ON STABLE CRITICAL POINTS FOR A SINGULAR PERTURBATION PROBLEM

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ON STABLE CRITICAL POINTS FOR A SINGULAR PERTURBATION PROBLEM

YOSHIHIRO TONEGAWA

ABSTRACT. We consider a singular perturbation problem arising in the scalar phase field model which Γ-converges to the area functional. Assuming the stability of the critical points for ε-problems, we show that the interface regions converge to a generalized stable minimal hypersurface as $\varepsilon \to 0$. The limit has L^2 generalized second fundamental form and the stability condition is expressed in terms of the corresponding inequality satisfied by stable minimal hypersurfaces. We show that the limit is a finite number of lines with no intersections when the dimension of the domain is 2.

1. INTRODUCTION

In this paper, we consider the variational problem associated with the functional

$$E_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $u : \Omega \to \mathbb{R}$ belongs to the Sobolev space $H^1(\Omega)$, $W : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ is a double-well potential function and $\varepsilon > 0$ is a small parameter. This is a typical energy modeling the phase separation phenomena within the van der Waals - Cahn - Hilliard theory [3]. In this context, u represents the density of a two-phase fluid, where the zero points ± 1 of W correspond to stable fluid phases, and the free energy $E_\varepsilon(\cdot)$ depends both on the density potential and the density gradient. The sequence of minimizers is expected to converge in an appropriate sense to a function u_0 as $\varepsilon \to 0$, where u_0 takes values ± 1 and the interface $\partial\{u_0 = 1\} \cap \Omega$ is a regular hypersurface with the least possible area with a given constraint, with a possible small singularities. The rigorous proof of this statement was given by Modica [10], Sternberg [14] with a volume constraint $\int_\Omega u = m$ for the sequence of global minimizers via the Γ-convergence technique.

One of the natural questions concerning $E_\varepsilon(\cdot)$ that we address in this paper is the following: given a sequence of *stable* critical points with uniform finite energy bound and $\varepsilon \to 0$, what can be said about the

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limit interface? Here we say u^ε is a critical point of E_ε if the Euler-Lagrange equation is satisfied:

$$\varepsilon \Delta u^\varepsilon = \frac{W'(u^\varepsilon)}{\varepsilon}. \tag{1.2}$$

We say that u^ε is stable if the second variation of E_ε is non-negative:

$$\frac{d^2}{dt^2}\bigg|_{t=0} E_\varepsilon(u^\varepsilon + t\phi) = \int_\Omega \varepsilon|\nabla\phi|^2 + \frac{W''(u^\varepsilon)}{\varepsilon}\phi^2 \geq 0 \tag{1.3}$$

for all $\phi \in C^1_c(\Omega)$. Since the functionals "converge" to the area functional, one naturally expects that the limit interface should be a stable minimal hypersurface in a suitable sense. Recall that the stability for smooth minimal hypersurface $M \subset \Omega$ is equivalent to the following inequality:

$$\int_M |\mathbf{B}|^2\phi^2 \leq \int_M |\nabla\phi|^2 \tag{1.4}$$

for any $\phi \in C^1_c(\Omega)$. Here, $\mathbf{B}$ is the second fundamental form of M [13]. In this paper, we show that there exists an L^2 second fundamental form defined on the limit interface satisfying the same inequality. Since we work in the setting of general critical points, the smoothness of the limit interfaces is not guaranteed in general. For this reason, we employ the notion of generalized second fundamental form for varifolds introduced by Hutchinson [7]. When $n = 2$, we show that the limit is finite number of lines with no intersections or junctions. Note that intersecting lines are in fact stationary and stable for smooth variations of the length functional, so this shows that the limits of stable phase interface possess a better regularity properties than stable critical points of the length functional in general.

There are numerous works related to the singular perturbation problem with double-well potential. Given a strict local minimizer for area functional with no constraint, Kohn and Sternberg [9] showed the existence of local minimizers for $E_\varepsilon(\cdot)$ which converge to the given limit. For such local minimizers, Córdoba and Caffarelli showed the local uniform convergence of the interface [4]. General stable critical points have been studied by Sternberg and Zumbrun, where the connectivity of interface for ε problem as well as the limit problem on strictly convex domains were proved [15, 16]. Up to the boundary uniform convergence of the interface is also studied in this case [18]. For general critical points, Hutchinson and the author showed that the interfaces of any critical points with finite energy and with or without volume constraint converge to a locally constant mean curvature hypersurface with possible integer multiplicities [8].

The organization of the paper is as follows. In section 2, we recall the notion of measure-function pair and generalized second fundamental form ([7]). In section 3, we state the assumptions and main results, and in section 4 show the stability property for the limit hypersurface M for general dimensions as well as the complete regularity for $n = 2$.

2. Generalized second fundamental form

In the next two subsections we collect definitions and theorems from [7] which we apply to the singular perturbation problems subsequently.

2.1. **Measure-function pairs.** Let U be a subset of $\mathbb{R}^\alpha$. In the following application, U will be $G_{n-1}(\Omega) = \Omega \times G(n, n-1)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $G(n, n-1)$ is the set of $n-1$-dimensional unoriented subspaces in $\mathbb{R}^n$. Via the identification of $G(n, n-1)$ as a set of projections, we regard $G(n, n-1)$ to be a subset of $\mathbf{R}^{n^2}$.

Definition 1. *Suppose μ is a Radon measure on U and $f : U \to \mathbb{R}^\beta$ is μ a.e. defined on U and in $L^1_{loc}(\mu)$. (μ, f) is called a measure-function pair over U.*

Definition 2. *([7, 4.2.1]) Suppose $\{(\mu_k, f_k)\}_{k=1}^\infty$ and (μ, f) are measure-function pairs over U. Suppose $\mu_k \to \mu$ on U as $k \to \infty$. Then we say (μ_k, f_k) converges to (μ, f) in the weak sense if*

$$\lim_{k\to\infty} \int_U < f_k, \phi > d\mu_k = \int_U < f, \phi > d\mu$$

for all $\phi \in C_c(U, \mathbb{R}^\beta)$.

Using the notion of *graph measures*, the following theorem was proved [17, 7]. The theorem holds for more general setting, but we only state it in the form necessary for our use:

Theorem 1. *([7, 4.4.2(i,ii)]) If (μ_k, f_k) is a sequence of measure-function pairs over U with $\liminf \int_U |f_k|^2 \, d\mu_k < \infty$, then some subsequence of (μ_k, f_k) converges in the weak sense to a measure-function pair (μ, f) for some f. Moreover,*

$$\int_U |f|^2 \, d\mu \le \liminf_{k\to\infty} \int_U |f_k|^2 \, d\mu_k.$$

2.2. **Generalized second fundamental form.** Suppose that $M \subset \mathbb{R}^n$ is a smooth hypersurface. For $x \in M$, let $S(x) = [S_{ij}(x)]$ be the $n \times n$ orthogonal projection matrix corresponding to projection onto T_xM, where T_xM is the tangent space to M at x. The *second fundamental form* of M at x is defined by

$$\mathbf{B} : T_xM \times T_xM \to (T_xM)^\perp, \qquad \mathbf{B}(v, w) = (D_v w)^\perp,$$

where $(T_x M)^\perp$ is the normal space and $D_v w$ is the covariant differentiation in $\mathbb{R}^n$. It is bilinear and it depends only on v and w at x. We also extend the domain of $\mathbf{B}$ to $\mathbb{R}^n \times \mathbb{R}^n$ by defining

$$\mathbf{B}(v, w) = \mathbf{B}(S(x)v, S(x)w).$$

For the standard orthonormal basis $\{\mathbf{e}_i\}_{i=1}^n$, define the component of $\mathbf{B}$ by

$$B_{ij}^k = < \mathbf{B}(\mathbf{e}_i, \mathbf{e}_j), \mathbf{e}_k > .$$

In the following, we use the usual summation convention whenever no ambiguity arises. The mean curvature vector is given by $B_{jj}^k \mathbf{e}_k$. The component of $\mathbf{B}$ is expressed in terms of the projection matrix S by

$$B_{ij}^k = S_{lj} \delta_i S_{kl},$$

where $\delta_i = S_{ij} \frac{\partial}{\partial x_j}$. This is because $B_{ij}^k = < D_{S\mathbf{e}_i} S\mathbf{e}_j, \mathbf{e}_k^\perp > = < \delta_i S_{lj} \mathbf{e}_l, \mathbf{e}_k - S_{sk}\mathbf{e}_s > = \delta_i S_{kj} - (\delta_i S_{lj}) S_{lk} = S_{lj} \delta_i S_{kl}$. We used $S_{lk} = S_{kl}$ and $S_{kj} = S_{kl} S_{lj}$. We also use the fact that

$$\delta_i S_{jk} = B_{ij}^k + B_{ik}^j.$$

This follows from $\delta_i S_{jk} = \delta_i (S_{jl} S_{lk}) = S_{jl} \delta_i S_{lk} + S_{lk} \delta_i S_{jl} = B_{ij}^k + B_{ik}^j$. If M is a level set $\{u = \text{const}\}$, $S = I - \nu \otimes \nu$, $\nu = \frac{\nabla u}{|\nabla u|} = (\nu_1, \cdots, \nu_n)$, one computes

$$B_{ij}^k = -\nu_k \delta_i \nu_j = -\frac{u_{x_k}}{|\nabla u|^2} (u_{x_i x_j} - \nu_j \nu_l u_{x_l x_i} - \nu_i \nu_l u_{x_l x_j} + \nu_i \nu_s \nu_j \nu_l u_{x_l x_s}).$$

If we choose a coordinate system such that $\nu(x) = (0, \cdots, 0, 1)$, we have

$$B_{ij}^n = -\frac{u_{x_i x_j}}{u_{x_n}} \qquad \text{for } i \neq n \text{ and } j \neq n$$

and $B_{ij}^k = 0$ otherwise. In particular, $2 \sum_{ijk} (B_{ij}^k)^2 = \sum_{ijk} (\delta_i S_{jk})^2$ in this case.

Now let $\phi \in C^1(\Omega \times \mathbb{R}^{n^2})$ be a "test function" to be used to define the generalized second fundamental form. Denote the differentiations with respect to x_i and S_{ij} by $D_i \phi$ and $D_{ij}^* \phi$, respectively. Apply the standard divergence theorem on submanifold M with $X = \phi(x, S(x))\mathbf{e}_i$ and $S(x) = T_x M$ to obtain

$$0 = \int_M \text{div}_M (X^\top) = \int_M \delta_r (S_{ir} \phi)$$

$$= \int_M S_{ij} D_j \phi + (\delta_i S_{jk}) D_{jk}^* \phi + (\delta_j S_{ij}) \phi.$$

Motivated by this we define a generalized second fundamental form for varifold. Here we briefly state the definitions and notations for varifolds. For more comprehensive account of rectifiable set and integral varifold, see [1, 13]. Radon measures on $G_{n-1}(\Omega)$ are called $(n-1)$-*varifold.* For $\phi \in C_c(\Omega)$, we define

$$||V||(\phi) = \int_{\Omega \times G(n,n-1)} \phi(x) dV(x,S).$$

We denote the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. V is called *integral varifold* if there exist an $(n-1)$-rectifiable set $M \subset \Omega$ and $\mathcal{H}^{n-1}$ measurable non-negative integer-valued function $\theta(x)$ on M such that, for $\phi \in C_c(G_{n-1}(\Omega))$,

$$V(\phi) = \int_M \phi(x, T_x M)\theta(x)\, d\mathcal{H}^{n-1}x.$$

Here, $T_x M$ is the approximate tangent plane which exists $\mathcal{H}^{n-1}$ a.e. on M. We say that V is *stationary* if, for any $g \in C^1_c(\Omega : \mathbf{R}^n)$,

$$\int_{G_{n-1}(\Omega)} Dg(x) \cdot S\, dV(x,S) = 0.$$

Here, $Dg(x)$ is the $n \times n$ first derivative matrix and $A \cdot B = \mathrm{tr}(AB)$.

Definition 3. *([7, 5.2.1]) A varifold V is said to have a generalized second fundamental form if there exist functions A_{ijk}, $1 \leq i,j,k \leq n$, defined V a.e. on $G_{n-1}(\Omega)$ such that*

1. *$(V, \{A_{ijk}\})$ is a measure-function pair,*
2. *$0 = \int_{G_{n-1}(\Omega)} (S_{ij} D_j \phi + A_{ijk} D^*_{jk}\phi + A_{jij}\phi)\, dV(x,S)$, $i = 1, \cdots, n$, for all $\phi \in C^1(\Omega \times \mathbb{R}^{n^2})$ with a compact support in the x variables.*

One proceeds to define:

Definition 4. *([7, 5.2.5]) The generalized second fundamental form $\mathbf{B} : G_{n-1}(\Omega) \to \mathbb{R}^{n^3}$ is $\mathbf{B} = \{B^k_{ij}\}$ defined V a.e. by*

$$B^k_{ij}(x,S) = S_{lj} A_{ikl}(x,S), \qquad 1 \leq i,j,k \leq n.$$

We write $|\mathbf{B}|^2 = \sum_{i,j,k=1}^n (B^k_{ij})^2$.

The generalized second fundamental form is uniquely determined V a.e. on $G_{n-1}(\Omega)$ if V is an integral varifold ([7, 5.2.2]).

3. Assumptions and main results

First we state the assumptions on W and u^ε and recall some definitions and known results.

We assume that

(1) $W \in C^3$, $W(\pm 1) = 0$, $W''(\pm 1) > 0$, $W \geq 0$ has only three critical points.
(2) A sequence of $C^3(\Omega)$ functions $\{u^{\varepsilon_i}\}$, where $\varepsilon_i > 0$ and $\lim_{i\to\infty} \varepsilon_i = 0$ satisfy (1.2) and (1.3) with ε there replaced by ε_i for all i.

(3) There exist C, $E_0 < \infty$ such that $E_{\varepsilon_i}(u^{\varepsilon_i}) \leq E_0$ and $\sup_\Omega |u^{\varepsilon_i}| \leq C$ for all i.

To characterize the limit interface, we next define a sequence of $(n-1)$-varifolds V^{ε_i} from u^{ε_i}. For $\phi \in C_c(G_{n-1}(\Omega))$, define

$$V^{\varepsilon_i}(\phi) = \frac{1}{\sigma}\int_{\Omega\cap\{|\nabla u^{\varepsilon_i}|>0\}} \phi(x, I - \nu^{\varepsilon_i}(x)\otimes\nu^{\varepsilon_i}(x))\frac{\varepsilon_i}{2}|\nabla u^{\varepsilon_i}|^2,$$

where $\nu^{\varepsilon_i} = \frac{\nabla u^{\varepsilon_i}}{|\nabla u^{\varepsilon_i}|}$ and $\sigma = \int_{-1}^1 \sqrt{W(s)/2}\,ds$. Due to the uniform energy bound (3), there always exists some converging subsequence in the sense of Radon measure. The following result applies to any such subsequence and the limit.

First, even without the stability condition (1.3), we have

Theorem 2. *([8, Theorem 1]) Suppose $V^{\varepsilon_i} \to V \in G_{n-1}(\Omega)$ as Radon measures on $G_{n-1}(\Omega)$. Then,*

(1) *V is a stationary integral varifold.*
(2) $\lim_{i\to\infty}\int_{\tilde\Omega}\left|\varepsilon_i\frac{|\nabla u^{\varepsilon_i}|^2}{2} - \frac{W(u^{\varepsilon_i})}{\varepsilon_i}\right| = 0$ *for all $\tilde\Omega \subset\subset \Omega$.*
(3) *For any $1 > s > 0$ and $\tilde\Omega \subset\subset \Omega$, $\{|u^{\varepsilon_i}| < 1 - s\} \cap \tilde\Omega$ converge to $\mathrm{spt}||V|| \cap \tilde\Omega$ in the Hausdorff distance sense.*

With stability condition (1.3), we have

Theorem 3. *The limit varifold V has a generalized second fundamental form $\mathbf{B}$ with*

$$\int_{G_{n-1}(\Omega)} |\mathbf{B}|^2\,dV \leq \int_\Omega |\nabla\phi|^2\,d||V|| \tag{3.1}$$

for all $\phi \in C_c^1(\Omega)$.

Due to the above properties of V, by defining $\mathbf{B}(x) = \mathbf{B}(x, T_xM)$ with $M = \mathrm{spt}||V||$, $\mathbf{B}$ may be regarded as a function defined on Ω instead of $G_{n-1}(\Omega)$. Hence with this implicitly assumed, we may write (3.1) as

$$\int_\Omega |\mathbf{B}|^2\phi^2\,d||V|| \leq \int_\Omega |\nabla\phi|^2\,d||V||$$

as well. For $n = 2$, we prove

Theorem 4. *For any open ball $B \subset\subset \Omega$, $\mathrm{spt}||V|| \cap B$ consists of finite line segments $\cup_{j=1}^N(\mathbf{a}_j, \mathbf{b}_j)$, with $\mathbf{a}_j, \mathbf{b}_j \in \partial B$ (boundary of B) and $[\mathbf{a}_j, \mathbf{b}_j] \cap [\mathbf{a}_k, \mathbf{b}_k] = \emptyset$ for $j \neq k$.*

For $n \geq 3$, there are no general regularity theory for stationary integral varifolds with L^2 second fundamental form. By Allard's regularity theory [1] for stationary varifolds, $\mathrm{spt}||V||$ is real analytic on a dense open set $O \subset \text{spt}||V||$. If we assume that $\text{spt}||V||$ is a regular hypersurface outside of a closed singular subset $X \subset \text{spt}||V||$ and that $\mathcal{H}^{n-3}(X) = 0$, then Schoen and Simon's result on stable hypersurfaces [12, Theorem 3] shows that $\text{spt}||V||$ is regular (i.e. X is empty) for $n \leq 7$ and $\mathcal{H}^{n-8+\delta}(X) = 0$ for all $\delta > 0$ in general dimensions.

4. Proof of the theorems

In the following proposition, we see a phase field analogue of (1.4). We omit i from ε_i in the following computations for simplicity.

Proposition 1. *([11, Proposition 2.6]) For $\phi \in C^1_c(\Omega)$ and u^ε satisfying (1.2) and (1.3),*

$$\int_{\Omega\cap\{|\nabla u^\varepsilon|>0\}} \varepsilon \left\{ \sum_{i,j=1}^{n} (u^\varepsilon_{x_i x_j})^2 - \frac{1}{|\nabla u^\varepsilon|^2} \sum_{j=1}^{n} (\sum_{i=1}^{n} u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j})^2 \right\} \phi^2$$

$$\leq \varepsilon \int_\Omega |\nabla\phi|^2 |\nabla u^\varepsilon|^2. \tag{4.1}$$

Proof We include the computation for the reader's convenience. In (1.3), replace ϕ by $\phi|\nabla u^\varepsilon|$. The computation shows that

$$\int_{\Omega\cap\{|\nabla u^\varepsilon|>0\}} \varepsilon \left(\phi^2 \sum_j \frac{(\sum_i u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j})^2}{|\nabla u^\varepsilon|^2} + 2\phi u^\varepsilon_{x_i} \phi_{x_j} u^\varepsilon_{x_i x_j} + |\nabla u^\varepsilon|^2 |\nabla\phi|^2 \right)$$

$$+ \frac{W''(u^\varepsilon)}{\varepsilon} |\nabla u^\varepsilon|^2 \phi^2 \geq 0. \tag{4.2}$$

Differentiate the equation (1.2) with respect to x_j and multiply $u^\varepsilon_{x_j}\phi^2$ to obtain

$$\varepsilon(\Delta u^\varepsilon_{x_j}) u^\varepsilon_{x_j} \phi^2 = \frac{W''(u^\varepsilon)}{\varepsilon} (u^\varepsilon_{x_j})^2 \phi^2.$$

After summing over j and integrating by parts, we have

$$\int_\Omega \frac{W''(u^\varepsilon)}{\varepsilon} |\nabla u^\varepsilon|^2 \phi^2 = -\varepsilon \int_\Omega \sum_{i,j} \{ (u^\varepsilon_{x_i x_j})^2 \phi^2 + 2 u^\varepsilon_{x_i x_j} u^\varepsilon_{x_j} \phi \phi_{x_i} \}.$$

By substituting this into (4.2), we obtain (4.1). □

Remark 1. *Note that the integrand on the left-hand side is invariant under orthogonal rotation, and if we choose the coordinate system at x with $\frac{\nabla u^\varepsilon(x)}{|\nabla u^\varepsilon(x)|} = (0, \cdots, 0, 1)$, it is*

$$\varepsilon \{ \sum_{i,j=1}^{n-1} (u^\varepsilon_{ij})^2 + \sum_{i=1}^{n-1} (u^\varepsilon_{ni})^2 \} \phi^2$$

at x. The first term divided by $\varepsilon|\nabla u^\varepsilon|^2$ corresponds to the length square of the second fundamental form of the level set $\{u^\varepsilon = const\}$. □

For $u^\varepsilon, (x,S) \in \Omega \times G(n, n-1)$, with $|\nabla u^\varepsilon(x)| \neq 0$ we define

$$\nu^\varepsilon(x) = \frac{\nabla u^\varepsilon(x)}{|\nabla u^\varepsilon(x)|},$$

$$A^\varepsilon_{ijk}(x,S) = \delta_i(-\nu^\varepsilon_j \nu^\varepsilon_k) = -S_{il}(\nu^\varepsilon_j \nu^\varepsilon_k)_{x_l},$$

$$(B^i_{jk})^\varepsilon(x,S) = S_{lj} A^\varepsilon_{ikl}(x,S),$$

$$H^\varepsilon_i = \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 - \frac{W(u)}{\varepsilon}\right)_{x_i} \frac{1}{\varepsilon|\nabla u^\varepsilon|^2},$$

$i = 1, \cdots, n$. $(B^k_{ij})^\varepsilon(x,S)$ with $S = I - \nu^\varepsilon \otimes \nu^\varepsilon$ corresponds to the second fundamental form of $\{u^\varepsilon = \text{const}\}$.

Proposition 2. *For $\phi \in C^1(\Omega \times \mathbf{R}^{n^2})$ with a compact support in the first set of variables, we have*

$$\int_{Gn-1(\Omega)} (S_{ij} D_j \phi + A^\varepsilon_{ijk} D^*_{jk}\phi + H^\varepsilon_i \phi) dV^\varepsilon(x,S) = 0,$$

$i = 1, \cdots, n.$

Proof Fix $i, \phi \in C^1_c(\Omega)$ and multiply the equation (1.2) by $\phi u^\varepsilon_{x_i}$. After two integrations by parts, one obtains

$$\int_\Omega \frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 \phi_{x_i} - \varepsilon u^\varepsilon_{x_i} u^\varepsilon_{x_j} \phi_{x_j} + \frac{W(u^\varepsilon)}{\varepsilon}\phi_{x_i} = 0$$

and consequently,

$$\int_\Omega (\phi_{x_i} - \nu^\varepsilon_i \nu^\varepsilon_j \phi_{x_j})\varepsilon|\nabla u^\varepsilon|^2 + \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 - \frac{W(u^\varepsilon)}{\varepsilon}\right)_{x_i} \phi = 0. \tag{4.3}$$

Now, for $\phi \in C^1_c(\Omega \times \mathbf{R}^{n^2})$ and $s > 0$, define $\phi^s(x) = \phi\left(x, I - \frac{\nabla u^\varepsilon \otimes \nabla u^\varepsilon}{s^2 + |\nabla u^\varepsilon|^2}\right)$. Then $\phi^s \in C^1_c(\Omega)$, and substitution in (4.3) with $s \to 0$ gives

$$\int_\Omega (I - \nu^\varepsilon \otimes \nu^\varepsilon)_{ij}(D_j\phi - (\nu^\varepsilon_l \nu^\varepsilon_k)_{x_j} D^*_{lk}\phi)\varepsilon|\nabla u^\varepsilon|^2$$
$$+ \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 - \frac{W(u^\varepsilon)}{\varepsilon}\right)_{x_i}\phi = 0.$$

Since $\psi(x,S)dV^\varepsilon(x,S) = \frac{1}{2\sigma}\psi(x, I - \nu^\varepsilon \otimes \nu^\varepsilon)\varepsilon|\nabla u^\varepsilon|^2 dx$, and by the definition of $A^\varepsilon_{ijk}, H^\varepsilon_i$, we obtain the stated identity. □

Proposition 3. *For $\phi \in C^1_c(\Omega)$, we have*

$$\int \sum_{i,j,k=1}^n |A^\varepsilon_{ijk}|^2 \phi^2 dV^\varepsilon \leq 2\int_\Omega |\nabla\phi|^2 d\|V^\varepsilon\|, \tag{4.4}$$

$$(4.5) \qquad \int \sum_{i,j,k=1}^{n} |(B^i_{jk})^\varepsilon|^2 \phi^2 dV^\varepsilon \le \int_\Omega |\nabla\phi|^2 d\|V^\varepsilon\|,$$

$$(4.6) \qquad \int_\Omega \sum_{i=1}^{n} |H^\varepsilon_i|^2 \phi^2 d\|V^\varepsilon\| \le C(n) \int_\Omega |\nabla\phi|^2 d\|V^\varepsilon\|.$$

Proof Since $\sum_{i,j,k} |A^\varepsilon_{ijk}|^2 dV^\varepsilon = \frac{1}{2\sigma}\varepsilon 2 \sum_{i,j=1}^{n-1} (u^\varepsilon_{x_i x_j})^2$ with a suitable coordinate system, Proposition 1 shows immediately (4.4). The inequality (4.5) is similar. For (4.6), using the equation (1.2),

$$\begin{aligned} \varepsilon|\nabla u^\varepsilon|^2 (H^\varepsilon_i)^2 &= (\varepsilon|\nabla u^\varepsilon|^2)^{-1}(\varepsilon u^\varepsilon_{x_j} u^\varepsilon_{x_i x_j} - \frac{W'}{\varepsilon} u_{x_i})^2 \\ &= \varepsilon|\nabla u^\varepsilon|^{-2}(u^\varepsilon_{x_j} u^\varepsilon_{x_i x_j} - \Delta u^\varepsilon u^\varepsilon_{x_i})^2 \\ &= \begin{cases} \varepsilon(\sum_{j=1}^{n-1} u^\varepsilon_{x_j x_j})^2 & i = n, \\ \varepsilon(u^\varepsilon_{x_i x_n})^2 & i \ne n, \end{cases} \end{aligned}$$

in the coordinate system with $\nabla u^\varepsilon(x) = (0, \cdots, 0, u^\varepsilon_{x_n})$. With a suitable choice of $C(n)$ with Proposition 1, (4.6) follows. □

Since the right-hand side of the above estimates are bounded by the energy bound E_0 and ϕ, Theorem 1 give weak limit functions $A_{ijk}, B^k_{ij}, H_i, 1 \le i,j,k \le n$ defined V a.e. with $(A^{\varepsilon_l}_{ijk}, V^{\varepsilon_l})$, $((B^k_{ij})^{\varepsilon_l}, V^{\varepsilon_l})$, $(H^{\varepsilon_l}_i, V^{\varepsilon_l})$ converging in the weak sense to (A_{ijk}, V), (B^k_{ij}, V) and (H_i, V), respectively. Moreover, for any $\phi \in C_c(\Omega)$

$$\int_{G_{n-1}(\Omega)} \phi^2 \sum_{i,j,k} |B^k_{ij}|^2 dV \le \liminf_{l\to\infty} \int_{G_{n-1}(\Omega)} \phi^2 \sum_{i,j,k} |(B^i_{jk})^{\varepsilon_l}|^2 dV^{\varepsilon_l}$$

$$\le \liminf_{l\to\infty} \int_\Omega |\nabla\phi|^2 d\|V^{\varepsilon_l}\| = \int_\Omega |\nabla\phi|^2 d\|V\|$$

by (4.5). Also by the definition of weak convergence, we have for $\phi \in C^1_c(\Omega \times \mathbf{R}^{n^2})$,

$$\int_{G_{n-1}(\Omega)} (S_{ij} D_j\phi + A_{ijk} D^*_{jk}\phi + H_i\phi) dV(x,S) = 0, \ i = 1, \cdots, n.$$

On the other hand, by (2) of Theorem 2, for $\phi \in C^1_c(\Omega)$,

$$\int_{G_{n-1}(\Omega)} H^{\varepsilon_l}_i(x,S)\phi(x) dV^{\varepsilon_l}(x,S) = \int_\Omega \left(\frac{\varepsilon_l}{2}|\nabla u^{\varepsilon_l}|^2 - \frac{W}{\varepsilon_l}\right)_{x_i} \phi$$
$$= -\int_\Omega \phi_{x_i} \left(\tfrac{\varepsilon_l}{2}|\nabla u^{\varepsilon_l}|^2 - \tfrac{W}{\varepsilon_l}\right) \to 0 \quad \text{as} \quad l \to \infty.$$

Hence,

$$\int_{G_{n-1}(\Omega)} H_i(x,S)\phi(x) dV(x,S) = 0 = \int_\Omega H_i(x, T_x M)\phi(x) d\|V\|,$$

where M is an $(n-1)$- rectifiable set (identified with spt $\|V\|$) and T_xM is the approximate tangent plane at x. This shows that $H_i(x, T_xM) = 0$ $\|V\|$ a.e. on Ω. Thus we obtain

$$\int_{G_{n-1}(\Omega)} (S_{ij}D_j\phi + A_{ijk}D^*_{jk}\phi)dV(x,S) = 0, \ i = 1, \cdots, n.$$

By the definition of the weak convergence, we also have

$$B^i_{jk} = S_{lj}A_{ikl} \quad V \ a.e. \quad \text{on} \quad G_{n-1}(\Omega).$$

With $\phi(x)S_{si}$ in place of ϕ, above we obtain

$$0 = \int_{G_{n-1}(\Omega)} (S_{ij}D_j\phi S_{si} + \phi A_{isi})dV = \int_{G_{n-1}(\Omega)} (S_{sj}D_j\phi + A_{isi}\phi)dV.$$

Since $\int_{G_{n-1}(\Omega)}(S_{ij}D_j\phi\, dV = 0$ by the stationarity of V, we have $A_{isi} = 0$ V a.e. on $G_{n-1}(\Omega)$, $s = 1, \cdots, n$, which is just another way of saying that V is stationary. Thus, we complete the proof of Theorem 3. □

For the proof of Theorem 4, we first show

Lemma 1. *Given $0 < s < 1$ and $\tilde{\Omega} \subset\subset \Omega$, there exists a positive constant c depending only on W and s such that, for all sufficiently small $\varepsilon > 0$,*

$$\frac{c}{\varepsilon} \le |\nabla u^\varepsilon(x)| \tag{4.7}$$

for all $x \in \{|u^\varepsilon| < 1 - s\} \cap \tilde{\Omega}$.

Proof Suppose for a contradiction that there exists $\bar{x} \in \{|u^\varepsilon| < 1 - s\} \cap \tilde{\Omega}$ with $|\nabla u^\varepsilon(\bar{x})| < \frac{c}{\varepsilon}$, where we set $c^2 = \frac{1}{4}\min_{|t|<1-s/2} W(t) > 0$. Then there exists $r = r(W)$ such that $|\nabla u^\varepsilon(x)| < \frac{2c}{\varepsilon}$ and $|u^\varepsilon(x)| \le 1-s/2$ for $x \in B_{\varepsilon r}(\bar{x})$, since we may obtain C^1 estimate for the rescaled equation $\Delta u = W'(u)$ after the change of variables $\tilde{x} = \frac{x-\bar{x}}{\varepsilon}$. Hence we have

$$\xi^\varepsilon = \left(\frac{W(u)}{\varepsilon} - \frac{\varepsilon}{2}|\nabla u|^2\right) \ge \frac{W(u)}{\varepsilon} - \frac{2c^2}{\varepsilon}$$

$$\ge \frac{W(u)}{2\varepsilon} \ge \frac{2c^2}{\varepsilon} \text{ on } B_{\varepsilon r}(\bar{x}). \tag{4.8}$$

On the other hand, by the Poincaré inequality and for $B_R(\bar{x})$ and $R < \frac{1}{2}\text{dist}(\tilde{\Omega}, \partial\Omega)$,

$$(\int_{B_R(\bar{x})} |\xi - \bar{\xi}|^2)^{\frac{1}{2}} \le c_0 \int_{B_R(\bar{x})} |\nabla\xi| = c_0 \int_{B_R(\bar{x})} |\varepsilon u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} - \frac{W'}{\varepsilon} u^\varepsilon_{x_j}|$$
$$\le c_0 \int_{B_R(\bar{x})} \varepsilon |u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} - \Delta u^\varepsilon u^\varepsilon_{x_j}| \quad \text{by (1.2)}.$$

By the same argument as before for H_i^ε, using (4.1) as well as the Hölder inequality,

(4.9)

$$\leq \; c_0 \left(\int_{B_R(\bar{x})} \varepsilon |\nabla u^\varepsilon|^2 \right)^{1/2} \left(\int_{B_R(\bar{x})} \varepsilon \{ (u^\varepsilon_{x_i x_j})^2 - \frac{1}{|\nabla u^\varepsilon|^2} (u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j})^2 \} \right)^{1/2}$$

$$\leq \; c_0 \left(\int_{B_R(\bar{x})} \varepsilon |\nabla u^\varepsilon|^2 \right)^{1/2} C(\Omega, \tilde{\Omega}) E_0^{1/2}.$$

By the monotonicity formula for the scaled energy on concentric balls [8, Proposition 1], we have

$$\frac{1}{R} \int_{B_R(\bar{x})} \varepsilon |\nabla u^\varepsilon|^2 \leq \frac{1}{R_1} \int_{B_{R_1}(\bar{x})} \left(\varepsilon |\nabla u|^2 + \frac{2W}{\varepsilon} \right) + c_1 R_1$$

$$\leq \frac{2E_0}{R_1} + c_1 R_1$$

for $R < R_1 = \mathrm{dist}(\tilde{\Omega}, \Omega)/2$. Then choose R small so that

$$c_0 (R(\frac{2E_0}{R_1} + c_1 R_1))^{1/2} c(\Omega, \tilde{\Omega}) E_0^{1/2} \leq c^2 r \sqrt{\pi}$$

is satisfied.

By (2) of Theorem 2, note that $\frac{1}{\pi R^2} \int_{B_R(\bar{x})} \xi = \bar{\xi} \to 0$ as $\varepsilon \to 0$, and by (4.8),

$$\left(\int_{B_R(\bar{x})} |\xi|^2 \right)^{1/2} \geq \left(\int_{B_{\varepsilon r}(\bar{x})} |\xi|^2 \right)^{1/2} \geq \sqrt{\pi} (\varepsilon r) \frac{2c^2}{\varepsilon} = 2c^2 r \sqrt{\pi}.$$

This contradicts (4.9) for small ε. □

Proposition 4. *For $0 < s < 1$ and all small ε,*

$$\int_{-1+s}^{1-s} \left(\int_{\{u^\varepsilon = t\} \cap \tilde{\Omega}} (\kappa^\varepsilon)^2 d\mathcal{H}^1 \right) dt \leq C(\Omega, \tilde{\Omega}, W, E_0).$$

Here, κ^ε is the geodesic curvature of the level curve of u^ε.

Proof Since $|\mathbf{B}^\varepsilon|^2 = (\kappa^\varepsilon)^2$ for $n = 2$, (4.5) and the coarea formula [5] yield the stated inequality. □

Proof of Theorem 4

By Fatou's Lemma, we have

$$\int_{-1+s}^{1-s} \liminf_{i \to \infty} \left(\int_{\{u^{\varepsilon_i} = t\}} (\kappa^{\varepsilon_i})^2 d\mathcal{H}^1 \right) dt \; < \; \infty,$$

so that we may choose $t \in [-1+s, 1-s]$ such that

$$\liminf_{i\to\infty} \int_{\{u^{\varepsilon_i}=t\}} (\kappa^{\varepsilon_i})^2 d\mathcal{H}^1 < \infty.$$

By (4.7), each curve $\tilde{\Omega} \cap \{u^{\varepsilon_i} = t\}$ is a finite number of curves with a uniform $C^{1,1/2}$ bound. Also by Theorem 2, $\{u^{\varepsilon_i} = t\} \cap \tilde{\Omega}$ converge to spt $\|V\|$ in the Hausdorff distance sense. Thus, locally, spt $\|V\|$ is expressed as $\cup_{j=1}^m$ graph g_j, $g_j \in C^{1,1/2}$ and $g_1 \le \cdots \le g_m$ over a suitable line segment. On the other hand, the support of one dimensional stationary integral varifold is locally either a line segment or a junction point ([2]). Since $g_1 \le \cdots \le g_m$ and $g_j \in C^{1,1/2}$, there cannot be any junction point in Ω. □

5. Remarks

1. Though we do not know how to utilize it, we point out that the following identity holds for u satisfying (1.2):

$$\sum_{i=1}^n (H_i^\varepsilon)_{x_i} = -\sum_{i=1}^n |H_i^\varepsilon|^2$$
$$+\left\{ \sum_{i,j=1}^n (u^\varepsilon_{x_i x_j})^2 - \frac{1}{|\nabla u^\varepsilon|^2} \sum_{j=1}^n (\sum_{i=1}^n u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j})^2 \right\} / |\nabla u^\varepsilon|^2.$$

Here, H_i^ε may be considered as an approximate mean curvature for the ε-problem (see Proposition 2). The identity can be checked by direct computation and the equation (1.2). For $n = 2$, the right-hand side is equal to 0. For stable critical points with a uniform energy bound, the integral with respect to $\|V^\varepsilon\|$ of the right-hand side is bounded locally uniformly due to (4.1) and (4.6), so there is a uniform estimate for the divergence of the approximate mean curvature vector in this sense.

2. One may speculate that the supports of limit varifolds of stable critical points are always smooth for $n \le 7$. For $n = 2$, we have a complete regularity results in this paper. For $n = 3$, with some extra work, one can show that the tangent cones of the limit varifolds are always 2-planes with integer multiplicities. This does not give a complete regularity for $n = 3$, since around a point of multiplicities greater than 1, no regularity theory is available even if the tangent cones are 2-planes at every point.

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