A SUBDIFFERENTIAL FORMULATION OF FOURTH ORDER SINGULAR DIFFUSION EQUATIONS

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A SUBDIFFERENTIAL FORMULATION OF FOURTH ORDER SINGULAR DIFFUSION EQUATIONS

YOHEI KASHIMA

Abstract

A fourth order equation with singular diffusivity, which is a model of relaxation dynamics for crystalline surfaces driven by surface diffusion, is formulated. The notion of subdifferentials enables us to formulate the singular diffusion equation mathematically as a gradient flow equation in the Sobolev space of negative power H^{-1}. The subdifferential of the singular energy in H^{-1} is calculated. Moreover, the speed of a special profile is calculated for one dimensional problem. It turns out that a seemingly natural free boundary formulation with facets is inconsistent with a subdifferential formulation which can be approximated by a smooth energy.

1 Introduction

We are concerned with a fourth order parabolic equation with a singular interfacial density $\sigma_\gamma(p) = |p| + |p|^\gamma/\gamma : \mathbb{R}^n \to \mathbb{R}$ of the form follows;

$$u_t = -\Delta \operatorname{div}(\nabla \sigma_\gamma(\nabla u)), \tag{EQ}$$

where $\gamma > 1$. An equivalent form of (EQ) is

$$u_t = -\Delta \operatorname{div}\left(\frac{\nabla u}{|\nabla u|} + |\nabla u|^{\gamma-1}\frac{\nabla u}{|\nabla u|}\right).$$

The equation (EQ) obviously has a singularity and does not make sense mathematically where $|\nabla u| = 0$. Our purpose is to formulate such a singular parabolic equation by interpreting its right hand as a subdifferential of a convex energy. Specifically, we are interested in formulating the initial-boundary value problem for (EQ) of the form

$$\begin{cases} u_t = -\Delta \operatorname{div}(\nabla \sigma_\gamma(\nabla u)) & \text{in } \Omega \times [0, +\infty), \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ u = \operatorname{div}(\nabla \sigma_\gamma(\nabla u)) = 0 & \text{on } \partial\Omega \times [0, +\infty), \end{cases} \tag{EQ1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \leq 4)$ is a bounded domain with a piecewise smooth boundary $\partial\Omega$ and supposed to locate one side of $\partial\Omega$. We are also interested in some profile of a solution of (EQ1).

The equation (EQ) comes from relaxation dynamics for crystalline surfaces caused by surface diffusion, where the function u stands for the height of the crystalline surface defined over Ω and $\sigma_\gamma(\nabla u)$ is the free energy of the surface at a given slope ∇u per unit area. It is generally explained that below roughening transition, the surface free energy of crystals includes the nondifferential term and this singular term plays a central role in the facetting of crystals. H. Spohn formulated this mechanism and derived (EQ) in [23]. Moreover, H. Spohn conjectured that the solution of (EQ) develops the spontaneous formation of facets (flat portion) for certain initial data in one dimensional case in [23], where he formulated (EQ) by a free boundary value problem.

While our equation is a fourth order PDE, the model of relaxation dynamics caused by evaporation was also proposed as a second order equation in [23] of the form

$$u_t = \operatorname{div}(\nabla\sigma_\gamma(\nabla u)). \tag{EQ'}$$

In [11] T. Fukui and Y. Giga formulated a similar model of crystalline motion as (EQ') with more general σ_γ in one dimension by using a method of subdifferential of a convex functional. In [10] C. M. Elliott, A. R. Gardiner and R. Schätzle proved that the solutions of the equation formulated in [11] describe the evolution of facets and simulated the motion of these solutions numerically. The results of these analysis corresponded to the theory of evaporation dynamics for crystalline surfaces.

In this paper, we formulate (EQ), the fourth order's version of (EQ'), by a similar approach developed in [11], that is to say, the *subdifferential formulation*. We impose two boundary conditions to fix the problem so that the equation we consider becomes (EQ1). Since the right hand of (EQ) is characterized by the variational derivative of the energy functional with respect to the metric of the Hilbert space $H^{-1}(\Omega)$ (the Sobolev space defined by a L^2-dual of $H_0^1(\Omega)$), we are able to formulate the formal variational derivative as the subdifferential of the energy functional with Dirichlet condition. Thus, (EQ1) can be formulated as an evolution equation written by the subdifferential of the energy in $H^{-1}(\Omega)$. Moreover, we know that the formulated equation has a unique global solution by the general nonlinear semigroup theory. The analysis of motions of the solution for a given initial data is also of our interest. To see the speed of the solution, we characterize the subdifferential of our convex energy when the space dimension is less than 4 by adjusting the argument by H. Attouch and A. Damlamian [2] in L^2 space to H^{-1} space. When the space dimension is one, we calculate the speed for a profile with one facet. It turns out that the speed is not constant on a facet contrary to the prediction that facets stay as a facet in [23], where (EQ) was formulated as a free boundary problem. This has a strong contrast to the evaporation model (EQ') where facets stay as a facet. The reader is referred to [12], [13] and review papers [14], [18] as well as [11].

It is difficult to characterize subdifferentials of convex functionals defined on a Sobolev space in general. In [2, Proposition (2.12)] H. Attouch and A. Damlamian proved that each value of an energy functional on a Sobolev space with zero Dirichlet condition can be approximated by the value of the functional at some smooth function with compact support. By a substantial use of this lemma the subdifferential of the functional was characterized in [2]. The reason why we are forced to impose zero Dirichlet condition on our energy functional and restrict the space dimension to be less than 4 is that we are forced to apply this lemma to our case.

Actually, we characterize subdifferentials of general energy functionals including our energy on $H^{-1}(\Omega)$ of the form $E(u) = \int_\Omega \tau(x, u(x), \nabla u(x))dx$ where the integrand $\tau : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ satisfies some properties such as convexity, integrability, and coerciveness (i.e. $\tau(x, r, s)/|s| \to +\infty$ as $|s| \to +\infty$), and so on (see the beginning of Section 3.2). In addition, we also give a simpler proof which is available for the characterization of the subdifferential of our energy functional under another assumption about the exponent γ of σ_γ.

The formulation of the fourth order equation (EQ") with the interfacial density $\sigma(p) = |p| : \mathbb{R}^n \to \mathbb{R}$ is also interesting topic.

$$u_t = -\Delta \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{EQ"}$$

The equation (EQ") is derived by the energy functional $\Phi(u) = \int_\Omega |\nabla u(x)|dx$ the total variation of each surface u on Ω. Our methods to formulate and analyze (EQ) are not applicable to (EQ"), since the interfacial density $\sigma(\cdot)$ of (EQ") is not coercive. In [11] the modified coercive energy $\Phi_K(u) = \int_\Omega \sigma_K(\nabla u)dx$ where $\sigma_K(p) = |p|$ if $|p| \leq K$ for a large $K > 0$ and $\sigma_K(\cdot)$ is supposed to be coercive (i.e. $\sigma_K(p)/|p| \to +\infty$ as $|p| \to +\infty$) was introduced to approximate its original non-coercive energy. In our case, however, solutions of the modified equation $u_t = -\Delta \operatorname{div}(\nabla\sigma_K(\nabla u))$ derived by this modified energy cannot be regarded as a solution of the equation (EQ") since the solution u is not expected to keep the Lipschitz bounded condition $\operatorname{ess\,sup}_{x\in\Omega} |\nabla u(x,t)| \leq K$ for any $t \geq 0$ therefore the solution u may be affected by the modification. This is a difficult point of the formulation of the fourth order equation (EQ") while each solution for a given Lipschitz bounded initial data is assured to retain the Lipschitz bounded condition globally in time in the second order's case [11]. We do not touch (EQ") any more in this paper.

As a model of the crystalline surface moved by surface diffusion, the following *surface diffusion flow equation* is typical;

$$V = -\Delta_s K, \tag{SD}$$

where V, K, and Δ_s represent the normal velocity of the surface, the surface curvature, and the surface Laplacian, respectively. The equation (SD) was introduced by W. W. Mullins in 1957 in [19], to describe evolutions of surface grooves produced by drift of surface atoms at grain boundaries of a polycrystal. There are several derivations of (SD) other than original [19], and some mathematical properties of (SD) are also known nowadays. About physical and mathematical background of (SD), see, for example, [7], [8], or the introduction of [15]. As one instance of fourth order parabolic equations represented by (SD), we are motivated to study (EQ).

The mathematical formulation of (EQ1) involving subdifferentials and some properties of the formulated equation will be argued in Chapter 2. To understand the formulated equation in detail, we explicitly calculate subdifferentials of convex functionals including our energy functional in Chapter 3. By using the result of Chapter 3, we will calculate the initial speed of the solution of the formulated (EQ1) in Chapter 4.

2 Subdifferential formulation

The purpose of this chapter is to formulate (EQ1) by using a subdifferential, an extended concept of differential, and to recall some fundamental properties of the formulated equation. The variational structure of (EQ) allows us to formulate the right hand of (EQ1) as the subdifferential of the surface free energy in the Sobolev space $H^{-1}(\Omega)$. By a general nonlinear semigroup theory, the formulated equation admits a unique global-in-time solution. Moreover, the general stability theorem implies that the solution can be approximated by solutions solving equations derived by properly smoothened energies. Finally, we will rewrite the subdifferential of our energy, which is a multi-valued operator in general, into a single-valued operator, the *canonical restriction* of the subdifferential.

2.1 Variational structure

The model (EQ) is derived by the total surface free energy $F_\gamma(u) = \int_\Omega \sigma_\gamma(\nabla u)dx$. To see the variational structure of (EQ), we calculate the first variation of the energy F_γ. The point is that we calculate the first variation with respect to the metric of $H^{-1}(\Omega)$. Let us review the properties of the Hilbert space $H^{-1}(\Omega)$ in advance;

The space $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ and the linear operator $-\Delta_D : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the isomorphism between these Hilbert spaces, where the scalar product of $H^{-1}(\Omega)$ is defined by $\langle \cdot, \cdot \rangle_{H^{-1}(\Omega)} := \langle (-\Delta_D)^{-1} \cdot, \cdot \rangle_{L^2(\Omega)}$.

Now we define our free energy F_γ as a convex functional on $H^{-1}(\Omega)$;

$$F_\gamma(u) := \begin{cases} \displaystyle\int_\Omega \sigma_\gamma(\nabla u)dx & u \in W_0^{1,1}(\Omega) \cap H^{-1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

The effective domain $D(F_\gamma)$ is defined by

$$D(F_\gamma) := \{u \in H^{-1}(\Omega) \mid F_\gamma(u) < +\infty\}.$$

Note that $D(F_\gamma)$ does not necessarily equal to $W_0^{1,1}(\Omega) \cap H^{-1}(\Omega)$, though $D(F_\gamma) \subset W_0^{1,1}(\Omega) \cap H^{-1}(\Omega)$ always holds.

Let us formally calculate the first variation of F_γ. For any $u \in D(F_\gamma)$ and $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \frac{dF_\gamma(u + \varepsilon\phi)}{d\varepsilon}\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon} \int_\Omega \sigma_\gamma(\nabla u + \varepsilon \nabla \phi)dx\Big|_{\varepsilon=0} \\ &= \int_\Omega \langle \nabla \sigma_\gamma(\nabla u), \nabla \phi \rangle dx \\ &= \int_\Omega (-\operatorname{div}(\nabla \sigma_\gamma(\nabla u)))\phi dx \\ &= \int_\Omega -(-\Delta_D)^{-1}(-\Delta_D) \operatorname{div}(\nabla \sigma_\gamma(\nabla u))\phi dx \\ &= \langle -(-\Delta_D) \operatorname{div}(\nabla \sigma_\gamma(\nabla u)), \phi \rangle_{H^{-1}(\Omega)}. \end{aligned}$$

Here we have postulated that $\operatorname{div}(\nabla\sigma_\gamma(\nabla u)) \in H_0^1(\Omega)$ so that identity $\operatorname{div}(\nabla\sigma_\gamma(\nabla u)) = (-\Delta_D)^{-1}(-\Delta_D)\operatorname{div}(\nabla\sigma_\gamma(\nabla u))$ holds.

Hence we see that if $u \in D(F_\gamma)$ and $\operatorname{div}(\nabla\sigma_\gamma(\nabla u)) \in H_0^1(\Omega)$, then,

$$\frac{\delta F_\gamma(u)}{\delta u}\|_{H^{-1}(\Omega)} = -(-\Delta_D)\operatorname{div}(\nabla\sigma_\gamma(\nabla u)),$$

where $\frac{\delta F_\gamma(u)}{\delta u}\|_{H^{-1}(\Omega)}$ means the variational derivative of F_γ at u with respect to the metric of $H^{-1}(\Omega)$.

Thus, the equation (EQ1) can be written in the gradient flow of the form

$$\begin{cases} u_t = -\dfrac{\delta F_\gamma(u)}{\delta u}\|_{H^{-1}(\Omega)}, \\ u|_{t=0} = u_0. \end{cases} \tag{EQ2}$$

This gradient flow equation (EQ2) requires the evolution of u to reduce the energy $F_\gamma(u)$ the most quickly with respect to the metric of $H^{-1}(\Omega)$. We next give a rigorous mathematical formulation of (EQ2) by using the subdifferential of F_γ.

2.2 Extended gradient flow equation

The equation (EQ2) is still a formal representation of (EQ1) since the functional derivative makes no sense at a point u where the functional $F_\gamma(\cdot)$ is not differentiable. The concept of subdifferential is useful to overcome this difficulty by interpreting the derivative of $F_\gamma(\cdot)$ as a multi-valued operator.

Definition 2.1. (*Subdifferential*) Let H be a real Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle_H$. For a proper convex function $f : H \to \mathbb{R}\cup\{+\infty\}$, a multi-valued operator $\partial f : H \to H$ called the *subdifferential* of f is defined by

$$\partial f(x) := \{y \in H \mid f(x+z) \geq f(x) + \langle y, z\rangle_H, \text{ for any } z \in H\}.$$

Example 2.2. In the case that $H = \mathbb{R}$ and $f(x) = |x|$, we see that

$$\partial f(x) = \begin{cases} 1 & (x > 0), \\ \in [-1, 1] & (x = 0), \\ -1 & (x < 0), \end{cases}$$

where we regard $\mathbb{R}$ as a Hilbert space with the standard inner product.

The next example will be used later.

Example 2.3. In the case that $H = \mathbb{R}$ and $\sigma_\gamma(x) = |x| + |x|^\gamma/\gamma$, we see that

$$\partial\sigma_\gamma(x) = \begin{cases} 1 + |x|^{\gamma-1} & (x > 0), \\ \in [-1, 1] & (x = 0), \\ -1 - |x|^{\gamma-1} & (x < 0). \end{cases}$$

Remark. Since $\partial f(u)$ corresponds to the gradient of f when f is differentiable at u, the subdifferential is an extended concept of differential (see, for example, [4]).

Now we are able to formulate (EQ2) as the *extended gradient flow equation* (EQ3) in $H^{-1}(\Omega)$ by using the subdifferential $\partial F_\gamma(u)$ of the form

$$\begin{cases} \dfrac{du}{dt} \in -\partial F_\gamma(u), \\ u|_{t=0} = u_0. \end{cases} \tag{EQ3}$$

Note that the evolution equation (EQ3) includes the boundary conditions $u|_{\partial\Omega} = \mathrm{div}(\nabla\sigma_\gamma(\nabla u))|_{\partial\Omega} = 0$ in a weak sense. Thus we obtained the mathematical formulation of (EQ1) as (EQ3).

2.3 Unique existence theorem

We are interested in the existence of a solution of the formulated equation ($EQ3$). The next theorem from the general nonlinear semigroup theory (initiated by Y. Kōmura [17]) is fundamental (see [3], [6]).

Theorem 2.4. *(Unique existence theorem) Let H be a real Hilbert space and ϕ be a proper, lower semicontinuous, and convex functional on H. Suppose that $u_0 \in \overline{D(\phi)}$. Then there exists a unique solution u of the following gradient flow equation;*

$$\begin{cases} \dfrac{du}{dt} \in -\partial\phi(u), \\ u|_{t=0} = u_0. \end{cases} \tag{2.1}$$

Here we regard $u : [0, +\infty) \to H$ as a solution of (2.1) if u satisfies following conditions;

(a) $u \in C([0, +\infty), H) \cap AC([\delta, +\infty), H)$ for any $\delta > 0$,

(b) u satisfies (2.1) for a.e. $t \in [0, +\infty)$,

where 'AC' stands for 'absolutely continuous'.

Remark. 'du/dt' in (2.1) means the time derivative of u with respect to the norm of H.

To apply this theorem to our (EQ3) we have to show

Proposition 2.5. *The functional F_γ is proper, lower semicontinuous, and convex on $H^{-1}(\Omega)$.*

Proof. The idea of the proof is essentially formal in [5] but we give it for completeness. Since σ_γ is convex and $\sigma_\gamma(0) = 0$, the functional F_γ is obviously proper and convex.

We have to show that F_γ is lower semicontinuous.

Suppose that $u_m \in D(F_\gamma)$, $u_m \to u$ as $m \to +\infty$ in $H^{-1}(\Omega)$ and $F_\gamma(u_m) \le \lambda$ holds for any $m \in \mathbb{N}$.

For any $\rho \ge 0$, there exists $C_\rho > 0$ such that

$$\rho|p| \le C_\rho + \sigma_\gamma(p) \text{ for any } p \in \mathbb{R}^n. \tag{2.2}$$

By (2.2) we see that for any $\rho \ge 0$, there exists $C_\rho > 0$ such that

$$\rho|\nabla u_m(x)| \le C_\rho + \sigma_\gamma(\nabla u_m(x)) \text{ for any } m \in \mathbb{N} \text{ and a.e.} x \in \Omega.$$

Moreover, for any measurable set $M \subset \Omega$,

$$\rho \int_M |\nabla u_m(x)|dx \leq C_\rho |M| + \int_M \sigma_\gamma(\nabla u_m(x))dx \\ \leq C_\rho |M| + \lambda \text{ for any } m \in \mathbb{N}, \tag{2.3}$$

where $|M|$ stands for the measure of M.

For each $\varepsilon > 0$ we take $\rho > 0$ large such that $\lambda/\rho \leq \varepsilon/2$ and set $\delta := \rho\varepsilon/2C_\rho$. Then, by (2.3) $\int_M |\nabla u_m(x)|dx \leq \varepsilon$ for any $m \in \mathbb{N}$ if $|M| < \delta$. Hence $\int_M |\nabla u_m(x)|dx$ $(m = 1, 2, \cdots)$ are uniformly absolutely continuous.

Moreover by using (2.3) for $\rho = 1$ and $M = \Omega$, we obtain that $\int_M |\nabla u_m(x)|dx$ $(m = 1, 2, \cdots)$ are bounded.

Thus Dunford-Pettis' theorem ([9], p.294) assures that $\{\nabla u_m\}_{m=1}^{+\infty}$ is weak sequentially compact in $L^1(\Omega, \mathbb{R}^n)$. Therefore by taking some subsequence, we see that there is $w \in L^1(\Omega, \mathbb{R}^n)$ such that $\nabla u_m \rightharpoonup w$ weakly as $m \to +\infty$ in $L^1(\Omega, \mathbb{R}^n)$.

Since $u_m \to u$ in $'(\Omega)$ (the space of Schwartz' distribution), we also have that $\partial u_m/\partial x^i \to \partial u/\partial x^i$ in $'(\Omega)$ for $i = 1, 2, \cdots, n$. Thus for any $\phi_i \in C_0^\infty(\Omega)$ $(i = 1, 2, \cdots, n)$,

$$\sum_{i=1}^n \int_\Omega \left(\frac{\partial u}{\partial x_i} - w_i\right) \phi_i dx = \sum_{i=1}^n \int_\Omega \left(\frac{\partial u}{\partial x_i} - \frac{\partial u_m}{\partial x_i}\right) \phi_i dx + \sum_{i=1}^n \int_\Omega \left(\frac{\partial u_m}{\partial x_i} - w_i\right) \phi_i dx \\ \to 0 \ (m \to +\infty).$$

We thus obtain that $\nabla u = w$.

Therefore, $\nabla u \in L^1(\Omega, \mathbb{R}^n)$ and $\nabla u_m \rightharpoonup \nabla u$ weakly in $L^1(\Omega, \mathbb{R}^n)$.

Moreover, by Mazur's theorem ([25], p.120) there exists a subsequence of $\{u_m\}_{m=1}^{+\infty}$ such that its suitable convex combination of the sequence denoted by v_m fulfills

$$v_m \to u \text{ strongly in } H^{-1}(\Omega), \\ \nabla v_m \to \nabla u \text{ strongly in } L^1(\Omega, \mathbb{R}^n), \\ \nabla v_m(x) \to \nabla u(x) \text{ for a.e. } x \in \Omega.$$

By the second convergence above and Poincaré's inequality, we obtain that $v_m \to u$ in $W_0^{1,1}(\Omega)$. By the convexity of F_γ, we see $F_\gamma(v_m) \leq \lambda$ for any $m \in \Omega$. Thus, by the fact that $u \in W_0^{1,1}(\Omega) \cap H^{-1}(\Omega)$ and Fatou's lemma, we obtain that $F_\gamma(u) \leq \lambda$. This implies that F_γ is lower semicontinuous in $H^{-1}(\Omega)$. □

We now apply Theorem 2.4 to our (EQ3) and see the unique existence of the solution of (EQ3) in the sense of Theorem 2.4.

2.4 Stability theorem

The solution of our (EQ3) can be approximated by solutions of approximated problems. Especially, our solution is a limit of solutions which solve modified problems by certain smooth energy. Let us see the general statement of this stability theorem;

Theorem 2.6. *(Stability theorem)(See [1] or [24]) Let Φ^m and Φ be proper, lower semicontinuous, convex functionals on a real Hilbert space H. Suppose that Φ^m converges to Φ as $m \to +\infty$ in the sense of Mosco, i.e.,*

(i) If $u_m \rightharpoonup u$ weakly in H, then $\Phi(u) \le \liminf_{m\to+\infty} \Phi^m(u_m)$ holds.

(ii) For any $u \in D(\Phi)$, there exists $u_m \in H$ such that $u_m \to u$ strongly in H and $\lim_{m\to+\infty} \Phi^m(u_m) = \Phi(u)$.

Suppose that $u_{0m} \to u_0$ strongly in H as $m \to +\infty$.
Let u_m be a solution of

$$\begin{cases} \dfrac{du}{dt} \in -\partial\Phi^m(u), \\ u|_{t=0} = u_{0m}, \end{cases}$$

and u be a solution of

$$\begin{cases} \dfrac{du}{dt} \in -\partial\Phi(u), \\ u|_{t=0} = u_0. \end{cases}$$

Then, we see that for any $T > 0$,

$$\lim_{m\to+\infty} \sup_{0\le t\le T} \|u_m(t) - u(t)\|_H = 0.$$

Let us give one example of the approximate smooth energies.

Proposition 2.7. *Define functionals F_γ^m $(m \in \mathbb{N})$ on $H^{-1}(\Omega)$ by*

$$F_\gamma^m(u) := \begin{cases} \displaystyle\int_\Omega \sigma_\gamma^m(\nabla u)dx & u \in W_0^{1,1}(\Omega) \cap H^{-1}(\Omega), \\ +\infty & \text{otherwise}, \end{cases}$$

$$\sigma_\gamma^m(p) := \left(|p|^2 + \frac{1}{m}\right)^{\frac{1}{2}} + \frac{1}{\gamma}\left(|p|^2 + \frac{1}{m}\right)^{\frac{\gamma}{2}} : \mathbb{R}^n \to \mathbb{R}.$$

Then, F_γ^m $(m \in \mathbb{N})$ are proper, lower semicontinuous, and convex functionals on $H^{-1}(\Omega)$. Moreover, F_γ^m converges to F_γ as $m \to +\infty$ in the sense of Mosco.

Proof. Since we can prove that F_γ^m $(m \in \mathbb{N})$ are proper, convex, and lower semicontinuous functionals on $H^{-1}(\Omega)$ and the condition (i) of Mosco convergence holds in the similar way as the proof of Proposition 2.5, we omit the proof.

We show that the condition (ii) of Mosco convergence holds.
Since we are able to prove that

$$\sigma_\gamma^m(p) \le 2^{\gamma-1}\sigma_\gamma(p) + \frac{1}{\sqrt{m}} + \frac{2^{\gamma-1}}{\gamma}\left(\frac{1}{\sqrt{m}}\right)^\gamma \quad \text{for any } p \in \mathbb{R}^n \text{ and any } m \in \mathbb{N},$$

we see that $D(F_\gamma) \subset D(F_\gamma^m)$. Therefore, it is enough to show that

$$\lim_{m\to+\infty} F_\gamma^m(u) = F_\gamma(u) \text{ for any } u \in D(F_\gamma). \tag{2.4}$$

For each $u \in D(F_\gamma)$, we see that

$$\begin{aligned}|\sigma_\gamma^m(\nabla u(x)) - \sigma_\gamma(\nabla u(x))| &\le 1 + \frac{2^{\gamma-1}}{\gamma} + \frac{1}{\gamma}(2^{\gamma-1}-1)|\nabla u(x)|^\gamma \\ &\in L^1(\Omega) \text{ for any } m \in \mathbb{N}.\end{aligned}$$

Thus, the convergence $\sigma_\gamma^m(\nabla u(x)) \to \sigma_\gamma(\nabla u(x))$ $(m \to +\infty)$ for a.e.$x \in \Omega$ and Lebesgue's convergence theorem yield (2.4). □

2.5 Canonical restriction

Since the right hand of (EQ3) is allowed to be multi-valued, the equation (EQ3) seems to have an ambiguity as an evolution equation. At least, it is difficult for us to calculate the speed of the solution of (EQ3) practically. By selecting a suitable element in the subdifferential at each point u, however, we can transform (EQ3) into an equivalent equation whose right hand is a single-valued operator. This single-valued operator is called a *canonical restriction* of the subdifferential of F_γ. Let us state the definition of *canonical restriction.*

Definition 2.8. (*Canonical restriction*)(See [3]) Let ϕ be a proper, lower semicontinuous, and convex functional on a real Hilbert space H. For any $u \in D(\phi)$ such that $\partial\phi(u) \neq \emptyset$, there uniquely exists $v \in \partial\phi(u)$ such that;

$$\|v\|_H = \min_{w\in\partial\phi(u)} \|w\|_H.$$

Thus if we write $\partial\phi^c(u) = v$, then $\partial\phi^c$ is well-defined as a single-value operator. We call $\partial\phi^c$ a *canonical restriction* of $\partial\phi$.

Remark. When ϕ is a proper convex functional on H, then its subdifferential $\partial\phi(u)$ is a closed convex set. Therefore the minimizer v uniquely exists.

The next theorem indicates that the canonical restriction is the desirable selection of the element in the subdifferential at each point u.

Theorem 2.9. *If ϕ is a proper, lower semicontinuous, and convex functional on H and $u_0 \in \overline{D(\phi)}$, then the gradient flow equation (2.1) is equivalent to the following equation;*

$$\begin{cases} \dfrac{d^+u}{dt} = -\partial\phi^c(u), \\ u|_{t=0} = u_0. \end{cases} \tag{2.5}$$

Here we regard $u : [0, +\infty) \to H$ as a solution of (2.5) if u satisfies following conditions;

(a) $u \in C([0, +\infty), H) \cap AC([\delta, +\infty), H)$ for any $\delta > 0$,

(b) u *satisfies (2.5) for any* $t \in [0, +\infty)$.

Remark. The notation 'd^+u/dt' stands for the right time derivative of u with respect to the norm of H. It is known that solutions of (2.1) are right differentiable at any $t \in [0, +\infty)$.

Theorem 2.9 allows us to consider (EQ4) instead of (EQ3);

$$\begin{cases} \dfrac{d^+u}{dt} = -\partial F_\gamma^c(u), \\ u|_{t=0} = u_0. \end{cases} \tag{EQ4}$$

Thus, the equation (EQ1) is formulated as an evolution equation whose evolution speed is expressed by the single-valued operator in the space of $H^{-1}(\Omega)$. In Chapter 4, by calculating the canonical restriction, we will see the speed of the solution of (EQ4) for a one space dimensional profile with a facet.

3 Subdifferential

In this chapter, we will determine subdifferentials of convex functionals defined on $H^{-1}(\Omega)$ whose forms are more general than our energy F_γ. We also give the simpler proof which is applicable to characterize the subdifferential of our energy F_γ if we impose certain restriction on the exponent γ of σ_γ. To carry out the proof, we need some notations and properties for convex functionals.

3.1 Preliminaries for convex functionals

Note that some assumptions and conclusions below are simplified for our purpose.

Proposition 3.1. *(See [22]) Let* $f : \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ *be a proper, convex, and lower semicontinuous function and* f^* *be its convex conjugate on* $\mathbb{R}^l$. *Then we see that* f^* *is coercive, i.e., for any* $\rho > 0$, *there exists* $C_\rho > 0$ *such that* $f^*(x) + C_\rho \geq \rho|x|$ *holds, if and only if* $D(f)(:= \{x \in \mathbb{R}^l | f(x) < +\infty\}) = \mathbb{R}^l$.

Proposition 3.2. *(See, for example, [4]) Let* f *be a proper, convex functional on a real Hilbert space* H. *Then* $y \in \partial f(x)$ *if and only if* $f(x) + f^*(y) = \langle x, y \rangle_H$ *holds, where* f^* *is the convex conjugate functional on* H *of* f *and* $\langle \cdot, \cdot \rangle_H$ *is the inner product of* H.

Let X be a real Banach space and X^* be its dual space.

Definition 3.3. (See [1]) For functionals f, g on X, we define *inf-convolution* $f \bigtriangledown g$ by $f \bigtriangledown g(x) := \inf_{x' \in X}(f(x - x') + g(x'))$ for any $x \in X$.

Definition 3.4. (See [2]) We call a functional f on X *inf-compact* if $\{x \in X \mid f(x) \leq \lambda\}$ is compact for any $\lambda \in \mathbb{R}$.

Proposition 3.5. *(See [2]) Let* f *and* g *be proper, lower semicontinuous, and convex functionals on* X. *If* f *and* g *have lower bounds on* X *and at least one of them is weak inf-compact, then* $f \bigtriangledown g = (f^\sharp + g^\sharp)^\sharp$ *holds, where* $f^\sharp$ *denotes the convex conjugate on* X^* *of* f *in* X *and* $(f^\sharp)^\sharp$ *denotes the convex conjugate on* X^{**} *of* $f^\sharp$ *in* X^*. *The identity holds on* X *which can be regarded as a subset of* X^{**} *(see [25]).*

3.2 Subdifferential of general energies

We now begin determining subdifferentials of energy functional E on $H^{-1}(\Omega)$ of the form

$$E(u) := \begin{cases} \displaystyle\int_\Omega \tau(x, u(x), \nabla u(x))dx & u \in W_0^{1,1}(\Omega) \cap H^{-1}(\Omega) \\ +\infty & \text{otherwise} \end{cases}.$$

The function $\tau : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following conditions;

(i) $\tau(x, \cdot, \cdot)$ is convex on $\mathbb{R} \times \mathbb{R}^n$ for a.e.$x \in \Omega$.

(ii) $\tau(\cdot, r, s)$ is integrable on Ω for any $(r, s) \in \mathbb{R} \times \mathbb{R}^n$.

(iii) For any $\rho \geq 0$, there exists a function $C_\rho \in L^1(\Omega)$, such that $\tau(x, r, s) + C_\rho(x) \geq \rho|s|$, for a.e.$x \in \Omega$ and any $(r, s) \in \mathbb{R} \times \mathbb{R}^n$.

(iv) There exist positive constants K, C_0', and a function $C_1' \in L^1(\Omega)$, such that $\tau(x, r, s) \leq K\{\tau(y, r, s) + C_0'|s| + C_1'(x) + C_1'(y)\}$ for a.e.$x, y \in \Omega$ and any $(r, s) \in \mathbb{R} \times \mathbb{R}^n$.

Remark. By the condition (i),(ii), we see that $\tau(\cdot, u(\cdot))$ is measurable for any measurable function $u : \Omega \to \mathbb{R}^{n+1}$(see [20] or [22]). Moreover, by the condition (iii), the integral of $\tau(\cdot, u(\cdot))$ over Ω makes sense and takes its value in $\mathbb{R} \cup \{+\infty\}$ for any $u \in L^1(\Omega, \mathbb{R}^{n+1})$.

Remark. Note that our energy functional F_γ is one instance of these energies.

We follow methods of [2] where subdifferential of convex functionals on $L^2(\Omega)$ is calculated. We begin with an approximation lemma.

Lemma 3.6. *For any $u \in D(E)$, there exists $\{u_m\}_{m=1}^{+\infty} \subset C_0^\infty(\Omega)$ such that*

$$\begin{aligned} u_m &\to u \text{ in } H^{-1}(\Omega), \\ u_m &\to u \text{ in } W_0^{1,1}(\Omega), \\ \text{and } E(u_m) &\to E(u) \text{ as } m \to +\infty. \end{aligned} \tag{3.1}$$

Proof. By the same way as [2, Proposition (2.12)], we can obtain that for any $u \in D(E)$ there exists $\{u_m\}_{m=1}^{+\infty} \subset C_0^\infty(\Omega)$ such that

$$\begin{aligned} u_m &\to u \text{ in } W_0^{1,1}(\Omega), \\ E(u_m) &\to E(u) \text{ as } m \to +\infty. \end{aligned}$$

Since we have assumed that the space dimension $n \leq 4$, Sobolev's embedding theorem (see, for example, [16]) assures that $W_0^{1,1}(\Omega) \subset H^{-1}(\Omega)$. Thus, u_m also converges to u in $H^{-1}(\Omega)$. □

Remark. The conditions imposed on the domain Ω, which are to be bounded with the piecewise smooth boundary and to locate one side of the boundary, and the condition (iii),(iv) are necessary to apply [2, Proposition (2.12)].

By the equivalence of Proposition 3.2 it is important to characterize the convex conjugate functional E^* of E in $H = H^{-1}(\Omega)$ to characterize ∂E.

Proposition 3.7. *For any $u \in H^{-1}(\Omega)$,*

$$E^*(u) = \sup_{v \in W_0^{1,\infty}(\Omega)} (\langle u, v\rangle_{H^{-1}(\Omega)} - E(v)).$$

Remark. Note that the Sobolev space $W_0^{1,\infty}(\Omega)$ is identified with the space of all Lipschitz functions in $\bar{\Omega}$ vanishing on $\partial\Omega$.

Proof. We fix $u \in H^{-1}(\Omega)$.
By the definitions of E and E^* we see that

$$\begin{aligned} E^*(u) &= \sup_{v \in H^{-1}(\Omega)} (\langle u, v\rangle_{H^{-1}(\Omega)} - E(v)) \\ &= \sup_{v \in D(E)} (\langle u, v\rangle_{H^{-1}(\Omega)} - E(v)). \end{aligned}$$

By Lemma 3.6 for each $\varepsilon > 0$ and each $v \in D(E)$, there exists $v' \in C_0^\infty(\Omega)$ such that

$$\langle u, v\rangle_{H^{-1}(\Omega)} - E(v) \le \langle u, v'\rangle_{H^{-1}(\Omega)} - E(v') + \varepsilon.$$

Thus, we see that

$$E^*(u) \le \sup_{v \in W_0^{1,\infty}(\Omega)} (\langle u, v\rangle_{H^{-1}(\Omega)} - E(v)) + \varepsilon.$$

Sending $\varepsilon \downarrow 0$, we obtain

$$E^*(u) \le \sup_{v \in W_0^{1,\infty}(\Omega)} (\langle u, v\rangle_{H^{-1}(\Omega)} - E(v)).$$

Since $W_0^{1,\infty}(\Omega) \subset H^{-1}(\Omega)$, the opposite inequality is immediate. □

In order to obtain more detailed characterization of E^*, we need to prepare some properties of the integrand conjugate $\tau^* : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of τ defined by

$$\tau^*(x, r', s') := \sup_{(r,s) \in \ \times \ ^n} \left(rr' + \langle s, s'\rangle - \tau(x, r, s) \right),$$

for a.e.$x \in \Omega$ and $(r', s') \in \mathbb{R} \times \mathbb{R}^n$.

Lemma 3.8. *The following statements hold.*

(1) The integral of $\tau^*(\cdot, u(\cdot))$ over Ω makes sense and takes its values in $\mathbb{R} \cup \{+\infty\}$ for any $u \in L^1(\Omega, \mathbb{R}^{n+1})$. In addition, the functional $\phi(\cdot)$ defined by $\phi(u) := \int_\Omega \tau^*(x, u(x))dx$ is proper, convex, lower semicontinuous, and weak inf-compact on $L^1(\Omega, \mathbb{R}^{n+1})$.

(2) The equality $\phi^\sharp(v) = \int_\Omega \tau(x, v(x))dx$ holds for any $v \in L^\infty(\Omega, \mathbb{R}^{n+1})$, where the functional $\phi^\sharp$ means the convex conjugate functional on $L^\infty(\Omega, \mathbb{R}^{n+1})$ with respect to L^2 inner product for the functional ϕ defined on $L^1(\Omega, \mathbb{R}^{n+1})$.

Proof. These are the consequences of Corollary 2A and Corollary 2B in [21]. □

Remark. The assumption that $\tau(\cdot, r, s)$ is integrable on Ω is necessary to apply these corollaries of [21].

Lemma 3.9. *For any $\rho > 0$, there exists $D_\rho \in L^1(\Omega)$ such that*

$$\tau^*(x, r', s') + D_\rho(x) \geq \rho|(r', s')| \text{ for a.e.} x \in \Omega \text{ and any } (r', s') \in \mathbb{R} \times \mathbb{R}^n.$$

Proof. The condition (iv) assures that there exists $a \in \Omega$ such that

$$\tau(x, r, s) \leq K(\tau(a, r, s) + C_0'|s| + C_1'(x) + C_1'(a)) \tag{3.2}$$

holds for a.e.$x \in \Omega$ and $(r, s) \in \mathbb{R} \times \mathbb{R}^n$.
Then, by the definition of τ^* and (3.2) we see that

$$\begin{aligned} \tau^*(x, r', s') &\geq -K(C_1'(x) + C_1'(a)) + \sup_{(r,s)\in \mathbb{R}\times\mathbb{R}^n} \{r'r + \langle s', s\rangle - K(\tau(a, r, s) + C_0'|s|)\} \\ &= -K(C_1'(x) + C_1'(a)) + \eta^*(r', s'), \end{aligned} \tag{3.3}$$

where we set $\eta(r, s) := K(\tau(a, r, s) + C_0'|s|)$ for any $(r, s) \in \mathbb{R} \times \mathbb{R}^n$.
Noting that $\tau(a, r, s) < +\infty$ for any $(r, s) \in \mathbb{R} \times \mathbb{R}^n$, we see that $D(\eta) = \mathbb{R} \times \mathbb{R}^n$. Thus, Proposition 3.1 yields that for any $\rho > 0$, there exists $C_\rho > 0$ such that

$$\eta^*(r', s') + C_\rho \geq \rho|(r', s')| \text{ for any } (r', s') \in \mathbb{R} \times \mathbb{R}^n. \tag{3.4}$$

Combining (3.4) with (3.3), we obtain the result. □

Now we give more detailed characterization of E^*.

Proposition 3.10. *If $u \in H^{-1}(\Omega)$ satisfies $E^*(u) < +\infty$, then*

$$E^*(u) = \min_{\substack{g\in L^1(\Omega,\mathbb{R}^n) \\ \operatorname{div} g\in L^1(\Omega)}} \int_\Omega \tau^*(x, (-\Delta_D)^{-1}u(x) + \operatorname{div} g(x), g(x))dx.$$

Proof. We fix $u \in H^{-1}(\Omega)$ satisfying $E^*(u) < +\infty$ and set $\tilde{u} := ((-\Delta_D)^{-1}u, 0, \cdots, 0) \in L^1(\Omega, \mathbb{R}^{n+1})$.

Define functionals G and θ on $L^\infty(\Omega, \mathbb{R}^{n+1})$ of the form

$$G(v) := \begin{cases} 0 & \text{if } v_0 \in W_0^{1,\infty}(\Omega),\ (v_1, \cdots, v_n) = \nabla v_0, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\theta(v) := \int_\Omega \tau(x, v_0(x), \cdots, v_n(x))dx,$$

for $v = (v_0, v_1, \cdots, v_n) \in L^\infty(\Omega, \mathbb{R}^{n+1})$. Then, we see that

$$E^*(u) = (\theta + G)^\sharp(\tilde{u}), \tag{3.5}$$

where $(\theta + G)^\sharp$ denotes the convex conjugate on the dual space of $L^\infty(\Omega, \mathbb{R}^{n+1})$ with respect to L^2 inner product of $\theta + G$ in $L^\infty(\Omega, \mathbb{R}^{n+1})$. Indeed, for any $w \in L^1(\Omega, \mathbb{R}^{n+1})$ we observe that

$$\begin{aligned}(\theta + G)^\sharp(w) &= \sup_{v \in L^\infty(\Omega, \ ^{n+1})} (\langle w, v\rangle_{L^2(\Omega, \ ^{n+1})} - (\theta + G)(v)) \\ &= \sup_{\substack{v \in L^\infty(\Omega, \ ^{n+1}) \\ v_0 \in W_0^{1,\infty}(\Omega) \\ (v_1, \cdots, v_n) = \nabla v_0}} (\langle w, v\rangle_{L^2(\Omega, \ ^{n+1})} - \theta(v)).\end{aligned}$$

Thus,

$$\begin{aligned}(\theta + G)^\sharp(\tilde{u}) &= \sup_{v_0 \in W_0^{1,\infty}(\Omega)} (\langle(-\Delta_D)^{-1}u, v_0\rangle_{L^2(\Omega)} - E(v_0)) \\ &= E^*(u).\end{aligned}$$

Here we have used the result of Proposition 3.7.

We next define a functional ϕ on $L^1(\Omega, \mathbb{R}^{n+1})$ by

$$\phi(w) := \int_\Omega \tau^*(x, w_0(x), w_1(x), \cdots, w_n(x))dx.$$

Then we see that for any $v \in L^\infty(\Omega, \mathbb{R}^{n+1})$,

$$\theta(v) = \phi^\sharp(v), \tag{3.6}$$

where $\phi^\sharp$ is the convex conjugate on $L^\infty(\Omega, \mathbb{R}^{n+1})$ with respect to L^2 inner product of ϕ in $L^1(\Omega, \mathbb{R}^{n+1})$. Indeed, Lemma 3.8.(1) assures that the functional ϕ can be defined and Lemma 3.8.(2) implies that the equality (3.6) holds.

We can also check that ϕ has a lower bound on $L^1(\Omega, \mathbb{R}^{n+1})$. Indeed, by the definition of τ^*, we see that for any $w \in L^1(\Omega, \mathbb{R}^{n+1})$,

$$\phi(w) \geq -\int_\Omega \tau(x, 0)dx > -\infty.$$

Next, we characterize the value of $G^\sharp$, which is the conjugate convex functional on the dual space of $L^\infty(\Omega, \mathbb{R}^{n+1})$ with respect to L^2 inner product of G in $L^\infty(\Omega, \mathbb{R}^{n+1})$, restricted to $L^1(\Omega, \mathbb{R}^{n+1})$. By the definition of G, for any $w \in L^1(\Omega, \mathbb{R}^{n+1})$,

$$\begin{aligned}G^\sharp(w) &= \sup_{v \in L^\infty(\Omega, \ ^{n+1})} (\langle v, w\rangle_{L^2(\Omega, \ ^{n+1})} - G(v)) \\ &= \sup_{v_0 \in W_0^{1,\infty}(\Omega)} \int_\Omega \left(v_0 w_0 + \sum_{i=1}^n \frac{\partial v_0}{\partial x_i} w_i\right) dx.\end{aligned}$$

If $w_0 \neq \sum_{i=1}^n \partial w_i / \partial x_i$ in $\ '(\Omega)$ then $G^\sharp(w) = +\infty$.
If $w_0 = \sum_{i=1}^n \partial w_i / \partial x_i$ in $\ '(\Omega)$ then $G^\sharp(w) = 0$ by integrated by parts.
As the result, for any $w \in L^1(\Omega, \mathbb{R}^{n+1})$,

$$G^\sharp(w) = \begin{cases} 0 & \text{if } w_0 = \displaystyle\sum_{i=1}^n \frac{\partial w_i}{\partial x_i}, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.7}$$

Set $H(w) := G^\sharp(w)$ for any $w \in L^1(\Omega, \mathbb{R}^{n+1})$. Then, we can also obtain that for any $v \in L^\infty(\Omega, \mathbb{R}^{n+1})$,

$$H^\sharp(v) = G(v), \tag{3.8}$$

where $H^\sharp$ denotes the conjugate functional on $L^\infty(\Omega, \mathbb{R}^{n+1})$ with respect to L^2 inner product of H in $L^1(\Omega, \mathbb{R}^{n+1})$. Indeed, by (3.7) we observe that

$$H^\sharp(v) = \sup_{\substack{w \in L^1(\Omega, {}^{n}) \\ \operatorname{div} w \in L^1(\Omega)}} \int_\Omega \left(v_0 \operatorname{div} w + \sum_{i=1}^{n} v_i w_i \right) dx.$$

If $\nabla v_0 \neq (v_1, \cdots, v_n)$ in $'(\Omega, \mathbb{R}^n)$, then $H^\sharp(v) = +\infty$. Therefore, if $H^\sharp(v) < +\infty$, then $v_0 \in W^{1,\infty}(\Omega)$ and $\nabla v_0 = (v_1, \cdots, v_n)$. Moreover, noting Remark below the statement of Proposition 3.7, we see that $H^\sharp(v) = 0$ if $v_0 \in W_0^{1,\infty}(\Omega)$ and $H^\sharp(v) = +\infty$ if $v_0 \in W^{1,\infty}(\Omega) \backslash W_0^{1,\infty}(\Omega)$. As the conclusion, we obtain (3.8).

By using (3.5),(3.6), and (3.8), we see that

$$\begin{aligned} E^*(u) &= (\theta + G)^\sharp(\tilde{u}) \\ &= (\phi^\sharp + H^\sharp)^\sharp(\tilde{u}) \\ &= (\phi \bigtriangledown H)(\tilde{u}), \end{aligned} \tag{3.9}$$

where $(\phi^\sharp + H^\sharp)^\sharp$ denotes the conjugate functional on the dual space of $L^\infty(\Omega, \mathbb{R}^{n+1})$ with respect to L^2 inner product of $\phi^\sharp + H^\sharp$ in $L^\infty(\Omega, \mathbb{R}^{n+1})$.
Here we have used the consequence of Proposition 3.5 for ϕ and H. Indeed, ϕ and H are proper, convex, and lower semicontinuous on $L^1(\Omega, \mathbb{R}^{n+1})$ by Lemma 3.8.(1), and both have lower bounds on $L^1(\Omega, \mathbb{R}^{n+1})$. In addition, Lemma 3.8.(2) assures that ϕ is weak inf-compact. Therefore ϕ and H satisfy the assumptions of Proposition 3.5 as functionals on $L^1(\Omega, \mathbb{R}^{n+1})$ and we obtain the last equality.

By the definition of inf-convolution and (3.7), we see that

$$\begin{aligned} (\phi \bigtriangledown H)(\tilde{u}) &= \inf_{w \in L^1(\Omega, {}^{n+1})} (\phi(\tilde{u} - w) + H(w)) \\ &= \inf_{\substack{w \in L^1(\Omega, {}^{n+1}) \\ w_0 = {}^{n}_{i=1} \partial w_i / \partial x_i}} \phi(\tilde{u} - w) \\ &= \inf_{\substack{g \in L^1(\Omega, {}^{n}) \\ \operatorname{div} g \in L^1(\Omega)}} \int_\Omega \tau^*(x, (-\Delta_D)^{-1} u(x) + \operatorname{div} g(x), g(x)) dx. \end{aligned} \tag{3.10}$$

Finally we show that the functional $\Phi(g) := \int_\Omega \tau^*(x, (-\Delta_D)^{-1} u(x) + \operatorname{div} g(x), g(x)) dx$ attains its minimum in

$$A := \{ g \in L^1(\Omega, \mathbb{R}^n) \mid \operatorname{div} g \in L^1(\Omega) \}.$$

Let $\{g_m\}_{m=1}^{+\infty} \subset A$ be a minimizing sequence such that $\lim_{m \to +\infty} \Phi(g_m) = \inf_{g \in A} \Phi(g)$.
Since $\Phi(g_m)$ $(m = 1, 2, \cdots)$ is bounded, Lemma 3.9 and the similar argument as the proof of Proposition 2.5 assure that there exist a subsequence of $\{g_m\}_{m=1}^{+\infty}$ and $(h_0, h) \in$

$L^1(\Omega, \mathbb{R}^{n+1})$ such that its suitable convex combination of the sequence denoted by h_m fulfills

$$\begin{aligned} ((-\Delta_D)^{-1}u + \operatorname{div} h_m, h_m) \to (h_0, h) \text{ strongly in } L^1(\Omega, \mathbb{R}^{n+1}), \\ ((-\Delta_D)^{-1}u(x) + \operatorname{div} h_m(x), h_m(x)) \to (h_0(x), h(x)) \text{ for a.e. } x \in \Omega. \end{aligned} \tag{3.11}$$

The first convergence of (3.11) yields that $h_0 = (-\Delta_D)^{-1}u + \operatorname{div} h$ and $\operatorname{div} h \in L^1(\Omega)$. This implies that $h \in A$.

Then, by the second convergence of (3.11), Fatou's lemma, and the convexity of Φ, we have

$$\Phi(h) \leq \liminf_{m\to+\infty} \Phi(h_m) \leq \inf_{g\in A} \Phi(g).$$

This means that Φ attains its minimum in A.

Thus, combining (3.9) with (3.10), we see that the desired result holds. □

Now we are ready to determine ∂E.

Theorem 3.11. *For any $u \in D(E)$ such that $\partial E(u) \neq \emptyset$, we observe that $f \in \partial E(u)$ in $H^{-1}(\Omega)$ if and only if there exists $g \in L^1(\Omega, \mathbb{R}^n)$ such that*

$$\operatorname{div} g \in L^1(\Omega),\ ((-\Delta_D)^{-1}f(x) + \operatorname{div} g(x), g(x)) \in \partial\tau_x(u(x), \nabla u(x)) \ \textit{a.e.} x \in \Omega,$$
$$\textit{and} \int_\Omega (u \cdot \operatorname{div} g + \langle g, \nabla u\rangle)dx = 0.$$

Remark. Note that $\partial\tau_x$ is defined by

$$\partial\tau_x(r,s) := \{(r', s') \in \mathbb{R}\times\mathbb{R}^n \mid \tau(x, r+h_1, s+h_2) \geq \tau(x,r,s) + r'h_1 + \langle s', h_2\rangle \text{ for any } (h_1,h_2) \in \mathbb{R}\times\mathbb{R}^n\} \text{ for a.e.} x \in \Omega.$$

Proof. ('⊂') Suppose $f \in \partial E(u)$.

Then Proposition 3.2 guarantees that

$$E(u) + E^*(f) = \langle f, u\rangle_{H^{-1}(\Omega)}. \tag{3.12}$$

By Proposition 3.10 there exists $g \in L^1(\Omega, \mathbb{R}^n)$ such that $\operatorname{div} g \in L^1(\Omega)$ and

$$E^*(f) = \int_\Omega \tau^*(x, (-\Delta_D)^{-1}f(x) + \operatorname{div} g(x), g(x))dx. \tag{3.13}$$

Combining (3.12) with (3.13), we have

$$\begin{aligned} \int_\Omega (\tau(x, u(x), \nabla u(x)) + \tau^*(x, (-\Delta_D)^{-1}f(x) + \operatorname{div} g(x), g(x)) \\ - (-\Delta_D)^{-1}f(x)\cdot u(x))dx = 0. \end{aligned} \tag{3.14}$$

Let $\{u_m\}_{m=1}^{+\infty} \subset C_0^\infty(\Omega)$ be the sequence of Lemma 3.6, then we see that

$$\begin{aligned} \int_\Omega (\tau(x, u_m(x), \nabla u_m(x)) + \tau^*(x, (-\Delta_D)^{-1}f(x) + \operatorname{div} g(x), g(x)) \\ - (-\Delta_D)^{-1}f(x)\cdot u_m(x))dx \to 0\ (m\to+\infty). \end{aligned}$$

Since $\int_\Omega(\operatorname{div} g\cdot u_m+\langle g,\nabla u_m\rangle)dx=0$ holds by integrated by parts, we obtain

$$\int_\Omega(\tau(x,u_m(x),\nabla u_m(x))+\tau^*(x,(-\Delta_D)^{-1}f(x)+\operatorname{div} g(x),g(x))-(-\Delta_D)^{-1}f(x)\cdot u_m(x)\\ -\operatorname{div} g(x)\cdot u_m(x)-\langle g(x),\nabla u_m(x)\rangle)dx\to 0\ (m\to+\infty). \tag{3.15}$$

By the definition of τ^*,

$$\begin{aligned}(\text{the integrand of (3.15)})&\geq \tau(x,u_m(x),\nabla u_m(x))+((-\Delta_D)^{-1}f(x)+\operatorname{div} g(x))u_m(x)\\ &\quad+\langle g(x),\nabla u_m(x)\rangle-\tau(x,u_m(x),\nabla u_m(x))\\ &\quad-(-\Delta_D)^{-1}f(x)\cdot u_m(x)-\operatorname{div} g(x)\cdot u_m(x)-\langle g(x),\nabla u_m(x)\rangle\\ &=0.\end{aligned}$$

Therefore,

$$\tau(x,u_m(x),\nabla u_m(x))+\tau^*(x,(-\Delta_D)^{-1}f(x)+\operatorname{div} g(x),g(x))-(-\Delta_D)^{-1}f(x)\cdot u_m(x)\\ -\operatorname{div} g(x)\cdot u_m(x)-\langle g(x),\nabla u_m(x)\rangle\to 0\ (m\to+\infty)\text{ in }L^1(\Omega). \tag{3.16}$$

Thus, by Lemma 3.6 and the convergence of (3.16), if we take some subsequence of $\{u_m\}_{m=1}^{+\infty}$ and let $m\to+\infty$, we have

$$\begin{aligned}&\tau(x,u(x),\nabla u(x))+\tau^*(x,(-\Delta_D)^{-1}f(x)+\operatorname{div} g(x),g(x))\\ &\quad-(-\Delta_D)^{-1}f(x)\cdot u(x)-\operatorname{div} g(x)\cdot u(x)-\langle g(x),\nabla u(x)\rangle=0,\ \text{a.e.}x\in\Omega.\end{aligned} \tag{3.17}$$

Moreover, Proposition 3.2 yields

$$((-\Delta_D)^{-1}f(x)+\operatorname{div} g(x),g(x))\in\partial\tau_x(u(x),\nabla u(x)),\ \text{a.e.}x\in\Omega.$$

Integrating (3.17) over Ω and subtracting it from (3.14), we obtain

$$\int_\Omega(u\cdot\operatorname{div} g+\langle g,\nabla u\rangle)dx=0.$$

('$\supset$') Suppose that $g\in L^1(\Omega,\mathbb{R}^n)$, $\operatorname{div} g\in L^1(\Omega)$,

$$((-\Delta_D)^{-1}f(x)+\operatorname{div} g(x),g(x))\in\partial\tau_x(u(x),\nabla u(x))\ \text{a.e.}x\in\Omega, \tag{3.18}$$

$$\int_\Omega(u\cdot\operatorname{div} g+\langle g,\nabla u\rangle)dx=0. \tag{3.19}$$

Proposition 3.2 and (3.18) yield,

$$\begin{aligned}&\tau(x,u(x),\nabla u(x))+\tau^*(x,(-\Delta_D)^{-1}f(x)+\operatorname{div} g(x),g(x))\\ &\quad=u(x)((-\Delta_D)^{-1}f(x)+\operatorname{div} g(x))+\langle\nabla u(x),g(x)\rangle\ \text{a.e.}x\in\Omega.\end{aligned} \tag{3.20}$$

Integrating (3.20) over Ω and combining with (3.19), we have

$$E(u) + \int_\Omega \tau^*(x, (-\Delta_D)^{-1} f(x) + \operatorname{div} g(x), g(x))dx = \langle u, f\rangle_{H^{-1}(\Omega)}.$$

Since $\operatorname{div} g \in L^1(\Omega)$, Proposition 3.10 assures that

$$E^*(f) \le \int_\Omega \tau^*(x, (-\Delta_D)^{-1} f(x) + \operatorname{div} g(x), g(x))dx.$$

Thus, we see that

$$E(u) + E^*(f) \le \langle u, f\rangle_{H^{-1}(\Omega)}.$$

Since the opposite inequality is obvious by the definition of E^*, we obtain $f \in \partial E(u)$ by Proposition 3.2 again. □

As a special case where τ does not depend on r, Theorem 3.11 can be stated as follows.

Corollary 3.12. *If $\tau(x, r, s)$ does not depend on r, we see that for any $u \in D(E)$ such that $\partial E(u) \neq \emptyset$, the set $\partial E(u)$ of the form*

$$\partial E(u) = \Bigg\{ -(-\Delta_D) \operatorname{div} g \;\Bigg|\; g \in L^1(\Omega, \mathbb{R}^n),\ \operatorname{div} g \in H_0^1(\Omega),$$
$$g(x) \in \partial\tau_x(\nabla u(x)) \ \text{a.e.} x \in \Omega, \quad \text{and} \int_\Omega (u \cdot \operatorname{div} g + \langle g, \nabla u\rangle)dx = 0 \Bigg\}.$$

Proof. In this case, by the definition of subdifferential, we see that $\partial\tau_x(r, s) \subset \{0\} \times \mathbb{R}^n$ for any $(r, s) \in \mathbb{R} \times \mathbb{R}^n$ and a.e.$x \in \Omega$.
Thus, Theorem 3.11 implies that $(-\Delta_D)^{-1} f(x) + \operatorname{div} g(x) = 0$, a.e.$x \in \Omega$. Therefore, we obtain that $\operatorname{div} g = -(-\Delta_D)^{-1} f \in H_0^1(\Omega)$ and $f = -(-\Delta_D) \operatorname{div} g$. □

The following corollary for one dimensional case will be used in the next chapter.

Corollary 3.13. *When $\tau(x, r, s)$ does not depend on r and the space dimension $n = 1$, for any $u \in D(E)$ such that $\partial E(u) \neq \emptyset$, the set $\partial E(u)$ is of the form*

$$\partial E(u) = \{-(-\Delta_D)g_x \mid g \in C^1(\bar\Omega), g_x \in H_0^1(\Omega),\ g(x) \in \partial\tau_x(u_x(x)) \ \text{a.e.} x \in \Omega\}.$$

Proof. In this case Sobolev's embedding theorem implies that $H_0^1(\Omega) \subset C^0(\bar\Omega)$. Thus, by the previous corollary $g_x \in C^0(\bar\Omega)$, thus, $g \in C^1(\bar\Omega)$.

Since $u \in W_0^{1,1}(\Omega)$, we can take $\{u_m\}_{m=1}^{+\infty} \subset C_0^\infty(\Omega)$ such that $u_m \to u$ in $W_0^{1,1}(\Omega)$. Then,

$$\int_\Omega (u \cdot g_x + g \cdot u_x)dx = \lim_{m\to+\infty} \int_\Omega (u_m \cdot g_x + g \cdot u_{mx})dx = 0.$$

□

3.3 Characterization of subdifferential of our energy

We give another proof to characterize the subdifferential of our energy F_γ. In the proof for general convex energies in the previous section, we considered the convex conjugate of functionals defined on L^1 space and L^∞ space which are not reflexive Banach spaces. We do not have to treat these non-reflexive Banach spaces, however, to characterize ∂F_γ if we impose extra assumption on the exponent γ of σ_γ. The argument becomes simpler since we only use the duality between reflexive Banach spaces.

We need following two propositions stated in [1] and [20].

Proposition 3.14. *(See [20] p.534)*
Let $J : \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous functional and J^* be its convex conjugate functional on $\mathbb{R}^l$. Suppose that Ω is a bounded domain of $\mathbb{R}^m$ and $p \in (1, +\infty)$.
Define functionals Φ_J on $L^p(\Omega, \mathbb{R}^l)$ and Φ_{J^*} on $L^{p/p-1}(\Omega, \mathbb{R}^l)$ by

$$\Phi_J(u) := \int_\Omega J(u(x))dx \text{ for } u \in L^p(\Omega, \mathbb{R}^l),$$
$$\Phi_{J^*} := \int_\Omega J^*(u(x))dx \text{ for } u \in L^{p/p-1}(\Omega, \mathbb{R}^l).$$

Then we see that

$$(\Phi_J)^\sharp = \Phi_{J^*} \text{ on } L^{p/p-1}(\Omega, \mathbb{R}^l) \text{ and } (\Phi_{J^*})^\sharp = \Phi_J \text{ on } L^p(\Omega, \mathbb{R}^l),$$

where $(\Phi_J)^\sharp$ denotes the convex conjugate functional on $L^{p/p-1}(\Omega, \mathbb{R}^l)$ of Φ_J and $(\Phi_{J^*})^\sharp$ denotes the convex conjugate functional on $L^p(\Omega, \mathbb{R}^l)$ of Φ_{J^*} with respect to the inner product of $L^2(\Omega, \mathbb{R}^l)$.

Proposition 3.15. *(See [1] p.268) Let Φ and Ψ be proper, convex, and lower semicontinuous functionals on a reflexive Banach space X.*
Suppose that $D(\Phi) - D(\Psi)$ is a neighbourhood of the origin, where $D(\Phi)$ and $D(\Psi)$ denote the effective domain of Φ and Ψ respectively.
Then we see that $(\Phi + \Psi)^\sharp = \Phi^\sharp \bigtriangledown \Psi^\sharp$ holds on the dual space X^* of X, where $\Phi^\sharp$, $\Psi^\sharp$, and $(\Phi + \Psi)^\sharp$ denote the convex conjugate functionals on X^* of Φ, Ψ, and $\Phi + \Psi$.

We have convenient properties of the conjugate functional ${\sigma_\gamma}^*$ of σ_γ.

Lemma 3.16. *We see that for any $\rho > 0$, there exists $B_\rho > 0$ such that*

$${\sigma_\gamma}^*(s) + B_\rho \geq \rho|s| \text{ holds for any } s \in \mathbb{R}^n.$$

In addition, for any measurable function $g : \Omega \to \mathbb{R}^n$, the following equivalence holds.

$$\int_\Omega {\sigma_\gamma}^*(g(x))dx < \infty \text{ if and only if } g \in L^{\gamma/\gamma-1}(\Omega, \mathbb{R}^n).$$

Proof. We can actually obtain the explicit form of ${\sigma_\gamma}^*$ as

$${\sigma_\gamma}^*(s) = \begin{cases} \left(1 - \frac{1}{\gamma}\right)(|s| - 1)^{\frac{\gamma}{\gamma-1}} & (|s| \geq 1), \\ 0 & (|s| < 1). \end{cases}$$

The statements are immediately assured by this form. $\square$

Now we assume that the space dimension $n(\leq 4)$ and the exponent $\gamma(>1)$ of σ_γ satisfy the following relation.

$$\begin{cases} \text{if } n \leq 2, & \text{then } \gamma > 1, \\ \text{if } n = 3, & \text{then } \gamma \geq 6/5, \\ \text{if } n = 4, & \text{then } \gamma \geq 4/3. \end{cases} \tag{$\natural$}$$

Then we see that

Proposition 3.17. *Suppose that the space dimension n and the exponent γ of σ_γ satisfy ($\natural$). If $u \in H^{-1}(\Omega)$ satisfies $F_\gamma^*(u) < +\infty$, then*

$$F_\gamma^*(u) = \min_{\substack{g \in L^{\gamma/\gamma-1}(\Omega, \ ^n) \\ \operatorname{div} g = -(-\Delta_D)^{-1}u}} \int_\Omega {\sigma_\gamma}^*(g)dx.$$

Proof. To carry out the proof we define a functional P on $\mathbb{R} \times \mathbb{R}^n$ of the form

$$P(r,s) := \sigma_\gamma(s) \text{ for any } r \in \mathbb{R}, \text{ and } s \in \mathbb{R}^n.$$

Then P is a proper, convex, and lower semicontinuous functional on $\mathbb{R} \times \mathbb{R}^n$. Moreover, since P does not depend on r, we see that if the convex conjugate $P^*(r', s') < +\infty$ then $r' = 0$ holds.

We fix $u \in H^{-1}(\Omega)$ satisfying $F_\gamma^*(u) < +\infty$ and set $\tilde{u} := ((-\Delta_D)^{-1}u, 0, \cdots, 0) \in L^{\gamma/\gamma-1}(\Omega, \mathbb{R}^{n+1})$. Note that by the assumption ($\natural$) Sobolev's embedding theorem assures the fact that $\tilde{u} \in L^{\gamma/\gamma-1}(\Omega, \mathbb{R}^{n+1})$.

Define functionals R and κ on $L^\gamma(\Omega, \mathbb{R}^{n+1})$ of the form

$$R(v) := \begin{cases} 0 & \text{if } v_0 \in W_0^{1,\gamma}(\Omega), \ (v_1, \cdots, v_n) = \nabla v_0, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\kappa(v) := \int_\Omega P(v_0(x), \cdots, v_n(x))dx,$$

for $v = (v_0, v_1, \cdots, v_n) \in L^\gamma(\Omega, \mathbb{R}^{n+1})$. Then, we see that R and κ are proper, convex, and lower semicontinuous on $L^\gamma(\Omega, \mathbb{R}^{n+1})$. In addition, we have that

$$F_\gamma^*(u) = (\kappa + R)^\sharp(\tilde{u}), \tag{3.21}$$

where $(\kappa + R)^\sharp$ denotes the convex conjugate on $L^{\gamma/\gamma-1}(\Omega, \mathbb{R}^{n+1})$ with respect to L^2 inner product of $\kappa + R$ in $L^\gamma(\Omega, \mathbb{R}^{n+1})$. Indeed, for any $w \in L^{\gamma/\gamma-1}(\Omega, \mathbb{R}^{n+1})$ we observe that

$$\begin{aligned} (\kappa + R)^\sharp(w) &= \sup_{v \in L^\gamma(\Omega, \ ^{n+1})} (\langle w, v \rangle_{L^2(\Omega, \ ^{n+1})} - (\kappa + R)(v)) \\ &= \sup_{\substack{v \in L^\gamma(\Omega, \ ^{n+1}) \\ v_0 \in W_0^{1,\gamma}(\Omega) \\ (v_1, \cdots, v_n) = \nabla v_0}} (\langle w, v \rangle_{L^2(\Omega, \ ^{n+1})} - \kappa(v)). \end{aligned}$$

Thus,

$$\begin{aligned}(\kappa + R)^\sharp(\tilde{u}) &= \sup_{v_0 \in W_0^{1,\gamma}(\Omega)} (\langle(-\Delta_D)^{-1}u, v_0\rangle_{L^2(\Omega)} - F_\gamma(v_0)) \\ &= F_\gamma^*(u).\end{aligned}$$

Here we have used the equality

$$F_\gamma^*(u) = \sup_{v_0 \in W_0^{1,\gamma}(\Omega)} (\langle(-\Delta_D)^{-1}u, v_0\rangle_{L^2(\Omega)} - F_\gamma(v_0)),$$

which can be proved by the same way as Proposition 3.7.

Since $D(\kappa) - D(R) = L^\gamma(\Omega, \mathbb{R}^{n+1})$ holds, the functionals κ and R satisfy the assumption of Proposition 3.15, thus we see that for any $w \in L^{\gamma/\gamma-1}(\Omega, \mathbb{R}^{n+1})$,

$$(\kappa + R)^\sharp(w) = (\kappa^\sharp \bigtriangledown R^\sharp)(w), \tag{3.22}$$

where $\kappa^\sharp$ and $R^\sharp$ denote the convex conjugate functionals on $L^{\gamma/\gamma-1}(\Omega, \mathbb{R}^{n+1})$ of κ and R with respect to L^2 inner product in $L^\gamma(\Omega, \mathbb{R}^{n+1})$ respectively.

Moreover, Proposition 3.14 assures that for any $w = (w_0, \cdots, w_n) \in L^{\gamma/\gamma-1}(\Omega, \mathbb{R}^{n+1})$,

$$\kappa^\sharp(w) = \int_\Omega P^*(w_0(x), \cdots, w_n(x))dx. \tag{3.23}$$

We next characterize $R^\sharp$. By the definition of R, for any $w \in L^{\gamma/\gamma-1}(\Omega, \mathbb{R}^{n+1})$,

$$\begin{aligned}R^\sharp(w) &= \sup_{v_0 \in W_0^{1,\gamma}(\Omega)} \int_\Omega \left(v_0 w_0 + \sum_{i=1}^n \frac{\partial v_0}{\partial x_i} w_i \right) dx \\ &= \begin{cases} 0 & \text{if } w_0 = \displaystyle\sum_{i=1}^n \frac{\partial w_i}{\partial x_i} \quad \text{in} \quad '(\Omega), \\ +\infty & \text{otherwise.} \end{cases}\end{aligned} \tag{3.24}$$

Now by (3.21),(3.22),(3.23), and (3.24), we see that

$$\begin{aligned}F_\gamma^*(u) &= (\kappa^\sharp \bigtriangledown R^\sharp)(\tilde{u}) \\ &= \inf_{\substack{w \in L^{\gamma/\gamma-1}(\Omega, \ {}^{n+1}) \\ w_0 = \ {}^{n}_{i=1} \partial w_i/\partial x_i}} \kappa^\sharp(\tilde{u} - w) \\ &= \min_{\substack{g \in L^{\gamma/\gamma-1}(\Omega, \ {}^{n}) \\ \operatorname{div} g = -(-\Delta_D)^{-1}u}} \int_\Omega {\sigma_\gamma}^*(g)dx.\end{aligned}$$

We changed 'inf' into 'min' since we can prove that $\int_\Omega {\sigma_\gamma}^*(\cdot)dx$ attains its minimum in

$$\left\{g \in L^{\gamma/\gamma-1}(\Omega, \mathbb{R}^n) \mid \operatorname{div} g = -(-\Delta_D)^{-1}u\right\}$$

by Lemma 3.16 and the similar argument of the proof of Proposition 3.10. $\square$

Remark. The condition (♮) is used to assure that $H_0^1(\Omega) \subset L^{\gamma/\gamma-1}(\Omega)$.

We can characterize ∂F_γ by Proposition 3.17. Since the proof is similar to Proposition 3.10, we omit it.

Theorem 3.18. *Suppose that the space dimension n and the exponent γ of σ_γ satisfy (♮). For any $u \in D(F_\gamma)$ such that $\partial F_\gamma(u) \neq \emptyset$ the set $\partial F_\gamma(u)$ of the form*

$$\partial F_\gamma(u) = \Bigg\{ -(-\Delta_D)\operatorname{div} g \;\Bigg|\; g \in L^{\gamma/\gamma-1}(\Omega, \mathbb{R}^n),\ \operatorname{div} g \in H_0^1(\Omega),$$
$$g(x) \in \partial\sigma_\gamma(\nabla u(x)) \ \textit{a.e.} x \in \Omega, \quad \textit{and} \ \int_\Omega (u \cdot \operatorname{div} g + \langle g, \nabla u \rangle) dx = 0 \Bigg\}.$$

Corollary 3.19. *When the space dimension $n = 1$, for any $u \in D(F_\gamma)$ such that $\partial F_\gamma(u) \neq \emptyset$ the set $\partial F_\gamma(u)$ is of the form*

$$\partial F_\gamma(u) = \{-(-\Delta_D) g_x \mid g \in C^1(\bar{\Omega}), g_x \in H_0^1(\Omega),\ g(x) \in \partial\sigma_\gamma(u_x(x)) \ \textit{a.e.} x \in \Omega\}.$$

4 Speed of a typical profile

We are interested in a speed for typical profile with a facet. For convenience, we set such a profile as an initial data. In this chapter, we will calculate an initial speed of a solution of (EQ4) for a typical initial data, a mountain with one plateau in one dimension. Through the initial speed obtained, we would like to consider how the solution will behave, especially, whether the facet will spread spontaneously or not.

4.1 Calculation of initial speed

First, let us set the situation.

Let the space dimension $n = 1$ and the domain $\Omega = (0, l)$. We think the case where our problem is derived by the energy $F_2(u) = \int_\Omega \sigma_2(u_x) dx$, $\sigma_2(p) = |p| + p^2/2$.

The initial data u_0 is defined by

$$u_0(x) = \begin{cases} u_{01}(x) & (0 < x \le a), \\ h(>0) & (a \le x \le b), \\ u_{02}(x) & (b \le x < l), \end{cases}$$

where $u_{01} \in C^4([0, a])$ is a strictly monotone increasing function with $u_{01}(0) = 0$, $u_{01}(a) = h$, and $u_{01}^{(i)}(0) = u_{01}^{(i)}(a) = 0$ $(i = 1, 2, 3, 4)$. Similarly $u_{01} \in C^4([b, l])$ is a strictly monotone decreasing function with $u_{02}(b) = h$, $u_{02}(l) = 0$, and $u_{02}^{(i)}(b) = u_{02}^{(i)}(l) = 0$ $(i = 1, 2, 3, 4)$.

We adopt this typical mountain-like function $u_0 \in C^4(\bar{\Omega})$ as an initial data of (EQ4) derived by the energy F_2 and we calculate the initial speed of the solution solving this initial value problem. Now since $u_0 \in D(F_2)$, the unique existence of the solution is assured by Theorem 2.4. Note that we can also obtain the initial speed of the solution for the general energy F_γ $(\gamma > 1)$ in the same way.

Theorem 4.1. *In the situation stated above, the initial speed V_0 of the solution of (EQ4) for the initial data u_0 is given by*

$$V_0 = -u_{0xxxx}\chi_{(0,a)\cup(b,l)} - \frac{24}{(b-a)^3}\chi_{(a,b)} + \frac{12}{(b-a)^2}\delta_a + \frac{12}{(b-a)^2}\delta_b, \tag{4.1}$$

where $\chi_{(0,a)\cup(b,l)}$ and $\chi_{(a,b)}$ are the characteristic functions of $(0,a)\cup(b,l)$ and (a,b), δ_a and δ_b stand for the Dirac distributions on a and b respectively.

Proof. As we saw in Chapter 2, the initial speed of the solution corresponds to $-\partial F_2^c(u_0)$.

By Corollary 3.13 (or Corollary 3.19) we know that

$$\partial F_2(u_0) = \{-(-\Delta_D)g_x \mid g \in C^1(\bar{\Omega}), g_x \in H_0^1(\Omega),\ g(x) \in \partial\sigma_2(u_{0x}(x)) \text{ a.e.} x \in \Omega\}.$$

Applying Example 2.3 for the case $\gamma = 2$, we see that

$$\begin{aligned} \partial\sigma_2(u_{0x}(x)) &= \begin{cases} 1 + u_{0x}(x) & (u_{0x}(x) > 0), \\ \in [-1,1] & (u_{0x}(x) = 0), \\ -1 + u_{0x}(x) & (u_{0x}(x) < 0) \end{cases} \\ &= \begin{cases} 1 + u_{01x}(x) & (0 < x < a), \\ \in [-1,1] & (a \le x \le b), \\ -1 + u_{02x}(x) & (b < x < l). \end{cases} \end{aligned}$$

Therefore,

$$\partial F_2(u_0) = \left\{ -(-\Delta_D)g_x \;\middle|\; g \in C^1(\bar{\Omega}),\ g_x \in H_0^1(\Omega),\ g(x) = \begin{cases} 1 + u_{01x}(x) & (0 \le x < a), \\ \in [-1,1] & (a \le x \le b), \\ -1 + u_{02x}(x) & (b < x \le l). \end{cases} \right\}.$$

Now we calculate $\partial F_2^c(u_0) = -(-\Delta_D)g_x^c$. Then, by Definition 2.8, $-(-\Delta_D)g_x^c$ is the unique minimizer which minimizes the norm of $H^{-1}(\Omega)$ in $\partial F_2(u_0)$. We see that

$$\begin{aligned} \| -(-\Delta_D)g_x\|^2_{H^{-1}(\Omega)} &= \langle g_x, -\frac{d}{dx}g_{xx}\rangle_{L^2(\Omega)} = \|g_{xx}\|^2_{L^2(\Omega)} \\ &= \|u_{01xx}\|^2_{L^2(0,a)} + \|u_{02xx}\|^2_{L^2(b,l)} + \|g_{xx}\|^2_{L^2(a,b)}. \end{aligned}$$

Thus, all we have to do is to find the function g^c which minimizes $\|g_{xx}\|_{L^2(a,b)}$ under the conditions that

$$g \in C^1(\bar{\Omega}),\ g_x \in H_0^1(\Omega), \text{ and } g(x) = \begin{cases} 1 + u_{01x}(x) & (0 \le x < a) \\ \in [-1,1] & (a \le x \le b) \\ -1 + u_{02x}(x) & (b < x \le l). \end{cases}$$

Eular-Lagrange equation of the variational problem $\|g_{xx}\|^2_{L^2(a,b)} \to \min$ is $g_{xxxx} = 0$ in (a,b).

Therefore, if we choose a polynomial $p(x)$ of degree 3 satisfying $p(a) = 1$, $p(b) = -1$, and $p'(a) = p'(b) = 0$ and set

$$g^c(x) = \begin{cases} 1 + u_{01x}(x) & (0 \leq x < a) \\ p(x) & (a \leq x \leq b) \\ -1 + u_{02x}(x) & (b < x \leq l), \end{cases}$$

then we see that g^c satisfies all of the above conditions and minimizes $||g_{xx}||_{L^2(a,b)}$ in the set of functions satisfying the conditions.

Such a polynomial $p(x)$ uniquely exists and is given by solving four linear equations related to the coefficients of $p(x)$ as following;

$$p(x) = \frac{4}{(b-a)^3}x^3 - \frac{6(b+a)}{(b-a)^3}x^2 + \frac{12ab}{(b-a)^3}x + 1 - \frac{2a^2(3b-a)}{(b-a)^3}.$$

Therefore, the canonical restriction $\partial F_2^c(u_0)$ is uniquely determined as the third derivative of this g^c.

Now we can calculate the initial speed $V_0 = -\partial F_2^c(u_0)$ and obtain (4.1). □

4.2 Conclusion

As we see the initial speed obtained in the previous section, the Dirac distributions on a and b appear though the initial speed is continuous in $(0, a) \cup (b, l)$ and constant in (a, b).

This does not cause any contradiction mathematically in our formulation, since we formulated (EQ1) in the space $H^{-1}(\Omega)$ which contains Dirac distributions. However, at least we can conclude that the initial speed indicates that our subdifferential formulation of (EQ) cannot be treated as a free boundary value problem with evolving facets. Also note that this conclusion implies that the free boundary formulation of (EQ) with facets in [23] cannot be approximated by smooth problems since our formulation is a limit of approximate problems derived by smooth energy.

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