# Momentum operators with a winding gauge potential

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# Momentum operators with a winding gauge potential

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Considered is a quantum system of $N(\geq 2)$ charged particles moving in the plane $\mathbf{R}^2$ under the influence of a perpendicular magnetic field. Each particle feels the magnetic field concenrated in the positions of the other particles. The gauge potential which gives this magnetic field is called a winding gauge potential. Properties of the momentum operators with the winding gauge potential are investigated. The momentum operators with the winding gauge potential are represented by the fibre direct integral of Arai's momentum operators [1]. Using this fibre direct integral decomposition, commutation properties of the momentum operators are investigated. A notion of local quantization of the magnetic flux is introduced to characterize the strong commutativity of the momentum operators. Aspects of the representation of the canonical commutation relations (CCR) are discussed. There is an interesting relation between the representation of the CCR with respect to this system and Arai's representation. Some applications of those results are also discussed.

## 1 Introduction

In Ref.[1, 2, 3, 4], A. Arai investigated commutation properties of two dimensional momentum operators with a strongly singular gauge potential. In those papers, he showed some interesting results. Especially, there exsists a beautiful relation between representations of the canonical commutation relations (CCR) and the local quantization of the magnetic flux.

The main aim of this paper is to analyze a quantum system of $N(\geq 2)$ particles moving in $\mathbf{R}^2$ under the influence of a special perpendicular magnetic field. In this system, each particle feels the magnetic field concenrated in the positions of the other particles. The gauge potential which gives this magnetic field is said to be a winding gauge potential, and is strongly singular. We show that the momentum operators with the winding gauge potential can be represented as the direct integral of Arai's momentum operator. In this sense, our result is a natural extension of Arai's work[1, 2, 3, 4]. It is also important whether the magnetic flux is locally quantized or not. Indeed, we see that the local quantization of the magnetic flux is closely connected with the Schrödinger representation of the CCR's. As an application of those results, we study a class of Schrödinger operators with the winding gauge potential. Moreover, there are some other important properties about this system. In particular, we see that there is an interesting correspondence between bosons and fermions, so called the statistical transformation.

The outline of the present paper is as follow. In Section 2, we introduce the winding gauge potential and show the self-adjointness of the momentum operators with the winding gauge potential. We also investigate the commutation relations (in the strong sense) of the momentum operators with the winding gauge potential. To do this, we express the momentum operators as direct integrals of Arai's momentum operator. By using this expression, we prove that the momentum operators strongly commute if and only if the magnetic flux is locally quantized. In Section 3, we apply the preceding results to the theory of reperesentaion of the CCR. We show that the momentum operators with the winding gauge potential and the position operators fulfills the Weyl relation if and only if the magnetic flux is locally quantized. Furthermore, we discuss a relation between direct integral representation of Arai's representation of the CCR and our system. In Section 4, we define the Hamiltonian with the winding gauge potential and investigate the properties of this Hamiltonian. We note that formal discussion of this Hamiltonian is foud in Ref [5, 6]. Moreover, we introduce the statistical transformation and disscus some applications. This transformation gives the correspondence between bosons and fermions, and comes from the two dimensionality of the sytem.

# 2 Momentum operators with the winding gauge potential

## 2.1 Definition of the momentum operators with the winding gauge potential

We consider a quantum system of $N(\geq 2)$ charged particles with charge $q \in \mathbf{R}\backslash\{0\}$, where each particle feels a perpendicular magnetic field $B_j$ $(j = 1, \cdots, N)$ given by a real distribution of the form

$$B_j(\mathbf{r}_1, \cdots, \mathbf{r}_N) = \gamma \sum_{i \neq j} \delta(\mathbf{r}_i - \mathbf{r}_j), \quad \mathbf{r}_1, \cdots, \mathbf{r}_N \in \mathbf{R}^2, \quad \mathbf{r}_j = (x_j, y_j),$$

where $\gamma \in \mathbf{R}$ and $\delta(\mathbf{r})$ is the Dirac's delta distribution. Gauge potentials $\mathbf{A}_j$ $(j = 1, \cdots, N)$ of the magnetic field $B_j$ are defined to be $\mathbf{R}^2$-valued functions $\mathbf{A}_j = (A_{j1}, A_{j2})$ on the domain

$$\mathcal{M}_N := \left\{ (\mathbf{r}_1, \cdots, \mathbf{r}_N) \in \mathbf{R}^{2N} \;\middle|\; \mathbf{r}_i \neq \mathbf{r}_j \ (i \neq j) \right\}$$

such that

$$B_j(\mathbf{r}_1, \cdots, \mathbf{r}_N) = D_{x_j} A_{j2} - D_{y_j} A_{j1}$$

in the sense of distribution on $\mathbf{R}^{2N}$, where $D_{x_j}$ and $D_{y_j}$ denote the distribution partial differential operators in $x_j$ and $y_j$, respectively.

We denote by $\Delta_j$ $(j = 1, \cdots, N)$ the Laplacian

$$\Delta_j := D_{x_j}^2 + D_{y_j}^2.$$

Using the well-known formula

$$\Delta_j \log |\mathbf{r}_j - \mathbf{r}_k| = 2\pi\delta(\mathbf{r}_j - \mathbf{r}_k) \ (k \neq j),$$

we see that the distribution

$$\phi_N(\mathbf{r}_1, \cdots, \mathbf{r}_N) := \sum_{i<j} \frac{\gamma}{2\pi} \log |\mathbf{r}_i - \mathbf{r}_j|$$

satisfies

$$\Delta_j \phi_N(\mathbf{r}_1, \cdots, \mathbf{r}_N) = B_j(\mathbf{r}_1, \cdots, \mathbf{r}_N).$$

From this fact, we can take as a gauge potential of the magnetic field

$$\mathbf{A}_j = (A_{j1}, A_{j2}) = (-D_{y_j}\phi_N, D_{x_j}\phi_N), \quad j = 1, \cdots, N.$$

Explicitely, we have

$$A_{j1}(\mathbf{r}_1, \cdots, \mathbf{r}_N) = -\frac{\gamma}{2\pi}\sum_{i\neq j}\frac{y_j - y_i}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \tag{1}$$

$$A_{j2}(\mathbf{r}_1, \cdots, \mathbf{r}_N) = \frac{\gamma}{2\pi}\sum_{i\neq j}\frac{x_j - x_i}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \tag{2}$$

for a.e. $(\mathbf{r}_1, \cdots, \mathbf{r}_N) \in \mathcal{M}_N$.

**Definition 2.1** Let $\mathbf{A}_j = (A_{j1}, A_{j2})$ $(j = 1, \cdots N)$ be given by (1) and (2). The mapping $\mathbf{A} = (\mathbf{A}_1, \cdots, \mathbf{A}_n) : \mathcal{M}_N \to \mathbf{R}^{2N}$ is called *a winding gauge potential.*

We use a system of units where the light speed $c$ and the Planck constant $\hbar$ are equal to 1. Let

$$p_{j1} := -\mathrm{i}D_{x_j}, \quad p_{j2} := -\mathrm{i}D_{y_j} \quad (j = 1, \cdots, N),$$

in $L^2(\mathbf{R}^{2N})$. The momentum operator $\mathbf{P}_j = (P_{j1},\ P_{j2})$ with the gauge potential $\mathbf{A}_j$ is defined by

$$P_{j\alpha} := p_{j\alpha} - qA_{j\alpha}, \quad (j = 1, \cdots, N, \ \alpha = 1, 2)$$

in $L^2(\mathbf{R}^{2N})$ with domain $\mathrm{dom}(P_{j\alpha}) = \mathrm{dom}(p_{j\alpha}) \cap \mathrm{dom}(A_{j\alpha})$.

## 2.2 Self-adjointness

Let

$$\mathcal{S}_1^{(N)} := \left\{(\mathbf{r}_1, \cdots, \mathbf{r}_N) \in \mathbf{R}^{2N} \,\middle|\, \mathbf{r}_i = (x_i, y_i),\ y_i \neq y_j\ (i \neq j)\right\},$$

$$\mathcal{S}_2^{(N)} := \left\{(\mathbf{r}_1, \cdots, \mathbf{r}_N) \in \mathbf{R}^{2N} \,\middle|\, \mathbf{r}_i = (x_i, y_i),\ x_i \neq x_j\ (i \neq j)\right\}$$

and

$$\psi_{j1}(\mathbf{r}_1, \cdots, \mathbf{r}_N) := -\frac{\gamma}{2\pi}\sum_{i\neq j}\mathrm{Arctan}\Big(\frac{x_j - x_i}{y_j - y_i}\Big),$$

$$\psi_{j2}(\mathbf{r}_1, \cdots, \mathbf{r}_N) := \frac{\gamma}{2\pi}\sum_{i\neq j}\mathrm{Arctan}\Big(\frac{y_j - y_i}{x_j - x_i}\Big).$$

Then it is easy to check that $\psi_{j\alpha} \in C^\infty(\mathcal{S}_\alpha^{(N)})$ and

$$A_{j1} = D_{x_j}\psi_{j1} \text{ on } \mathcal{S}_1^{(N)}, \tag{3}$$

$$A_{j2} = D_{y_j}\psi_{j2} \text{ on } \mathcal{S}_2^{(N)}. \tag{4}$$

**Theorem 2.2** *For each $j = 1, \cdots, N$ and $\alpha = 1, 2$, $P_{j\alpha}$ is essentially self-adjoint on $C^\infty(\mathcal{S}_\alpha^{(N)})$.*

*Proof.* Since $\psi_{j\alpha} \in C^\infty(\mathcal{S}_\alpha^{(N)})$, $\mathrm{e}^{\mathrm{i}q\psi_{j\alpha}}$ is a unitary operator such that

$$\mathrm{e}^{\mathrm{i}q\psi_{j\alpha}} C_0^\infty(\mathcal{S}_\alpha^{(N)}) = C_0^\infty(\mathcal{S}_\alpha^{(N)}).$$

By (3), we have

$$P_{j\alpha} = \mathrm{e}^{\mathrm{i}q\psi_{j\alpha}} p_{j\alpha} \mathrm{e}^{-\mathrm{i}q\psi_{j\alpha}} \quad \text{on } C_0^\infty(\mathcal{S}_\alpha^{(N)}).$$

On the other hand, it is easy to see that $p_{j\alpha}$ is essentially self-adjoint on $C_0^\infty(\mathcal{S}_\alpha^{(N)})$. Hence we have the desired result. □

## 2.3 Commutation relations of the momentum operators with winding gauge potential

For $\mathbf{r} = (x, y) \in \mathbf{R}^2$, $s, t \in \mathbf{R}$, we introduce a path $C(\mathbf{r}; s, t)$ which is the rectangular curve:

$$\mathbf{r} \to \mathbf{r} + (s, 0) \to \mathbf{r} + (s, t) \to \mathbf{r} + (0, t) \to \mathbf{r}.$$

Let $D(\mathbf{r}; s, t)$ be the interior of $C(\mathbf{r}; s, t)$ and

$$\epsilon(s) := \begin{cases} 1 & (s \geq 0) \\ -1 & (s < 0) \end{cases}.$$

We denote the closure of $P_{j\alpha}$ by $\overline{P}_{j\alpha}$

**Theorem 2.3** *For each $s, t \in \mathbf{R}$ and $j, k = 1, \cdots, N$, we have*

(i) $\mathrm{e}^{\mathrm{i}s\overline{P}_{j1}}\, \mathrm{e}^{\mathrm{i}t\overline{P}_{k2}} = \exp(-\mathrm{i}q\Phi_{j,k}^{(s,t)})\, \mathrm{e}^{\mathrm{i}t\overline{P}_{k2}}\, \mathrm{e}^{\mathrm{i}s\overline{P}_{j1}}$,

(ii) $\mathrm{e}^{\mathrm{i}s\overline{P}_{j\alpha}}\, \mathrm{e}^{\mathrm{i}t\overline{P}_{k\alpha}} = \mathrm{e}^{\mathrm{i}t\overline{P}_{k\alpha}}\, \mathrm{e}^{\mathrm{i}s\overline{P}_{j\alpha}}$ $(\alpha = 1, 2)$,

*where, for each $(\mathbf{r}_1, \cdots, \mathbf{r}_N) \in \mathbf{R}^{2N}$ with $\mathbf{r}_i = (x_i, y_i) \in \mathbf{R}_i^2$, we define*

$$\Phi_{j,k}^{(s,t)}(\mathbf{r}_1, \cdots, \mathbf{r}_N) := \begin{cases} \gamma\epsilon(s)\epsilon(t)\#\Big\{i \neq k \mid \mathbf{r}_i \in D(\mathbf{r}_k; s, t)\Big\} & (j = k) \\ \gamma\epsilon(s)\epsilon(t)\#\Big\{(j,k) \mid (x_k, -y_j) \in D((x_j, -y_k); -s, t)\Big\} & (j \neq k) \end{cases}$$

**Definition 2.4** ([1]) We say that the magnetic flux associated with the winding gauge potential is *locally quantized* if $\Phi_{j,k}^{(s,t)}$ is a $2\pi\mathbf{Z}/q$-valued function for all $s, t \in \mathbf{R}$.

**Corollary 2.5** *The momentum operators $\overline{P}_{j\alpha}$ strongly commute to each other if ond only if $\frac{\gamma}{\theta_0} \in \mathbf{Z}$, where $\theta_0 := \frac{2\pi}{q}$ the flux quanta, equivalently, the magnetic flux associated with the winding gauge potential is locally quantized.*

To prove Theorem 3.2, we need some preparations. Let

$$\mathbf{R}^{2N} = \mathbf{R}_1^2 \times \cdots \times \mathbf{R}_N^2,$$

where each $\mathbf{R}_i^2$ $(i = 1, \cdots, N)$ is a copy of $\mathbf{R}^2$. For each $j, k = 1, \cdots, N$, we define

$$\Omega_{jk} := \begin{cases} \mathbf{R}_1^2 \times \cdots \times \widehat{\mathbf{R}_k^2} \times \cdots \times \widehat{\mathbf{R}_k^2} \times \cdots \times \mathbf{R}_N^2 & (j \neq k) \\ \mathbf{R}_1^2 \times \cdots \times \widehat{\mathbf{R}_j^2} \times \cdots \times \mathbf{R}_N^2 & (j = k) \end{cases},$$

where $\widehat{\mathbf{R}_i^2}$ indicates the omission of $\mathbf{R}_i^2$.

Let $\omega_{jk} := (\mathbf{a}_1, \cdots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \cdots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_N) \in \Omega_{jk}$ (if $j = k$, then $\omega_{jj}$ is given by $\omega_{jj} = (\mathbf{a}_1, \cdots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \cdots, \mathbf{a}_N) \in \Omega_{jj}$). Then we define the multiplication operators $\tilde{A}_{j\alpha}(\omega_{jk})$ on $L^2(\mathbf{R}_j^2 \times \mathbf{R}_k^2)$ $(j \neq k)$ and $\tilde{A}_{j\alpha}(\omega_{jj})$ on $L^2(\mathbf{R}_j^2)$ by

$$\begin{aligned}
\tilde{A}_{j\alpha}(\omega_{jk})(\mathbf{r}_j, \mathbf{r}_k) &:= A_{j\alpha}(\mathbf{a}_1, \cdots, \mathbf{a}_{j-1}, \mathbf{r}_j, \mathbf{a}_{j+1}, \cdots, \mathbf{a}_{k-1}, \mathbf{r}_k, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_N) \quad (j \neq k) \\
\tilde{A}_{j\alpha}(\omega_{jj})(\mathbf{r}_j) &:= A_{j\alpha}(\mathbf{a}_1, \cdots, \mathbf{a}_{j-1}, \mathbf{r}_j, \mathbf{a}_{j+1}, \cdots, \mathbf{a}_N),
\end{aligned}$$

respectively. We set

$$\mathcal{H}_{jk} = \begin{cases} L^2(\mathbf{R}_j^2 \times \mathbf{R}_k^2) & (j \neq k) \\ L^2(\mathbf{R}_j^2) & (j = k) \end{cases}.$$

Then relative to the direct integral decomposition

$$L^2(\mathbf{R}^{2N}) = \int_{\Omega_{jk}}^{\oplus} \mathcal{H}_{jk} \, \mathrm{d}\omega_{jk}, \tag{5}$$

we can represent the multiplication operators $A_{j\alpha}$, $A_{k\alpha}$ as

$$A_{j\alpha} = \int_{\Omega_{jk}}^{\oplus} \tilde{A}_{j\alpha}(\omega_{jk}) \, \mathrm{d}\omega_{jk}, \quad A_{k\alpha} = \int_{\Omega_{jk}}^{\oplus} \tilde{A}_{k\alpha}(\omega_{jk}) \, \mathrm{d}\omega_{jk}.$$

On the other hand, it is clear that

$$p_{j\alpha} = \int_{\Omega_{jk}}^{\oplus} p_{j\alpha} \, \mathrm{d}\omega_{jk}, \quad p_{k\alpha} = \int_{\Omega_{jk}}^{\oplus} p_{k\alpha} \, \mathrm{d}\omega_{jk}$$

for $\alpha = 1, 2$.

For each $\omega_{jk} \in \Omega_{jk}$, we define

$$\begin{aligned}
P_{j\alpha}(\omega_{jk}) &:= p_{j\alpha} - q\tilde{A}_{j\alpha}(\omega_{jk}), \\
\mathrm{dom}(P_{j\alpha}(\omega_{jk})) &:= \mathrm{dom}(p_{j\alpha}) \cap \mathrm{dom}(\tilde{A}_{j\alpha}(\omega_{jk}))
\end{aligned}$$

and

$$\begin{aligned}
P_{k\alpha}(\omega_{jk}) &:= p_{k\alpha} - q\tilde{A}_{k\alpha}(\omega_{jk}), \\
\mathrm{dom}(P_{k\alpha}(\omega_{jk})) &:= \mathrm{dom}(p_{k\alpha}) \cap \mathrm{dom}(\tilde{A}_{k\alpha}(\omega_{jk})).
\end{aligned}$$

**Remark 2.6** If $j = k$, then the operator $P_{j\alpha}(\omega_{jj})$ is called Arai's momentum operator ([1]).

Now, we have a following useful lemma.

**Lemma 2.7** *Let $P_{j\alpha}(\omega_{jk})$ and $P_{k\alpha}(\omega_{jk})$ be as above.*

(i) *For all $\omega_{jk} \in \Omega_{jk}$ and $\alpha = 1, 2$, $P_{j\alpha}(\omega_{jk})$ and $P_{k\alpha}(\omega_{jk})$ are essentially self-adjoint.*

(ii) *The mappings $\omega_{jk} \in \Omega_{jk} \to P_{j\alpha}(\omega_{jk})$, $P_{k\alpha}(\omega_{jk})$ are measurable.*

*Proof.* (i) If $j = k$, then we can apply Theorem 3.2 in Ref. [1]. Hence we only prove the assertion in the case $j \neq k$. For each $\omega_{jk} = (\mathbf{a}_1, \cdots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \cdots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_N) \in \Omega_{jk}$, let

$$\tilde{S}_{j1}^{(N)}(\omega_{jk}) := \left\{ (\mathbf{r}_j, \mathbf{r}_k) \in \mathbf{R}_j^2 \times \mathbf{R}_k^2 \;\middle|\; y_j \neq y_k, \; y_j \neq a_{i2}, \; (i = 1, \cdots, \hat{j} \cdots, \hat{k}, \cdots, N) \right\}.$$

We introduce the function $\tilde{\psi}_{j1}(\omega_{jk})$ by

$$\begin{aligned}\tilde{\psi}_{j1}(\omega_{jk}) &:= \psi_{j1}(\mathbf{a}_1,\cdots,\mathbf{r}_j,\cdots,\mathbf{r}_k,\cdots,\mathbf{a}_N)\\ &= -\frac{\gamma}{2\pi}\Big\{\mathrm{Arctan}\Big(\frac{x_j-x_k}{y_j-y_k}\Big)+\sum_{i\neq j,k}\mathrm{Arctan}\Big(\frac{x_j-a_{i1}}{y_j-a_{i2}}\Big)\Big\}\end{aligned}$$

for each $(\mathbf{r}_j,\mathbf{r}_k)\in\tilde{S}_{j1}^{(N)}(\omega_{jk})$. Then it is easy to check that $\tilde{\psi}_{j1}(\omega_{jk})\in C^\infty(\tilde{S}_{j1}^{(N)}(\omega_{jk}))$ and

$$D_{x_j}\tilde{\psi}_{j1}(\omega_{jk})(\mathbf{r}_j,\mathbf{r}_k)=\tilde{A}_{j1}(\omega_{jk})(\mathbf{r}_j,\mathbf{r}_k)\quad((\mathbf{r}_j,\mathbf{r}_k)\in\tilde{S}_{j1}^{(N)}(\omega_{jk})).$$

Hence $e^{iq\tilde{\psi}_{j1}(\omega_{jk})}$ is a unitary operator such that

$$e^{iq\tilde{\psi}_{j1}(\omega_{jk})}C_0^\infty(\tilde{S}_{j1}^{(N)}(\omega_{jk}))=C_0^\infty(\tilde{S}_{j1}^{(N)}(\omega_{jk}))$$

and

$$P_{j1}(\omega_{j1})=e^{iq\tilde{\psi}_{j1}(\omega_{jk})}p_{j\alpha}e^{-iq\tilde{\psi}_{j1}(\omega_{jk})}$$

on $C_0^\infty(\tilde{S}_{j1}^{(N)}(\omega_{jk}))$. Since $C_0^\infty(\tilde{S}_{j1}^{(N)}(\omega_{jk}))\supseteq C_0^\infty(\mathbf{R}^1_{jx})\otimes_{\mathrm{alg}}C_0^\infty(L_{jk,1})$ ($\otimes_{\mathrm{alg}}$ denotes algebraic tensor product), where

$$L_{jk,1}:=\Big\{(y_j,\mathbf{r}_k)\in\mathbf{R}^1_{jy}\times\mathbf{R}^2_k\ \Big|\ y_j\neq y_k,\ y_j\neq a_{i2}\ (i=1,\cdots,\hat{j},\cdots,\hat{k},\cdots,N)\Big\},$$

$p_{j1}$ is essentially self-adjoint on $C_0^\infty(\tilde{S}_{j1}^{(N)}(\omega_{jk}))$. Hence we have the desired result. By the similar way, we can prove the assertions about $P_{j2}(\omega_{jk}),P_{k1}(\omega_{jk})$ and $P_{k2}(\omega_{jk})$.

(ii) Let $\mathcal{A}_{jk}$ be the algebra of decomposable bounded operators whose fibres are all multiples of the identity of $L^2(\mathbf{R}_j\times\mathbf{R}_k)$. Then it is clear that $(P_{j\alpha}(\cdot)+i)^{-1},(P_{k\alpha}(\cdot)+i)^{-1}\in\mathcal{A}'_{jk}$, where $\mathcal{A}'_{jk}$ is the commutant of $\mathcal{A}_{jk}$. Therefore applying Theorem XIII 84 in Ref. [9], we have the desired result. □

By the above lemma, we can define the following two self-adjoint operators:

$$\int_{\Omega_{jk}}^\oplus\overline{P_{j\alpha}(\omega_{jk})}\,d\omega_{jk},\qquad\int_{\Omega_{jk}}^\oplus\overline{P_{k\alpha}(\omega_{jk})}\,d\omega_{jk},$$

where we denote the closure of $P_{j\alpha}(\omega_{jk})$ (resp. $P_{k\alpha}(\omega_{jk})$) by $\overline{P_{j\alpha}(\omega_{jk})}$ (resp. $\overline{P_{k\alpha}(\omega_{jk})}$).

**Proposition 2.8** *For each $j,k=1,\cdots,N,\alpha=1,2$, we have*

$$\overline{P}_{j\alpha}=\int_{\Omega_{jk}}^\oplus\overline{P_{j\alpha}(\omega_{jk})}\,d\omega_{jk},\quad\overline{P}_{k\alpha}=\int_{\Omega_{jk}}^\oplus\overline{P_{k\alpha}(\omega_{jk})}\,d\omega_{jk}.$$

*Proof.* We denote the operator in the right hand side of the equations on the proposition by $C_{j\alpha}$ and $C_{k\alpha}$, respectively. Suppose that $\Psi$ is an element of $C_0^\infty(\mathcal{S}_\alpha^{(N)})$. For each $\omega_{jk}=(\mathbf{a}_1,\cdots,\mathbf{a}_{j-1},\mathbf{a}_{j+1},\cdots,\mathbf{a}_{k-1},\mathbf{a}_{k+1},\cdots,\mathbf{a}_N)\in\Omega_{jk}$, we introduce $\tilde{\Psi}(\omega_{jk})\in L^2(\mathbf{R}^2_j\times\mathbf{R}^2_k)$ defined by

$$\tilde{\Psi}(\omega_{jk})(\mathbf{r}_j,\mathbf{r}_k)=\Psi(\mathbf{a}_1,\cdots,\mathbf{r}_j,\cdots,\mathbf{r}_k,\cdots,\mathbf{a}_N),\ (\mathbf{r}_j,\mathbf{r}_k)\in\mathbf{R}^2_j\times\mathbf{R}^2_k.$$

Then relative to the direct integral decomposition (5), we have

$$\Psi=\int_{\Omega_{jk}}^\oplus\tilde{\Psi}(\omega_{jk})d\omega_{jk}.$$

It is not difficult to check that $\tilde{\Psi}(\omega_{jk}) \in \mathrm{dom}(\overline{P_{j\alpha}(\omega_{jk})})$ and

$$P_{j\alpha}\Psi = \int_{\Omega_{jk}}^{\oplus} P_{j\alpha}(\omega_{jk})\tilde{\Psi}(\omega_{jk})\mathrm{d}\omega_{jk} = C_{j\alpha}\Psi.$$

Since $C_0^\infty(\mathcal{S}_\alpha^{(N)})$ is a core of $\overline{P}_{j\alpha}$, we have the desired result. Similarly, we can prove the assertion about $\overline{P}_{k\alpha}$. □

*Proof of Theorem* 2.3: By Propostion 2.8, we have

$$\exp(\mathrm{i}t\overline{P}_{l\alpha}) = \int_{\Omega_{jk}}^{\oplus} \exp(\mathrm{i}t\overline{P_{l\alpha}(\omega_{jk})})\ \mathrm{d}\omega_{jk} \quad (l = j, k)$$

for each $t \in \mathbf{R}$. Hence it suffices to disscus the commutation relations in the theorem at each fibre.

If $j = k$, then we can apply Thoerem 2.1 in Ref. [1] and obtain

$$\begin{aligned}&\exp(\mathrm{i}s\overline{P_{j1}(\omega_{jj})})\exp(\mathrm{i}t\overline{P_{j2}(\omega_{jj})})\\ &= \exp\Big( - \mathrm{i}q\Phi_{j,j}^{(s,t)}(\omega_{jj})\Big) \exp(\mathrm{i}t\overline{P_{j2}(\omega_{jj})})\exp(\mathrm{i}s\overline{P_{j1}(\omega_{jj})}),\end{aligned}$$

where $\Phi_{j,j}^{(s,t)}(\omega_{jj})$ is a multipliation oprator defined by

$$\begin{aligned}&\Phi_{j,j}^{(s,t)}(\omega_{jj})(\mathbf{r}_j)\\ :=\ &\gamma\epsilon(s)\epsilon(t)\#\Big\{i \neq j \mid \omega_{jj}(i) \in D_{jj,12}(\mathbf{r}_j; s, t)\Big\},\end{aligned}$$

where $\omega_{jj} = (\omega_{jj}(1), \cdots, \omega_{jj}(N-1)) \in \Omega_{jj}$. Hence we have the desired result in this case.

Next we prove the assertion in the case $j \neq k$. We can apply the Trotter product formula (e.g., Theorem VIII 31 in Ref. [8]) to each $\overline{P_{l\alpha}(\omega_{jk})}$ $(l = j, k)$ to obtain

$$\exp(\mathrm{i}t\overline{P_{l\alpha}(\omega_{jk})}) = \text{s-}\lim_{n\to\infty} \Big(\exp(\mathrm{i}tp_{l\alpha}/n)\exp(-\mathrm{i}tq\tilde{A}_{l\alpha}(\omega_{jk})/n)\Big)^n \quad (l = j, k)$$

for each $t \in \mathbf{R}$. Using the fact that

$$(\mathrm{e}^{\mathrm{i}tp_{j1}}\Psi)(\mathbf{r}_j, \mathbf{r}_k) = \Psi(\mathbf{r}_j + (t, 0), \mathbf{r}_k) \ \text{a.e.}(\mathbf{r}_j, \mathbf{r}_k) \in \mathbf{R}_j^2 \times \mathbf{R}_k^2,\ s \in \mathbf{R},$$

we can show that

$$\begin{aligned}&(\exp(\mathrm{i}t\overline{P}_{j1}(\omega_{jk}))\Psi)(\mathbf{r}_j, \mathbf{r}_k)\\ &= \exp\Big( - \mathrm{i}q\int_0^t \tilde{A}_{j1}(\omega_{jk})(\mathbf{r}_j + (x_j', 0), \mathbf{r}_k)\ \mathrm{d}x_j'\Big)\Psi(\mathbf{r}_j + (t, 0), \mathbf{r}_k),\ \text{a.e.}(\mathbf{r}_j, \mathbf{r}_k) \in \mathbf{R}_j^2 \times \mathbf{R}_k^2\end{aligned}$$

for each $\Psi \in L^2(\mathbf{R}_j^2 \times \mathbf{R}_k^2)$. Similarly we have

$$\begin{aligned}&(\exp(\mathrm{i}t\overline{P}_{k2}(\omega_{jk}))\Psi)(\mathbf{r}_j, \mathbf{r}_k)\\ &= \exp\Big( - \mathrm{i}q\int_0^t \tilde{A}_{k2}(\omega_{jk})(\mathbf{r}_j, \mathbf{r}_k + (0, y_k'))\ \mathrm{d}y_k'\Big)\Psi(\mathbf{r}_j, \mathbf{r}_k + (0, t)),\ \text{a.e.}(\mathbf{r}_j, \mathbf{r}_k) \in \mathbf{R}_j^2 \times \mathbf{R}_k^2\end{aligned}$$

for each $\Psi \in L^2(\mathbf{R}_j^2 \times \mathbf{R}_k^2)$. Using these formulas, we obtain

$$\begin{aligned}&\exp(\mathrm{i}s\overline{P}_{j1}(\omega_{jk}))\exp(\mathrm{i}t\overline{P}_{k2}(\omega_{jk}))\\ &= \exp\Big( - \mathrm{i}q\Phi_{j,k}^{(s,t)}(\omega_{jk})\Big) \exp(\mathrm{i}t\overline{P}_{k2}(\omega_{jk}))\exp(\mathrm{i}s\overline{P}_{j1}(\omega_{jk})),\end{aligned}$$

where

$$
\begin{aligned}
&\Phi^{(s,t)}_{j,k}(\omega_{jk})(\mathbf{r}_j,\mathbf{r}_k)\\
&=\int_0^s \tilde{A}_{j1}((\mathbf{r}_j+(x'_j,0),\mathbf{r}_k))(\omega_{jk})\ \mathrm{d}x'_j+\int_0^t \tilde{A}_{k2}((\mathbf{r}_j+(s,0),\mathbf{r}_k+(0,y'_k)))(\omega_{jk})\ \mathrm{d}y'_k\\
&-\int_0^t \tilde{A}_{k2}((\mathbf{r}_j,\mathbf{r}_k+(0,y'_k)))(\omega_{jk})\ \mathrm{d}y'_k-\int_0^s \tilde{A}_{j1}((\mathbf{r}_j+(x'_j,0),\mathbf{r}_k+(0,s)))(\omega_{jk})\ \mathrm{d}x'_j
\end{aligned}
$$

for a.e.$(\mathbf{r}_j,\mathbf{r}_k)\in\mathbf{R}^2_j\times\mathbf{R}^2_k$. To calculate $\Phi^{(s,t)}_{j,k}(\omega_{jk})$, we introduce some notations:

$$
\begin{aligned}
a_{jk,1}(\mathbf{r}_j,\mathbf{r}_k)&:=-\frac{\gamma}{2\pi}\frac{y_j-y_k}{|\mathbf{r}_j-\mathbf{r}_k|^2},\quad a_{jk,2}(\mathbf{r}_j,\mathbf{r}_k):=\frac{\gamma}{2\pi}\frac{x_k-x_j}{|\mathbf{r}_j-\mathbf{r}_k|^2},\\
b_{jk,1}(\omega_{jk})(\mathbf{r}_j,\mathbf{r}_k)&:=-\frac{\gamma}{2\pi}\sum_{i\neq j,k}\frac{y_j-\omega_{jk}(i)_2}{|\mathbf{r}_j-\omega_{jk}(i)|^2},\quad b_{jk,2}:=\frac{\gamma}{2\pi}\sum_{i\neq j,k}\frac{x_k-\omega_{jk}(i)_1}{|\mathbf{r}_k-\omega_{jk}(i)|^2},
\end{aligned}
$$

where we use the notations $\omega_{jk}=(\omega_{jk}(1),\cdots,\widehat{\omega_{jk}(j)},\cdots,\widehat{\omega_{jk}(k)},\cdots,\omega_{jk}(N))\in\Omega_{jk}$, $\omega_{jk}(i)=(\omega_{jk}(i)_1,\omega_{jk}(i)_2)\in\mathbf{R}^2_i$. Then it is clear that

$$
\tilde{A}_{j1}(\omega_{jk})=a_{jk,1}+b_{jk,1},\quad \tilde{A}_{k2}(\omega_{jk})=a_{jk,2}+b_{jk,2}(\omega_{jk}).
$$

Next, we introduce the new coordinate:

$$
\mathbf{r}_{jk}:=(x_j,-y_k),\quad \overline{\mathbf{r}}_{jk}:=(x_k,-y_j)
$$

for each $\mathbf{r}_j=(x_j,y_j)\in\mathbf{R}^2_j$, $\mathbf{r}_k=(x_k,y_k)\in\mathbf{R}^2_k$. Then we can regard $a_{jk,\alpha}$ $(\alpha=1,2)$ as the function with two variable $\mathbf{r}_{jk},\overline{\mathbf{r}}_{jk}$:

$$
\tilde{a}_{jk,\alpha}(\mathbf{r}_{jk},\overline{\mathbf{r}}_{jk})=a_{jk,\alpha}(\mathbf{r}_j,\mathbf{r}_k),\ \alpha=1,2.
$$

On the one hand, we have

$$
\begin{aligned}
&\int_0^s a_{jk,1}(\mathbf{r}_j+(x'_j,0),\mathbf{r}_k)\ \mathrm{d}x'_j+\int_0^t a_{jk,2}(\mathbf{r}_j+(s,0),\mathbf{r}_k+(0,y'_k))\ \mathrm{d}y'_k\\
&-\int_0^t a_{jk,2}(\mathbf{r}_j,\mathbf{r}_k+(0,y'_k))\ \mathrm{d}y'_k-\int_0^s a_{jk,1}(\mathbf{r}_j+(x'_j,0),\mathbf{r}_k+(0,s))\ \mathrm{d}x'_j\\
&=\int_{C_{jk,12}(\mathbf{r}_{jk};s,-t)}\tilde{\mathbf{a}}_{jk}(\mathbf{r}'_{jk},\overline{\mathbf{r}}_{jk})\cdot\mathrm{d}\mathbf{r}'_{jk}\quad(\tilde{\mathbf{a}}_{jk}:=(\tilde{a}_{jk,1},\tilde{a}_{jk,2}))\\
&=\int_{D_{jk,12}(\mathbf{r}_{jk};s,-t)}(D_{r_{jk,2}}\tilde{a}_{jk,1}+D_{r_{jk,1}}\tilde{a}_{jk,2})\mathrm{d}\mathbf{r}'_{jk}\quad\text{(by Green's theorem)}\\
&=\gamma\epsilon(s)\epsilon(t)\#\Big\{(j,k)\mid \overline{\mathbf{r}}_{jk}\in D_{jk,12}(\mathbf{r}_{jk};s,-t)\Big\},
\end{aligned}
$$

where we use the notation $r_{jk,1}=x_j,\ r_{jk,2}=-y_k$ and the fact

$$
D_{r_{jk,2}}\tilde{a}_{jk,1}(\mathbf{r}_{jk},\overline{\mathbf{r}}_{jk})+D_{r_{jk,1}}\tilde{a}_{jk,2}(\mathbf{r}_{jk},\overline{\mathbf{r}}_{jk})=-\gamma\delta(\mathbf{r}_{jk}-\overline{\mathbf{r}}_{jk}).
$$

On the other hand, we obtain

$$
\begin{aligned}
&\int_0^s b_{jk,1}(\omega_{jk})(\mathbf{r}_j+(x'_j,0),\mathbf{r}_k)\ \mathrm{d}x'_j+\int_0^t b_{jk,2}(\omega_{jk})(\mathbf{r}_j,\mathbf{r}_k+(0,y'_k))\ \mathrm{d}y'_k\\
&-\int_0^t b_{jk,2}(\omega_{jk})(\mathbf{r}_j,\mathbf{r}_k+(0,y'_k))\ \mathrm{d}y'_k-\int_0^s b_{jk,1}(\omega_{jk})(\mathbf{r}_j+(x'_j,0),\mathbf{r}_k)\ \mathrm{d}x'_j\\
&=0.
\end{aligned}
$$

Combining these facts, we have

$$\Phi_{j,k}^{(s,t)}(\omega_{jk})(\mathbf{r}_j,\mathbf{r}_k) = \gamma\epsilon(s)\epsilon(t)\#\Big\{(j,k) \mid \bar{\mathbf{r}}_{jk} \in D_{jk,12}(\mathbf{r}_{jk}; s, -t)\Big\}.$$

Therefore we have the assertion about $\overline{P}_{j1}, \overline{P}_{k2}$. By similar a way, we have the assertions about $\overline{P}_{j1}, \overline{P}_{k1}$ and $\overline{P}_{j2}, \overline{P}_{k2}$. □

# 3 Representation of the CCR

## 3.1 Schrödinger representation and local quantization of magnetic flux

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ and $\{p_j, q_j\}_{j=1}^n$ be a set of self-adjoint operators on $\mathcal{H}$. The set

$$\pi := \{\mathcal{H}, \mathcal{D}, \{p_j, q_j\}_{j=1}^n\}$$

is called a *representation of the CCR with n degree of freedom on $\mathcal{D}$* if it satisfies

(i) $\mathrm{dom}(p_j), \mathrm{dom}(q_j) \subset \mathcal{D}$ $(j = 1, \cdots, n)$;

(ii) $p_j\mathcal{D} \subset \mathcal{D},\ q_j\mathcal{D} \subset \mathcal{D}$ $(j = 1, \cdots, n)$;

(iii) $\{p_j, q_j\}_{j=1}^n$ satisfy the CCR on $\mathcal{D}$:

$$[p_j, q_k]\psi = -\mathrm{i}\delta_{jk}\psi,$$
$$[p_j, p_k]\psi = 0 = [q_j, q_k]\psi \qquad (j, k = 1, \cdots, N).$$

for all $\psi$ in $\mathcal{D}$.

We often write $\mathcal{D}(\pi)$ for $\mathcal{D}$. Our main aim of this section is to investigate the momentum operators with winding quage potential from the viewpoint of the representation of the CCR.

**Proposition 3.1** *Let $Q_{j1}, Q_{j2}$ $(j = 1, \cdots, N)$ be the multiplication operators by the coordinate functions $x_j$ and $y_j$, respectively. Then*

$$\pi_{\mathbf{A}} := \Big\{L^2(\mathbf{R}^{2N}), C_0^\infty(\mathcal{M}_N), \{\overline{P}_{j\alpha}, Q_{j\alpha} \mid j = 1, \cdots, N, \alpha = 1, 2\}\Big\}$$

*is an irreducible representation of the CCR of $2N$ degree of freedom.*

*Proof.* Easy. □

As for the CCRs in the Weyl form, we have the following result.

**Theorem 3.2** *The set $\{\overline{P}_{j\alpha}, Q_{j\alpha} \mid j = 1, \cdots, N, \alpha = 1, 2\}$ of self-adjoint operators fullfills the Weyl relations if and only if $\frac{\gamma}{\theta_0} \in \mathbf{Z}$, i.e., the magnetic flux is locally quantized.*

*Proof.* In a similar way to the proof of Theorem 2.3, we can prove

$$\exp(\mathrm{i}sQ_{j\alpha})\exp(\mathrm{i}t\overline{P}_{k\beta}) = \exp(-\mathrm{i}st\delta_{jk}\delta_{\alpha\beta})\exp(\mathrm{i}t\overline{P}_{k\beta})\exp(\mathrm{i}sQ_{j\alpha})$$

for all $s, t \in \mathbf{R},\ j, k = 1, \cdots, N,\ \alpha, \beta = 1, 2$. Combining these facts with Theorem 2.3, we have the desired assertion. □

Let $\pi_1 = \{\mathcal{H}_{\pi_1}, \mathcal{D}(\pi_1), \{p_{\pi_1 j}, q_{\pi_1 j}\}_{j=1}^n\}$ and $\pi_2 = \{\mathcal{H}_{\pi_2}, \mathcal{D}(\pi_2), \{p_{\pi_2 j}, q_{\pi_2 j}\}_{j=1}^n\}$ be representation of the CCR of $n$ degree of freedom. If there exists a unitary operator $U$ from $\mathcal{H}_{\pi_1}$ onto $\mathcal{H}_{\pi_2}$ such that

$$Up_{\pi_1}U^* = p_{\pi_2}, \quad Uq_{\pi_1}U^* = q_{\pi_2},$$

for each $j = 1, \cdots, n$, then we say that $\pi_1$ and $\pi_2$ are *unitarily equivalent.*

To state a corollay of the above theorem, we need some notations. Let $p_{j\alpha}$ be the free momentum operators defined in the preveous section. Then it is not difficult to check that $p_{j\alpha}, Q_{j\alpha}$ are self-adjoint and satisfy the CCR on $C_0^\infty(\mathcal{M}_N)$. Hence

$$\pi_{\mathrm{S}} = \left\{L^2(\mathbf{R}^{2N}), C_0^\infty(\mathcal{M}_N), \{p_{j\alpha}, Q_{j\alpha} | j = 1, \cdots, N, \alpha = 1, 2\}\right\}$$

is an irreducible representation of the CCRs. $\pi_{\mathrm{S}}$ is said to be the *Schödinger representation on* $C_0^\infty(\mathcal{M}_N)$.

**Corollary 3.3** *$\pi_{\mathbf{A}}$ is unitarily equivalent to $\pi_{\mathrm{S}}$ if and only if $\frac{\gamma}{\theta_0} \in \mathbf{Z}$. Furtheremore, the unitary operator which gives this unitarily equivalence has the follwing formula:*

$$U_N(\theta_0, \gamma) = \exp(\mathrm{i}\frac{\gamma}{\theta_0}\eta_N),$$
$$\eta_N(\mathbf{r}_1, \cdots, \mathbf{r}_N) := \frac{1}{i}\sum_{k<j} \log \frac{z_k - z_j}{|z_k - z_j|},$$

*where $z_k = x_k + \mathrm{i}y_k$. That is, for each $j = 1, \cdots, N, \alpha = 1, 2$,*

$$U_N(\theta_0, \gamma) p_{j\alpha} U_N(\theta_0, \gamma)^* = \overline{P}_{j\alpha}, \quad U_N(\theta_0, \gamma) Q_{j\alpha} U_N(\theta_0, \gamma)^* = Q_{j\alpha}.$$

*Proof.* The first half immediately follows from Proposition 3.1, Theorem 3.2 and Theorem VIII 14 in Ref. [8] .

It is easy to see that $\mathrm{e}^{\mathrm{i}\theta_0 \eta_N} C_0^\infty(\mathcal{S}_1^{(N)}) = C_0^\infty(\mathcal{S}_1^{(N)})$. By direct calculation, we can check that

$$D_{x_k}\eta_N(\mathbf{r}_1, \cdots, \mathbf{r}_N) = \frac{2\pi}{\gamma} A_{k1}(\mathbf{r}_1, \cdots, \mathbf{r}_N), \quad (\mathbf{r}_1, \cdots, \mathbf{r}_N) \in \mathcal{S}_1^{(N)}.$$

Hence we have

$$\mathrm{e}^{\mathrm{i}\frac{\gamma}{\theta_0}\eta_N} p_{k1} \mathrm{e}^{-\mathrm{i}\frac{\gamma}{\theta_0}\eta_N} = P_{k1} \text{ on } C_0^\infty(\mathcal{S}_1^{(N)}).$$

By a similar way, we can show the assertion about $P_{k2}$. □

## 3.2 Fibre direct integral representation of the CCR and Arai's representation

Let $\pi = \{\mathcal{H}, \mathcal{D}, \{p_j, q_j\}_{j=1}^n\}$ be a representation of the CCR of $n$ degree of freedom on $\mathcal{D}$. For each subset $I = \{i_1, \cdots, i_k\} \subset \{1, \cdots, n\}$ $(i_1 < \cdots < i_k)$, we introduce

$$\pi(I) := \{\mathcal{H}, \mathcal{D}, \{p_j, q_j | j \in I\}\}.$$

Then it is clear that $\pi(I)$ is a representation of the CCR of $\#I$ degree of freedom ($\#\ I$ means the cardinality of the set $I$).

In this subsection, we derive the direct integral decomposition of $\pi_{\mathbf{A}}(I)$. To do that, we need some prepartions. Let $I = \{i_1, \cdots, i_k\} \subset \{1, \cdots, N\}$ be fixed. Then we introduce

$$\begin{aligned} \Omega_I &:= \mathbf{R}_1^2 \times \cdots \times \widehat{\mathbf{R}}_{i_1}^2 \times \cdots \times \widehat{\mathbf{R}}_{i_k}^2 \times \cdots \times \mathbf{R}_N^2, \\ \mathbf{R}^2(I) &:= \mathbf{R}_{i_1}^2 \times \cdots \times \mathbf{R}_{i_k}^2. \end{aligned}$$

For each $\omega_I = (\mathbf{a}_1, \cdots, \widehat{\mathbf{a}}_{i_1}, \cdots, \widehat{\mathbf{a}}_{i_k}, \cdots, \mathbf{a}_N) \in \Omega_I$, we define the multiplication operators $\tilde{A}_{i_\alpha}(\omega_I)$ $(i \in I)$ on $L^2(\mathbf{R}^2(I))$ by

$$\tilde{A}_{i\alpha}(\omega_I)(\mathbf{r}_{i_1}, \cdots, \mathbf{r}_{i_k}) := A_{i_\alpha}(\mathbf{a}_1, \cdots, \mathbf{r}_{i_1}, \cdots, \mathbf{r}_{i_k}, \cdots, \mathbf{a}_N).$$

Relative to the direct integral decomposition

$$L^2(\mathbf{R}^{2N}) = \int_{\Omega_I}^{\oplus} L^2(\mathbf{R}^2(I))\ \mathrm{d}\omega_I,$$

it is not difficult to see that

$$A_{i\alpha} = \int_{\Omega_I}^{\oplus} \tilde{A}_{i\alpha}(\omega_I)\ \mathrm{d}\omega_I,\ \ p_{i\alpha} = \int_{\Omega_I}^{\oplus} p_{i\alpha}\ \mathrm{d}\omega_I$$

for each $\alpha = 1, 2,\ i \in I$. For each $\omega_I \in \Omega_I$, we define

$$\begin{aligned} P_{i\alpha}(\omega_I) &:= p_{i\alpha} - q\tilde{A}_{i\alpha}(\omega_I), \\ \mathrm{dom}(P_{i\alpha}(\omega_I)) &:= \mathrm{dom}(p_{i\alpha}) \cap \mathrm{dom}(\tilde{A}_{i\alpha}(\omega_I)). \end{aligned}$$

Then by a similar way to the proof of Lemma 2.7, we can prove that for each $\omega_I \in \Omega_I$, $P_{i\alpha}(\omega_I)$ is essentially self-adjoint and measurable. Moreover,

$$\overline{P}_{i\alpha} = \int_{\Omega_I}^{\oplus} \overline{P_{i\alpha}(\omega_I)}\ \mathrm{d}\omega_I,\ (i \in I,\ \alpha = 1, 2).$$

Let $I = \{i_1, \cdots, i_k\}$ $(k \le N)$ be a subset of $\{1, \cdots, N\}$. For each $\omega = (\mathbf{a}_1, \cdots, \mathbf{a}_{N-k}) \in \Omega(I)$, we introduce

$$\mathcal{M}_N(\omega) := \left\{ (\mathbf{r}_{i_1}, \cdots, \mathbf{r}_{i_k}) \in \mathcal{M}_k \ \middle|\ \mathbf{r}_{i_m} \neq \mathbf{a}_j\ (m = 1, \cdots, k,\ j = 1, \cdots, N-k) \right\}.$$

The following proposition can be easily proven.

**Proposition 3.4** *For each* $\omega \in \Omega_I$,

$$\pi_{\mathbf{A}}^I(\omega) := \{L^2(\mathbf{R}^2(I)), C_0^\infty(\mathcal{M}_N(\omega)), \{P_{i\alpha}(\omega), Q_{i\alpha} \mid i \in I, \alpha = 1, 2\}\}$$

*is a representation of the CCR of* $\#I \times 2$ *degree of freedom.*

**Remark 3.5** If $\#I = 1$ , then $\pi_{\mathbf{A}}^I(\omega)$ is called Arai's representation.

Let $\mathcal{K} = \int_\Lambda^\oplus \mathcal{H}\ \mathrm{d}\mu(\lambda)$ be the direct intgral of $\mathcal{H}$ over a measure space $(\Lambda, \mu)$. Suppose that

$$\pi = \{\mathcal{K}, \mathcal{D}, \{p_i, q_i | i = 1, \cdots, N\}\}$$

be a representation of the CCR of $N$ degree of freedom. If for $\mu$-a.e.$\lambda \in \Lambda$, there exsists a representation of the CCR of $N$ degree of freedom

$$\pi_\lambda = \{\mathcal{H}, \mathcal{D}_\lambda, \{p_j(\lambda), q_j(\lambda) | j = 1, \cdots, N\}\}$$

such that

(i) for each $j = 1, \cdots, N$, the fields $\lambda \in \Lambda \to p_j(\lambda),\ q_j(\lambda)$ are measurable and

$$p_j = \int_\Lambda^\oplus p_j(\lambda)\ \mathrm{d}\mu(\lambda),\ \ q_j = \int_\Lambda^\oplus q_j(\lambda)\ \mathrm{d}\mu(\lambda),$$

(ii) for all $\Psi \in \mathcal{D}$, $\Psi(\lambda) \in \mathcal{D}_\lambda$ $\mu$-a.e., then we say that $\pi$ is *decomposable* or *direct integal of* $\{\pi_\lambda\}_{\lambda\in\Lambda}$ and write

$$\pi = \int_\Lambda^\oplus \pi_\lambda \, \mathrm{d}\mu(\lambda).$$

**Theorem 3.6** *Let $\pi_{\mathbf{A}}$ be the representation of the CCR defined in the preceeding subsection. Then, for each $I \subset \{1, \cdots, N\}$, we have*

$$\pi_{\mathbf{A}}(I) = \int_{\Omega_I}^\oplus \pi_{\mathbf{A}}^I(\omega_I) \, \mathrm{d}\omega_I.$$

*Especially, if $\# I = 1$, then $\pi_{\mathbf{A}}(I)$ is a direct integral of Arai's representations $\{\pi_{\mathbf{A}}^I(\omega)\}_{\omega\in\Omega_I}$.*

# 4 Applications

## 4.1 Schrödinger operators for systems with winding gauge potential

Let $V(\mathbf{r}_1, \cdots, \mathbf{r}_N)$ be a real valued Borel measurable function on $\mathbf{R}^{2N}$. Here, we investigate the Schrödinger operator on $L^2(\mathbf{R}^{2N})$ defined by

$$H = \sum_{j=1}^N \Big( -\mathrm{i}D_{x_j} + \frac{\gamma q}{2\pi} \sum_{k\neq j} \frac{y_j - y_k}{|\mathbf{r}_j - \mathbf{r}_k|^2} \Big)^2 + \sum_{j=1}^N \Big( -\mathrm{i}D_{y_j} - \frac{\gamma q}{2\pi} \sum_{k\neq j} \frac{x_j - x_k}{|\mathbf{r}_j - \mathbf{r}_k|^2} \Big)^2 + V(\mathbf{r}_1, \cdots, \mathbf{r}_N).$$

For this purpose, we introduce an operator $H_0$ defined by

$$H_0 := -\Delta + V,$$

where $\Delta$ is the $2N$ dimensional Laplacian.

We assume the following conditions:

(A.1) $H_0$ is essentially self-adjoint.

(A.2) $\frac{\gamma}{\theta_0} \in \mathbf{Z}$.

**Theorem 4.1** *Under the assumption (A.1) and (A.2), we have the followings.*

(i) *$H$ is essentially self-adjoint.*

(ii) *$\sigma(\overline{H}) = \sigma(\overline{H}_0)$, where $\sigma(A)$ denots the spectrum of the linear operator $A$.*

(iii) *$\sigma_\mathrm{p}(\overline{H}) = \sigma_\mathrm{p}(\overline{H}_0)$, where $\sigma_\mathrm{p}(A)$ denots the point spectrum of $A$. Especially, if $E \in \sigma_\mathrm{p}(\overline{H}_0)$, then*

$$\ker(H - E) = \Big\{ \Pi_{k<j} (z_k - z_j)^{\gamma/\theta_0} |z_k - z_j|^{-\gamma/\theta_0} \Psi \;\Big|\; \Psi \in \ker(H_0 - E) \Big\}.$$

*Proof.* (i) The Hamiltonian $H$ can be expressed as

$$H = \sum_{j=1}^N \mathbf{P}_j^2 + V,$$

where $\mathbf{P}_j := (P_{j1}, P_{j2})$. Hence, by Cororally 3.3, we have

$$H = U_N(\theta_0, \gamma) H_0 U_N(\theta_0, \gamma)^*$$

on $\mathrm{dom}(H)$. Hence

$$\overline{H} = U_N(\theta_0, \gamma) \overline{H_0} U_N(\theta_0, \gamma)^*. \tag{6}$$

Since $\overline{H}_0$ is self adjoint, it follows that $\overline{H}$ is self adjoint. Parts (ii) and (iii) follows from (6). □

## 4.2 Statistical transformation

Throughout this subsection, we assume the following condition:

$$\frac{\gamma}{\theta_0} \in \mathbf{Z}. \tag{7}$$

Under this condition, the representation of the CCR

$$\pi_{\mathbf{A}} = \left\{ L^2(\mathbf{R}^{2N}), C_0^\infty(\mathcal{M}_N), \{\overline{P}_{j\alpha}, Q_{j\alpha} \mid j = 1, \cdots, N, \alpha = 1, 2\} \right\}$$

is unitarily equivalent to the Schrödinger representation $\pi_S$ by Cororally 3.3. Hence, the system satisfying (7) seems to be trivial at first glance. But there are some interesting structures in this system.

Let $\mathcal{H} := L^2(\mathbf{R}^2)$. For each $N \geq 2$, it is well-known that $L^2(\mathbf{R}^{2N}) = \otimes^N \mathcal{H}$. We introduce the following closed subspaces of $\otimes^N \mathcal{H}$:

$$\begin{aligned} \otimes_{\mathrm{s}}^N \mathcal{H} &:= S_N(\otimes^N \mathcal{H}), \\ \otimes_{\mathrm{as}}^N \mathcal{H} &:= A_N(\otimes^N \mathcal{H}), \end{aligned}$$

where we denote by $S_N$ ( resp.$A_N$ ) the symmetrizer ( resp. the antisymmetrizer ) on $\otimes^N \mathcal{H}$.

**Proposition 4.2** *Suppose that the condtion (7) is satisfied. Then we have*

(i) *if* $\frac{\gamma}{\theta_0}$ *is even, then*

$$U_N(\theta_0, \gamma) S_N = S_N U_N(\theta_0, \gamma), \quad U_N(\theta_0, \gamma) A_N = A_N U_N(\theta_0, \gamma);$$

(ii) *if* $\frac{\gamma}{\theta_0}$ *is odd, then*

$$U_N(\theta_0, \gamma) A_N = S_N U_N(\theta_0, \gamma).$$

*Hence, the unitary operator* $U_N(\theta_0, \gamma)$ *gives a natural correspondence between* $\otimes_{\mathrm{s}}^N \mathcal{H}$ *and* $\otimes_{\mathrm{as}}^N \mathcal{H}$.

*Proof.* Let $\mathsf{S}_N$ be the group of permutations of a set of cardinality $N$. For each $\phi_1, \cdots, \phi_N \in \mathcal{H}$ and $\sigma \in \mathsf{S}_N$, we define

$$U_\sigma \phi_1 \otimes \cdots \otimes \phi_N := \phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(N)}.$$

Then it is easy to see that $U_\sigma$ can be extended to a unitary operator on $\otimes^N \mathcal{H}$. We denote it by the same symbol $U_\sigma$.

For each $(\mathbf{r}_1, \cdots, \mathbf{r}_N) \in \mathcal{M}_N$, we have

$$U_N(\theta_0, \gamma)(\mathbf{r}_1, \cdots, \mathbf{r}_N) = \Pi_{i<j} \left( \frac{z_i - z_j}{|z_i - z_j|} \right)^{\gamma/\theta_0}.$$

Hence if $\frac{\gamma}{\theta_0}$ is even, then for each $\Phi \in C_0^\infty(\mathcal{M}_N)$, we have

$$U_N(\theta_0, \gamma) U_\sigma \Phi = U_\sigma U_N(\theta_0, \gamma) \Phi.$$

Since $C_0^\infty(\mathcal{M}_N)$ is dense in $\otimes^N \mathcal{H}$, we obtain

$$U_N(\theta_0, \gamma) U_\sigma = U_\sigma U_N(\theta_0, \gamma).$$

On the other hand, if $\frac{\gamma}{\theta_0}$ is odd, we can easily check that

$$U_N(\theta_0,\gamma)U_\sigma = \mathrm{sgn}(\sigma)U_\sigma U_N(\theta_0,\gamma).$$

From these facts, we have the desired results. □

Let $A$ be a self-adjoint operator acting in $\otimes^N\mathcal{H}$. We denote the pure point spectrum of $A$ by $\sigma_{\mathrm{p}}(A)$. Then we introduce the closed subspaces of $\otimes^N\mathcal{H}$ by

$$\mathcal{H}(A) := \bigoplus_{\lambda\in\sigma_{\mathrm{p}}(A)} \ker(A-\lambda)$$

and

$$\mathcal{H}_{\mathrm{s}}(A) := S_N\mathcal{H}(A), \quad \mathcal{H}_{\mathrm{as}}(A) := A_N\mathcal{H}(A).$$

It is clear that

$$\begin{aligned}\mathcal{H}_{\mathrm{s}}(A) &= \bigoplus_{\lambda\in\sigma_{\mathrm{p}}(A)} \ker_{\mathrm{s}}(A-\lambda),\\ \mathcal{H}_{\mathrm{as}}(A) &= \bigoplus_{\lambda\in\sigma_{\mathrm{p}}(A)} \ker_{\mathrm{as}}(A-\lambda),\end{aligned}$$

where $\ker_{\mathrm{s}}(A-\lambda) := S_N\ker(A-\lambda)$ and $\ker_{\mathrm{as}}(A-\lambda) := A_N\ker(A-\lambda)$.

**Proposition 4.3** *Let $\overline{H}$ and $\overline{H}_0$ be the Schrödinger operators defined in the preceding subsection. Suppose that the conditions (A.1) are satisfied. Moreover, if $\frac{\theta_0}{\gamma}$ is odd, we have the following.*

(i) *For each $\lambda\in\sigma_{\mathrm{p}}(\overline{H}_0)$,*

$$\begin{aligned}\ker_{\mathrm{s}}(\overline{H}-\lambda) &= U_N(\theta_0,\gamma)\ker_{\mathrm{as}}(\overline{H}_0-\lambda),\\ \ker_{\mathrm{as}}(\overline{H}-\lambda) &= U_N(\theta_0,\gamma)\ker_{\mathrm{s}}(\overline{H}_0-\lambda).\end{aligned}$$

(ii)

$$\begin{aligned}\mathcal{H}_{\mathrm{s}}(\overline{H}) &= U_N(\theta_0,\gamma)\mathcal{H}_{\mathrm{as}}(\overline{H}_0),\\ \mathcal{H}_{\mathrm{as}}(\overline{H}) &= U_N(\theta_0,\gamma)\mathcal{H}_{\mathrm{s}}(\overline{H}_0).\end{aligned}$$

*Proof.* These are simple applications of Theorem 4.1 and Proposition 4.2. □


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# References


[1] A. Arai, *Momentum operators with gauge potential, local quantization of magnetic flux, and representation of canonical commutation relations*, J. Math. Phys. **33**, 3374-3378 (1992).

[2] A. Arai, *gauge theory on a nonsimply connected domain and representations of canonical commutation relations*, J. Math. Phys. **36**, 2569-2580 (1995).

[3] A. Arai, *Canonical commutation relations, the Weierstrass Zeta function, and infinite dimensional Hilbert space representation of the quantum group $U_q(sl_2)$*, J. Math. Phys. **37**, 4203-4218 (1996).

[4] A. Arai, *Representation-theoretic aspects of two-dimensional quantum system in singular vector potentials: Canonical commutation relations, quantum algebras, and reduction to lattice quantum system*, J. Math. Phys. **39**, 2476-2498 (1998).

[5] C. B. Hanna, R. B. Laughlin and A. L. Fetter, *Quantum mechanics of the fractional-statistics gas: Hartree-Fock approximation*, Phys. Rev. B. **40**, 8745-8758 (1989).

[6] C. B. Hanna, R. B. Laughlin and A. L. Fetter, *Quantum mechanics of the fractional statistics gas: Particle-hole interaction*, Phys. Rev. B. **43**, 309-319 (1991).

[7] H. Kurose and H. Nakazato, *Geometric construction of $*$-representation of the Weyl algebra with degree 2*, Publ. RIMS Kyoto Univ. **32**, 555-579 (1996).

[8] M. Reed and B. Simon, "Methods of Modern Mathematical Physics Vol.I," Academic Press, New York, 1972.

[9] M. Reed and B. Simon, "Methods of Modern Mathematical Physics Vol.IV," Academic Press, New York, 1975.

[10] K. Schmüdgen, "Unbounded Operator Algebra and Representation Theory" Birkhäuser Verlag, Basel. Boston. Berlin, 1990.