A CLASS OF NONLINEAR EVOLUTION EQUATIONS GOVERNED BY TIME-DEPENDENT OPERATORS OF SUBDIFFERENTIAL TYPE

Dedicated to Professor N. Kenmochi *on the Occasion of His 60th Birthday*

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Abstract. Recently there are so many mathematical models which describe nonlinear phenomena. In some phenomena, the free energy functional is not convex. So, the existence-uniqueness question is sometimes difficult. In order to study such phenomena, let us introduce the new class of abstract nonlinear evolution equations governed by time-dependent operators of subdifferential type. In this paper we shall show the existence and uniqueness of solution to nonlinear evolution equations with time-dependent constraints in a real Hilbert space. Moreover we apply our abstract results to a parabolic variational inequality with time-dependent double obstacles constraints.

AMS Subject Classification 34A60, 35K55, 35K90, 47J35:
Keywords Nonlinear evolution equation, subdifferential, parabolic PDE's, variational inequality

1 Introduction

We study an abstract nonlinear evolution equation in a real Hilbert space H of the form

$$u'(t) + \partial\varphi^t(u(t);u(t)) + G(t,u(t)) \ni f(t) \quad \text{in } H, \text{ a.e. } t \in (0,T), \tag{1}$$

where $u'(t) := \frac{d}{dt}u(t)$, $G(t,\cdot)$ is a single valued perturbation small relative to φ^t, and f is a given H-valued function. For each $t \in [0,T]$, a function $\varphi^t(\cdot;\cdot) : H \times H \to \mathbb{R} \cup \{\infty\}$ is given such that for all $w \in H$, $\varphi^t(w;\cdot) : H \to \mathbb{R} \cup \{\infty\}$ is a proper, l.s.c. (lower semi-continuous) and convex function, and $\partial\varphi^t(w;\cdot)$ is its subdifferential operator, i.e., $z^* \in \partial\varphi^t(w;z)$ if and only if

$$z \in D(\varphi^t(w;\cdot)) \quad \text{and} \quad (z^*, y-z) \le \varphi^t(w;y) - \varphi^t(w;z) \text{ for all } y \in H.$$

For a proper, l.s.c. and convex function $\psi^t(\cdot) : H \to \mathbb{R} \cup \{\infty\}$, many mathematicians studied the nonlinear evolution equation of the form

$$u'(t) + \partial\psi^t(u(t)) \ni f(t) \quad \text{in } H, \text{ a.e. } t \in (0,T). \tag{2}$$

For various aspects of (2), we refer to [2, 5, 6, 8, 9, 11, 18, 19]. For instance, Kenmochi [6] showed the existence-uniqueness, stability and convergence of solutions to (2).

For the nonmonotone perturbation $G(t,\cdot)$, Ôtani [16] has already shown the existence of solution to

$$u'(t) + \partial\psi^t(u(t)) + G(t,u(t)) \ni f(t) \quad \text{in } H, \text{ a.e. } t \in (0,T). \tag{3}$$

The large-time behavior of solutions for (3) was discussed by [20] from the view-point of attractors. For another works of (3), we refer to [10, 16, 17, 20, 21, 22], for instance.

The main object of this paper is to establish abstract results on existence-uniqueness of solutions to (1). Note that the function $\varphi^t(u;u)$ is not convex in u, hence we can not apply the theory established by Ôtani [16]. So, by using the idea of Kenmochi-Kubo [7] and Kubo-Yamazaki [12, 13], we shall show the existence of solution to (1) in this paper. Namely, for the given function $w : [0,T] \to H$, let us consider the problem

$$u'(t) + \partial\varphi^t(w(t);u(t)) \ni f(t) - G(t,w(t)) \quad \text{in } H, \text{ a.e. } t \in (0,T). \tag{4}$$

Assuming some appropriate conditions on the t- and w-dependence of the function $\varphi^t(w;z)$, we can apply the result of Kenmochi [6]. Then we see that the equation (4) has a unique solution u for each w, and that the mapping $w \mapsto u$ has some compactness property. Hence, by using a fixed point argument, we can get the existence of solution to (1).

In Section 2 we present our main results on existence and uniqueness of solution to (1), and then the uniqueness (Theorem 3) is proved. In Section 3 we prove the local existence result (Theorem 1). In Section 4, the global existence result (Theorem 2) is proved. In the final Section 5 we apply our abstract results to a parabolic variational inequality with time-dependent double obstacle constraints.

Notation

Throughout this paper, let H be a real Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot,\cdot)$. For a proper l.s.c. convex function ψ on H we use the notation $D(\psi)$, $\partial\psi$ and $D(\partial\psi)$ to indicate the effective domain, subdifferential and its domain of $\partial\psi$, respectively. For their precise definitions and basic properties, see a monograph by Brézis [4].

2 Assumptions and main results

We consider a Cauchy problem $\mathrm{CP}(u_0)$ for (1) of the following form:

$$\mathrm{CP}(u_0)\begin{cases} u'(t) + \partial\varphi^t(u(t);u(t)) + G(t,u(t)) \ni f(t) \quad \text{ in } H \text{ a.e. } t \in (0,T), \\ u(0) = u_0, \end{cases}$$

where T is a given positive number, a function $\varphi^t(u(t);u(t))$ is introduced in Section 1, $G(t,\cdot)$ is a single valued perturbation small relative to φ^t, $f \in L^2(0,T;H)$ is a given function, and $u_0 \in H$ is given data.

Definition 1. Given $u_0 \in H$ and $f \in L^2(0,T;H)$, the function $u : [0,T] \to H$ will be called a solution to $\mathrm{CP}(u_0)$, if $u \in W^{1,2}(0,T;H)$, $u(0) = u_0$, $u(t) \in D(\partial\varphi^t(u(t);\cdot)$ and $f(t) - u'(t) - G(t,u(t)) \in \partial\varphi^t(u(t);u(t))$ for a.e. $t \in [0,T]$, namely

$$(f(t) - u'(t) - G(t,u(t)),\ y - u(t)) \leq \varphi^t(u(t);y) - \varphi^t(u(t);u(t))$$

$$\text{for any } y \in H, \text{ a.e. } t \in [0,T].$$

For a given positive number T, let $\{\alpha_r\} := \{\alpha_r;\ r > 0\}$ be a family of functions $\alpha_r \in W^{1,2}(0,T)$, with parameter $r > 0$. With this family $\{\alpha_r\}$, we specify a class $\Phi(\{\alpha_r\})$ of all families $\{\varphi^t\} := \{\varphi^t;\ t \in [0,T]\}$ of time-dependent functions $\varphi^t(\cdot;\cdot)$ on $H \times H$ as follows.

Definition 2. We denote by $\{\varphi^t\} \in \Phi(\{\alpha_r\})$ the set of all time-dependent functions $\varphi^t(\cdot;\cdot)$ from $H \times H$ into $\mathbb{R} \cup \{\infty\}$ satisfying the following seven conditions:

(Φ1) For each $w \in H$ and $t \in [0,T]$, $\varphi^t(w;\cdot) : H \to \mathbb{R} \cup \{\infty\}$ is a proper l.s.c. convex function;

(Φ2) There exists a positive constant $C_1 > 0$ such that

$$\varphi^t(w;z) \geq C_1|z|_H^2, \quad \forall t \in [0,T],\ \forall w \in H,\ \forall z \in D(\varphi^t(w;\cdot));$$

(Φ3) For each $t \in [0,T]$, $w \in H$ and $k > 0$, the level set $\{z \in H; \varphi^t(w;z) \leq k\}$ is compact in H;

(Φ4) $D(\varphi^t(w;\cdot))$ is independent of $w \in H$ for any $t \in [0,T]$;

(Φ5) For each $r > 0$, $s,\ t \in [0,T]$ with $s \le t$, $w \in D(\varphi^s(0;\cdot))$ with $|w|_H \le r$ and $z \in D(\varphi^s(w;\cdot))$ with $|z|_H \le r$ there exists an element $\tilde{z} \in D(\varphi^t(w;\cdot))$ such that

$$|\tilde{z} - z|_H \le |\alpha_r(t) - \alpha_r(s)| \left(1 + \varphi^s(0;z)^{\frac{1}{2}}\right)$$

and

$$\varphi^t(w;\tilde{z}) - \varphi^s(w;z) \le |\alpha_r(t) - \alpha_r(s)| \left(1 + \varphi^s(0;z) + \varphi^s(0;w)^{\frac{1}{2}}\varphi^s(0;z)^{\frac{1}{2}} + \varphi^s(0;w)^{\frac{1}{2}}\right);$$

(Φ6) For each $r > 0$ there is a positive constant $C_r > 0$ such that

$$|\varphi^t(w_1;z) - \varphi^t(w_2;z)| \le C_r |w_1 - w_2|_H \varphi^t(0;z)^{\frac{1}{2}},$$

$$\forall t \in [0,T],\ \forall w_i \in H \text{ with } |w_i|_H \le r,\ (i=1,2), \text{ and } \forall z \in D(\varphi^t(0,\cdot));$$

(Φ7) There is a function $h \in W^{1,2}(0,T;H)$ with $C_h := \sup_{t \in [0,T]} \varphi^t(0;h(t)) < +\infty$.

Next, we introduce the class $\mathcal{G}(\{\varphi^t\})$ of time-dependent perturbation $G(t,\cdot)$ associated with $\{\varphi^t\} \in \Phi(\{a_r\})$.

Definition 3. $\{G(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$ if and only if $G(t,\cdot)$ is a single valued operator from $D(G(t,\cdot)) \subset H$ into H which fulfills the following conditions (G1)-(G3):

(G1) $D(\varphi^t(0;\cdot)) \subset D(G(t,\cdot)) \subset H$ for all $t \in [0,T]$ and $G(\cdot, v(\cdot))$ is (strongly) measurable on J for any interval $J \subset [0,T]$ and $v \in L^2_{loc}(J;H)$ with $v(t) \in D(\varphi^t(0;\cdot))$ for a.e. $t \in J$.

(G2) There are positive constants $C_2 > 0,\ C_3 > 0$ such that

$$|G(t,z)|_H^2 \le C_2 \varphi^t(z;z) + C_3, \quad \forall t \in [0,T],\ \forall z \in D(\varphi^t(0;\cdot)).$$

(G3) (Demi-closedness) If $\{t_n\} \subset [0,T]$, $\{z_n\} \subset H$, $t_n \to t$, $z_n \to z$ in H (as $n \to +\infty$) and $\{\varphi^{t_n}(0,z_n)\}$ is bounded, then $G(t_n,z_n) \to G(t,z)$ weakly in H as $n \to +\infty$.

Now let us mention our main local existence result in this paper. In Section 3 we shall prove Theorem 1.

Theorem 1. *Let T be any positive number. Assume $\{\varphi^t\} \in \Phi(\{\alpha_r\})$, $\{G(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $f \in L^2(0,T;H)$. Then, for each $u_0 \in D(\varphi^0(0;\cdot))$ there exists a positive constant $T_0(\le T)$ such that $CP(u_0)$ has at least one solution u on $[0,T_0]$.*

The next main theorem is concerned with the global existence result in this paper. In Section 4 we shall prove Theorem 2.

Theorem 2. *Let T be any positive number. Assume $\{\varphi^t\} \in \Phi(\{\alpha_r\})$, $\{G(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $f \in L^2(0,T;H)$. Additionally, assume that*

(Φ8) *There is a positive constant $C_4 > 0$ such that*

$$\varphi^t(w; z) \le C_4(1 + |w|_H^2 + \varphi^t(0; z)), \quad \forall t \in [0, T],\ \forall w \in H,\ \forall z \in D(\varphi^t(0; \cdot)).$$

Then, for each $u_0 \in D(\varphi^0(0; \cdot))$ there exists at least one solution u to CP(u_0) on $[0, T]$.

To show the uniqueness of solution to CP(u_0), we shall introduce subclasses of $\Phi(\{\alpha_r\})$ and $\mathcal{G}(\{\varphi^t\})$.

Definition 4. Let γ be a non-negative continuous and convex function on H such that $\gamma(z) + \gamma(-z) = 0$ if and only if $z = 0$. Then
(1) $\{\varphi^t\} \in \Phi_\gamma(\{\alpha_r\})$ if and only if $\{\varphi^t\} \in \Phi(\{\alpha_r\})$ satisfies the γ-accretiveness $(\star)$ for φ^t as follows:

($\star$) For any $z_i \in D(\partial\varphi^t(z_i; \cdot))$ and $z_i^* \in \partial\varphi^t(z_i; z_i)$ $(i = 1,\ 2)$, there is an element $w_0 \in \partial\gamma(z_1 - z_2)$ so that $(z_1^* - z_2^*, w_0) \ge 0$, where $\partial\gamma$ is the subdifferential of γ in H.

(2) $\{G(t, \cdot)\} \in \mathcal{G}_\gamma(\{\varphi^t\})$ if and only if for any positive number $\varepsilon > 0$, there is a positive constant $C_\varepsilon > 0$ such that

$$|(G(t, z_1) - G(t, z_2), w_0)| \le \varepsilon(z_1^* - z_2^*, w_0) + C_\varepsilon\{\gamma(z_1 - z_2) + \gamma(z_2 - z_1)\},$$

whenever $t \in [0, T],\ z_i \in D(\partial\varphi^t(z_i; \cdot)),\ z_i^* \in \partial\varphi^t(z_i; z_i)\ (i = 1, 2)$, and
$w_0 \in \partial\gamma(z_1 - z_2)$ with $(z_1^* - z_2^*, w_0)_H \ge 0$.

Now let us mention our main uniqueness result in this paper.

Theorem 3. *Let T be any positive number. Assume $\{\varphi^t\} \in \Phi_\gamma(\{\alpha_r\})$, $\{G(t, \cdot)\} \in \mathcal{G}_\gamma(\{\varphi^t\})$ and $f \in L^2(0, T; H)$. Then, for each $u_0 \in H$ the solution u to CP(u_0) is unique.*

Proof. Let u and v be solutions to CP(u_0). By the γ-accretiveness of φ^t, for a.e. $\tau \in [0, T]$ there exists $z^*(\tau) \in \partial\gamma(u(\tau) - v(\tau))$ such that

$$(u^*(\tau) - v^*(\tau), z^*(\tau)) \ge 0 \tag{5}$$

for any $u^*(\tau) \in \partial\varphi^\tau(u(\tau); u(\tau))$ and $v^*(\tau) \in \partial\varphi^\tau(v(\tau); v(\tau))$.

By $\{G(t, \cdot)\} \in \mathcal{G}_\gamma(\{\varphi^t\})$, for a number $\varepsilon \in (0, 1]$ there is a constant $C_\varepsilon > 0$ such that

$$\begin{aligned} & |\,(G(\tau, u(\tau)) - G(\tau, v(\tau)), z^*(\tau))\,| \\ \le\ & \varepsilon(u^*(\tau) - v^*(\tau), z^*(\tau)) + C_\varepsilon\{\gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau))\} \end{aligned} \tag{6}$$

for a.e. $\tau \in [0, T]$.

From (5) and (6) it follows that

$$\begin{aligned} 0 &\le (u^*(\tau) - v^*(\tau), z^*(\tau)) \\ &= ([f(\tau) - u'(\tau) - G(\tau, u(\tau))] - [f(\tau) - v'(\tau) - G(\tau, v(\tau))], z^*(\tau)) \\ &\le (-u'(\tau) + v'(\tau), z^*(\tau)) + |\,(-G(\tau, u(\tau)) + G(\tau, v(\tau)), z^*(\tau))\,| \\ &\le -\frac{d}{d\tau}\gamma(u(\tau) - v(\tau)) \\ &\qquad + \varepsilon(u^*(\tau) - v^*(\tau), z^*(\tau)) + C_\varepsilon\{\gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau))\}, \end{aligned}$$

which implies that

$$\frac{d}{d\tau}\gamma(u(\tau)-v(\tau)) \le C_\varepsilon\{\gamma(u(\tau)-v(\tau))+\gamma(v(\tau)-u(\tau))\} \quad \text{for a.e. } \tau\in[0,T].$$

Similarly we have

$$\frac{d}{d\tau}\gamma(v(\tau)-u(\tau)) \le C_\varepsilon\{\gamma(u(\tau)-v(\tau))+\gamma(v(\tau)-u(\tau))\},$$

hence we have

$$\frac{d}{d\tau}\{\gamma(u(\tau)-v(\tau))+\gamma(v(\tau)-u(\tau))\} \le 2C_\varepsilon\{\gamma(u(\tau)-v(\tau))+\gamma(v(\tau)-u(\tau))\}, \tag{7}$$

for a.e. $\tau\in[0,T]$.

Now, applying Gronwall's inequality to (7), we get

$$e^{-2C_\varepsilon t}\{\gamma(u(t)-v(t))+\gamma(v(t)-u(t))\}\le 0 \quad \text{for any } 0\le t\le T,$$

which implies that $u(t)=v(t)$ for all $t\in[0,T]$. Thus Theorem 3 has been proved. □

3 Proof of Theorem 1

In this section we shall show Theorem 1 by the fixed point argument. To do so, for a given positive number $T>0$, we put a Banach space

$$E(T)\equiv\left\{w\in W^{1,2}(0,T;H)\ ;\ \sup_{t\in[0,T]}\varphi^t(0;w(t))<+\infty\right\}.$$

By the assumption (Φ7) we note that $E(T)\neq\emptyset$.

Now, for each $w\in E(T)$ let us consider a following Cauchy problem CP$(w;u_0)$:

$$\mathrm{CP}(w;u_0)\begin{cases} u'(t)+\partial\varphi^t(w(t);u(t))\ni f(t)-G(t,w(t)) & \text{in } H,\ \text{a.e. } t\in(0,T),\\ u(0)=u_0.\end{cases}$$

To show the existence-uniqueness of solution to CP$(w;u_0)$, we prepare the key lemma.

Lemma 1. *For each $w\in E(T)$ we take a positive constant $R>0$ such that $\sup_{t\in[0,T]}|w(t)|_H\le R$. Put*

$$\psi_w^t(z):=\varphi^t(w(t);z) \text{ for } z\in H.$$

Then, there is a positive constant $N_1>0$ independent of w satisfying the following: for any $s,\ t\in[0,T]$ with $s\le t$ and $z\in D(\psi_w^s)$ with $|z|_H\le R$, there exists $\tilde z\in D(\psi_w^t)$ such that

$$|\tilde z-z|_H\le N_1(1+C_R)^4(1+R)^6|\alpha_R(t)-\alpha_R(s)|\left(1+\psi_w^s(z)^{\frac12}\right), \tag{8}$$

$$\begin{aligned}&\psi_w^t(\tilde z)-\psi_w^s(z)\\ \le\ & N_1(1+C_R)^4(1+R)^6\Big\{|\alpha_R(t)-\alpha_R(s)|(1+\psi_w^s(z))+|w(t)-w(s)|_H\,(1+\psi_w^s(z))^{\frac12}\\ &+|\alpha_R(t)-\alpha_R(s)|\varphi^s(0;w(s))^{\frac12}\,(1+\psi_w^s(z))^{\frac12}\Big\}.\end{aligned} \tag{9}$$

Proof. Taking $w = w(s)$ in (Φ5), then for any $s,\ t \in [0,T]$ with $s \le t$ and $z \in D(\varphi^s(w(s);\cdot)$ with $|z|_H \le R$, there exists $\tilde{z} \in D(\varphi^t(w(s);\cdot))$ such that

$$|\tilde{z} - z|_H \le |\alpha_R(t) - \alpha_R(s)| \left(1 + \varphi^s(0;z)^{\frac{1}{2}}\right), \tag{10}$$

$$\begin{aligned} & \varphi^t(w(s);\tilde{z}) - \varphi^s(w(s);z) \\ \le\ & |\alpha_R(t) - \alpha_R(s)| \left(1 + \varphi^s(0;z) + \varphi^s(0;w(s))^{\frac{1}{2}}\varphi^s(0;z)^{\frac{1}{2}} + \varphi^s(0;w(s))^{\frac{1}{2}}\right). \end{aligned} \tag{11}$$

It follows from (Φ4) that

$$z \in D(\varphi^s(w(s);\cdot) = D(\psi_w^s), \quad \tilde{z} \in D(\varphi^t(w(s);\cdot)) = D(\psi_w^t). \tag{12}$$

Note that by (Φ6) and $w \in E(T)$ we have

$$\varphi^s(0;z) \le 2\varphi^s(w(s);z) + C_R^2|w(s)|_H^2 \le 2\psi_w^s(z) + C_R^2R^2. \tag{13}$$

Then, by (10) and (13) there is a positive number $N_2 > 0$ independent of w satisfying

$$\begin{aligned} |\tilde{z} - z|_H \ &\le\ |\alpha_R(t) - \alpha_R(s)| \left(1 + \sqrt{2}\psi_w^s(z)^{\frac{1}{2}} + C_RR\right) \\ &\le\ N_2(1 + C_RR)|\alpha_R(t) - \alpha_R(s)| \left(1 + \psi_w^s(z)^{\frac{1}{2}}\right). \end{aligned} \tag{14}$$

Moreover, we observe that by (11), (13), (Φ6) there is a positive number $N_3 > 0$ independent of w satisfying the following:

$$\begin{aligned} & \psi_w^t(\tilde{z}) - \psi_w^s(z) \ \left(= \varphi^t(w(t);\tilde{z}) - \varphi^t(w(s);\tilde{z}) + \varphi^t(w(s);\tilde{z}) - \varphi^s(w(s);z)\right) \\ \le\ & N_3(1 + C_R)^2(1 + R)^2 \Big\{ |w(t) - w(s)|_H\psi_w^t(\tilde{z})^{\frac{1}{2}} + |w(t) - w(s)|_H \\ & +|\alpha_R(t) - \alpha_R(s)|(1 + \psi_w^s(z)) + |\alpha_R(t) - \alpha_R(s)|\varphi^s(0;w(s))^{\frac{1}{2}}\left(1 + \psi_w^s(z)\right)^{\frac{1}{2}} \Big\}. \end{aligned} \tag{15}$$

From $\alpha_R \in W^{1,2}(0,T)$, $w \in E(T)$ and (15) it follows that

$$\psi_w^t(\tilde{z}) \le N_4(1 + C_R)^4(1 + R)^6 \left\{1 + \psi_w^s(z) + |\alpha_R(t) - \alpha_R(s)|^2\varphi^s(0;w(s))\right\} \tag{16}$$

for some constant $N_4 > 0$. Therefore, using (16) in the right hand side of (15), and by (12)-(14), we get this Lemma for some constant $N_1 > 0$ independent of w. □

Proposition 1. *For each $w \in E(T)$, CP($w;u_0$) has a unique solution u on $[0,T]$.*

Proof. We note that CP($w;u_0$) can be regarded as the Cauchy problem for the nonlinear evolution equation of the form:

$$\begin{cases} u'(t) + \partial\psi_w^t(u(t)) \ni f(t) - G(t,w(t)) & \text{in } H \text{ a.e. } t \in (0,T), \\ u(0) = u_0. \end{cases}$$

Here, from (Φ6) and (G2) we see that for each $w \in E(T)$ with $\sup_{t\in[0,T]}|w(t)|_H \le R$

$$\begin{aligned} \int_0^T |G(t,w(t))|_H^2 dt \ &\le\ \int_0^T \left\{C_2\varphi^t(w(t);w(t)) + C_3\right\} dt \\ &\le\ T\left\{2C_2\sup_{t\in[0,T]}\varphi^t(0;w(t)) + \frac{C_2C_R^2R^2}{4} + C_3\right\} < +\infty, \end{aligned}$$

which implies that $f - G(\cdot, w(\cdot)) \in L^2(0,T;H)$. Moreover, by Lemma 1 we get the time-dependence of ψ_w^t. Therefore taking account of the assumption (Φ1), we can apply the abstract theory established by Kenmochi [6]. Thus we get the existence-uniqueness of solution u for CP$(w;u_0)$. For detail proofs, see [6, Theorems 1.1.1, 1.1.2]. $\square$

By Proposition 1, the boundedness (cf. [6, Theorem 1.1.2]) of solution to CP$(w;u_0)$, and (13), we can define a mapping , we can define a mapping $Q : E(T) \longrightarrow E(T)$ by $Qw = u$ for each $w \in E(T)$, where u is a solution for CP$(w;u_0)$.

Lemma 2. *There are positive constants T_0, M_0 and R_0 such that Q is a self-mapping on $E(T_0,M_0,R_0)$, i.e., $Qw(=u) \in E(T_0,M_0,R_0)$ for any $w \in E(T_0,M_0,R_0)$, where*

$$E(T_0,M_0,R_0) \equiv \left\{ w \in E(T_0) \ ; \ \begin{array}{ll} \sup\limits_{t\in[0,T_0]} \varphi^t(0;w(t)) \le M_0, & |w'|^2_{L^2(0,T_0;H)} \le M_0 \\ \sup\limits_{t\in[0,T_0]} |w(t)|_H \le R_0, & w(0) = u_0 \end{array} \right\}.$$

Proof. Fix $R > 0$ for a while and take $w \in E(T)$ with $\sup\limits_{t\in[0,T]} |w(t)|_H \le R$. We shall give a boundedness of solution u to the problem CP$(w;u_0)$.

Now, multiplying CP$(w;u_0)$ by $u(t) - h(t)$, we get

$$\begin{aligned} & (u'(t), u(t)-h(t)) + \varphi^t(w(t);u(t)) - \varphi^t(w(t);h(t)) \\ \le \ & (f(t) - G(t,w(t)), u(t)-h(t)) \qquad \text{a.e. } t \in (0,T), \end{aligned} \tag{17}$$

where h is the function in (Φ7). Taking account of (Φ2), (Φ6) and (G2), we have

$$\begin{aligned} & \frac{d}{dt}|u(t)-h(t)|_H^2 - |u(t)-h(t)|_H^2 \\ \le \ & N_5 \left(|f(t)|_H^2 + |h'(t)|_H^2 + \varphi^t(0;h(t)) + \varphi^t(0;w(t)) + C_R^2 R^2 + 1 \right), \end{aligned} \tag{18}$$

for a constant $N_5 = N_5(C_2, C_3) > 0$. By applying Gronwall's inequality to (18), we obtain

$$\begin{aligned} & \sup_{t\in[0,T]} |u(t)|_H \\ \le \ & \sup_{t\in[0,T]} |h(t)|_H + e^{\frac{T}{2}}|u_0 - h(0)|_H + e^{\frac{T}{2}} N_5^{\frac{1}{2}} \left\{ |f|_{L^2(0,T;H)} + |h'|_{L^2(0,T;H)} \right\} \\ & + e^{\frac{T}{2}} N_5^{\frac{1}{2}} T^{\frac{1}{2}} \left\{ \sup_{t\in[0,T]} \varphi^t(0;h(t))^{\frac{1}{2}} + \sup_{t\in[0,T]} \varphi^t(0;w(t))^{\frac{1}{2}} + C_R R + 1 \right\}. \end{aligned} \tag{19}$$

Moreover, by Lemma 1 and arguments of [6, section 1], we see that the function $\psi_w^t(u(t)) = \varphi^t(w(t);u(t))$ is of bounded variation on $[0,T]$ and satisfies

$$\begin{aligned} & \psi_w^t(u(t)) - \psi_w^s(u(s)) + \int_s^t (u'(\tau) - f(\tau) + G(\tau, w(\tau)), u'(\tau)) d\tau \\ \le \ & N_1(1+C_R)^4(1+R)^6 \int_s^t |\alpha_R'(\tau)||u'(\tau) - f(\tau) + G(\tau,w(\tau))| \left\{ 1 + \psi_w^\tau(u(\tau))^{\frac{1}{2}} \right\} d\tau \\ & + N_1(1+C_R)^4(1+R)^6 \int_s^t \left[|\alpha_R'(\tau)|(1 + \psi_w^\tau(u(\tau)) + |w'(\tau)|_H \left\{ 1 + \psi_w^\tau(u(\tau)) \right\}^{\frac{1}{2}} \right] d\tau \\ & + N_1(1+C_R)^4(1+R)^6 \int_s^t |\alpha_R'(\tau)| \varphi^\tau(0;w(\tau))^{\frac{1}{2}} \left\{ 1 + \psi_w^\tau(u(\tau)) \right\}^{\frac{1}{2}} d\tau \end{aligned} \tag{20}$$

for $0 \le s \le t \le T$ and $w \in E(T)$ with $\sup_{t\in[0,T]} |w(t)|_H \le R$.

Here we notice the following relations:

$$(u'(\tau) - f(\tau) + G(\tau, w(\tau)), u'(\tau)) \ge \frac{1}{2}|u'(\tau)|_H^2 - |f(\tau)|_H^2 - |G(\tau, w(\tau))|_H^2, \tag{21}$$

$$\begin{aligned} & |\alpha_R'(\tau)||u'(\tau) - f(\tau) + G(\tau, w(\tau))|_H \left\{1 + \psi_w^\tau(u(\tau))^{\frac{1}{2}}\right\} \\ \le\ & \delta|u'(\tau) - f(\tau) + G(\tau, w(\tau))|_H^2 + \delta^{-1}|\alpha_R'(\tau)|^2 \{1 + \psi_w^\tau(u(\tau))\} \\ \le\ & 3\delta|u'(\tau)|_H^2 + 3\delta|f(\tau)|_H^2 + 3\delta|G(\tau, w(\tau))|_H^2 + \delta^{-1}|\alpha_R'(\tau)|^2 \{1 + \psi_w^\tau(u(\tau))\}, \end{aligned} \tag{22}$$

where in (22) we put $\delta := \dfrac{1}{12N_1(1 + C_R)^4(1 + R)^6}$. Using (21)-(22) in (20), we obtain

$$\begin{aligned} & \psi_w^t(u(t)) - \psi_w^s(u(s)) + \frac{1}{4}\int_s^t |u'(\tau)|_H^2 d\tau \\ \le\ & N_6(1 + C_R)^8(1 + R)^{12} \int_s^t \Big\{X(\tau)(1 + \psi_w^\tau(u(\tau))) + Y(\tau)\{1 + \psi_w^\tau(u(\tau))\}^{\frac{1}{2}} \\ & \qquad + |G(\tau, w(\tau))|_H^2\Big\} d\tau \end{aligned} \tag{23}$$

for $0 \le s \le t \le T$, where the constant $N_6 > 0$ is determined only by N_1, and we put

$$X(\tau) := |f(\tau)|_H^2 + |\alpha_R'(\tau)|^2 + 1, \qquad Y(\tau) := |w'(\tau)|_H + |\alpha_R'(\tau)|\varphi^\tau(0; w(\tau))^{\frac{1}{2}}.$$

By (Φ6), (G2) and (23), we obtain

$$\begin{aligned} & \psi_w^t(u(t)) - \psi_w^s(u(s)) + \frac{1}{4}\int_s^t |u'(\tau)|_H^2 d\tau \\ \le\ & N_7(1 + C_R)^{10}(1 + R)^{14} \int_s^t \{X(\tau) + Y(\tau) + \varphi^\tau(0; w(\tau))\}\{1 + \psi_w^\tau(u(\tau))\} d\tau \end{aligned} \tag{24}$$

for $0 \le s \le t \le T$, where $N_7 > 0$ depends on N_6, C_2 and C_3.

Applying Gronwall's inequality to (24), we obtain

$$\begin{aligned} & \sup_{0\le t\le T} \psi_w^t(u(t)) + \frac{1}{4}\int_0^T |u'(t)|_H^2 dt \\ & \le e^{N_7(1+C_R)^{10}(1+R)^{14}\left(|X|_{L^1(0,T)} + |Y|_{L^1(0,T)} + |\varphi^t(0;w(t))|_{L^1(0,T)}\right)} \\ & \times \left\{\psi_w^0(u_0) + N_7(1 + C_R)^{10}(1 + R)^{14}\left(|X|_{L^1(0,T)} + |Y|_{L^1(0,T)} + |\varphi^t(0; w(t))|_{L^1(0,T)}\right)\right\}. \end{aligned} \tag{25}$$

Now we show that Q is the self-mapping on $E(T_0, M_0, R_0)$ for some chosen constants $T_0 > 0$, $M_0 > 0$ and $R_0 > 0$.

Note that by (Φ6) we have

$$\varphi^t(0; u(t)) \le 2\varphi^t(w(t); u(t)) + C_R^2R^2 \left(= 2\psi_w^t(u(t)) + C_R^2R^2\right) \tag{26}$$

for any $w \in E(T)$ with $\sup_{t\in[0,T]} |w(t)|_H \le R$.

Here, we take $R_0 > 0,\ M_0 > 0$ so large that

$$2\left[\sup_{t\in[0,T]} |h(t)|_H + e^{\frac{T}{2}}|u_0 - h(0)|_H + e^{\frac{T}{2}} N_5^{\frac{1}{2}} \left\{|f|_{L^2(0,T;H)} + |h'|_{L^2(0,T;H)}\right\}\right] \le R_0,$$

$$\begin{aligned} & 4e^{2N_7(1+C_{R_0})^{10}(1+R_0)^{14}} \left\{\psi_w^0(u_0) + 2N_7(1+C_{R_0})^{10}(1+R_0)^{14}\right\} + C_{R_0}^2 R_0^2 + C_h \\ \le\ & 4e^{2N_7(1+C_{R_0})^{10}(1+R_0)^{14}} \left\{2\varphi^0(0;u_0) + \frac{C_{R_0}^2 R_0^2}{4} + 2N_7(1+C_{R_0})^{10}(1+R_0)^{14}\right\} \\ & \qquad + C_{R_0}^2 R_0^2 + C_h \\ \le\ & M_0. \end{aligned}$$

Next, we choose $T_0 > 0$ so small that $T_0 \le T$, $|h'|^2_{L^2(0,T_0;H)} \le M_0$, $|X|_{L^1(0,T_0)} \le 1$,

$$|Y|_{L^1(0,T_0)} + |\varphi^t(0;w(t))|_{L^1(0,T_0)} \le T_0^{\frac{1}{2}} M_0^{\frac{1}{2}} + M_0^{\frac{1}{2}} T_0^{\frac{1}{2}} |\alpha'_{R_0}|_{L^2(0,T_0)} + T_0 M_0 \le 1,$$

$$\begin{aligned} & \sup_{t\in[0,T_0]} |h(t)|_H + e^{\frac{T_0}{2}}|u_0 - h(0)|_H + e^{\frac{T_0}{2}} N_5^{\frac{1}{2}} \left\{|f|_{L^2(0,T_0;H)} + |h'|_{L^2(0,T_0;H)}\right\} \\ & \qquad + e^{\frac{T_0}{2}} N_5^{\frac{1}{2}} T_0^{\frac{1}{2}} \left\{\sup_{t\in[0,T_0]} \varphi^t(0;h(t))^{\frac{1}{2}} + M_0^{\frac{1}{2}} + C_{R_0} R_0 + 1\right\} \le R_0. \end{aligned}$$

Then, the estimates (19), (25) with (26) implies that $Qw(= u)$ belongs to the set $E(T_0, M_0, R_0)$ for $w \in E(T_0, M_0, R_0)$, thus Q is the self-mapping on $E(T_0, M_0, R_0)$. □

Lemma 3. *Let $M_0 > 0$, $R_0 > 0$ and $T_0 > 0$ be constants obtained in Lemma 2. Let $\{w_n\} \subset E(T_0, M_0, R_0)$, $w \in E(T_0, M_0, R_0)$ and u_n be the solution of CP($w_n; u_0$). Suppose $w_n \longrightarrow w$ in $C([0,T_0];H)$ as $n \to +\infty$. Then, there is a solution u of CP($w; u_0$) on $[0,T_0]$ such that $u \in E(T_0, M_0, R_0)$ and $u_n \longrightarrow u$ in $C([0,T_0];H)$ as $n \to +\infty$.*

Proof. Since $\{w_n\} \subset E(T_0, M_0, R_0)$ and Lemma 2, we have

$$\sup_{t\in[0,T_0]} \varphi^t(0;u_n(t)) \le M_0, \qquad |u_n'|^2_{L^2(0,T_0;H)} \le M_0, \qquad \forall n = 1,2,\cdots, \tag{27}$$

$$\sup_{t\in[0,T_0]} |u_n(t)|_H \le R_0, \qquad \forall n = 1,2,\cdots. \tag{28}$$

By (Φ3), (27), (28) there are a subsequence $\{n_k\}$ of $\{n\}$ and a function $u \in W^{1,2}(0,T_0;H)$ such that

$$u_{n_k} \longrightarrow u \quad \text{strongly in } C([0,T_0];H), \tag{29}$$

$$u'_{n_k} \rightharpoonup u' \quad \text{weakly in } L^2(0,T_0;H) \tag{30}$$

as $k \to +\infty$. By (Φ1), (27)-(30) and the uniqueness of u_n, we easily observe that $u \in E(T_0, M_0, R_0)$ and $u_n \longrightarrow u$ in $C([0,T_0];H)$ as $n \to +\infty$.

Now, let us show that u is a solution of CP($w; u_0$) on $[0,T_0]$. To do so, we define $\Phi(w;z) = \int_0^{T_0} \varphi^t(w(t);z(t))dt$. Then by the assumption (Φ6) we see that

$$\Phi(w_n;z) \longrightarrow \Phi(w;z) \text{ as } n \to +\infty \tag{31}$$

for any $z \in L^2(0,T_0;H)$ with $\varphi^{(\cdot)}(0;z(\cdot)) \in L^1(0,T_0)$. From (27), (29), ($\Phi$1), ($\Phi$2), ($\Phi$6) and the Fatou's lemma, it follows that

$$\begin{aligned}\liminf_{k\to+\infty} \Phi(w_{n_k};u_{n_k}) &= \liminf_{k\to+\infty}\{\Phi(w_{n_k};u_{n_k}) - \Phi(w;u_{n_k}) + \Phi(w;u_{n_k})\} \\ &\geq \liminf_{k\to+\infty} \Phi(w;u_{n_k}) \geq \Phi(w;u). \end{aligned} \tag{32}$$

Moreover, by $\{w_n\} \subset E(T_0,M_0,R_0)$ and the demi-closedness (G3) we see that

$$G(\cdot,w_{n_k}(\cdot)) \rightharpoonup G(\cdot,w(\cdot)) \quad \text{weakly in } L^2(0,T_0;H),$$

hence

$$f - G(\cdot,w_{n_k}(\cdot)) \rightharpoonup f - G(\cdot,w(\cdot)) \quad \text{weakly in } L^2(0,T_0;H) \tag{33}$$

as $k \to +\infty$.

Now, let z be any function in $L^2(0,T_0;H)$ with $\varphi^{(\cdot)}(0;z(\cdot)) \in L^1(0,T_0)$. Since u_{n_k} is the unique solution of CP($w_{n_k};u_0$), then the following inequality holds:

$$\int_0^{T_0} \big(f(t) - G(t,w_{n_k}(t)) - u'_{n_k}(t), z(t) - u_{n_k}(t)\big)\,dt \leq \Phi(w_{n_k};z) - \Phi(w_{n_k};u_{n_k}). \tag{34}$$

Taking account of (29)-(33) and letting $k \to +\infty$ in (34), we get

$$\int_0^{T_0} (f(t) - G(t,w(t)) - u'(t), z(t) - u(t))\,dt \leq \Phi(w;z) - \Phi(w;u),$$

which implies that $f(t) - G(t,w(t)) - u'(t) \in \partial\varphi^t(w(t);u(t))$ for a.e. $t \in [0,T_0]$ (cf. [1, Proposition 3.3]). Thus u is the solution of CP($w;u_0$) on $[0,T_0]$. □

Proof. **[Proof of Theorem 1; Local existence]** By Lemma 2, we can define a self-mapping $Q : E(T_0,M_0,R_0) \longrightarrow E(T_0,M_0,R_0)$ by $Qw = u$ for each $w \in E(T_0,M_0,R_0)$, where u is a solution of CP($w;u_0$). Clearly, $E(T_0,M_0,R_0)$ is compact in $C([0,T_0];H)$. Moreover, it follows from Lemma 3 that Q is continuous with respect to the topology of $C([0,T_0];H)$. Therefore, the Schauder's fixed point theorem implies that the self-mapping Q has a fixed point u in $E(T_0,M_0,R_0)$, i.e. $Qu = u$. Clearly u is the solution of CP(u_0), thus we can construct the local solution u of CP(u_0) on $[0,T_0]$. □

4 Proof of Theorem 2

In this section we shall prove Theorem 2, which is concerned with the global existence of solution to CP(u_0).

First, we consider the inequality (17). By the local existence result in Section 3, we can take $w = u \in E(T_0,M_0,R_0)$, u being the solution of CP(u_0) on a small time interval $[0,T_0]$ with $0 < T_0 \leq T$. Hence, by taking $w = u$ in (17) it follows from (G2) and the additional assumption (Φ8) that

$$\begin{aligned}&\frac{d}{dt}|u(t) - h(t)|_H^2 + \varphi^t(u(t);u(t)) \\ \leq\ & N_8|u(t) - h(t)|_H^2 + N_9\big(|f(t)|_H^2 + |h'(t)|_H^2 + \varphi^t(0;h(t)) + 1\big)\end{aligned} \tag{35}$$

for some constants $N_8 > 0$ and $N_9 > 0$ depending only on $C_1, C_2,\ C_3,\ C_4$. By applying Gronwall's inequality to (35), we obtain

$$\begin{aligned}
&\sup_{t\in[0,T_0]} |u(t)|_H \\
\le\ & \sup_{t\in[0,T]} |h(t)|_H + \sqrt{e^{N_8 T}}|u_0 - h(0)|_H + \sqrt{N_9 e^{N_8 T}}\left\{|f|_{L^2(0,T;H)} + |h'|_{L^2(0,T;H)}\right\} \\
&\qquad + \sqrt{N_9 T e^{N_8 T}}\left\{\sup_{t\in[0,T]} \varphi^t(0;h(t))^{\frac{1}{2}} + 1\right\} \equiv N_{10}. \qquad (36)
\end{aligned}$$

Next, take a number $R > 0$ with $R \ge N_{10}$, and we now consider the inequality (23). Applying Schwarz inequality to the term $Y(\tau)\left\{1 + \psi_w^\tau(u(\tau))\right\}^{\frac{1}{2}}$ and using (G2), (Φ8), we obtain

$$\begin{aligned}
&\psi_w^t(u(t)) - \psi_w^s(u(s)) + \frac{1}{4}\int_s^t |u'(\tau)|_H^2 d\tau \\
\le\ & N_{11}(1+C_R)^{16}(1+R)^{24}\int_s^t X(\tau)(1+\psi_w^\tau(u(\tau)))d\tau + \frac{1}{8}\int_s^t |w'(\tau)|_H^2 d\tau \\
&\qquad + N_{12}(1+C_R)^8(1+R)^{12}\int_s^t \varphi^\tau(0;w(\tau))d\tau \qquad (37)
\end{aligned}$$

for $0 \le s \le t \le T$, where $N_{11} > 0$ and $N_{12} > 0$ depend on $C_1, C_2,\ C_3,\ C_4,\ N_6$.

Applying Gronwall's inequality to (37), we obtain

$$\begin{aligned}
&\psi_w^t(u(t)) + \frac{1}{4}\int_0^t e^{N_{11}(1+C_R)^{16}(1+R)^{24}\int_\tau^t X(s)ds}|u'(\tau)|_H^2 d\tau \\
\le\ & e^{N_{11}(1+C_R)^{16}(1+R)^{24}\int_0^T X(s)ds}\left\{\psi_w^0(u_0) + N_{11}(1+C_R)^{16}(1+R)^{24}\int_0^T X(s)ds\right\} \\
&\quad + \frac{1}{8}\int_0^t e^{N_{11}(1+C_R)^{16}(1+R)^{24}\int_\tau^t X(s)ds}|w'(\tau)|_H^2 d\tau \\
&\quad + N_{12}(1+C_R)^8(1+R)^{12}\int_0^t e^{N_{11}(1+C_R)^{16}(1+R)^{24}\int_\tau^t X(s)ds}\varphi^\tau(0;w(\tau))d\tau. \qquad (38)
\end{aligned}$$

Here, we can take $w = u \in E(T_0, M_0, R_0), u$ being the solution of CP(u_0) on a small time interval $[0, T_0]$ with $0 < T_0 \le T$. Then, by using (26), (36), (38) we get

$$\begin{aligned}
&\varphi^t(u(t);u(t)) + \frac{1}{8}\int_0^t e^{N_{11}(1+C_R)^{16}(1+R)^{24}\int_\tau^t X(s)ds}|u'(\tau)|_H^2 d\tau \\
\le\ & N_{13}(1+C_R)^{16}(1+R)^{24}e^{N_{14}(1+C_R)^{16}(1+R)^{24}}\left(1 + \int_0^t \varphi^\tau(u(\tau);u(\tau))d\tau\right), \qquad (39)
\end{aligned}$$

for $0 \le t \le T_0$, where $N_{13} > 0, N_{14} > 0$ are dependent only on the given data. By applying Gronwall's inequality to (39), we conclude that

$$\begin{aligned}
&\varphi^t(u(t);u(t)) + \frac{1}{8}\int_0^{T_0} |u'(t)|_H^2 dt \\
\le\ & N_{15}(1+C_R)^{32}(1+R)^{48}\exp(N_{16}(1+C_R)^{16}(1+R)^{24}e^{N_{14}(1+C_R)^{16}(1+R)^{24}}), \qquad (40)
\end{aligned}$$

where $N_{15} > 0$ and $N_{16} > 0$ depends only on the given data and are independent of $T_0(\leq T)$ and $R(\geq N_{10})$.

Now we shall prove Theorem 2 by employing the estimates (36) and (40).

Proof. [**Proof of Theorem 2; Global existence**] Assume that

$$T^* := \sup\{T_0;\ \mathrm{CP}(u_0) \text{ has a solution on } [0,T_0]\} < +\infty.$$

By the local existence result in Section 3, we note $T^* > 0$. By the definition of T^*, there is a function $u : [0,T^*) \to H$ such that for any T_0 $(< T^*)$ u is the solution of $\mathrm{CP}(u_0)$ on $[0,T_0]$. By (36) and (40) we have

$$u \in W^{1,2}(0,T^*;H), \quad \varphi^{(\cdot)}(u(\cdot);u(\cdot)) \in L^\infty(0,T^*).$$

Hence by assumptions (Φ1), (Φ3), (Φ5), (Φ6) , we observe that the limit $u_0^* := \lim_{t\uparrow T^*} u(t)$ exists strongly in H such that

$$u_0^* \in D(\varphi^{T^*}(0;\cdot)).$$

Now, taking u_0^* as the initial value at $t = T^*$, we can get the solution u beyond the time interval $[0,T^*]$. Thus we observe that the solution to $\mathrm{CP}(u_0)$ exists on the whole time interval $[0,T]$. □

5 Application to a double obstacle problem

In this section we apply our abstract results (Theorems 1, 2, 3) to a parabolic variational inequality with time-dependent double obstacles.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary. Let g_1, g_2 be prescribed obstacle functions on $[0,T]\times\Omega$ so that

$$g_i \in L^\infty(0,T;H^1(\Omega)) \cap L^\infty([0,T]\times\Omega), \quad g_i' \in L^2(0,T;H^1(\Omega)) \cap L^2(0,T;L^\infty(\Omega))$$

for $i = 1,2$, and

$$g_2 - g_1 \geq C_g \quad \text{a.e. on } [0,T]\times\Omega \text{ for some constant } C_g > 0.$$

For each $t \in [0,T]$, we define the convex set $K(t)$ by

$$K(t) := \{z \in H^1(\Omega); g_1(t) \leq z \leq g_2(t) \text{ a.e. on } \Omega\}.$$

Now, let us consider the following interior time-dependent double obstacle problem.

Problem (P): Find a function $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$ such that

$$u(t) \in K(t) \quad \text{for a.e. } t \in [0,T],$$

$$(u'(t) + b(t,\cdot,u(t)) - f(t), u(t) - z) + \int_\Omega a(x,u(t),\nabla u(t))\cdot\nabla(u(t)-z)dx \leq 0$$
$$\text{for all } z \in K(t),$$

$$u(0) = u_0 \quad \text{in } \Omega,$$

where $(\cdot,\cdot)$ is a usual inner product of $L^2(\Omega)$, $a = (a_1, ..., a_N)$ is an elliptic vector field, b and f are given functions.

The aim of this section is to consider the problem (P) as an application of the abstract evolution equation CP(u_0). To do so, we suppose that

(A1) $a(x, s, p)$ is continuous on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ such that $a(x, s, p) = \partial_p A(x, s, p)$ for some potential function $A(x, s, p)$. Moreover, there exist constants $\mu > 0$, $\nu_1 = \nu_1(a) > 0$ and $\nu_2 = \nu_2(a) > 0$ such that

$$[a(x, s, p) - a(x, s, \hat{p})] \cdot (p - \hat{p}) \geq \mu |p - \hat{p}|^2,$$

$$|a(x, s, p)|^2 + |A(x, s, p)| + |\partial_s A(x, s, p)|^2 \leq \nu_1(1 + |s|^2 + |p|^2),$$
$$|a(x, s, p) - a(x, \hat{s}, p)| \leq \nu_2(1 + |p|)|s - \hat{s}|$$

for all $x \in \Omega,\ s, \hat{s} \in \mathbb{R},\ p, \hat{p} \in \mathbb{R}^N$.

(A2) $b(t, x, s)$ is continuous on $[0, T] \times \Omega \times \mathbb{R}$ satisfying the following properties: there exist a constant $L_b > 0$ and a function $d \in L^1(0, T)$ such that

$$|b(t, x, s) - b(t, x, \hat{s})] \leq L_b |s - \hat{s}|, \quad \forall t \in [0, T],\ \forall x \in \Omega,\ \forall s, \hat{s} \in \mathbb{R},$$

$$\sup_{x \in \Omega} \left| \frac{\partial}{\partial t} b(t, x, 0) \right| \leq d(t) \quad \text{for a.e. } t \geq 0.$$

As a direct application of Theorems 1, 2 and 3, we have:

Proposition 2. *Assume (A1) and (A2). Then, for each $f \in L^2(0, T; L^2(\Omega))$ and $u_0 \in K(0)$, the problem (P) has a unique solution u on $[0, T]$.*

Proof. To apply Theorems 1, 2 and 3 to the problem (P), we choose $L^2(\Omega)$ as a real Hilbert space H, and define a function $\varphi^t(\cdot;\cdot) : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ by

$$\varphi^t(w; z) := \begin{cases} \displaystyle\int_\Omega A(x, w(x), \nabla z(x)) dx + C_\mu(1 + |w|^2_{L^2(\Omega)}), & \text{if } z \in K(t), \\ +\infty, & \text{otherwise}, \end{cases}$$

where $C_\mu > 0$ is a constant such that $\varphi^t(w; z) \geq \dfrac{\mu}{4}|z|^2_{L^2(\Omega)} + 1$ for all $t \geq 0$, $w \in L^2(\Omega)$ and $z \in K(t)$ (cf. [13, Lemma 3.1]).

Let us define an operator $G(t, \cdot) : L^2(\Omega) \to L^2(\Omega)$ by $G(t, z) := b(t, \cdot, z(\cdot))$ in $L^2(\Omega)$. And we define a function γ by $\gamma(z) := \displaystyle\int_\Omega z^+(x) dx$ for $z \in L^2(\Omega)$, where $z^+ := \max\{z, 0\}$.

Now we put for any $t \in [0, T]$ and $r > 0$

$$\alpha_r(t) = k \int_0^t \left\{ |g_1'|_{L^\infty(\Omega)} + |g_2'|_{L^\infty(\Omega)} + |g_1'|_{H^1(\Omega)} + |g_2'|_{H^1(\Omega)} \right\} d\tau,$$

where $k > 0$ is a (sufficient large) positive constant. Then, we easily verify $\{\varphi^t\} \in \Phi_\gamma(\{\alpha_r\})$. For instance, we can show (Φ5) by taking

$$\tilde{z} := (z - g_1(s))\frac{g_2(t) - g_1(t)}{g_2(s) - g_1(s)} + g_1(t)$$

for given $z \in K(s)$. Then, by the slight modification of [22, Lemma 5.1], we can show (Φ5).

Moreover we easily see that $G(t, \cdot) \in \mathcal{G}_\gamma(\{\varphi^t\})$ and the assumption (Φ8) hold.

Clearly, the problem (P) can be reformulated in the evolution equation CP(u_0). Thus, by applying Theorems 1, 2 and 3, we see that (P) has a unique global solution u. □

6 Acknowledgements

The author wish to thank Professor Masahiro KUBO for his useful discussions and comments.

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