitter hyperquadrics are the candidates of totally umbilic timelike hypersurfaces in de Sitter space (cf., §3).

On the other hand, we now review some properties of contact manifolds and Legendrian submanifolds. Let N be a $(2n+1)$-dimensional smooth manifold and K be a tangent hyperplane field on N. Locally such a field is defined as the field of zeros of a 1-form α. The tangent hyperplane field K is *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of N. We say that (N, K) is a *contact manifold* if K is a non-degenerate hyperplane filed. In this case K is called a *contact structure* and α is a *contact form.* Let $\phi : N \longrightarrow N'$ be a mapping between contact manifolds (N, K) and (N', K'). We say that ϕ is a *contact morphism* if $d\phi(K) = K'$. Any contact morphism is a local diffeomorphism. If it is a diffeomorphism, we call it a *contact diffeomorphism.* Two contact manifolds (N, K) and (N', K') are *contact diffeomorphic* if there exists a contact diffeomorphism $\phi : N \longrightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if dim $L = n$ and $di_x(T_xL) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \to M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \to M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \to M$ is called a *Legendrian map.* The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of i which is denoted by $W(i)$. Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$. For any $p \in E$, it is known that there is a local coordinate system $(x_1, \ldots, x_m, p_1, \ldots, p_m, z)$ around p such that

$$\pi(x_1, \ldots, x_m, p_1, \ldots, p_m, z) = (x_1, \ldots, x_m, z)$$

and the contact structure is given by the 1-form

$$\alpha = dz - \sum_{i=1}^{m} p_i dx_i$$

(cf. [1], 20.3).

One of the examples of Legendrian fibrations is given by the projective cotangent bundle of over a manifold. Let $\pi : PT^*(M) \longrightarrow M$ be the projective cotangent bundle over an n-dimensional manifold M. This fibration can be considered as a Legendrian fibration with the canonical contact structure K on $PT^*(M)$. We now review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*(M) \to PT^*(M)$ and the differential map $d\pi : TPT^*(M) \to N$ of π. For any $X \in TPT^*(M)$, there exists an element $\alpha \in T^*(M)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(M)$, the property $\alpha(V) = \mathbf{0}$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(M)$ by

$$K = \{X \in TPT^*(M) | \tau(X)(d\pi(X)) = 0\}.$$

For a local coordinate neighborhood $(U, (x_1, \ldots, x_n))$ on M, we have a trivialization

$$PT^*(U) \cong U \times P(\mathbb{R}^{n-1})^*$$

and we call $((x_1, \ldots, x_n), [\xi_1 : \cdots : \xi_n])$ *homogeneous coordinates*, where $[\xi_1 : \cdots : \xi_n]$ are homogeneous coordinates of the dual projective space $P(\mathbb{R}^{n-1})^*$. It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=1}^n \mu_i \xi_i = 0$, where $d\tilde{\pi}(X) = \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i}$.

We consider the following five double fibrations:

(1) (a) $H^n(-1)\times S^n_1\supset\Delta_1=\{(\boldsymbol{v},\boldsymbol{w})\mid\langle\boldsymbol{v},\boldsymbol{w}\rangle=0\ \}$,
(b) $\pi_{11}:\Delta_1\longrightarrow H^n(-1)$,$\pi_{12}:\Delta_1\longrightarrow S^n_1$,
(c) $\theta_{11}=\langle d\boldsymbol{v},\boldsymbol{w}\rangle|\Delta_1$, $\theta_{12}=\langle\boldsymbol{v},d\boldsymbol{w}\rangle|\Delta_1$.

(2) (a) $H^n(-1)\times LC^*\supset\Delta_2=\{(\boldsymbol{v},\boldsymbol{w})\mid\langle\boldsymbol{v},\boldsymbol{w}\rangle=-1\ \}$,
(b) $\pi_{21}:\Delta_2\longrightarrow H^n(-1)$,$\pi_{22}:\Delta_2\longrightarrow LC^*$,
(c) $\theta_{21}=\langle d\boldsymbol{v},\boldsymbol{w}\rangle|\Delta_2$, $\theta_{22}=\langle\boldsymbol{v},d\boldsymbol{w}\rangle|\Delta_2$.

(3) (a) $LC^*\times S^n_1\supset\Delta_3=\{(\boldsymbol{v},\boldsymbol{w})\mid\langle\boldsymbol{v},\boldsymbol{w}\rangle=1\ \}$,
(b) $\pi_{31}:\Delta_3\longrightarrow LC^*$,$\pi_{32}:\Delta_3\longrightarrow S^n_1$,
(c) $\theta_{31}=\langle d\boldsymbol{v},\boldsymbol{w}\rangle|\Delta_3$, $\theta_{32}=\langle\boldsymbol{v},d\boldsymbol{w}\rangle|\Delta_3$.

(4) (a) $LC^*\times LC^*\supset\Delta_4=\{(\boldsymbol{v},\boldsymbol{w})\mid\langle\boldsymbol{v},\boldsymbol{w}\rangle=-2\ \}$,
(b) $\pi_{41}:\Delta_4\longrightarrow LC^*$,$\pi_{42}:\Delta_4\longrightarrow LC^*$,
(c) $\theta_{41}=\langle d\boldsymbol{v},\boldsymbol{w}\rangle|\Delta_4$, $\theta_{42}=\langle\boldsymbol{v},d\boldsymbol{w}\rangle|\Delta_4$.

(5) (a) $S^n_1\times S^n_1\supset\Delta_5=\{(\boldsymbol{v},\boldsymbol{w})\mid\langle\boldsymbol{v},\boldsymbol{w}\rangle=0\ \}$,
(b) $\pi_{11}:\Delta_1\longrightarrow S^n_1$,$\pi_{12}:\Delta_1\longrightarrow S^n_1$,
(c) $\theta_{51}=\langle d\boldsymbol{v},\boldsymbol{w}\rangle|\Delta_5$, $\theta_{52}=\langle\boldsymbol{v},d\boldsymbol{w}\rangle|\Delta_5$.

Here,$\pi_{i1}(\boldsymbol{v},\boldsymbol{w})=\boldsymbol{v}$, $\pi_{i2}(\boldsymbol{v},\boldsymbol{w})=\boldsymbol{w}$, $\langle d\boldsymbol{v},\boldsymbol{w}\rangle=-w_0dv_0+\sum_{i=1}^n w_idv_i$ and $\langle\boldsymbol{v},d\boldsymbol{w}\rangle=-v_0dw_0+\sum_{i=1}^n v_idw_i$.

We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i. In [12], we have shown the following basic Legendrian duality theorem:

Theorem 2.1 *Under the same notations as the previous paragraph, each* (Δ_i,K_i) $(i=1,2,3,4)$ *is a contact manifold and both of* π_{ij} $(j=1,2)$ *are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic each other.*

We also have the similar result for the case (5), but (Δ_5,K_5) is not contact diffeomorphic to other (Δ_i,K_i).

Proposition 2.2 *Under the same notations as the above,* (Δ_5,K_5) *is a contact manifold and both of* π_{5j} $(j=1,2)$ *are Legendrian fibrations.*

Proof. We consider a coordinate neighborhood $W_n^+=\{\boldsymbol{w}=(w_0,w_1,\dots,w_n)\in S^n_1\mid w_n>0\}$ on which we have

$$w_n=\sqrt{w_0^2-\sum_{i=1}^{n-1}w_i^2+1}.$$

Therefore, we regards that $(w_0,\dots,w_{n-1})$ is the local coordinates on W_n^+. We consider a mapping $\Psi:\Delta_5(W_n^+)=(\pi_{52})^{-1}(W_n^+)\longrightarrow PT*(S^n_1)|W_n^+$ defined by

$$\Psi(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{w},[-w_nv_0+w_0v_n:w_nv_1-v_nw_1:\cdots:w_nv_{n-1}-v_nw_{n-1}]).$$

Let $((w_0,\dots,w_{n-1}),[\xi_1:\cdots:\xi_n])$ be the homogeneous coordinates of $PT*(S^n_1)$ over W_n^+. We have the canonical contact form $\theta=\sum_{i=1}^n\xi_iw_{i-1}$ on $PT^*(S^n_1)|W_n^+$. It follows that

$$\Psi^*\theta=(-w_nv_0+w_0v_n)dw_0+\sum_{i=1}^{n-1}(w_nv_i-v_nw_i)dw_i=w_n\langle\boldsymbol{v},d\boldsymbol{w}\rangle|\Delta_5=w_n\theta_{52}.$$

This means that θ_{52} is a contact structure such that Ψ is a contact morphism. We have the similar calculation as the above on the other coordinate neighborhoods. $\square$

Given n vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathbb{R}^{n+1}_1$, we can define the wedge product $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n$ as follows:

$$\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ a_0^1 & a_1^1 & \cdots & a_n^1 \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{vmatrix},$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{a}_i = (a_0^i, a_1^i, \ldots, a_n^i)$. We can easily check that

$$\langle \boldsymbol{a}, \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n \rangle = \det(\boldsymbol{a}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n),$$

so that $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n$ is pseudo orthogonal to $\boldsymbol{a}_i$, $\forall i = 1, \ldots, n$.

3 Geometry of timelike hypersurfaces in de Sitter space

In this section we construct the basic tools for the study of the extrinsic differential geometry on timelike hypersurfaces in de Sitter space S^n_1.

Let $\boldsymbol{x} : U \longrightarrow S^n_1$ be a regular timelike hypersurface (i.e., an embedding with timelike tangent space), where $U \subset \mathbb{R}^{n-1}$ is an open subset. We denote that $M = \boldsymbol{x}(U)$ and identify M with U by the embedding $\boldsymbol{x}$. Since $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \equiv 1$, we have $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x} \rangle \equiv 0 \quad (i = 1, \ldots, n-1)$, where $u = (u_1, \ldots u_{n-1}) \in U$. Define a vector

$$\boldsymbol{x}^d(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}(u)\|},$$

then we have

$$\langle \boldsymbol{x}^d, \boldsymbol{x}_{u_i} \rangle \equiv \langle \boldsymbol{x}^d, \boldsymbol{x} \rangle \equiv 0, \quad \langle \boldsymbol{x}^d, \boldsymbol{x}^d \rangle \equiv 1.$$

Therefore the vector $\boldsymbol{x}^d$ is spacelike. We call the map

$$\boldsymbol{x}^d : U \longrightarrow S^n_1$$

the *de Sitter Gauss image* of M. We study the extrinsic differential geometry of $\boldsymbol{x}(U) = M$ by using the de Sitter Gauss image $\boldsymbol{x}^d$ like as the unit normal of a hypersurface in Euclidean space. We define a mapping

$$\mathcal{L}_5 : U \longrightarrow \Delta_5$$

by $\mathcal{L}(u) = (\boldsymbol{x}(u), \boldsymbol{x}^d(u))$. Since $\langle \boldsymbol{x}(u), \boldsymbol{x}^d(u) \rangle = \langle d\boldsymbol{x}(u), \boldsymbol{x}^d(u) \rangle = 0$, the mapping $\mathcal{L}_5$ is a Legendrian embedding. It follows that, we have $\langle \boldsymbol{x}(u), \boldsymbol{x}^d(u) \rangle = 0$. Since $\langle \boldsymbol{x}^d(u), \boldsymbol{x}^d(u) \rangle = 0$, we also have $\langle \boldsymbol{x}^d(u), d\boldsymbol{x}^d(u) \rangle = 0$. This means that $d\boldsymbol{x}^d(u_0)(\boldsymbol{v})$ is a tangent vector of M at $p = \boldsymbol{x}(u_0)$ for any $\boldsymbol{v} \in T_{u_0}U$.

We now identify U and M through the embedding $\boldsymbol{x}$. Under the identification, the derivative $d\boldsymbol{x}^d(u_0)$ can be considered as a linear transformation on the tangent space T_pM where $p = \boldsymbol{x}(u_0)$. We call the linear transformation $S^d_p = -d\boldsymbol{x}^d(u_0) : T_pM \longrightarrow T_pM$ the *de Sitter shape operator*. We denote the eigenvalue of S^d_p by $\kappa_d(p)$, which we call the *de Sitter principal curvature* of M at p. We remark that all de Sitter principal curvatures are real numbers (c.f., Proposition 3.3).

We define the notion of de Sitter curvatures of $\boldsymbol{x}(U)=M$ at $p=\boldsymbol{x}(u_0)$ as follows:

$$\begin{aligned} K_d(u_0) &= \det S_p^d\ ; \quad \textit{The de Sitter Gauss-Kronecker curvature}, \\ H_d(u_0) &= \frac{1}{n-1}\operatorname{Trace} S_p^d\ ; \quad \textit{The de Sitter mean curvature}. \end{aligned}$$

We can define the notion of umbilicity like as the case of hypersurfaces in Euclidean space. A point $p=\boldsymbol{x}(u_0)$ (or u_0) is said to be an *umbilic point* if $S_p^d=\kappa_d(p)1_{T_pM}$. We say that $M=\boldsymbol{x}(U)$ is *totally umbilic* if all points on M are umbilic. We have the following classification of totally umbilic hypersurfaces in S^n.

Proposition 3.1 *Suppose that $M=\boldsymbol{x}(U)$ is totally umbilic. Then $\kappa_\ell(p)$ is constant κ_d. Under this condition, we have the following classification.*

(1) *If $\kappa_d\neq 0$, then M is a part of a hyperbolic de Sitter hyperquadric $DH(\boldsymbol{c},\kappa_d/\sqrt{\kappa_d^2+1})$, where*

$$\boldsymbol{c}=\frac{1}{\sqrt{\kappa_d^2+1}}(\kappa_d\boldsymbol{x}(u)+\boldsymbol{x}^d(u))\in S_1^n$$

is a constant spacelike vector.

(2) *If $\kappa_d=0$, then M is a part of a hyperbolic de Sitter hyperquadric $DH(\boldsymbol{c},0)$, where $\boldsymbol{c}=\boldsymbol{x}^d(u)\in S_1^n$ is a constant lightlike vector.*

Proof. By definition, we have $-(\boldsymbol{x}^d)_{u_i}=\kappa_d(u)\boldsymbol{x}_{u_i}$ for $i=1,\dots,n-1$. Therefore, we have

$$-(\boldsymbol{x}^d)_{u_iu_j}=(\kappa_d)_{u_j}(u)\boldsymbol{x}_{u_i}+\kappa_d(u)\boldsymbol{x}_{u_iu_j}.$$

Since $-(\boldsymbol{x}^d)_{u_iu_j}=-(\boldsymbol{x}^d)_{u_ju_i}$ and $\kappa_d(u)\boldsymbol{x}_{u_iu_j}=\kappa_d(u)\boldsymbol{x}_{u_ju_i}$, we have $(\kappa_d)_{u_j}(u)\boldsymbol{x}_{u_i}-(\kappa_d)_{u_i}(u)\boldsymbol{x}_{u_j}$. By definition $\{\boldsymbol{x}_{u_1},\dots,\boldsymbol{x}_{u_{n-1}}\}$ is linearly independent, so that κ_d is constant. Under this condition, we distinguish two cases.

(Case 1). We assume that $\kappa_d\neq 0$: By definition, we have $-d\boldsymbol{x}^d=\kappa_d d\boldsymbol{x}$. Since κ_d is constant, it follows from the above equality that $d(\kappa_d\boldsymbol{x}+\boldsymbol{x}^d)=\boldsymbol{0}$. Therefore $\boldsymbol{c}=\frac{1}{\sqrt{\kappa_d^2+1}}(\kappa_d\boldsymbol{x}(u)+\boldsymbol{x}^d(u))$ is constant and we have $\langle\boldsymbol{c},\boldsymbol{c}\rangle=1$. On the other hand, we have

$$\langle\boldsymbol{x}(u),\boldsymbol{c}\rangle=\frac{1}{\sqrt{\kappa_d^2+1}}\langle\boldsymbol{x}(u),\kappa_d\boldsymbol{x}(u)+\boldsymbol{x}^\ell(u)\rangle=\frac{\kappa_d}{2\sqrt{\kappa_d^2+1}}.$$

This means that $M=\boldsymbol{x}(U)\subset DH(\boldsymbol{c},\kappa_d/\sqrt{\kappa_d^2+1})$.

(Case 2). We assume that $\kappa_d=0$: By definition, we have $d\boldsymbol{x}^d(u)=\boldsymbol{0}$, so that $\boldsymbol{c}=\boldsymbol{x}^d$ is constant. We also have $\langle\boldsymbol{x}(u),\boldsymbol{c}\rangle=\langle\boldsymbol{x}(u),\boldsymbol{x}^d(u)\rangle=0$. This means that $M=\boldsymbol{x}(U)\subset DH(\boldsymbol{c},0)$. This completes the proof. □

By the above proposition, we can classify the umbilic point as follows. Let $p=\boldsymbol{x}(u_0)\in\boldsymbol{x}(U)=M$ be an umbilic point; we say that p is a *timelike umbilic point* if $\kappa_d\neq 0$ or a *timelike flat point* if $\kappa_d=0$. Especially we call $DH(\boldsymbol{c},0)$ a *flat timelike hyperquadric.*

In the last part of this section, we prove the de Sitter Weingarten formula. We induce the Lorentz metric (the *de Sitter first fundamental form*) $ds^2=\sum_{i=1}^{n-1}g_{ij}du_idu_j$ on $M=\boldsymbol{x}(U)$, where $g_{ij}(u)=\langle\boldsymbol{x}_{u_i}(u),\boldsymbol{x}_{u_j}(u)\rangle$ for any $u\in U$. We also define the *de Sitter second fundamental invariant* by $h_{ij}^d(u)=\langle-(\boldsymbol{x}^d)_{u_i}(u),\boldsymbol{x}_{u_j}(u)\rangle$ for any $u\in U$.

Proposition 3.2 *Under the above notations, we have the following de Sitter Weingarten formula:*

$$\left(\boldsymbol{x}^d\right)_{u_i} = -\sum_{j=1}^{n-1} \left(h^d\right)_i^j \boldsymbol{x}_{u_j},$$

where $\left(h^d\right)_i^j = \left(h_{ik}^d\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = \left(g_{kj}\right)^{-1}$.

Proof. By Lemma 3.1, there exist real numbers Γ_i^j such that

$$\left(\boldsymbol{x}^d\right)_{u_i} = \sum_{j=1}^{n-1} \Gamma_i^j \boldsymbol{x}_{u_j}.$$

By definition, we have

$$-h_{i\beta}^d = \sum_{\alpha=1}^{n-1} \Gamma_i^\alpha \langle \boldsymbol{x}_{u_\alpha}, \boldsymbol{x}_{u_\beta} \rangle = \sum_{\alpha=1}^{n-1} \Gamma_i^\alpha g_{\alpha\beta}.$$

Hence, we have

$$-\left(h^d\right)_i^j = -\sum_{\beta=1}^{n-1} h_{i\beta}^d g^{\beta j} = \sum_{\beta=1}^{n-1}\sum_{\alpha=1}^{n-1} \Gamma_i^\alpha g_{\alpha\beta} g^{\beta j} = \Gamma_i^j.$$

This completes the proof of the de Sitter Weingarten formula. □

As a corollary of the above proposition, we have an explicit expression of the de Sitter Gauss-Kronecker curvature by Lorentz metric and the de Sitter second fundamental invariant.

Corollary 3.3 *Under the same notations as in the above proposition, the de Sitter Gauss-Kronecker curvature is given by*

$$K_d = \frac{\det\left(h_{ij}^d\right)}{\det\left(g_{\alpha\beta}\right)}.$$

Proof. By the de Sitter Weingarten formula, the representation matrix of the de Sitter shape operator with respect to the basis $\{\boldsymbol{x}_{u_1}, \ldots, \boldsymbol{x}_{u_{n-1}}\}$ is $\left(\left(h^d\right)_i^j\right) = \left(h_{i\beta}^d\right)\left(g^{\beta j}\right)$. It follows from this fact that

$$K_d = \det S_p^d = \det\left(\left(h^d\right)_i^j\right) = \det\left(h_{i\beta}^d\right)\left(g^{\beta j}\right) = \frac{\det\left(h_{ij}^d\right)}{\det\left(g_{\alpha\beta}\right)}.$$

□

We say that a point $p = \boldsymbol{x}(u)$ is a *de Sitter parabolic point* if $K_d(u) = 0$ and a *de Sitter flat point* if it is an umbilic point and $K_d(u) = 0$.

4 De Sitter height functions

In this section we introduce a family of functions on a timelike hypersurface in the de Sitter space which are useful for the study of singularities of de Sitter Gauss images. Let $\boldsymbol{x} : U \longrightarrow S_1^n$ be a timelike hypersurface. We define a family of functions

$$H : U \times S_1^n \longrightarrow \mathbb{R}$$

by $H(u, \boldsymbol{v}) = \langle \boldsymbol{x}(u), \boldsymbol{v} \rangle$. We call H a *de Sitter height function* on $\boldsymbol{x} : U \longrightarrow S_1^n$.

Proposition 4.1 *Let $H : U \times S^n_1 \longrightarrow \mathbb{R}$ be a de Sitter height function on $\boldsymbol{x} : U \longrightarrow S^n_1$. Then*

(1) $H(u, \boldsymbol{v}) = 0$ *if and only if* $(\boldsymbol{x}(u), \boldsymbol{v}) \in \Delta_5$.

(2) $H(u, \boldsymbol{v}) = \dfrac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = 0$ $(i = 1, \ldots, n-1)$ *if and only if* $\boldsymbol{v} = \boldsymbol{x}^d(u)$.

Proof. The assertion (1) follows from the definition of H and Δ_5.

(2) There exist real numbers λ, μ, ξ_i $(i = 1, \ldots, n-1)$ such that $\boldsymbol{v} = \lambda\boldsymbol{x}^d + \mu\boldsymbol{x} + \sum_{i=1}^{n-1} \xi_i \boldsymbol{x}_{u_i}$. Since $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1$, we have $\langle \boldsymbol{x}, \boldsymbol{x}_{u_i}\rangle = 0$. Therefore $0 = H(u, \boldsymbol{v}) = \langle \boldsymbol{x}, \lambda\boldsymbol{x}^d\rangle + \langle \boldsymbol{x}, \mu\boldsymbol{x}\rangle = \mu$ if and only if $\lambda = 1$. Since $\dfrac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = \langle \boldsymbol{x}_{u_i}, \boldsymbol{v}\rangle$, we have $0 = \langle \boldsymbol{x}_{u_i}, \boldsymbol{v}\rangle + \sum_{j=1}^{n-1} \xi_j g_{ij}(u)$. The equation $\langle d\boldsymbol{x}, \boldsymbol{x}^d\rangle = 0$ means that $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x}^d\rangle = 0$. It follows that $\sum_{j=1}^{n-1} \xi_j g_{ij}(u) = 0$. Since g_{ij} is non-degenerate, we have $\xi_j = 0$ $(j = 1, \ldots, n-1)$. We also have $1 = \langle \boldsymbol{v}, \boldsymbol{v}\rangle = \mu^2 + 1$. Therefore $\mu = 0$. This completes the proof. □

We denote the *Hessian matrix* of the de Sitter height function $h_{v_0}(u) = H(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(h_{v_0})(u_0)$.

Proposition 4.2 *Let $\boldsymbol{x} : U \longrightarrow S^n_1$ be a timelike hypersurface in de Sitter space and $\boldsymbol{v}_0 = \boldsymbol{x}^d(u_0)$. Then*

(1) $p = \boldsymbol{x}(u_0)$ *is a timelike parabolic point if and only if* $\det \mathrm{Hess}(h_{v_0})(u_0) = 0$.

(2) $p = \boldsymbol{x}(u_0)$ *is a timelike flat point if and only if* $\mathrm{rank}\, \mathrm{Hess}(h_{v_0})(u_0) = 0$.

Proof. By definition and $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x}^d\rangle = 0$, we have

$$\mathrm{Hess}(h_{v_0})(u_0) = \left(\langle \boldsymbol{x}_{u_i u_j}(u_0), \boldsymbol{x}^d(u_0)\rangle\right) = \Big(-\langle \boldsymbol{x}_{u_i}(u_0), \boldsymbol{x}^d_{u_j}(u_0)\rangle\Big) = h^d_{ij},$$

so that we have $-\langle \boldsymbol{x}_{u_i}, \boldsymbol{x}^d_{u_j}\rangle = h^d_{ij}$, so that we have

$$K_d(u_0) = \frac{\det \mathrm{Hess}(h_{v_0})(u_0)}{\det\ (g_{\alpha\beta}(u_0))}.$$

The first assertion follows from this formula.

For the second assertion, by the de Sitter Weingarten formula, $p = \boldsymbol{x}(u_0)$ is an umbilic point if and only if there exists an orthogonal matrix A such that ${}^tA \left(\left(h^d\right)^\alpha_i\right) A = \kappa_d I$. Therefore, we have $\left(\left(h^d\right)^\alpha_i\right) = A\kappa_d I {}^tA = \kappa_d I$, so that

$$\mathrm{Hess}(h_{v_0}) = \left(h^d_{ij}\right) = \left(\left(h^d\right)^\alpha_i\right)(g_{\alpha j}) = \kappa_d\,(g_{ij})\,.$$

Thus, p is a timelike flat point (i.e., $\kappa_d(u_0) = 0$) if and only if $\mathrm{rank}\, \mathrm{Hess}(h_{v_0})(u_0) = 0$. □

5 De Sitter Gauss images as wave fronts

In this section we naturally interpret the de Sitter Gauss image of a timelike hypersurface in de Sitter space as a wave front set in the framework of contact geometry. We now give a quick survey on the theory of Legendrian singularities. For notions and some detailed results on generating families, please refer to [1, 19]. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0}) \longrightarrow (\mathbb{R}, \boldsymbol{0})$ be a function germ. We say that F is a *Morse family of hypersurfaces* if the mapping

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_k}\right) : (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \boldsymbol{0})$$

is non-singular, where $(q,x)=(q_1,\dots,q_k,x_1,\dots,x_n)\in(\mathbb{R}^k\times\mathbb{R}^n,\mathbf{0})$. In this case we have a smooth $(n-1)$-dimensional submanifold

$$\Sigma_*(F)=\left\{(q,x)\in(\mathbb{R}^k\times\mathbb{R}^n,\mathbf{0})\mid F(q,x)=\frac{\partial F}{\partial q_1}(q,x)=\cdots=\frac{\partial F}{\partial q_k}(q,x)=0\right\}$$

and the map germ $\Phi_F:(\Sigma_*(F),\mathbf{0})\longrightarrow PT^*\mathbb{R}^n$ defined by

$$\Phi_F(q,x)=\left(x,[\frac{\partial F}{\partial x_1}(q,x):\cdots:\frac{\partial F}{\partial x_n}(q,x)]\right)$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd-Zakalyukin [1, 19].

Proposition 5.1 *All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.*

We call F a *generating family* of $\Phi_F(\Sigma_*(F))$. Therefore the wave front is

$$W(\Phi_F)=\left\{x\in\mathbb{R}^n\mid\exists\ q\in\mathbb{R}^k;\ F(q,x)=\frac{\partial F}{\partial q_1}(q,x)=\cdots=\frac{\partial F}{\partial q_k}(q,x)=0\right\}.$$

We sometime denote $\mathcal{D}_F=W(\Phi_F)$ and call it the *discriminant set* of F.

Proposition 5.2 *The de Sitter height function $H:U\times S^n_1\longrightarrow\mathbb{R}$ is a Morse family of hypersurfaces.*

Proof. For any $\boldsymbol{v}=(v_0,v_1,\dots,v_n)\in S^n_1$, we have $-v_0^2+v_1^2+\cdots+v_n^2=1$. Without loss of the generality, we might assume that $v_n>0$. We have $v_n=\sqrt{1+v_0^2-v_1^2-\cdots-v_{n-1}^2}$, so that

$$H(u,\boldsymbol{v})=-x_0(u)v_0+x_1(u)v_1+\cdots+x_{n-1}(u)v_{n-1}\pm x_n(u)\sqrt{1+v_0^2-v_1^2-\cdots-v_{n-1}^2}.$$

We prove that the mapping

$$\Delta^*H=\big(H,\frac{\partial H}{\partial u_1},\dots,\frac{\partial H}{\partial u_{n-1}}\big)$$

is non-singular at any point on $(\Delta^*H)^{-1}(0)$. The Jacobian matrix of Δ^*H is given as follows:

$$\begin{pmatrix}\langle\boldsymbol{x}_{u_1},\boldsymbol{v}\rangle & \cdots & \langle\boldsymbol{x}_{u_{n-1}}\rangle & -x_0+x_n\frac{v_0}{v_n} & \cdots & x_{n-1}-x_n\frac{v_{n-1}}{v_n}\\ \langle\boldsymbol{x}_{u_1u_1},\boldsymbol{v}\rangle & \cdots & \langle\boldsymbol{x}_{u_1u_{n-1}},\boldsymbol{v}\rangle & -x_{0u_1}+x_{nu_1}\dfrac{v_0}{v_n} & \cdots & x_{n-1u_1}-x_{nu_1}\dfrac{v_{n-1}}{v_n}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \langle\boldsymbol{x}_{u_{n-1}u_1},\boldsymbol{v}\rangle & \cdots & \langle\boldsymbol{x}_{u_{n-1}u_{n-1}},\boldsymbol{v}\rangle & -x_{0u_{n-1}}+x_{nu_{n-1}}\dfrac{v_0}{v_n} & \cdots & x_{n-1u_{n-1}}-x_{nu_{n-1}}\dfrac{v_{n-1}}{v_n}\end{pmatrix}.$$

We now show that the determinant of the matrix

$$A=\begin{pmatrix}-x_{0u_1}+x_{nu_1}\dfrac{v_0}{v_n} & x_{1u_1}-x_{nu_1}\dfrac{v_1}{v_n} & \cdots & x_{n-1u_1}-x_{nu_1}\dfrac{v_{n-1}}{v_n}\\ \vdots & \vdots & \vdots & \vdots\\ -x_{0u_{n-1}}+x_{nu_{n-1}}\dfrac{v_0}{v_n} & x_{1u_{n-1}}-x_{nu_{n-1}}\dfrac{v_1}{v_n} & \cdots & x_{n-1u_{n-1}}-x_{nu_{n-1}}\dfrac{v_{n-1}}{v_n}\end{pmatrix}$$

does not vanish on $\Sigma_*(H) = (\Delta^* H)^{-1}(0)$. We denote that

$$\boldsymbol{a}_0 = \begin{pmatrix} x_0 \\ x_{0u_1} \\ \vdots \\ x_{0u_{n-1}} \end{pmatrix}, \boldsymbol{a}_1 = \begin{pmatrix} x_1 \\ x_{1u_1} \\ \vdots \\ x_{1u_{n-1}} \end{pmatrix}, \dots, \boldsymbol{a}_n = \begin{pmatrix} x_n \\ x_{nu_1} \\ \vdots \\ x_{nu_{n-1}} \end{pmatrix}.$$

It follow that we have

$$A = \big(-\boldsymbol{a}_0 + \boldsymbol{a}_n \frac{v_0}{v_n}, \boldsymbol{a}_1 - \boldsymbol{a}_n \frac{v_1}{v_n}, \dots, \boldsymbol{a}_{n-1} - \boldsymbol{a}_n \frac{v_{n-1}}{v_n}\big).$$

Therefore

$$\begin{aligned} \mathrm{detA} &= (-1)^{n-1}\big\{\frac{v_0}{v_n}\det(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) - \frac{v_1}{v_n}\det(\boldsymbol{a}_0, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n) \\ &\qquad + \dots + (-1)^n \frac{v_n}{v_n}\det(\boldsymbol{a}_0, \dots, \boldsymbol{a}_{n-1})\big\} \\ &= (-1)^{n-1}\left\langle \left(\frac{v_0}{v_n}, \dots, \frac{v_n}{v_n}\right), \boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \dots \wedge \boldsymbol{x}_{u_{n-1}} \right\rangle \\ &= \frac{(-1)^{n-1}}{v_n} \langle \boldsymbol{x}^d, \|\boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \dots \wedge \boldsymbol{x}_{u_{n-1}}\| \boldsymbol{x}^d \rangle \\ &= \frac{(-1)^{n-1}}{v_n} \|\boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \dots \wedge \boldsymbol{x}_{u_{n-1}}\| \neq 0 \end{aligned}$$

for $(u, \boldsymbol{v}) \in \Sigma_*(H)$. We have the same calculations as the above on the other local coordinates of S^n_1. This completes the proof of proposition. □

We now show that H is a generating family of $\mathcal{L}_5(U) \subset \Delta_5$.

Theorem 5.3 *For any timelike hypersurface $\boldsymbol{x} : U \longrightarrow S^n_1$, the de Sitter height function $H : U \times S^n_1 \longrightarrow \mathbb{R}$ of $\boldsymbol{x}$ is a generating family of the Legendrian embedding $\mathcal{L}_5$.*

Proof. We consider a coordinate neighborhood $W^+_n = \{\boldsymbol{w} = (w_0, w_1, \dots, w_n) \in S^n_1 \mid w_n > 0\}$. Remember the contact morphism $\Psi : \Delta_5(W^+) \longrightarrow PT^*(S^n_1)|\Delta_5$ defined in the proof of Proposition 2.2. Since the de Sitter height function $H : U \times S^n_1 \longrightarrow \mathbb{R}$ is a Morse family of hypersurfaces, we have a Legendrian immersion

$$\mathcal{L}_H : \Sigma_*(H)|(U \times w^+_n) \longrightarrow PT^*(S^n_1)|W^+_n$$

defined by

$$\mathcal{L}_H(u, \boldsymbol{w}) = \left(\boldsymbol{w}, \left[\frac{\partial H}{\partial w_0} : \dots : \frac{\partial H}{\partial w_n}\right]\right),$$

where $\boldsymbol{w} = (w_0, \dots, w_n)$. By Proposition 4.1, we have

$$\Sigma_*(H) = \{(u, \boldsymbol{x}^d(u)) \in U \times S^n_1 \mid u \in U\}.$$

Since $\boldsymbol{w} = \boldsymbol{x}^d(u)$ and $w_n = \sqrt{w_0^2 - \sum_{i=1}^{n-1} w_i^2 - 1}$, we have

$$\begin{aligned} \frac{\partial H}{\partial w_i}(u, \boldsymbol{x}^d(u)) &= -x_0(u) + x_n(u)\frac{x^d_0(u)}{x^d_n(u)}, \\ \frac{\partial H}{\partial w_i}(u, \boldsymbol{x}^d(u)) &= x_i(u) - \frac{x^d_i(u)}{x^d_n(u)} \ (i = 1, \dots, n-1), \end{aligned}$$

where $\boldsymbol{x}(u) = (x_0(u), \ldots, x_n(u))$ and $\boldsymbol{x}^d(u) = (x_0^d(u), \ldots, x_n^d(u))$. It follows that

$$\mathcal{L}_H(u, \boldsymbol{x}^d(u)) = (\boldsymbol{x}^d(u), [-x_n^d(u)x_0(u) + x_0^d(u)x_1(u) : \cdots : x_n^d(u)x_{n-1}(u) - x_n(u)x_{n-1}^d(u)]).$$

Therefore we have $\Psi \circ \mathcal{L}_5(u) = \mathcal{L}_H(u)$ on W_n^+. We also have the same relation as the above on the other local coordinates. This means that H is a generating family of $\mathcal{L}_5(U) \subset \Delta_5$. $\Box$

6 Contact with flat timelike de Sitter hyperquadrics

Before we start to consider the contact of timelike hypersurfaces with de Sitter flat timelike hyperquadrics, we briefly review the theory of contact due to Montaldi [14]. Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact of* X_1 and Y_1 at y_1 is the same type as the *contact of* X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. In his paper [14], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory. Two function germs $f, g : (\mathbb{R}^m, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ are said to be $\mathcal{K}$-*equivalent* if there exists a diffeomorphism germ $\phi : (\mathbb{R}^m, \mathbf{0}) \longrightarrow (\mathbb{R}^m, \mathbf{0})$ such that $\phi^*\langle f\rangle_{\mathcal{E}_m} = \langle g\rangle_{\mathcal{E}_m}$ Here, $\mathcal{E}_m$ is the local ring of function germs $(\mathbb{R}^m, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_m = \{h \in \mathcal{E}_m \mid h(0) = 0\}$ and $\phi^* : \mathcal{E}_m \longrightarrow \mathcal{E}_m$ is the pull back homomorphism.

Theorem 6.1 *Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

We also need some preparations from the theory of Legendrian singularities. Let us introduce an equivalence relation among Legendrian immersion germs. Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$ such that H preserves fibers of π and that $H(L) = L'$. A Legendrian immersion germ $i : (L.p) \subset PT^*\mathbb{R}^n$ (or, a *Legendrian map* $\pi \circ i$) at a point is said to be *Legendrian stable* if for every map with the given germ there is a neighborhood in the space of Legendrian immersions (in the Whitney C^∞ topology) and a neighborhood of the original point such that each Legendrian immersion belonging to the first neighborhood has in the second neighborhood a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs:

Proposition 6.2 *Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs such that the representative of both of germs are proper mappings and the regular sets of the projections $\pi \circ i, \pi \circ i'$ are dense. Then i, i' are Legendrian equivalent if and only if wave front sets $W(i), W(i')$ are diffeomorphic as set germs.*

This result has been firstly pointed out by Zakalyukin [20]. The assumption in the above proposition is a generic condition for i, i'. Specially, if i, i' are Legendrian stable, then these satisfy the assumption. We can interpret the Legendrian equivalence by using the notion of generating families. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F\rangle_{\mathcal{E}_{k+n}}) = \langle G\rangle_{\mathcal{E}_{k+n}}$. For any map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define the *local ring with degree ℓ of f* by $Q_\ell(f) = \mathcal{E}_n/f^*(\mathfrak{M}_p)\mathcal{E}_n + \mathfrak{M}_n^{\ell+1}$. One of the main results in the theory of Legendrian singularities is the following classification theorem (cf., [1], §21, see also [3], the appendix).

Theorem 6.3 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Suppose that Φ_F, Φ_G are Legendrian stable. The the following conditions are equivalent.*

(1) *$(W(\Phi_F), \mathbf{0})$ and $(W(\Phi_G), \mathbf{0})$ are diffeomorphic as germs.*

(2) *Φ_F and Φ_G are Legendrian equivalent.*

(3) *F and G are $\mathcal{K}$-equivalent.*

(4) *$Q_{n+2}(f)$ and $Q_{n+2}(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$, $g = G|\mathbb{R}^k \times \{\mathbf{0}\}$.*

If we do not assume that Φ_F and Φ_G are Legendrian stable, then the conditions (2) and (3) are equivalent.

We do not need the stable $\mathcal{K}$-equivalence in the above theorem because we fix the number of parameters n of generating families.

We now consider a function $\mathcal{H} : S^n_1 \times S^n_1 \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v}\rangle$. For any $\boldsymbol{v}_0 \in S^n_1$, we denote that $\mathfrak{h}_{v_0}(\boldsymbol{u}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v}_0)$ and we have a de Sitter flat timelike hyperquadric $\mathfrak{h}^{-1}_{v_0}(0) = HP(\boldsymbol{v}_0, 0) \cap S^n_1 = DH(\boldsymbol{v}_0.0)$. For any $u_0 \in U$, we consider the spacelike vector $\boldsymbol{v}_0 = \boldsymbol{x}^d(u_0)$, then we have

$$\mathfrak{h}_{v_0} \circ \boldsymbol{x}(u_0) = \mathcal{H} \circ (\boldsymbol{x} \times 1_{LC^*})(u_0, \boldsymbol{v}_0) = H(u_0, \boldsymbol{x}^d(u_0)) = 0.$$

By Proposition 4.1, we also have relations that

$$\frac{\partial \mathfrak{h}_{v_0} \circ \boldsymbol{x}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0, \boldsymbol{x}^d(u_0)) = 0.$$

for $i = 1, \dots, n-1$. This means that the de Sitter flat timelike hyperquadric $\mathfrak{h}^{-1}_{v_0}(0) = DH(\boldsymbol{v}_0, 0)$ is tangent to $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. In this case, we call $DH(\boldsymbol{v}_0, 0)$ the *tangent de Sitter flat hyperquadric* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ (or, u_0), which we write $TDH(\boldsymbol{x}, u_0)$. We call a germ $(\boldsymbol{x}^{-1}(TDH(\boldsymbol{x}, u_0)), u_0)$ the *tangent de Sitter indicatrix germ* of $\boldsymbol{x}(U) = M$ at $\boldsymbol{x}(u_0)$. Then we have the following simple lemma.

Lemma 6.4 *Let $\boldsymbol{x} : U \longrightarrow S^n_1$ be a timelike hypersurface. Consider two points $u_1, u_2 \in U$. Then $\boldsymbol{x}^d(u_1) = \boldsymbol{x}^d(u_2)$ if and only if $TDH(\boldsymbol{x}, u_1) = TDH(\boldsymbol{x}, u_2)$.*

Eventually, we have tools for the study of the contact between hypersurfaces and de Sitter flat timelike hyperquadrics.

Let $\boldsymbol{x}^d_i : (U, u_i) \longrightarrow (S^n_1, \boldsymbol{v}_i)$ $(i = 1, 2)$ be de Sitter Gaussian image germs of timelike hypersurface germs $\boldsymbol{x}_i : (U, u_i) \longrightarrow (S^n_1, \boldsymbol{u}_i)$. We say that $\boldsymbol{x}^d_1$ and $\boldsymbol{x}^d_2$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\phi : (U, u_1) \longrightarrow (U, u_2)$ and $\Phi : (S^n_1, \boldsymbol{v}_1) \longrightarrow (S^n_1, \boldsymbol{v}_2)$ such that $\Phi \circ \boldsymbol{x}^d_1 = \boldsymbol{x}^d_2 \circ \phi$. If both the regular sets of $\boldsymbol{x}^d_i$ are dense in (U, u_i), it follows from Proposition 6.2 that $\boldsymbol{x}^d_1$ and $\boldsymbol{x}^d_2$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian immersion germs $\mathcal{L}^1_5 : (U, u_1) \longrightarrow (\Delta_5, z_1)$ and $\mathcal{L}^2_5 : (U, u_2) \longrightarrow (\Delta_5, z_2)$ are Legendrian equivalent. This condition is also equivalent to the condition that two generating families H_1 and H_2 are P-$\mathcal{K}$-equivalent by Theorem 6.3. Here, $H_i : (U \times S^n_1, (u_i, \boldsymbol{v}_i)) \longrightarrow \mathbb{R}$ is the de Sitter height function germ of $\boldsymbol{x}_i$.

On the other hand, we denote that $h_{i,v_i}(u) = H_i(u, \boldsymbol{v}_i)$, then we have $h_{i,v_i}(u) = \mathfrak{h}_{v_i} \circ \boldsymbol{x}_i(u)$. By Theorem 6.1, $K(\boldsymbol{x}_1(U), TDH(\boldsymbol{x}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TDH(\boldsymbol{x}_2, u_2), \boldsymbol{v}_2)$ if and only if h_{1,v_1} and h_{1,v_2} are $\mathcal{K}$-equivalent. Therefore, we can apply Theorem 6.3 to our situation. We denote $Q_{n+2}(\boldsymbol{x}, u_0)$ the local ring with degree $n+1$ of the function germ $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$, where $\boldsymbol{v}_0 = \boldsymbol{x}^\ell(u_0)$. We remark that we can explicitly write the local ring as follows:

$$Q_{n+1}(\boldsymbol{x}, u_0) = \frac{C^\infty_{u_0}(U)}{\langle\langle \boldsymbol{x}(u), \boldsymbol{x}^d(u_0)\rangle\rangle_{C^\infty_{u_0}(U)} + \mathfrak{M}^{n+2}_{u_0}(U)},$$

where $C^\infty_{u_0}(U)$ is the local ring of function germs at u_0 with the unique maximal ideal $\mathfrak{M}_{u_0}(U)$.

Theorem 6.5 *Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (S^n_1, \boldsymbol{u}_i)$ $(i = 1, 2)$ be hypersurfaces germs such that the corresponding Legendrian map germs $\pi_{5,2} \circ \mathcal{L}^i_5 = \boldsymbol{x}^d_i : (U, u_i) \longrightarrow (S^n_1, \boldsymbol{v}_i)$ are Legendrian stable. Then the following conditions are equivalent:*

(1) *de Sitter Gauss image germs $\boldsymbol{x}^d_1$ and $\boldsymbol{x}^d_2$ are $\mathcal{A}$-equivalent.*

(2) $K(\boldsymbol{x}_1(U), TDH(\boldsymbol{x}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TDH(\boldsymbol{x}_2, u_2), \boldsymbol{v}_2)$.

(3) *H_1 and H_2 are P-$\mathcal{K}$-equivalent.*

(3) *$Q_{n+1}(\boldsymbol{x}_1, u_1)$ and $Q_{n+1}(\boldsymbol{x}_2, u_2)$ are isomorphic as $\mathbb{R}$-algebras.*

In the next section, we will prove that the assumption of the theorem is generic in the case when $n \leq 6$. In general we have the following proposition.

Proposition 6.6 *Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (S^n_1, \boldsymbol{x}_i(u_i))$ $(i = 1, 2)$ be timelike hypersurface germs such that their de Sitter parabolic sets have no interior points as subspaces of U. If de Sitter Gauss image germs $\boldsymbol{x}^d_1$, $\boldsymbol{x}^d_2$ are $\mathcal{A}$-equivalent, then*

$$K(\boldsymbol{x}_1(U), TDH(\boldsymbol{x}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TDH(\boldsymbol{x}_2, u_2), \boldsymbol{v}_2).$$

In this case, $(\boldsymbol{x}^{-1}_1(TDH(\boldsymbol{x}_1, \boldsymbol{v}_1)), u_1)$ and $(\boldsymbol{x}^{-1}_2(TDH(\boldsymbol{x}_2, \boldsymbol{v}_2)), u_2)$ are diffeomorphic as set germs.

Proof. The de Sitter parabolic set is the set of singular points of the de Sitter Gauss image. So the corresponding Legendrian embeddings $\mathcal{L}^i_5$ satisfy the hypothesis of Proposition 6.2. If de Sitter Gauss image germs $\boldsymbol{x}^d_1$, $\boldsymbol{x}^d_2$ are $\mathcal{A}$-equivalent, then $\mathcal{L}^1_5$, $\mathcal{L}^2_5$ are Legendrian equivalent, so that H_1, H_2 are P-$\mathcal{K}$-equivalent. Therefore, h_{1,v_1}, h_{1,v_2} are $\mathcal{K}$-equivalent. By Theorem 6.1, this condition is equivalent to the condition that $K(\boldsymbol{x}_1(U), TDH(\boldsymbol{x}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TDH(\boldsymbol{x}_2, u_2), \boldsymbol{v}_2)$.

On the other hand, we have $(\boldsymbol{x}^{-1}_i(TDH(\boldsymbol{x}_i, u_i)), u_i) = (h^{-1}_{i,v_i}(0), u_i)$. It follows from this fact that $(\boldsymbol{x}^{-1}_1(TDH(\boldsymbol{x}_1, u_1)), u_1)$ and $(\boldsymbol{x}^{-1}_2(TDH(\boldsymbol{x}_2, u_2)), u_2)$ are diffeomorphic as set germs because the $\mathcal{K}$-equivalence preserves the zero level sets. □

By Proposition 6.6, the diffeomorphism type of the tangent de Sitter indicatrix germ is an invariant of the $\mathcal{A}$-classification of the de Sitter Gauss image germ of $\boldsymbol{x}$. Moreover, we can borrow some basic invariants from the singularity theory on function germs. We need $\mathcal{K}$-invariants for function germ. The local ring of a function germ is a complete $\mathcal{K}$-invariant for generic function germs. It is, however, not a numerical invariant. The $\mathcal{K}$-codimension (or, Tyurina number) of a function germ is a numerical $\mathcal{K}$-invariant of function germs [13]. We denote that

$$\text{P-ord}(\boldsymbol{x}, u_0) = \dim \frac{C^\infty_{u_0}(U)}{\langle\langle \boldsymbol{x}(u), \boldsymbol{x}^d(u_0)\rangle, \langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}^d(u_0)\rangle\rangle_{C^\infty_{u_0}}}.$$

Usually $\text{P-ord}(\boldsymbol{x}, u_0)$ is called the *$\mathcal{K}$-codimension of* h_{v_0}. However, we call it the *order of contact with the tangent de Sitter flat hyperquadric* at $\boldsymbol{x}(u_0)$. We also have the notion of corank of function germs.

$$\text{P-corank}(\boldsymbol{x}, u_0) = (n-1) - \operatorname{rank}\operatorname{Hess}(h_{v_0}(u_0)),$$

where $\boldsymbol{v}_0 = \boldsymbol{x}^d(u_0)$.

By Proposition 4.2, $\boldsymbol{x}(u_0)$ is a de Sitter parabolic point if and only if $\text{P-corank}(\boldsymbol{x}, u_0) \geq 1$. Moreover $\boldsymbol{x}(u_0)$ is a lightcone flat point if and only if $\text{P-corank}(\boldsymbol{x}, u_0) = n-1$.

On the other hand, a function germ $f : (\mathbb{R}^{n-1}, \boldsymbol{a}) \longrightarrow \mathbb{R}$ has the A_k-type singularity if f is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 \pm \cdots \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If $\text{P-corank}(\boldsymbol{x}, u_0) = 1$, the de Sitter height function h_{v_0} has the A_k-type singularity at u_0 in generic. In this case we have $\text{P-ord}(\boldsymbol{x}, u_0) = k$. This number is equal to the order of contact in the classical sense (cf., [2]). This is the reason why we call $\text{P-ord}(\boldsymbol{x}, u_0)$ the order of contact with the tangent de Sitter flat hyperquadric at $\boldsymbol{x}(u_0)$.

7 Generic properties

In this section we consider generic properties of timelike hypersurfaces in S^n_1. The main tool is a kind of transversality theorems. We consider the space of timelike embeddings $\mathrm{Emb}_{\mathrm{T}}(U, S^n_1)$ with Whitney C^∞-topology. We also consider the function $\mathcal{H} : S^n_1 \times S^n_1 \longrightarrow \mathbb{R}$ which is given in §6. We claim that $\mathcal{H}_u$ is a submersion for any $\boldsymbol{u} \in S^n_1$, where $\mathcal{H}_u(\boldsymbol{v}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v})$. For any $\boldsymbol{x} \in \mathrm{Emb}_{\mathrm{T}}(U, S^n_1)$, we have $H = \mathcal{H} \circ (\boldsymbol{x} \times 1_{S^n_1})$. We also have the k-jet extension

$$j^k_1 H : U \times S^n_1 \longrightarrow J^k(U, \mathbb{R})$$

defined by $j^k_1 H(u, \boldsymbol{v}) = j^k h_v(u)$. We consider the trivialization $J^k(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^k(n-1, 1)$. For any submanifold $Q \subset J^k(n-1, 1)$, we denote that $\widetilde{Q} = U \times \{0\} \times Q$. Then we have the following proposition as a corollary of Lemma 6 in Wassermann [17]. (See also Montaldi [15]).

Proposition 7.1 *Let Q be a submanifold of $J^k(n-1, 1)$. Then the set*

$$T_Q = \{\boldsymbol{x} \in \mathrm{Emb}_{\mathrm{T}}(U, S^n_1) \mid j^k_1 H \text{ is transversal to } \widetilde{Q}\ \}$$

is a residual subset of $\mathrm{Emb}_{\mathrm{T}}(U, S^n_1)$. *If Q is a closed subset, then T_Q is open.*

We have a characterization of the stability of Legendrian immersion germs. Let $F : (\mathbb{R}^k \times \mathbb{R}^3, \boldsymbol{0}) \longrightarrow (\mathbb{R}, \boldsymbol{0})$ be a function germ. We say that F is a *$\mathcal{K}$-versal deformation of* $f = F|\mathbb{R}^k \times \{\boldsymbol{0}\}$ if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{\boldsymbol{0}\}, \dots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{\boldsymbol{0}\} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

(See [13].) It has been shown in [1, 19] that Φ_F is Legendrian stable if and only if F is a *$\mathcal{K}$-versal deformation of* $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$. We need the following characterization of $\mathcal{K}$-versality of generating families. Let $J^\ell(\mathbb{R}^k,\mathbb{R})$ be the ℓ-jet bundle of n-variable functions which has the canonical decomposition: $J^\ell(\mathbb{R}^k,\mathbb{R}) \equiv \mathbb{R}^k \times \mathbb{R} \times J^\ell(k,1)$. For any Morse family of hypersurfaces $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$, we define a map germ

$$j_1^\ell F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow J^\ell(\mathbb{R}^k,\mathbb{R})$$

by $j_1^\ell F(q,x) = j^\ell F_x(q)$, where $F_x(q) = F(q,x)$. We denote $\mathcal{K}^\ell(z)$ the $\mathcal{K}$-orbit through $z = j^\ell f(\mathbf{0}) \in J^\ell(k,1)$. (cf., [13]). If $f(q) = F(q,\mathbf{0})$ is ℓ-determined relative to $\mathcal{K}$, then F is a $\mathcal{K}$-versal deformation of f if and only if $j_1^\ell F$ is transversal to $\mathbb{R}^k \times \{0\} \times \mathcal{K}^\ell(z)$ (cf., [13]). Therefore we can apply this characterization to the de Sitter height function. By the classification of stable Legendrian singularities of $n < 6$ and Proposition 7.1, we have the following proposition.

Proposition 7.2 *Assume that* $n < 6$. *Then there exists an open dense subset* $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{T}}(U, S^n_1)$ *such that for any* $\boldsymbol{x} \in \mathcal{O}$ *the corresponding Legendrian embedding germ* $\mathcal{L}_5 : (U,u) \longrightarrow \Delta_5$ *at any point* $u \in U$ *is Legendrian stable*

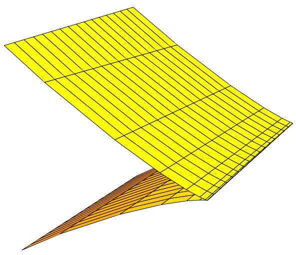
As a corollary of the above theorem and the classification results on wave fronts (cf., [1, 19]), we have the following local classification for the de Sitter Gauss image for a generic timelike surface in S^3_1.

Theorem 7.3 *Let* $\mathrm{Emb}_{\mathrm{T}}\,(U, S^3_1)$ *be the space of timelike embeddings from an open region* $U \subset \mathbb{R}^2$ *into* S^3_1 *equipped with the Whitney* C^∞*-topology. There exists an open dense subset* $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{T}}\,(U, S^3_1)$ *such that for any* $\boldsymbol{x} \in \mathcal{O}$, *the following conditions hold:*

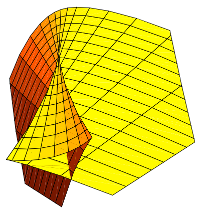
(1) *The de Sitter parabolic set* $K_d^{-1}(0)$ *is a regular curve. We call such a curve the de Sitter parabolic curve.*

(2) *The de Sitter Gauss image* $\boldsymbol{x}^\ell$ *along the de Sitter parabolic curve is locally diffeomorphic to the cuspidaledge except at isolated points. At such isolated points,* $\boldsymbol{x}^d$ *is locally diffeomorphic to the swallowtail.*

Here, the cuspidaledge is $C = \{(x_1,x_2,x_3)|{x_1}^2 = {x_2}^3\}$ *and the swallowtail is* $SW = \{(x_1,x_2,x_3)|x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ *(cf., Fig.1).*



cuspidaledge



swallowtail

Fig. 1.

Following the terminology of Whitney[18], we say that a timelike surface $\boldsymbol{x} : U \longrightarrow S^3_1$ has *the excellent de Sitter Gauss image* $\boldsymbol{x}^d$ if $\mathcal{L}_5$ is a stable Legendrian embedding at each point. In this case, the de Sitter Gauss image $\boldsymbol{x}^d$ has only cuspidaledges and swallowtails as singularities.

Theorem 7.2 asserts that a timelike surface with the excellent de Sitter Gauss image is generic in the space of all timelike surfaces in S^3_1. Since the diffeomorphism type of tangent de Sitter indicatrix germ is an invariant of the $\mathcal{A}$-classification of the de Sitter Gauss image, we have the following properties.

Proposition 7.4 *Let $\boldsymbol{x}^d : (U, u_0) \longrightarrow (S^3_1, \boldsymbol{v}_0)$ be the excellent lightcone Gauss image germ of a timelike surface $\boldsymbol{x}$ and $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$ be the de Sitter height function germ at $\boldsymbol{v}_0 = \boldsymbol{x}^d(u_0)$. Suppose that u_0 is a de Sitter parabolic point of $\boldsymbol{x}$.*

(1) *The following conditions are equivalent:*

(a) $\boldsymbol{x}^d$ *has a cuspidaledge at* u_0

(b) h_{v_0} *has the* A_2*-type singularity.*

(c) $\text{P-ord}(\boldsymbol{x}, u_0) = 2$.

(d) *The tangent de Sitter indicatrix germ is a ordinary cusp, where a curve $C \subset \mathbb{R}^2$ is called a ordinary cusp if it is diffeomorphic to the curve given by $\{(u_1, u_2) \mid u_1^2 - u_2^3 = 0\ \}$.*

(2) *The following conditions are equivalent:*

(a) $\boldsymbol{x}^d$ *has a swallowtail at* u_0

(b) h_{v_0} *has the* A_3*-type singularity.*

(c) $\text{P-ord}(\boldsymbol{x}, u_0) = 3$.

(d) *The tangent de Sitter indicatrix germ is a point or a tachnodal, where a curve $C \subset \mathbb{R}^2$ is called a tachnodal if it is diffeomorphic to the curve given by $\{(u_1, u_2) \mid u_1^2 - u_2^4 = 0\ \}$.*

Proof. We have shown that u_0 is a de Sitter parabolic point if and only if $\text{P-corank}(\boldsymbol{x}, u_0) \geq 1$. Since $n = 3$, we have $\text{P-corank}(\boldsymbol{x}, u_0) \leq 2$. Since the de Sitter height function germ $H : (U \times S^3_1, (u_0, \boldsymbol{v}_0)) \longrightarrow \mathbb{R}$ can be considered as a generating family of the Legendrian immersion germ $\mathcal{L}_5$, h_{v_0} has only the A_k-type singularities ($k = 1, 2, 3$). This means that the corank of the Hessian matrix of h_{v_0} at a de Sitter parabolic point is 1. Therefore, the conditions (1);(a),(b),(c) (respectively, (2); (a),(b),(c)) are equivalent. If the height function germ h_{v_0} has the A_2-type singularity, it is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 + u_2^3$. Since the $\mathcal{K}$-equivalence send the zero level sets, the tangent de Sitter indicatrix germ is diffeomorphic to the curve given by $\pm u_1^2 + u_2^3 = 0$. This is the ordinary cusp. The normal form for the A_3-type singularity is given by $\pm u_1^2 + u_2^4$, so that the tangent de Sitter indicatrix germ is diffeomorphic to the curve $\pm u_1^2 + u_2^4 = 0$. This means that the condition (1),(d) (respectively, (2),(d)) is also equivalent to the other conditions. This completes the proof. $\square$

8 Examples

In this section we give some examples. We consider a function germ $f(u_0, u_1)$ around the origin with $f(0) = 0$ and $f_{u_i}(0) = 0$ $(i = 0, 1)$. We only consider on the local coordinates

$$W_3^+ = \{(x_0, x_1, x_2, x_3) \in S^3_1 \mid x_3 > 0\}.$$

We define a function $g(u_0, u_1)$ by

$$g(u_0, u_1) = \sqrt{u_0^2 - u_1^2 - f^2(u_0, u_1) + 1}.$$

Then we have a time like surface

$$\boldsymbol{x}_f(u_0, u_1) = (u_0, u_1, f(u_0, u_1), g(u_0, u_1))$$

in de Sitter space S^3_1. Since $\boldsymbol{x}_f(0,0) = (0,0,0,1)$, the tangent de Sitter flat hyperquadrics is

$$TDH_f(u_0,u_1) = (u_0,u_1,0,\sqrt{u_0^2 - u_1^2 + 1}).$$

Therefore the tangent de Sitter indicatrix germ at the origin is

$$\{(u_0,u_1) \in \mathbb{R}^2 \mid f(u_0,u_1) = 0\}.$$

If we try to draw pictures of the de Sitter Gauss image, it might be very hard to give a parameterization. However, the tangent de Sitter indicatrix germ is very useful and easy to detect the type of singularities of the de Sitter Gauss image.

Example 8.1 Consider the function given by

$$f(u_0,u_1) = \left(\frac{1}{3}u_0^3 - \frac{1}{2}u_1^2\right).$$

Then the tangent de Sitter indicatrix germ at the origin is the ordinary cusp. By Proposition 7.4, $\boldsymbol{x}_f(0)$ is a de Sitter parabolic point and $\boldsymbol{x}^d_f(0)$ might be the cuspidaledge.

Example 8.2 Consider the function given by

$$f(u_0,u_1) = \left(\frac{1}{4}u_0^4 - \frac{1}{2}u_1^2\right).$$

Then the tangent de Sitter indicatrix germ at the origin is the tachnode. Therefore, $\boldsymbol{x}_f(0)$ is a de Sitter parabolic point and $\boldsymbol{x}^d_f(0)$ might be the swallowtail.

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