Monge's problem with a quadratic cost by the zero-noise limit of h-pass processes

Toshio Mikami

Series #596. July, 2003

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#570 H. Ishii and T. Mikami, Convexified Gauss curvature flow of bounded open sets in an anisotropic external field: a stochastic apporoximation and PDE, 37 pages. 2002.

#571 Y. Nakano, Minimization of shortfall risk in a jump-diffusion model, 10 pages. 2002.

#572 K. Izuchi and T. Nakazi, Backward shift invariant subspaces in the bidisc, 8 pages. 2002.

#573 S. Izumiya, D. Pei and M. C. Romero-Fuster, The horospherical geometry of surfaces in Hyperbolic 4-space, 17 pages. 2002.

#574 S. Izumiya and M. C. Romero-Fuster, The hyperbolic Gauss-Bonnet type theorem, 10 pages. 2002.

#575 S. Izumiya and S. Janeczko, A symplectic framework for multiplane gravitational lensing, 19 pages. 2002.

#576 S. Izumiya, M. Kossowski, D. Pei and M. C. Romero-Fuster, Singularities of C^∞-lightlike hypersurfaces in Minkowski 4-space, 18 pages. 2002.

#577 S. Izumiya, D. Pei and M.Takahashi, Evolutes of hypersurfaces in Hyperbolic space, 21 pages. 2002.

#578 Y. Giga, S. Matsui and S. Sasayama, Blow up rate for semilinear heat equation with subcritical nonlinearity, 29 pages. 2002.

#579 M. Tsujii, Physical measures for partially hyperbolic surface endomorphisms, 71 pages. 2003.

#580 Y. Giga and K. Yamada, On viscous Burgers-like equations with linearly growing initial data, 19 pages. 2003.

#581 T. Nakazi and T. Osawa, Spectra of Toeplitz Operators and Uniform Algebras, 9 pages. 2003.

#582 Y. Daido, M. Ikehata and G. Nakamura, Reconstruction of Inclusions for the Inverse Boundary Value Problem with Mixed Type Boundary Condition, 18 pages. 2003.

#583 Y. Daido and G. Nakamura, Reconstruction of Inclusions for the Inverse Boundary Value Problem with Mixed Type Boundary Condition and Source Term, 26 pages. 2003.

#584 M.-H. Giga and Y. Giga, A PDE approach for motion of phase-boundaries by a singular interfacial energy, 19 pages. 2003.

#585 A.A. Davydov, G. Ishikawa, S. Izumiya and W.-Z. Sun, Generic singularities of implicit systems of first order differential equations on the plane, 28 pages. 2003.

#586 K. Yamauchi, On an underlying structure for the consistency of viscosity solutions, 12 pages. 2003.

#587 T. Miyao, Momentum operators with a winding gauge potential, 15 pages. 2003.

#588 Y. Giga and R. Kobayashi, On constrained equations with singular diffusivity, 35 pages. 2003.

#589 O. Sawada, On analyticity rate estimates of the solutions to the Navier-Stokes equations in Bessel-potential spaces, 13 pages. 2003.

#590 T. Nakazi, Exposed points and extremal problems in H^1 on a bidisc, 11 pages. 2003.

#591 Y. Tsai and Y. Giga, A numerical study of anisotropic crystal growth with bunching under very singular vertical diffusion, 9 pages. 2003.

#592 K. Izuchi, T. Nakazi and M. Seto, Backward shift invariant subspaces in the bidisc II, 17 pages. 2003.

#593 S. Jimbo, Singular perturbation of domains and Semilinear elliptic equations III, 26 pages. 2003.

#594 Y. Giga, S. Matsui and S. Sasayama, On blow up rate for sign-changing solutions in a convex domain, 12 pages. 2003.

#595 T. Nakazi and T. Osawa, Invariant subspaces of finite codimension and uniform algebras, 6 pages. 2003.

Monge's problem with a quadratic cost by the zero-noise limit of *h-pass* processes

Toshio Mikami
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan

July 10, 2003

Abstract

We study the asymptotic behavior, in the zero noise limit, of solutions to Schrödinger's functional equations and that of *h-pass* processes, and give a new proof of the existence of the minimizer of Monge's problem with a quadratic cost.

1 Introduction.

Let $L : \mathbf{R}^d \mapsto [0, \infty)$ be convex, P_0 and P_1 be Borel probability measures on $\mathbf{R}^d$, and put

$$V(P_0, P_1) := \inf\{\int_{\mathbf{R}^d} L(\psi(X)-X)P_0(dx) : P_0(\psi(X) \in dx) = P_1(dx)\}. \quad (1.1)$$

The study of the minimizer of (1.1) can be considered as a special case of Monge's problem.

Kantorovich's approach to Monge's problem is to study the minimizer of the following relaxed problem:

$$\tilde{V}(P_0,P_1) := \inf\{\iint_{\mathbf{R}^d\times\mathbf{R}^d} L(y-x)\mu(dxdy) \quad : \mu(dx\times\mathbf{R}^d)=P_0(dx), \mu(\mathbf{R}^d\times dy)=P_1(dy)\}. \tag{1.2}$$

If there exists a Borel measurable function ψ, on $\mathbf{R}^d$, such that the minimizer of (1.2) is $P_0(dx)\delta_{\psi(x)}(dy)$, then $V(P_0,P_1)=\tilde{V}(P_0,P_1)$ and ψ is a minimizer of (1.1).

This is called the Monge-Kantorovich problem and plays a crucial role in many fields and has been studied by many authors (see [8, 20, 25] and the references therein).

It is easy to see that the following holds:

$$\tilde{V}(P_0,P_1) = \inf\Big\{E\Big[\int_0^1 L\Big(\frac{d\phi(t)}{dt}\Big)dt\Big]\Big\}, \tag{1.3}$$

where the infimum is taken over all absolutely continuous stochastic processes $\{\phi(t)\}_{0\le t\le 1}$ for which $P(\phi(t)\in dx)=P_t(dx)$ $(t=0,1)$. (In this paper we use the same notation P for different probability measures for the sake of simplicity when it is not confusing.) Indeed, the minimizer of (1.3) is linear in t (see e.g. [5], [10, p. 35]).

This implies that Monge's problem with a quadratic cost $L(u)=|u|^2$ should be the zero noise limit of *h-pass* processes (see [7, p. 566]), which enables us not to use Kantorovich's approach to study (1.1).

To make the point clearer, we introduce Schrödinger's functional equation and then describe the *h-pass* process briefly. For $\varepsilon>0$ and $x\in\mathbf{R}^d$, put

$$g_\varepsilon(x) := \frac{1}{\sqrt{2\pi\varepsilon}^{\,d}}\exp\Big(-\frac{|x|^2}{2\varepsilon}\Big), \tag{1.4}$$

$$P_{1,\varepsilon}(dy) := \Big(\int_{\mathbf{R}^d} g_\varepsilon(z-y)P_1(dz)\Big)dy. \tag{1.5}$$

The following is a special case of Schrödinger's functional equations: find nonnegative Borel measures $(\nu_{0,\varepsilon},\nu_{1,\varepsilon})$ for which

$$\begin{aligned} P_0(dx) &= \Big(\int_{\mathbf{R}^d} g_\varepsilon(x-y)\nu_{1,\varepsilon}(dy)\Big)\nu_{0,\varepsilon}(dx), \\ P_{1,\varepsilon}(dy) &= \Big(\int_{\mathbf{R}^d} g_\varepsilon(x-y)\nu_{0,\varepsilon}(dx)\Big)\nu_{1,\varepsilon}(dy). \end{aligned} \tag{1.6}$$

It is known that there exists a unique solution $(\nu_{0,\varepsilon}, \nu_{1,\varepsilon})$ to (1.6) (see [13], and also [22] for the recent development).

Let $(\Omega, \mathbf{B}, P)$ be a probability space, $\{\mathbf{B}_t\}_{t\geq 0}$ be a right continuous, increasing family of sub σ-fields of $\mathbf{B}$, X_o be a $\mathbf{R}^d$-valued, $\mathbf{B}_0$-adapted random variable such that $P(X_o)^{-1} = P_0$, and $\{W(t)\}_{t\geq 0}$ denote a d-dimensional $(\mathbf{B}_t)$-Wiener process (see e.g. [7], [10] or [12]).

The *h-pass* process on $[0,1]$ with an initial distribution P_0 and a terminal one $P_{1,\varepsilon}$, and with the diffusion matrix $\sqrt{\varepsilon}\times$(Identity matrix) is the unique weak solution to the following (see [14]): for $t \in [0,1]$,

$$X_\varepsilon(t) = X_o + \int_0^t b_\varepsilon(s, X_\varepsilon(s))ds + \sqrt{\varepsilon}W(t), \tag{1.7}$$

where

$$b_\varepsilon(s,x) := \varepsilon D_x \log\Big(\int_{\mathbf{R}^d} g_{\varepsilon(1-s)}(x-y)\nu_{1,\varepsilon}(dy)\Big) \quad ((s,x) \in [0,1) \times \mathbf{R}^d). \tag{1.8}$$

Here $D_x := (\partial/\partial x_i)_{i=1}^d$.

It is known that

$$P((X_\varepsilon(0), X_\varepsilon(1)) \in dxdy) = \mu_\varepsilon(dxdy) := \nu_{0,\varepsilon}(dx) g_\varepsilon(x-y)\nu_{1,\varepsilon}(dy). \tag{1.9}$$

It is also known that the minimizer of the following is the *h-pass* process in (1.7) (see [11]):

$$V_\varepsilon(P_0, P_{1,\varepsilon}) := \inf\Big\{E\Big[\int_0^1 |u(t)|^2 dt\Big]\Big\}, \tag{1.10}$$

where the infimum is taken over all $\mathbf{R}^d$-valued, $(\mathbf{B}_t)$-progressively measurable $\{u(t)\}_{0\leq t\leq 1}$ for which the distribution of $X_o + \int_0^1 u(s)ds + \sqrt{\varepsilon}W(1)$ is $P_{1,\varepsilon}$, provided that the right hand side of (1.10) is finite.

It seems likely that the limit of h-path processes as $\varepsilon \to 0$ is the minimizer of (1.3) with $L(u) = |u|^2$. But it is not trivial that this limit is a function of t and X_o since a continuous strong Markov process which is of bounded variation in time is not always a function of the initial point and time (see [23] and also [19]). Therefore we prove that the limit of $X_\varepsilon(1)$ as $\varepsilon \to 0$ is a function of X_o.

If $P_0(dx)$ is absolutely continuous with respect to dx and $L(u) = |u|^2$, then (1.1) and (1.2) have the unique minimizers $D\varphi(x)$ and $P_0(dx)\delta_{D\varphi(x)}(dy)$ respectively, where $\varphi : \mathbf{R}^d \mapsto (-\infty, \infty]$ is convex (see [3, 4], and also [8, 15, 16, 20, 21, 25] and the reference therein, and also [18, 19] for the continuum limit of (1.3)).

In this paper, independently of known results on the Monge-Kantorovich problem, we show that $V_\varepsilon(P_0, P_{1,\varepsilon})$ converges to $V(P_0, P_1)$ and $X_\varepsilon(1)$ converges, in L^2, to the minimizer of (1.1) as $\varepsilon \to 0$, when $L(u) = |u|^2$. As a by-product, we give a new proof of the existence of the minimizer of (1.1) with $L(u) = |u|^2$.

From a probabilistic interest, replacing $P_{1,\varepsilon}$ by P_1 in (1.6)-(1.10), we also show the similar result to above.

When $L(u) = |u|$, in [9] they studied (1.2) by the "$p \to \infty$" limit of the minimization problem for which the Euler-Lagrange equation is the p-Laplacian PDE under the assumption that P_0 and P_1 have disjoint compact supports, and in [6] and [24] they studied (1.2) by the "$q \downarrow 1$" limit of (1.2) with $L(u) = |u|^q$ under the assumption that P_0 and P_1 have compact supports (see also [1]).

In future we would like to study the zero noise limit of the minimizer of (1.10) with a more general cost function $L(u)$, instead of $|u|^2$, and then apply the result to Monge's problem.

In section 2 we give our main result which will be proved in section 3.

2 Main Result.

In this section we give our main result. We first state assumptions.
(A.0) P_0 and P_1 are Borel probability measures on $\mathbf{R}^d$ such that the following holds:

$$\int_{\mathbf{R}^d} |x|^2 (P_0(dx) + P_1(dx)) < \infty.$$

(A.1) $p_0(x) := P_0(dx)/dx$ exists.

Then the following holds.

Theorem 2. 1 *Suppose that (A.0) holds. Then $\{\mu_\varepsilon\}_{\varepsilon>0}$ is tight, and any weak limit point of $\{\mu_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0$ is supported on a cyclically monotone set.*

For the readers' convenience, we introduce the following.

Definition 2. 1 *The set $A \in \mathbf{R}^d \times \mathbf{R}^d$ is called cyclically monotone if for any $n \geq 1$ and any $(x_i, y_i) \in A$ $(i = 1, \cdots, n)$,*

$$\sum_{i=1}^{n} < y_i, x_{i+1} - x_i > \leq 0 \tag{2.1}$$

(see e. g. [25, p. 80]), where $x_{n+1} := x_1$, and $< \cdot, \cdot >$ denotes the inner product in $\mathbf{R}^d$.

Since a cyclically monotone set in $\mathbf{R}^d \times \mathbf{R}^d$ is contained in the subdifferential of a proper lower semicontinuous convex function on $\mathbf{R}^d$ and since a proper convex function is differentiable dx-a.e. in the interior of its domain (see [25, pp. 52, 82]), we obtain the following.

Corollary 2. 1 *Suppose that (A.0) and (A.1) hold. Then for any weak limit point μ of $\{\mu_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0$, there exists a proper lower semicontinuous convex function $\varphi : \mathbf{R}^d \mapsto (-\infty, \infty]$ such that*

$$\mu(dxdy) = P_0(dx)\delta_{D\varphi(x)}(dy). \tag{2.2}$$

Remark 2. 1 *If (A.1) holds and $p_1(y) := P_1(dy)/dy$ exists, then Corollary 2.1 gives a new proof of the existence to the following Monge-Ampère equation:*

$$p_0(x) = p_1(D\varphi(x)) \det(D^2\varphi(x)) \tag{2.3}$$

in the sense that $P_0(D\varphi)^{-1} = P_1$, where $D^2 := (\partial^2/\partial x_i \partial x_j)_{i,j=1}^d$.

The following which can be proved from Theorem 2.1 and Corollary 2.1, independently of known results on the Monge-Kantorovich problem [1, 3, 4, 15, 16, 21], is our main result.

Theorem 2. 2 *Suppose that (A.0) and (A.1) hold, and that $L(u) = |u|^2$. Then*

$$\lim_{\varepsilon \to 0} V_\varepsilon(P_0, P_{1,\varepsilon}) = V(P_0, P_1) < \infty. \tag{2.4}$$

In particular, $D\varphi$ in Corollary 2.1 is the unique minimizer of (1.1), and the following holds:

$$\lim_{\varepsilon\to 0} E[\int_0^1 |b_\varepsilon(t, X_\varepsilon(t)) - (D\varphi(X_o) - X_o)|^2 dt] = 0, \tag{2.5}$$

$$\lim_{\varepsilon\to 0} E[\sup_{0\le t\le 1} |X_\varepsilon(t) - \{X_o + t(D\varphi(X_o) - X_o)\}|^2] = 0. \tag{2.6}$$

The following is known on (1.1)-(1.3) with $L(u) = |u|^2$.
(i) Suppose that (A.0) holds. Then a probability measure supported on a cyclically monotone set in $\mathbf{R}^d \times \mathbf{R}^d$ is a minimizer of (1.2) (see [15, 16] and also [25, pp. 66, 82], [1, Theorem 3.2]).
(ii) Suppose that (A.0) and (A.1) hold. Then there exists a convex function φ such that $P_0(dx)\delta_{D\varphi(x)}(dy)$ is the unique minimizer of (1.2) (see [3, 4]).
Using these facts, we have the following.

Corollary 2. 2 *(i) Suppose that (A.0) holds and that $L(u) = |u|^2$. Then any weak limit point of $\{\mu_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0$ is a minimizer of (1.2). (ii) Suppose in addition that (A.1) holds. Then μ_ε weakly converges to the unique minimizer of (1.2) as $\varepsilon \to 0$, and $X_o + t(D\varphi(X_o) - X_o)$ in (2.6) is the unique minimizer of (1.3).*

To replace a terminal distribution $P_{1,\varepsilon}$ by P_1 in Theorems 2.1-2.2, we need extra assumptions.
(A.2) $p_1(x) := P_1(dx)/dx$ exists.
(A.3)

$$\int_{\mathbf{R}^d} \log\Big(\frac{P_1(dx)}{dx}\Big) P_1(dx) < \infty.$$

Repalce $P_{1,\varepsilon}$ by P_1 in (1.6)-(1.7). Then there exists the unique solution $\tilde{X}_\varepsilon(t)$ to (1.7) from (A.2), and $V_\varepsilon(P_0, P_1)$ is finite from (A.3) (see Lemma 3.4). Besides, the following holds.

Proposition 2. 1 *Suppose that (A.0)-(A.3) hold, and replace X_ε by $\tilde{X}_\varepsilon$ in (2.6). Then (2.6) still holds.*

3 Proof.

In this section we prove our results stated in section 2.

We first state and prove technical lemmas to prove Theorem 2.1. For x, $y \in \mathbf{R}^d$, $m \ge 1$ and $\varepsilon > 0$, put

$$\begin{aligned} H_{m,\varepsilon}(x,y) \quad &:= \quad \varepsilon \log\Big\{ \iint_{U_m(o)\times U_m(o)} \exp\Big(\frac{< x, z_2 > + < y, z_1 >}{\varepsilon} \\ &\qquad - \frac{< z_1, z_2 >}{\varepsilon} \Big) \mu_\varepsilon(dz_1 dz_2) \Big\}, \end{aligned} \tag{3.1}$$

$$H_{i,m,\varepsilon}(x) := \varepsilon \log\Big(\int_{U_m(o)} g_\varepsilon(x-y)\nu_{j,\varepsilon}(dy) \Big) + \frac{|x|^2}{2} \quad (i,j = 0,1,\ i \ne j), \tag{3.2}$$

$$\mu_{0,m,\varepsilon}(dx) := \mu_\varepsilon(dx \times U_m(o)), \quad \mu_{1,m,\varepsilon}(dy) := \mu_\varepsilon(U_m(o) \times dy), \tag{3.3}$$

where $U_m(o) := \{x \in \mathbf{R}^d : |x| < m\}$. Then the following holds.

Lemma 3. 1 *(i) For x, $y \in \mathbf{R}^d$, $m \ge 1$ and $\varepsilon > 0$,*

$$\begin{aligned} H_{m,\varepsilon}(x,y) \quad &= \quad H_{0,m,\varepsilon}(x) + H_{1,m,\varepsilon}(y) + \varepsilon \log \sqrt{2\pi\varepsilon}^{\,d} \\ &= \quad \varepsilon \log\Big\{ \iint_{U_m(o)\times U_m(o)} \exp\Big(\frac{< x, z_2 > + < y, z_1 >}{\varepsilon} \\ &\qquad - \frac{H_{m,\varepsilon}(z_1, z_2)}{\varepsilon} \Big) \mu_{0,m,\varepsilon}(dz_1)\mu_{1,m,\varepsilon}(dz_2) \}, \end{aligned} \tag{3.4}$$

$$\mu_\varepsilon(dxdy) = \exp\Big(\frac{1}{\varepsilon}(< x, y > - H_{m,\varepsilon}(x,y)) \Big) \mu_{0,m,\varepsilon}(dx)\mu_{1,m,\varepsilon}(dy), \tag{3.5}$$

provided that $\mu_\varepsilon(U_m(o) \times U_m(o)) > 0$.
(ii) For $m \ge 1$ and $\varepsilon > 0$, $(x,y) \mapsto H_{m,\varepsilon}(x,y)$ is convex, and for any x and $y \in \mathbf{R}^d$,

$$|H_{m,\varepsilon}(x,y)| \le (|x| + |y|)m + m^2 - \varepsilon \log \mu_\varepsilon(U_m(o) \times U_m(o)). \tag{3.6}$$

Proof. The first equality in (3.4) and (3.6) can be obtained from (3.1)-(3.2) easily. (3.5) holds from (1.9), the first equality in (3.4) and the following: for i, $j = 0$, 1 for which $i \neq j$,

$$\frac{\mu_{i,m,\varepsilon}(dx)}{\nu_{i,\varepsilon}(dx)} = \int_{U_m(o)} g_\varepsilon(x-y)\nu_{j,\varepsilon}(dy) = \exp\Big(\frac{1}{\varepsilon}\Big(H_{i,m,\varepsilon}(x) - \frac{|x|^2}{2}\Big)\Big). \quad (3.7)$$

The second equality in (3.4) can be obtained from (3.1) and (3.5).

Q. E. D.

Remark 3. 1 *For $x \in \mathbf{R}^d$, $m \geq 1$, $\varepsilon > 0$, and i, $j = 0$, 1 $(i \neq j)$,*

$$H_{i,m,\varepsilon}(x) = \varepsilon \log\Big(\int_{U_m(o)} \frac{1}{\sqrt{2\pi\varepsilon}^d} \exp\Big(\frac{1}{\varepsilon}(< x, y > - H_{j,m,\varepsilon}(y))\Big)\mu_{j,m,\varepsilon}(dy)\Big)$$

from (3.2) and (3.7).

Lemma 3. 2 *Suppose that (A.0) holds. Then for any sequence $\{\varepsilon_n\}_{n\geq 1}$ for which $\varepsilon_n \to 0$ as $n \to \infty$, there exist $m_0 \geq 1$ and subsequences $\{\{\varepsilon_{m,n}\}_{n\geq 1}\}_{m\geq m_0}$ such that $H_{m,\varepsilon_{m,n}}$ is convergent in $C(\mathbf{R}^d \times \mathbf{R}^d)$ as $n \to \infty$ for all $m \geq m_0$, and such that*

$$\{\varepsilon_{m+1,n}\}_{n\geq 1} \subset \{\varepsilon_{m,n}\}_{n\geq 1} \quad (m \geq m_0). \quad (3.8)$$

In particular, $m \mapsto H_m := \lim_{n\to\infty} H_{m,\varepsilon_{m,n}}$ is nondecreasing on $\{m_0, m_0 + 1, \cdots\}$,

$$(x, y) \mapsto H(x, y) := \lim_{m\to\infty} H_m(x, y) \in (-\infty, \infty] \quad (3.9)$$

is a lower semicontinuous convex function,

$$< x, y > -H(x, y) \leq 0 \quad ((x, y) \in supp(P_0) \times supp(P_1)), \quad (3.10)$$

and the following set is cyclically monotone:

$$S := \{(x, y) \in supp(P_0) \times supp(P_1) | < x, y >= H(x, y)\}. \quad (3.11)$$

Proof. There exist $m_0 \geq 1$ such that for any $m \geq m_0$, $\{H_{m,\varepsilon_n}\}_{n\geq 1}$ is bounded in $U_{\ell+1}(o) \times U_{\ell+1}(o)$ for any $\ell \geq 1$ from (3.6) and from the following:

$$
\begin{aligned}
& 1 - \mu_\varepsilon(U_m(o) \times U_m(o)) && (3.12)\\
\leq\ & \frac{\iint_{\mathbf{R}^d\times\mathbf{R}^d}(|x|^2+|y|^2)\mu_\varepsilon(dxdy)}{m^2} = \frac{\int_{\mathbf{R}^d}|x|^2P_0(dx)+\int_{\mathbf{R}^d}|y|^2P_{1,\varepsilon}(dy)}{m^2}\\
=\ & \frac{\int_{\mathbf{R}^d}|x|^2P_0(dx)+\iint_{\mathbf{R}^d\times\mathbf{R}^d}|x+y|^2 g_\varepsilon(x)dxP_1(dy)}{m^2}\\
\leq\ & \frac{\int_{\mathbf{R}^d}|x|^2P_0(dx)+2(\varepsilon d+\int_{\mathbf{R}^d}|y|^2P_1(dy))}{m^2} \to 0 \quad (\text{as } m\to\infty \text{ from (A.0)}).
\end{aligned}
$$

Hence for any $m \geq m_0$, $\{H_{m,\varepsilon_n}\}_{n\geq 1}$ contains a uniformly convergent subsequence on $U_\ell(o) \times U_\ell(o)$ (see [2, p. 21, Theorem 3.2]). By the diagonal argument, $\{H_{m,\varepsilon_n}\}_{n\geq 1}$ contains a convergent subsequence $\{H_{m,\varepsilon_{m,n}}\}_{n\geq 1}$ in $C(\mathbf{R}^d \times \mathbf{R}^d)$. In particular, we can take $\{\varepsilon_{m,n}\}_{n\geq 1}$ so that (3.8) holds.

$m \mapsto H_m$ is nondecreasing on $\{m_0, m_0+1, \cdots\}$ since

$$H_{m+1,\varepsilon_{m+1,n}} \geq H_{m,\varepsilon_{m+1,n}}$$

for all $m \geq m_0$ from (3.1), and since $H_{m,\varepsilon_{m+1,n}} \to H_m$ as $n \to \infty$ from (3.8). Hence for any $(x,y) \in \mathbf{R}^d \times \mathbf{R}^d$, $H_m(x,y)$ is convergent or diverges to ∞ as $m \to \infty$.

As the limit of convex functions, H in (3.9) is convex in $\mathbf{R}^d \times \mathbf{R}^d$. H is also lower semicontinuous. Indeed, if $(x_n, y_n) \to (x,y)$ as $n \to \infty$, then

$$
\begin{aligned}
H(x_n,y_n) \geq H_m(x_n,y_n) &\to H_m(x,y) \quad (\text{as } n\to\infty, \text{ for all } m \geq m_0)\\
&\to H(x,y) \quad (\text{as } m \to \infty)
\end{aligned}
$$

since $H_m \in C(\mathbf{R}^d \times \mathbf{R}^d)$ as a finite convex function (see (3.6)).

For any $(x,y) \in \operatorname{supp}(P_0) \times \operatorname{supp}(P_1)$, $r > 0$, $m \geq r + |x| + |y| + m_0$ and $n \geq 1$, from the second equality of (3.4),

$$
\begin{aligned}
& H_{m,\varepsilon_{m,n}}(x,y) && (3.13)\\
\geq\ & \inf_{(z_1,z_2)\in U_r(x)\times U_r(y)}\{< x, z_2 > + < y, z_1 > - H_{m,\varepsilon_{m,n}}(z_1,z_2)\}\\
& +\varepsilon \log\{\mu_{0,m,\varepsilon_{m,n}}(U_r(x))\mu_{1,m,\varepsilon_{m,n}}(U_r(y))\}.
\end{aligned}
$$

Since $H_{m,\varepsilon_{m,n}}$ converges to H_m as $n \to \infty$, uniformly on every compact subset of $\mathbf{R}^d \times \mathbf{R}^d$,

$$\begin{aligned} & \inf_{(z_1,z_2)\in U_r(x)\times U_r(y)} (< x, z_2 > + < y, z_1 > - H_{m,\varepsilon_{m,n}}(z_1, z_2)) \qquad (3.14) \\ \to & \inf_{(z_1,z_2)\in U_r(x)\times U_r(y)} (< x, z_2 > + < y, z_1 > - H_m(z_1, z_2)) \quad (\text{as } n \to \infty) \\ \to & \ 2 < x, y > - H_m(x, y) \quad (\text{as } r \to 0) \\ \to & \ 2 < x, y > - H(x, y) \quad (\text{as } m \to \infty). \end{aligned}$$

From (A.0), for sufficiently large $m \geq 1$,

$$\liminf_{\varepsilon\to 0} \{\mu_{0,m,\varepsilon}(U_r(x))\mu_{1,m,\varepsilon}(U_r(y))\} > 0. \qquad (3.15)$$

Indeed,

$$\begin{aligned} & \mu_{0,m,\varepsilon}(U_r(x))\mu_{1,m,\varepsilon}(U_r(y)) \\ = & \ \{P_0(U_r(x)) - \mu_\varepsilon(U_r(x) \times U_m(o)^c)\}\{P_{1,\varepsilon}(U_r(y)) - \mu_\varepsilon(U_m(o)^c \times U_r(y))\}. \end{aligned}$$

$$\mu_\varepsilon(U_r(x) \times U_m(o)^c) \leq \frac{1}{m^2} \int_{\mathbf{R}^d} |z|^2 P_{1,\varepsilon}(dz) \leq \frac{2(\varepsilon + \int_{\mathbf{R}^d} |z|^2 P_1(dz))}{m^2}$$

as in (3.12), and

$$\mu_\varepsilon(U_m(o)^c \times U_r(y)) \leq \frac{1}{m^2} \int_{\mathbf{R}^d} |z|^2 P_0(dz).$$

$$\liminf_{\varepsilon\to 0} P_{1,\varepsilon}(U_r(y)) \geq P_1(U_r(y))$$

since $P_{1,\varepsilon}$ weakly converges to P_1 as $\varepsilon \to 0$, and

$$(P_0 \times P_1)(U_r(x) \times U_r(y)) > 0.$$

(3.13)-(3.15) implies (3.10).

The set S is cyclically monotone. Indeed, for any $k, n \geq 1$, $(x_1, y_1), \cdots, (x_k, y_k) \in S$ and $m \geq m_0$, putting $x_{k+1} := x_1$,

$$\sum_{i=1}^{k}(H_{m,\varepsilon_{m,n}}(x_{i+1},y_i)-H_{m,\varepsilon_{m,n}}(x_i,y_i))=0 \tag{3.16}$$

from the first equality of (3.4). Let $n\to\infty$ and then $m\to\infty$. Then from (3.10),

$$\sum_{i=1}^{k} < y_i, x_{i+1}-x_i > \le \sum_{i=1}^{k}(H(x_{i+1},y_i)-H(x_i,y_i))=0. \tag{3.17}$$

(Notice that $H(x_i,y_i)$ is finite for all $i=1,\cdots,k$.)

Q. E. D.

Remark 3. 2 *If $H(x,y)$ and $H(a,b)$ are finite, then $H(x,b)$ and $H(a,y)$ are also finite since for sufficienlty large $m\ge 1$, from (3.9) and (3.16),*

$$-\infty < H_m(x,b)+H_m(a,y)\le H(x,b)+H(a,y)=H(x,y)+H(a,b)<\infty.$$

In particular,

$$H(x,y)=H(x,b)+H(a,y)-H(a,b).$$

(Proof of Theorem 2.1.) $\{\mu_\varepsilon\}_{\varepsilon>0}$ is tight from (3.12) (see e.g. [12, p. 7]). Take a weakly convergent subsequence $\{\mu_{\varepsilon_n}\}_{n\ge 1}$ and denote by μ its weak limit, where $\varepsilon_n\to 0$ as $n\to\infty$.

By taking $m_0\ge 1$ and subsequences $\{\varepsilon_{m,n}\}_{n\ge 1}$ $(m\ge m_0)$, construct a convex function H as in Lemma 3.2.

From (3.10)-(3.11), we only have to show the following to complete the proof:

$$\mu(\{(x,y)| < x,y > -H(x,y)<0\})=0. \tag{3.18}$$

By the monotone convergence theorem and Lemma 3.2,

$$\begin{aligned} & \mu(\{(x,y)| < x,y > -H(x,y)<0\}) \qquad (3.19)\\ = & \lim_{r\downarrow 0}(\lim_{m\uparrow\infty}\mu(\{(x,y)| < x,y > -H_m(x,y)< -r\})). \end{aligned}$$

For any $m \geq m_0$, $H_{m,\varepsilon_{m,n}}$ converges to H_m as $n \to \infty$, uniformly on every compact subset of $\mathbf{R}^d \times \mathbf{R}^d$. Therefore for any $R > 0$,

$$\begin{aligned}
&\mu(\{(x,y)| < x,y > -H_m(x,y) < -r, |x|,|y| < R\}) \qquad (3.20)\\
\leq\ & \liminf_{n\to\infty} \mu_{\varepsilon_{m,n}}(\{(x,y)| < x,y > -H_m(x,y) < -r, |x|,|y| < R\})\\
\leq\ & \liminf_{n\to\infty} \mu_{\varepsilon_{m,n}}(\{(x,y)| < x,y > -H_{m,\varepsilon_{m,n}}(x,y) < -r/2, |x|,|y| < R\})\\
\leq\ & \liminf_{n\to\infty} \exp\Big(-\frac{r}{2\varepsilon_{m,n}}\Big) = 0 \quad \text{(from (3.5))}.
\end{aligned}$$

Notice that the set $\{(x,y)| < x,y > -H_m(x,y) < -r, |x|,|y| < R\}$ is open since $H_m \in C(\mathbf{R}^d \times \mathbf{R}^d)$ from Lemma 3.1, (ii).

Letting $R \to \infty$ in (3.20), we obtain (3.18) from (3.19).

Q. E. D.

Next we prove Theorem 2.2.

(Proof of Theorem 2.2). The proof of (2.4) is devided into the following:

$$\liminf_{\varepsilon\to 0} V_\varepsilon(P_0, P_{1,\varepsilon}) \geq V(P_0,P_1), \qquad (3.21)$$

$$\limsup_{\varepsilon\to 0} V_\varepsilon(P_0, P_{1,\varepsilon}) \leq V(P_0,P_1) < \infty. \qquad (3.22)$$

To prove (3.21), we only have to show that for any $\{\varepsilon_n\}_{n\geq 1}$ for which $\varepsilon_n \to 0$ and $E[\int_0^1 |b_{\varepsilon_n}(s, X_{\varepsilon_n}(s))|^2 ds]$ is convergent as $n \to \infty$,

$$\lim_{n\to\infty} E[\int_0^1 |b_{\varepsilon_n}(s, X_{\varepsilon_n}(s))|^2 ds] \geq V(P_0,P_1) \qquad (3.23)$$

(see (1.7) for notation). (3.23) holds since $\{X_{\varepsilon_n}(\cdot)\}_{n\geq 1}$ is tight in $C([0,1])$, any weak limit point $X(\cdot)$ of $\{X_{\varepsilon_n}(\cdot)\}_{n\geq 1}$ is an absolutely continuous stochastic process (see e.g. [19, Lemmas 2-3]), and

$$\begin{aligned}
&\lim_{n\to\infty} E[\int_0^1 |b_{\varepsilon_n}(s, X_{\varepsilon_n}(s))|^2 ds] \qquad (3.24)\\
\geq\ & E[\int_0^1 \Big|\frac{dX(s)}{ds}\Big|^2 ds] \geq E[|X(1)-X(0)|^2] \geq V(P_0,P_1)
\end{aligned}$$

from (1.9) and (2.2) (see e.g. [19, the proof of (3.17)]).

Next we prove (3.22). Take ψ for which $P_0\psi^{-1} = P_1$, which is possible from (2.2). Then from (A.0),

$$V(P_0, P_1) \leq E[|\psi(X_o) - X_o|^2] \leq 2\int_{\mathbf{R}^d} |x|^2(P_0(dx) + P_1(dx)) < \infty. \quad (3.25)$$

Put

$$X_{\varepsilon,\psi}(t) := X_o + t(\psi(X_o) - X_o) + \sqrt{\varepsilon}W(t). \quad (3.26)$$

Then $P(X_{\varepsilon,\psi}(1))^{-1} = P_{1,\varepsilon}$, which implies (3.22).

By (2.2), (2.4) and (3.24), $D\varphi$ in Corollary 2.1 is a minimizer of (1.1) with $L(u) = |u|^2$. In particular, $D\varphi$ is the unique minimizer. Indeed, if ψ is a minimizer of (1.1) with $L(u) = |u|^2$, then

$$\begin{aligned} & E[< X_o, \psi(X_o) >] = E[< X_o, D\varphi(X_o) >] \\ = \ & E[\varphi(X_o) + \varphi^*(D\varphi(X_o))] = E[\varphi(X_o) + \varphi^*(\psi(X_o))], \end{aligned}$$

which implies that $\psi(X_o) \in \partial\varphi(X_o)$ a.s., where

$$\varphi^*(y) := \sup_{x \in \mathbf{R}^d} \{< x, y > -\varphi(x)\},$$

$$\partial\varphi(x) := \{p \in \mathbf{R}^d | \varphi(y) \geq \varphi(x)+ < p, y - x > \text{ for all } y \in \mathbf{R}^d\}.$$

Here we used the fact that for any $(x, y) \in \mathbf{R}^d \times \mathbf{R}^d$,

$$< x, y > \leq \varphi(x) + \varphi^*(y),$$

where the equality holds if and only if $y \in \partial\varphi(x)$ (see e.g. [25]). From (A.1), $\psi(X_o) = D\varphi(X_o)$ a.s. since a proper convex function is differentiable dx-a.e. in the interior of its domain (see [25, pp. 52]).

(2.5)-(2.6) is an easy consequence of (2.4). For $t \in [0, 1]$,

$$\begin{aligned} & |X_\varepsilon(t) - \{X_o + t(D\varphi(X_o) - X_o)\}| \quad (3.27) \\ \leq \ & \int_0^1 |b_\varepsilon(s, X_\varepsilon(s)) - (D\varphi(X_o) - X_o)|ds + \sqrt{\varepsilon} \sup_{0 \leq t \leq 1} |W(t)|. \end{aligned}$$

$$E[\sup_{0\le t\le 1}|W(t)|^2]\le 4d \tag{3.28}$$

(see e.g. [12, p. 34]), and from (2.4),

$$\begin{aligned}
&E[\int_0^1 |b_\varepsilon(s,X_\varepsilon(s))-(D\varphi(X_o)-X_o)|^2 ds] \qquad (3.29)\\
=\ &E[\int_0^1 |b_\varepsilon(s,X_\varepsilon(s))|^2 ds+|D\varphi(X_o)-X_o|^2]\\
&\quad -2E[<X_\varepsilon(1)-X_o-\sqrt{\varepsilon}W(1),D\varphi(X_o)-X_o>]\\
\to\ &2V(P_0,P_1)-2E[<D\varphi(X_o)-X_o,D\varphi(X_o)-X_o>]=0 \quad \text{as } \varepsilon\to 0.
\end{aligned}$$

Indeed,

$$E[<W(1),D\varphi(X_o)-X_o>]=<E[W(1)],E[D\varphi(X_o)-X_o]>=0.$$

For any $R>0$, taking $f_R\in C(\mathbf{R}^d:[0,1])$ for which $f_R(x)=1$ ($|x|\le R$) and $f_R(x)=0$ ($|x|\ge R+1$),

$$\begin{aligned}
&E[<X_\varepsilon(1),D\varphi(X_o)-X_o>]\\
=\ &E[<X_\varepsilon(1),D\varphi(X_o)-X_o>(1-f_R(X_\varepsilon(1))f_R(X_o))]\\
&\quad +E[<X_\varepsilon(1),D\varphi(X_\varepsilon(0))-X_\varepsilon(0)>f_R(X_\varepsilon(1))f_R(X_\varepsilon(0))].
\end{aligned}$$

$$\begin{aligned}
&E[|<X_\varepsilon(1),D\varphi(X_o)-X_o>|(1-f_R(X_\varepsilon(1))f_R(X_o))]\\
\le\ &\sqrt{E[|D\varphi(X_o)-X_o|^2]E[|X_\varepsilon(1)|^2:|X_\varepsilon(1)|\ge R]}\\
&\quad +\sqrt{E[|X_\varepsilon(1)|^2]E[|D\varphi(X_o)-X_o|^2:|X_o|\ge R]}\to 0 \quad \text{as } R\to 0
\end{aligned}$$

uniformly in $\varepsilon\in[0,1]$. Since $(X_\varepsilon(0),X_\varepsilon(1))$ weakly converges to $(X_o,D\varphi(X_o))$ as $\varepsilon\to 0$ by the uniqueness of the minimizer of $V(P_0,P_1)$, one can assume, by taking a new probability space $(\tilde\Omega,\tilde{\mathbf{B}},\tilde P)$, that $(X_\varepsilon(0),X_\varepsilon(1))$ converges to $(X_o,D\varphi(X_o))$ as $\varepsilon\to 0$, $\tilde P$-a.s., by Skhorohod's theorem (see e.g. [12, p. 9]). Put

$$A := \{y \in \mathbf{R}^d | \varphi(y) < \infty, \partial\varphi(y) = \{D\varphi(y)\}\}.$$

Then $X_o \in A$ a.s. from (A.1) and $\cap_{r>0}\partial\varphi(U_r(x)) = \{D\varphi(x)\}$ for any $x \in A$ (see [25, p. 54]), from which the following holds:

$$\begin{aligned}
& E[< X_\varepsilon(1), D\varphi(X_\varepsilon(0)) - X_\varepsilon(0) > f_R(X_\varepsilon(1)) f_R(X_\varepsilon(0))] \\
= \; & \tilde{E}[< X_\varepsilon(1), D\varphi(X_\varepsilon(0)) - X_\varepsilon(0) > f_R(X_\varepsilon(1)) f_R(X_\varepsilon(0)) : X_o \in A] \\
\to \; & \tilde{E}[< D\varphi(X_o), D\varphi(X_o) - X_o > f_R(D\varphi(X_o)) f_R(X_o) : X_o \in A] \\
& \quad (\text{as } \varepsilon \to 0) \\
\to \; & E[< D\varphi(X_o), D\varphi(X_o) - X_o >] \quad (\text{as } R \to \infty).
\end{aligned}$$

(3.27)-(3.29) implies (2.5)-(2.6).

Q. E. D.

We give technical lemmas and then prove Proposition 2.1.

Lemma 3. 3 *(see [17, Lemma 2.5]). Suppose that (A.2) holds and replace $P_{1,\varepsilon}$ by P_1 in (1.6). Then for any $\varepsilon > 0$,*

$$V_\varepsilon(P_0, P_1) = 2\varepsilon E\Big[\log \frac{h_\varepsilon(1, \tilde{X}_\varepsilon(1))}{h_\varepsilon(0, \tilde{X}_\varepsilon(0))}\Big], \tag{3.30}$$

where $\tilde{X}_\varepsilon$ is the unique weak solution to (1.7),

$$h_\varepsilon(t, x) := \int_{\mathbf{R}^d} g_{\varepsilon(1-t)}(x - y)\tilde{\nu}_{1,\varepsilon}(dy),$$

and $(\tilde{\nu}_{0,\varepsilon}, \tilde{\nu}_{1,\varepsilon})$ is a solution to (1.6). $V_\varepsilon(P_0, P_1)$ is also the infimum of

$$\int_0^1 \int_{\mathbf{R}^d} |b(t, x)|^2 q(t, x) dt dx \tag{3.31}$$

over all (b, q) for which

$$q(t, x) \geq 0 \quad dx - a.e., \quad \int_{\mathbf{R}^d} q(t, x) dx = 1 \quad \textit{for all } t \in [0, 1], \tag{3.32}$$

$$q(0, x)dx = P_0(dx), \quad q(1, x)dx = P_1(dx), \tag{3.33}$$

and for which the following holds: for any $f \in C_o^\infty(\mathbf{R}^d)$ and any $t \in [0,1]$,

$$\begin{aligned}&\int_{\mathbf{R}^d} f(x)(q(t,x)-q(0,x))dx \\ = &\int_0^t ds \int_{\mathbf{R}^d}\Big(\frac{\varepsilon}{2}\triangle f(x) + < b(t,x), Df(x) >\Big)q(s,x)dx,\end{aligned} \tag{3.34}$$

where $\triangle := \sum_{i=1}^d \partial^2/\partial x_i^2$.

Remark 3. 3 *Suppose that (A.1) and (A.2) hold and that $supp(P_0) \cup supp(P_1)$ is bounded. Then it is known that $\tilde{V}(P_0,P_1)$ is the infimum of (3.31) over all (b,q) for which (3.32)-(3.34) hold for $\varepsilon = 0$ and for which $\cup_{0\le t\le 1} supp(q(t,\cdot))$ is bounded (see [5] or [25, p. 239]).*

Lemma 3. 4 *Suppose that (A.0), (A.2) and (A.3) hold. Then for any $\varepsilon > 0$, $V_\varepsilon(P_0,P_1)$ is finite. In particular, $V_1(P_{1,1},P_1)$ is finite.*

Proof. Put $\tilde{\mu}_\varepsilon(dxdy) := \tilde{\nu}_{0,\varepsilon}(dx)g_\varepsilon(x-y)\tilde{\nu}_{1,\varepsilon}(dy)$ (see (3.30) for notation). Replace μ_ε by $\tilde{\mu}_\varepsilon$ in (3.1) and denote by $\tilde{H}_{m,\varepsilon}$ a function obtained from (3.1). Then, from (3.4), (3.7) and (3.30),

$$\begin{aligned}V_\varepsilon(P_0,P_1) &= E[|\tilde{X}_\varepsilon(0)|^2 + |\tilde{X}_\varepsilon(1)|^2 - 2\tilde{H}_{\infty,\varepsilon}(\tilde{X}_\varepsilon(0),\tilde{X}_\varepsilon(1))] \\ &\quad +2\varepsilon\int_{\mathbf{R}^d}\log\Big(\frac{dP_1}{dx}\Big)P_1(dx) + 2\varepsilon\log\sqrt{2\pi\varepsilon}^d.\end{aligned} \tag{3.35}$$

From (3.1), (3.6), (3.12) and (A.0), for sufficiently large $m \ge 1$,

$$E[\tilde{H}_{\infty,\varepsilon}(\tilde{X}_\varepsilon(0),\tilde{X}_\varepsilon(1))] \ge E[\tilde{H}_{m,\varepsilon}(\tilde{X}_\varepsilon(0),\tilde{X}_\varepsilon(1))] > -\infty. \tag{3.36}$$

Q. E. D.

(Proof of Proposition 2.1). Most part of the proof is almost the same as that of Theorem 2.2. The only thing we have to prove is the following:

$$\limsup_{\varepsilon\to 0} V_\varepsilon(P_0,P_1) \le V(P_0,P_1). \tag{3.37}$$

Take ψ for which $P_0\psi^{-1} = P_1$, which is possible from (2.2). For $r \in (0, 1/2)$, solve Schrödinger's functional equation:

$$\begin{aligned} P_{1,\varepsilon(1-r)}(dx) &= \Big(\int_{\mathbf{R}^d} g_{\varepsilon r}(x-y)\nu_{1,r,\varepsilon}(dy)\Big)\nu_{0,r,\varepsilon}(dx), \\ P_1(dy) &= \Big(\int_{\mathbf{R}^d} g_{\varepsilon r}(x-y)\nu_{0,r,\varepsilon}(dx)\Big)\nu_{1,r,\varepsilon}(dy). \end{aligned} \tag{3.38}$$

For $t \in [0, 1-r]$, put

$$X_{r,\varepsilon}(t) := X_o + t\frac{\psi(X_o) - X_o}{1-r} + \sqrt{\varepsilon}W(t), \tag{3.39}$$

and solve the following: for $t \in [1-r, 1]$

$$X_{r,\varepsilon}(t) = X_{r,\varepsilon}(1-r) + \int_{1-r}^t b_{r,\varepsilon}(s, X_{r,\varepsilon}(s))ds + \sqrt{\varepsilon}(W(t) - W(1-r)), \tag{3.40}$$

where

$$b_{r,\varepsilon}(s,x) := D_x \log\Big(\int_{\mathbf{R}^d} g_{\varepsilon(1-s)}(x-y)\nu_{1,r,\varepsilon}(dy)\Big).$$

Then, from Lemma 3.3,

$$V_\varepsilon(P_0, P_1) \le \frac{E[|\psi(X_o) - X_o|^2]}{1-r} + E[\int_{1-r}^1 |b_{r,\varepsilon}(s, X_{r,\varepsilon}(s))|^2 ds] \tag{3.41}$$

since $X_{r,\varepsilon}(0) = X_o$ and $PX_{r,\varepsilon}(1)^{-1} = P_1$.

We prove the following to complete the proof: for any $r \in (0, 1/2)$,

$$\lim_{\varepsilon\to 0} E[\int_{1-r}^1 |b_{r,\varepsilon}(s, X_{r,\varepsilon}(s))|^2 ds] = 0. \tag{3.42}$$

$$p_{r,\varepsilon}(t,x) := \int_{\mathbf{R}^d} g_{\frac{\varepsilon(1-r)(1-t)}{r}}(x-y)P_1(dy) \tag{3.43}$$

is a weak solution to the following: for $t \in [1-r, 1)$,

$$\frac{\partial p_{r,\varepsilon}(t,x)}{\partial t} = \frac{\varepsilon}{2}\triangle p_{r,\varepsilon}(t,x) - div\Big\{\Big(\frac{\varepsilon}{2r}\Big)\frac{D_x p_{r,\varepsilon}(t,x)}{p_{r,\varepsilon}(t,x)} p_{r,\varepsilon}(t,x)\Big\}. \tag{3.44}$$

Hence, from Lemmas 3.3 and 3.4, for $\varepsilon < 1$,

$$
\begin{aligned}
& E[\int_{1-r}^{1} |b_{r,\varepsilon}(s, X_{r,\varepsilon}(s))|^2 ds] \qquad (3.45) \\
\leq \; & \int_{1-r}^{1} dt \int_{\mathbf{R}^d} \left| \left(\frac{\varepsilon}{2r}\right) \frac{D_x p_{r,\varepsilon}(t,x)}{p_{r,\varepsilon}(t,x)} \right|^2 p_{r,\varepsilon}(t,x) dx \\
= \; & \frac{\varepsilon}{4r(1-r)} \int_{1-\varepsilon(1-r)}^{1} ds \int_{\mathbf{R}^d} \left| \frac{D_x p_{\frac{1}{2},1}(s,x)}{p_{\frac{1}{2},1}(s,x)} \right|^2 p_{\frac{1}{2},1}(s,x) dx \to 0 \quad (\text{as } \varepsilon \to 0),
\end{aligned}
$$

where we used the following change of variable:

$$
\frac{\varepsilon(1-r)(1-t)}{r} = 1 - s,
$$

and the following:

$$
\begin{aligned}
& \int_{1-\varepsilon(1-r)}^{1} ds \int_{\mathbf{R}^d} \left| \frac{D_x p_{\frac{1}{2},1}(s,x)}{p_{\frac{1}{2},1}(s,x)} \right|^2 p_{\frac{1}{2},1}(s,x) dx \\
\leq \; & \int_{0}^{1} ds \int_{\mathbf{R}^d} \left| \frac{D_x p_{\frac{1}{2},1}(s,x)}{p_{\frac{1}{2},1}(s,x)} \right|^2 p_{\frac{1}{2},1}(s,x) dx = V_1(P_{1,1}, P_1) < \infty.
\end{aligned}
$$

Q. E. D.

References

[1] Ambrosio L and Pratelli A, Existence and stability results in the L^1 theory of optimal transportation, preprint.

[2] Bakelman IJ, Convex Analysis and Nonlinear Geometric Elliptic Equations (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1994).

[3] Brenier Y, Décomposition polaire et réarrangement monotone des champs de vecteurs, C. R. Acad. Sci. Paris S屍. I Math. 305, (1987) 805–808.

[4] Brenier Y, Polar factorization and monotone rearrangement of vector-valued functions, Comm. Pure Appl. Math. 44, (1991) 375–417.

[5] Brenier Y and Benamou JD, A numerical method for the optimal mass transport problem and related problems, Contemporary Mathematics 226, (1999) 1–11.

[6] Caffarelli LA, Feldman M and McCann RJ, Constructing optimal maps for Monge's transport problem as a limit of strictly convex costs, J. of the Amer. Math. Soc. 15, (2001) 1–26.

[7] Doob JL, Classical potential theory and its probabilistic counterpart (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984).

[8] Evans LC, Partial differential equations and Monge-Kantorovich mass transfer, Current developments in mathematics, 1997 (Cambridge, MA) (Int. Press, Boston, MA, 1999) 65–126.

[9] Evans LC and Gangbo W, Differential equations methods for the Monge-Kantorovich mass transfer problem, Mem. Amer. Math. Soc. 137, (1999) No. 653.

[10] Fleming WH and Soner HM, Controlled Markov Processes and Viscosity Solutions (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1993).

[11] Föllmer H, Random fields and diffusion processes, in École d'Été de Probabilités de Saint-Flour XV–XVII, 1985–87 (Hennequin PL, ed.), Lecture Notes in Math. 1362 (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1988) 101–203.

[12] Ikeda N and Watanabe S, Stochastic differential equations and diffusion processes (North-Holland/Kodansha, Amsterdam, New York, Oxford, Tokyo, 1981).

[13] Jamison B, Reciprocal processes, Z. Wahrsch. Verw. Gebiete 30, (1974) 65–86.

[14] Jamison B, The Markov process of Schrödinger, Z. Wahrsch. Verw. Gebiete 32, (1975) 323–331.

[15] Knott M and Smith CS, On the optimal mapping of distributions, J. of Optimization Theory and Applications 43, (1984) 39–49.

[16] Knott M and Smith CS, Note on the optimal transportation of distributions, J. of Optimization Theory and Applications 52, (1987) 323–329.

[17] Mikami T, Variational processes from the weak forward equation, Commun. Math. Phys. 135, (1990) 19–40.

[18] Mikami T, Dynamical systems in the variational formulation of the Fokker-Planck equation by the Wasserstein metric, Appl. Math. Optim. 42, (2000) 203–227.

[19] Mikami T, Optimal control for absolutely continuous stochastic processesand the mass transportation problem, Elect. Comm. in Probab. 7, (2002) 199–213.

[20] Rachev ST and Rüschendorf L, Mass transportation problems, Vol. I: Theory (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1999).

[21] Rüschendorf L and Rachev ST, A characterization of random variables with minimum L^2-distance, J. of Multivarite Analysis 32, (1990) 48–54.

[22] Rüschendorf L and Thomsen W, Note on the Schrödinger equation and I-projections, Statist. Probab. Lett. 17, (1993) 369–375.

[23] Salisbury TS, An increasing diffusion, in Seminar on Stochastic Processes 1984 (Cinlar E, Chung KL, Getoor RK, eds.) (Birkhäuser, Boston, MA, 1986) 173–194.

[24] Trudinger NS and Wang Xu-Jia, On the Monge mass transfer problem, Calc. Var. 13, (2001) 19–31.

[25] Villani C, Topics in Optimal Transportation, Graduate Studies in Mathematics Vol. 58 (Amer. Math. Soc., Providence, RI, 2003).