Reconstruction of Inclusions for the Inverse Boundary Value Problem with Mixed Type Boundary Condition and Source Term

Yuki Daido and Gen Nakamura

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Reconstruction of Inclusions for the Inverse Boundary Value Problem with Mixed Type Boundary Condition and Source Term

Yuki Daido and Gen Nakamura*
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan

1 Introduction

Let $\Omega \subset \boldsymbol{R}^n$ $(n \geq 2)$ be a bounded domain with C^2 boundary Γ. Ω is considered as a conductive medium with conductivity:

$$\gamma = \gamma_0 + \chi_D \gamma_1 \tag{1.1}$$

with matrices $\gamma_0 = (\gamma_{0ij}) \in C^{0,1}(\overline{\Omega})$, $\gamma_1 = (\gamma_{1ij}) \in L^\infty(\Omega)$. Here D is a bounded domain with Lipschitz boundary ∂D such that $\overline{D} \subset \Omega$, $\Omega \setminus \overline{D}$ is connected, χ_D is the characteristic function of D and $C^{0,1}(\overline{\Omega})$ is the space of functions which are Lipschitz continuous on $\overline{\Omega}$. We assume that $\gamma = \big(\gamma_{ij}(x)\big)$ and $\gamma_0 = \big(\gamma_{0ij}(x)\big)$ are symmetric matrices satisfying

$$\begin{cases} \displaystyle\sum_{i,j=1}^n \gamma_{0ij}(x)\xi_i\xi_j \geq C_1|\xi|^2 & \big(\xi = (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n,\ x \in \overline{\Omega}\big) \\ \displaystyle\sum_{i,j=1}^n \gamma_{ij}(x)\xi_i\xi_j \geq C_1|\xi|^2 & \big(\xi = (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n,\ \text{a.e. } x \in \overline{\Omega}\big) \end{cases} \tag{1.2}$$

for some constant $C_1 > 0$. Moreover, we assume that for any $a \in \partial D$, there exists a $\delta > 0$ such that either

$$\sum_{i,j=1}^n \gamma_{1ij}\xi_i\xi_j \geq C_2|\xi|^2 \quad \big(\xi = (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n,\ \text{a.e. } x \in B_\delta(a) \cap D\big) \tag{1.3}$$

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or

$$\sum_{i,j=1}^{n} \gamma_{1ij}\xi_i\xi_j \le -C_2|\xi|^2 \quad \big(\xi = (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n, \text{ a.e. } x \in B_\delta(a) \cap D\big) \tag{1.4}$$

holds for some constant $C_2 > 0$, where $B_\delta(a) := \{x \in \boldsymbol{R}^n; |x-a| < \delta\}$.

Let Γ consist of two parts. That is

$$\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}, \tag{1.5}$$

where Γ_D, Γ_N are open subsets of Γ such that $\Gamma_D \cap \Gamma_N = \phi$, $\Gamma_D \ne \phi$, $\Gamma_N \ne \phi$ and for $n \ge 3$, the boundaries $\partial\Gamma_D$ of Γ_D and $\partial\Gamma_N$ of Γ_N are C^2.

Consider the mixed type boundary value problem:

$$\begin{cases} \big(L_\gamma u\big)(x) := \displaystyle\sum_{i,j=1}^{n} \partial_i\big(\gamma_{ij}(x)\partial_j u(x)\big) = F(x) \text{ in } \Omega \\ u = f \text{ on } \Gamma_D, \quad \partial_{L_\gamma} u = g \text{ on } \Gamma_N \end{cases} \tag{1.6}$$

for given $f \in \overline{H}^{\frac{1}{2}}(\Gamma_D), g \in \overline{H}^{-\frac{1}{2}}(\Gamma_N), F \in L^2(\Omega)$ where $x = (x_1, \cdots, x_n)$, $\partial_i := \partial/\partial x_i$ and

$$\big(\partial_{L_\gamma} u\big)(x) := \sum_{i,j=1}^{n} \nu_i\gamma_{ij}(x)\partial_i u(x) \tag{1.7}$$

with the unit outer normal vector $\nu = (\nu_1, \cdots, \nu_n)$ of Γ. Here we have used the notations given in [3] to denote Sobolev spaces.

By Appendix A, there exists a unique solution $u = u(f, g, F) \in \overline{H}^1(\Omega)$ to (1.6) with the estimate:

$$\|u\|_{\overline{H}^1(\Omega)} \le C\Big(\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} + \|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|F\|_{L^2(\Omega)}\Big), \tag{1.8}$$

where the constant $C > 0$ does not depend on f, g, F.

Moreover, even for $F \in W^*$ with $W := \big\{w \in \overline{H}^1(\Omega);\ w = 0 \text{ on } \Gamma_D\big\}$ and $\operatorname{supp} F \subset \Omega$, we have a similar result except that $\|F\|_{L^2(\Omega)}$ in (1.8) has to be replaced by $\|F\|_{W^*}$. Hereafter, the norm $\|\cdot\|_W$ and inner product $(\ ,\ )$ of W are those of $\overline{H}^1(\Omega)$, and the norm of the dual space W^* of W is denoted by $\|\cdot\|_{W^*}$.

Next, we define the Dirichlet to Neumann map Λ_γ and the Neumann to Dirichlet map Π_γ as follows.

Definition 1.1 *Let $u(f, g, F)$ be the solution to (1.8).*

(i) Fixing g and F, define $\Lambda_\gamma : \overline{H}^{\frac{1}{2}}(\Gamma_D) \to \overline{H}^{-\frac{1}{2}}(\Gamma_D)$ by

$$\Lambda_\gamma f := \partial_{L_\gamma} u(f, g, F) \ \text{ on } \Gamma_D. \tag{1.9}$$

(ii) Fixing f and F, define $\Pi_\gamma : \overline{H}^{-\frac{1}{2}}(\Gamma_N) \to \overline{H}^{\frac{1}{2}}(\Gamma_N)$ by

$$\Pi_\gamma g := u(f, g, F) \ \text{ on } \Gamma_N. \tag{1.10}$$

Remark 1.2 *The trace of $\partial_{L_\gamma} u(f, g, F) \in \overline{H}^{\frac{1}{2}}(\Omega)$ exists, because $F \in L^2(\Omega)$ or $F \in W^*$ with* $\operatorname{supp} F \subset \Omega$.

Now, we consider the two kinds of inverse problems *(IP1)* and *(IP2)*:

(IP1) Suppose γ_0 is known and γ_1, D are unknown. Reconstruct D from Λ_γ.
(IP2) Suppose γ_0 is known and γ_1, D are unknown. Reconstruct D from Π_γ.

Theorem 1.3 *There are reconstruction procedures for the both inverse problems (IP1) and (IP2).*

Remark 1.4

Let Ω_1 be subdomain of Ω such that $D \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega$, $\Omega \setminus \overline{\Omega_1}$ and $\Omega_1 \setminus \overline{D}$ are connected and its boundary $\partial\Omega_1$ is Lipschitz smooth. Define the Dirichlet to Neumann map $\Lambda_{1\gamma} : \overline{H}^{\frac{1}{2}}(\partial\Omega_1) \to \overline{H}^{-\frac{1}{2}}(\partial\Omega_1)$ by $\Lambda_{1\gamma}\varphi := \partial_{L_\gamma} v(\varphi)$ on $\partial\Omega_1$ for any $\varphi \in \overline{H}^{\frac{1}{2}}(\partial\Omega_1)$ where $v = v(\varphi) \in \overline{H}^1(\Omega_1)$ is the solution to $L_\gamma v = 0$ in Ω_1, $v = \varphi$ on $\partial\Omega_1$. Knowing $\Lambda_{1\gamma}, D$ can be reconstructed from $\Lambda_{1\gamma}$ by an argument analogous to that given [5]. However, to relate $\Lambda_{1\gamma}$ to Λ_γ or Π_γ, the usual way is to solve Cauchy problem iteratively which is very ill-posed. Therefore, we focuss on obtaining a reconstruction procedure which directly uses Λ_γ or Π_γ.

The probe method for the inverse boundary value problem with mixed type boundary condition was shown in [6] for identifying cracks. Also, when γ_1 is conformal to γ_0 and γ_1 satisfies some Hölder continuity near the boundary ∂D of the inclusion in two or three dimensional medium Ω, an analogous result was obtained in [1]. A new ingredient of this paper is an application of the Green function obtained in [2] (see Appendix B) for analyzing the behavior of the indicator function given later in the next section. If $\Gamma_N = \phi$ and there is no source term, we can analyze the behavior of the indicator function without using this Green function. We also have to point out the closely related works done by Potthast and his collaborators ([10]) which use singular solutions for reconstructing unknown scatterer.

The inverse boundary value problem for identifying inclusions inside a conductive medium was initiated by V. Isakov [8]. He proved the uniqueness for identifying D and γ_1 when $\Gamma_N = \phi, F = \phi$ and γ_0, γ_1 are isotropic.

The unique continuation property is essentially used in most of argument for identifying the unknown boundary inside a known medium. $\gamma_0 \in C^{0,1}(\overline{\Omega})$ is the minimum regularity assumption for L_{γ_0} to have the unique continuation property.

Therefore, our result given here is almost a final result about the uniqueness and reconstruction for identifying D.

2 Reconstruction procedure

Definition 2.1 (needle) *We call a nonselfintersecting piecewise C^1 curve $\mathcal{C} := \{c(t); 0 \le t \le 1\}$ joining $c(0), c(1) \in \Gamma$ needle if it satisfies $\mathcal{C} \setminus \{c(0), c(1)\} \subset \Omega$.*

Definition 2.2 (singular solution)

(i) Fix $x^0 \in \Omega$ and $G(x - x^0) \in \mathcal{D}'(\boldsymbol{R}^n)$ be a fundamental solution of

$$\nabla \cdot \big(\gamma_0(x^0)\nabla G(x - x^0)\big) + \delta(x - x^0) = 0 \ \ in \ \boldsymbol{R}^n. \tag{2.1}$$

(ii) Let $H_j(x, x^0) \in \mathcal{D}'(\boldsymbol{R}^n_x)$ $(j = 1, 2)$ be solutions of

$$L_{\gamma_0} H_j(x, x^0) + \delta(x - x^0) = 0 \ \ in \ \Omega \tag{2.2}$$

such that

$$H_j(x, x^0) - G(x - x^0) \in \overline{H}^1(\Omega) \tag{2.3}$$

and

$$\begin{cases} \partial_{L_{\gamma_0}} H_1(x, x^0) = 0 \ \ on \ \Gamma_N \\ H_2(x, x^0) = 0 \ \ on \ \Gamma_D. \end{cases} \tag{2.4}$$

We call each $H_j(x, x^0)$ singular solution.

Remark 2.3 *The construction of singular solution can be done similarly as Lemma 3 in [7]*

Let $\mathcal{C} := \{c(t); 0 \le t \le 1\}$ be a needle. By the Runge's approximation theorem given in Appendix C, there exist sequences of approximate functions $\{v_{1k}\}, \{v_{2k}\} \subset \overline{H}^1(\Omega)$ such that $v_{jk} \to v_j' + H_j\big(\,\cdot\,, c(t)\big)$ $(k \to \infty)$ in $\overline{H}^1_{\mathrm{loc}}(\Omega \setminus \mathcal{C}_t)$ for each j $(j = 1, 2)$,

$$\begin{cases} L_{\gamma_0} v_{1k} = F \text{ in } \Omega \\ \partial_{L_{\gamma_0}} v_{1k} = g \text{ on } \Gamma_N \end{cases} \tag{2.5}$$

and

$$\begin{cases} L_{\gamma_0} v_{2k} = F \text{ in } \Omega \\ v_{2k} = f \text{ on } \Gamma_D \end{cases} \tag{2.6}$$

where $\mathcal{C}_t := \{c(s); 0 \le s \le t\}$ and $v_j' \in \overline{H}^1(\Omega)$ $(j = 1, 2)$ are the solutions to

$$\begin{cases} L_{\gamma_0} v_1' = F \quad \text{in } \Omega \\ v_1' = 0 \quad \text{on } \Gamma_D, \quad \partial_{L_{\gamma_0}} v_1' = g \quad \text{on } \Gamma_N \end{cases} \tag{2.7}$$

and

$$\begin{cases} L_{\gamma_0} v_2' = F \quad \text{in } \Omega \\ v_2' = f \quad \text{on } \Gamma_D, \quad \partial_{L_{\gamma_0}} v_2' = 0 \quad \text{on } \Gamma_N \end{cases} \tag{2.8}$$

(see Appendix C for the details).

Definition 2.4 (indicator function) *Let $\mathcal{C} = \{c(t); 0 \le t \le 1\}$ be a needle, t $(0 < t < 1)$ satisfy $\mathcal{C}_t \cap \overline{D} = \phi$ and $\{v_{jk}\} \subset \overline{H}^1(\Omega)$ $(j = 1, 2)$ be the sequences of approximate functions given above. Then, for t satisfying $\mathcal{C}_t \cap \overline{D} = \phi$, we define two indicator functions $I_1(t, \mathcal{C})$ and $I_2(t, \mathcal{C})$ associated with (IP1) and (IP2):*

$$I_1(t, \mathcal{C}) := \lim_{k \to \infty} \big\langle (\Lambda_\gamma - \Lambda_{\gamma_0})\big(v_{1k}\big|_{\Gamma_D}\big), v_{1k}\big|_{\Gamma_D} \big\rangle_1 \tag{2.9}$$

and

$$I_2(t, \mathcal{C}) := \lim_{k \to \infty} \big\langle \big(\partial_{L_\gamma} v_{2k}\big)\big|_{\Gamma_N}, (\Pi_\gamma - \Pi_{\gamma_0})\big((\partial_{L_\gamma} v_{2k})\big|_{\Gamma_N}\big) \big\rangle_2 \tag{2.10}$$

where $\langle\,,\,\rangle_1$ and $\langle\,,\,\rangle_2$ are the pairings for the pair $\{\dot{H}^{-\frac{1}{2}}(\overline{\Gamma_D}), \overline{H}^{\frac{1}{2}}(\Gamma_D)\}$ and for the pair $\{\overline{H}^{-\frac{1}{2}}(\Gamma_N), \dot{H}^{\frac{1}{2}}(\overline{\Gamma_N})\}$, respectively.

Remark 2.5 *From (3.6) and (D.8) given later, we can see that the definitions of the indicator functions do not depend on the choice of* $\{v_{jk}\}$.

Definition 2.6 (first hitting time) *Let* $\mathcal{C} = \{c(t); 0 \le t \le 1\}$ *be a needle such that* $\mathcal{C} \cap D \neq \phi$. *We define* $T(\mathcal{C}, D)$ *by*

$$T(\mathcal{C}, D) := \sup\{\, t\,; 0 < t < 1,\ c(s) \notin \overline{D}\ \ (0 \le s < t)\}. \tag{2.11}$$

We call $T(\mathcal{C}, D)$ *the first hitting time of* $\mathcal{C}$ *to* D.

Definition 2.7 (detecting time) *Let* $\mathcal{C}$ *be as in Definition 2.6. For the indicator functions* $I_j(t, \mathcal{C})$ $(j = 1, 2)$, *we define their detecting times* $t_j(t, \mathcal{C})$ $(j = 1, 2)$ *by*

$$t_j(\mathcal{C}, D) := \sup\Big\{0 < t < 1;\ \sup_{0<s<t} |I_j(s, \mathcal{C})| < \infty\Big\}. \tag{2.12}$$

Then, we have our main theorem.

Theorem 2.8 *For each* j $(j = 1, 2)$, *we have*

$$T(\mathcal{C}, D) = t_j(\mathcal{C}, D) \quad \text{if } \mathcal{C} \cap \overline{D} \neq \phi. \tag{2.13}$$

Since we can reconstruct D by knowing $t_j(\mathcal{C}, D)$ for all possible $\mathcal{C}$, Theorem 2.8 implies Theorem 1.3.

3 Estimates of indicator functions

In this Section we give some estimates for the indicator functions $I_j(t, \mathcal{C})$ $(j = 1, 2)$.

Let $u_{jk} \in \overline{H}^1(\Omega)$ $(j = 1, 2;\ k \in \boldsymbol{N})$ be

$$\begin{cases} u_{1k} := u\big(v_{1k}\big|_{\Gamma_D}, g, F\big) \\ u_{2k} := u\big(f, (\partial_{L_{\gamma_0}} v_{2k})\big|_{\Gamma_N}, F\big), \end{cases} \tag{3.1}$$

where $u = u(f, g, F)$ is the solution to (1.6). Also, let

$$w_{jk} := u_{jk} - v_{jk} \quad (j = 1, 2;\ k \in \boldsymbol{N}). \tag{3.2}$$

Then, we have

$$\begin{cases} L_\gamma w_{jk} = \nabla \cdot \big((\gamma_0 - \gamma)\nabla v_{jk}\big) \text{ in } \Omega \\ w_{jk} = 0 \text{ on } \Gamma_D, \quad \partial_{L_\gamma} w_{jk} = 0 \text{ on } \Gamma_N. \end{cases} \tag{3.3}$$

More precisely, $w_{jk} \in W$ is the solution of the variational equation:

$$\int_\Omega \gamma \nabla w_{jk} \cdot \nabla \varphi \, dx = \int_\Omega (\gamma_0 - \gamma) \nabla v_{jk} \cdot \nabla \varphi \, dx \quad (\varphi \in W). \tag{3.4}$$

Since

$$\sup_{\|\varphi\|_W \le 1} \left| \int_\Omega (\gamma_0 - \gamma) \nabla (v_{jk} - v_{jl}) \cdot \nabla \varphi \, dx \right| \le \|\gamma_1\|_{L^\infty(D)} \|v_{jk} - v_{jl}\|_{\overline{H}^1(D)} \to 0 \tag{3.5}$$

as $k, l \to \infty$ by $\mathcal{C}_t \cap \overline{D} = \phi$ and $v_{jk} \to v_j' + H_j(\,\cdot\,, c(t)) \quad (k \to \infty)$ in $\overline{H}^1_{\text{loc}}(\Omega \setminus \mathcal{C}_t)$, we have from (1.8)

$$\|w_{jk} - w_{jl}\|_{\overline{H}^1(\Omega)} \to 0 \quad (k, l \to \infty). \tag{3.6}$$

Hence, there exist limits $w_j := \lim_{k \to \infty} w_{jk} \in \overline{H}^1(\Omega)$ $(j = 1, 2)$ and they satisfy

$$\begin{cases} L_\gamma w_j = \nabla \cdot \Big((\gamma_0 - \gamma)\nabla \Big(v_j' + H_j\big(\,\cdot\,, c(t)\big)\Big)\Big) \quad \text{in } \Omega \\ w_j = 0 \quad \text{on } \Gamma_D, \quad \partial_{L_\gamma} w_j = 0 \quad \text{on } \Gamma_N. \end{cases} \tag{3.7}$$

Therefore, by Theorem D.1 in Appendix D, we have

$$\begin{aligned} \int_D {\gamma_0}^{-1} \gamma_1 \gamma^{-1} (\gamma_0 \nabla V_1) \cdot (\gamma_0 \nabla V_1) \, dx - \int_\Omega F w_1 \, dx + \langle g, w_1 \rangle_2 \le I_1(t, \mathcal{C}) \\ \le \int_D \gamma_1 \nabla V_1 \cdot \nabla V_1 \, dx - \int_\Omega F w_1 \, dx + \langle g, w_1 \rangle_2 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \int_D {\gamma_0}^{-1} \gamma_1 \gamma^{-1} (\gamma_0 \nabla V_2) \cdot (\gamma_0 \nabla V_2) \, dx - \int_\Omega F w_2 \, dx + \langle \partial_{L_{\gamma_0}} w_2, f \rangle_1 \le I_2(t, \mathcal{C}) \\ \le \int_D \gamma_1 \nabla V_2 \cdot \nabla V_2 \, dx - \int_\Omega F w_2 \, dx - \langle \partial_{L_{\gamma_0}} w_2, f \rangle_1, \end{aligned} \tag{3.9}$$

where V_j $(j = 1, 2)$ are defined by $V_j := v_j' + H_j(\,\cdot\,, c(t))$ $(j = 1, 2)$.

4 Behavior of the indicator functions

In this Section we analyze the behavior of the indicator functions $I_j(t, \mathcal{C})$ as $t \uparrow T(\mathcal{C}, D)$ when $\mathcal{C} \cap \overline{D} \neq \phi$. Hereafter, constants C, C' which will appear in the estimates are general constants.

Let $\mathcal{C} \cap \overline{D} \neq \phi$ and $0 < t < 1$ satisfy $\mathcal{C}_t \cap \overline{D} = \phi$.

Lemma 4.1 *There exists a constant $M > 0$ independent of t such that*

$$\|w_j\|_{L^2(\Omega)} \leq M \quad (j = 1, 2) \quad \text{as } t \uparrow T(\mathcal{C}, D). \tag{4.1}$$

Proof For simplicity, put $t_0 := T(\mathcal{C}, D)$ and $a = c(t_0)$. For $0 < t < t_0$, $w = w\big(x, c(t)\big)$ be the Green function given in Appendix B for $A = L_\gamma$.
w satisfies

$$\begin{cases} L_\gamma w + \delta\big(\,\cdot\, - c(t)\big) = 0 & \text{in } \Omega \\ w = 0 \quad \text{on } \Gamma. \end{cases} \tag{4.2}$$

Define V_j $(j = 1, 2)$ by

$$V_j = w_j + v_j' + H_j\big(\,\cdot\, - c(t)\big) \quad (j = 1, 2). \tag{4.3}$$

Then, from (2.2), (2.4), (2.7), (2.8) and (3.7), we have

$$\begin{cases} L_\gamma V_1 + \delta\big(\,\cdot\, - c(t)\big) = F & \text{in } \Omega \\ V_1 = H_1\big(\,\cdot\,, c(t)\big) \quad \text{on } \Gamma_D, \quad \partial_{L_\gamma} V_1 = g & \text{on } \Gamma_N \end{cases} \tag{4.4}$$

and

$$\begin{cases} L_\gamma V_2 + \delta\big(\,\cdot\, - c(t)\big) = F & \text{in } \Omega \\ V_2 = f \quad \text{on } \Gamma_D, \quad \partial_{L_\gamma} V_2 = \partial_{L_\gamma} H_2\big(\,\cdot\,, c(t)\big) & \text{on } \Gamma_N. \end{cases} \tag{4.5}$$

Hence, defining Z_j $(j = 1, 2)$ by

$$Z_j = V_j - w, \tag{4.6}$$

we have

$$\begin{cases} L_\gamma Z_1 = F & \text{in } \Omega \\ Z_1 = H_1\big(\,\cdot\,, c(t)\big) \quad \text{on } \Gamma_D, \quad \partial_{L_\gamma} Z_1 = g - \partial_{L_\gamma} w & \text{on } \Gamma_N \end{cases} \tag{4.7}$$

and

$$\begin{cases} L_\gamma Z_2 = F & \text{in } \Omega \\ Z_2 = f \quad \text{on } \Gamma_D, \quad \partial_{L_\gamma} Z_2 = \partial_{L_\gamma} H_2\big(\,\cdot\,, c(t)\big) - \partial_{L_\gamma} w & \text{on } \Gamma_N. \end{cases} \tag{4.8}$$

Next we prove that $\partial_{L_\gamma} w$ is uniformly bounded in $\overline{H}^{-\frac{1}{2}}(\Gamma)$ as $t \uparrow t_0$. In order to do that let $\eta \in C_0^\infty(\Omega)$, $\eta = 1$ in an open neighborhood of $\overline{D}$ and $\zeta := 1 - \eta$. Then, we have

$$L_\gamma(\zeta w) = \sum_{i,j=1}^{n} (2\gamma_{0ij}\partial_i\zeta\partial_j w + \partial_i\gamma_{0ij}\partial_j\zeta w). \tag{4.9}$$

Here, we can assume $c(t) \notin \operatorname{supp}\zeta$. Hence, from (B.6), the right hand side of (4.9) is uniformly bounded in $L^2(\Omega)$ as $t \uparrow t_0$. Then, $(\zeta w)\big|_\Gamma = 0$ and the well-posedness of the Dirichlet boundary value problem imply $\partial_{L_\gamma} w = \partial_{L_\gamma}(\zeta w)$ is uniformly bounded in $\overline{H}^{-\frac{1}{2}}(\Gamma)$ as $t \uparrow t_0$.

Now, by (1.6) and what we have just proven, we have that for each j $(j = 1, 2)$, Z_j is uniformly bounded in $\overline{H}^1(\Omega)$ as $t \to t_0$. Hence, by (2.3), (4.3), (4.6) and (B.12), $w_j = Z_j + w - v_j' - H(\,\cdot\,, c(t))$ is uniformly bounded in $L^2(\Omega)$ as $t \uparrow t_0$. □

Let $\alpha \in C_0^\infty(\Omega)$ satisfy $\alpha = 1$ in an open neighborhood of $\overline{D}$ and $\widetilde{w}_j := w_j - \alpha w_j$ $(j = 1, 2)$. From (3.7) and $\operatorname{supp}(\gamma - \gamma_0) \subset \overline{D}$, we have

$$(1 - \alpha)L_\gamma w_j = 0 \quad \text{in } \Omega \tag{4.10}$$

and

$$L_\gamma \alpha = L_{\gamma_0}\alpha. \tag{4.11}$$

Then $\widetilde{w}_j$ satisfies

$$\begin{cases} L_\gamma \widetilde{w}_j = F_j & \text{in } \Omega \\ \widetilde{w}_j = 0 \quad \text{on } \Gamma_D, & \partial_{L_\gamma}\widetilde{w}_j = 0 \quad \text{on } \Gamma_N, \end{cases} \tag{4.12}$$

where $F_j := -(L_\gamma\alpha)w_j - 2\gamma\nabla\alpha\cdot\nabla w_j$ satisfies $\operatorname{supp} F_j \subset \Omega$ and $\|F_j\|_{W^*}$ is uniformly bounded for any t $(0 < t - t_0 \le \eta)$. Therefore, by the continuity of the trace and $\partial_{L_{\gamma_0}} w_2 = \partial_{L_\gamma}\widetilde{w}_2$,

$$\|w_1\|_{\overline{H}^{\frac{1}{2}}(\Gamma_N)} + \|\partial_{L_{\gamma_0}} w_2\|_{\dot{H}^{-\frac{1}{2}}(\overline{\Gamma_D})} \le M \quad \text{as } t \uparrow T(\mathcal{C}, D) \tag{4.13}$$

for another constant $M > 0$ independent of t.

Now it is easy to see that the dominant parts of $\int_D \gamma_0^{-1}\gamma_1\gamma^{-1}(\gamma_0\nabla V_j)\cdot(\gamma_0\nabla V_j)\,dx$ and $\int_D \gamma_1\nabla V_j\cdot\nabla V_j\,dx$ are $\int_{D\cap B_\delta(a)} \gamma_0^{-1}\gamma_1\gamma^{-1}\Big(\gamma_0\nabla G_j\big(\,\cdot\,-c(t)\big)\Big)\cdot\Big(\gamma_0\nabla G_j\big(\,\cdot\,-c(t)\big)\Big)\,dx$ and $\int_{D\cap B_\delta(a)} \gamma_1\nabla G_j\big(\,\cdot\,-c(t)\big)\cdot\nabla G_j\big(\,\cdot\,-c(t)\big)\,dx$ which are positive or negative according to (1.3) or (1.4) and blow up as $t \uparrow t_0$. Here we have used the identity:

$$\gamma_0^{-1}\gamma_1(\gamma_0+\gamma_1)^{-1} = (\gamma_0+\gamma_1)^{-1}\gamma_1(\gamma_0+\gamma_1)^{-1} + (\gamma_0+\gamma_1)^{-1}\gamma_1\gamma_0^{-1}\gamma_1(\gamma_0+\gamma_1)^{-1} \tag{4.14}$$

Therefore, by (3.8), (3.9), (4.1) and (4.13),

$$|I_j(t, \mathcal{C})| \to \infty \quad (t \uparrow t_0). \tag{4.15}$$

Finally, (2.13) can be proven by the standard argument given in [5], So we omit its proof.

Appendix

A Boundary value problem for forward problem

In this Appendix we give the proof of the well-posedness of the boundary value problem (1.6).

Theorem A.1 *If $\gamma \in L^\infty(\Omega)$ satisfies $\gamma \geq \delta$ in Ω, there exists a unique solution of (1.6). Moreover u satisfies (1.8).*

Proof For $f \in \overline{H}^{\frac{1}{2}}(\Gamma_D)$, there exists $\widetilde{f} \in \overline{H}^{\frac{1}{2}}(\Gamma)$ which is the extension of f.
Let $\widetilde{u} \in \overline{H}^1(\Omega)$ be the solution to

$$\text{(A.1)} \qquad \begin{cases} L_\gamma \widetilde{u} = 0 \text{ in } \Omega \\ \widetilde{u} = \widetilde{f} \text{ on } \Gamma. \end{cases}$$

Then, it is well known that

$$\text{(A.2)} \qquad \|\widetilde{u}\|_{\overline{H}^1(\Omega)} \leq C\|\widetilde{f}\|_{\overline{H}^{\frac{1}{2}}(\Gamma)} \leq C'\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)}$$

and

$$\text{(A.3)} \qquad \|\partial_{L_\gamma}\widetilde{u}\|_{\overline{H}^{-\frac{1}{2}}(\Gamma)} \leq C\|\widetilde{u}\|_{\overline{H}^1(\Omega)} \leq C'\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)}$$

for some constant $C, C' > 0$ independent of f.
Put $v := u - \widetilde{u} \in W$. This v has to satisfy

$$\text{(A.4)} \qquad \begin{cases} L_\gamma v = F \quad \text{in } \Omega \\ v = 0 \quad \text{on } \Gamma_D, \quad \partial_{L_\gamma} v = \widetilde{g} \quad \text{on } \Gamma_N, \end{cases}$$

where $\widetilde{g} = g - \partial_{L_\gamma}\widetilde{u}$. Define

$$\text{(A.5)} \qquad \langle G, w\rangle := \langle F, w\rangle - \int_{\Gamma_N} \widetilde{g} w \, d\Gamma$$

and

$$\text{(A.6)} \qquad B[v,w] := \int_\Omega \gamma \nabla v \cdot \nabla w \, dx$$

for any $v, w \in W$.
By the Schwarz inequality,

$$\text{(A.7)} \qquad |B[v,w]| \leq \int_\Omega |\gamma||\nabla v||\nabla w| \, dx \leq M\|v\|_{\overline{H}^1(\Omega)}\|w\|_{\overline{H}^1(\Omega)}.$$

By the Poincaré inequality

$$B[v,v] \geq \int_\Omega \gamma|\nabla v|^2\,dx \geq \delta\|\nabla v\|_{L^2(\Omega)} \geq \delta'\|v\|_{\overline{H}^1(\Omega)} \tag{A.8}$$

for some constant $\delta' > 0$ independent of v, w.

Now we remind the Lax-Milgram theorem.

Theorem A.2 (Lax-Milgram) *Let X be a real Hilbert space and $B : X \times X \to \boldsymbol{R}$ be a bilinear map satisfying*

$$|B[x,y]| \leq \gamma\|x\|\|y\| \tag{A.9}$$

and

$$B[x,x] \geq \delta\|x\|^2, \tag{A.10}$$

then there exists a unique bounded linear bijective operator $S : X \to X$ such that

$$(x,y) = B[Sx,y] \quad \textit{with } \|S\| \leq \delta^{-1},\ \|S^{-1}\| \leq \gamma. \tag{A.11}$$

By applying Theorem A.2, there exists a unique bounded linear bijective operator $S : W \to W$ such that

$$(S^{-1}v, w) = B[v,w] \quad (v,w \in W) \quad \text{with } \|S\| \leq (\delta')^{-1},\ \|S^{-1}\| \leq M. \tag{A.12}$$

As immediate estimates, we have

$$|\langle F, w\rangle| \leq \|F\|_{W^*}\|w\|_W \tag{A.13}$$

and

$$\begin{aligned} \left|\int_{\Gamma_N} \widetilde{g}w\,d\Gamma\right| &\leq C\left(\|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|\partial_{L_\gamma}\widetilde{u}\|_{\overline{H}^{-\frac{1}{2}}(\Gamma)}\right)\|w\|_W \\ &\leq C'\left(\|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)}\right)\|w\|_W \end{aligned} \tag{A.14}$$

for some constants $C, C' > 0$ independent of f, g, F. Hence, by (A.13) and (A.14),

$$|\langle G, w\rangle| \leq C\left(\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} + \|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|F\|_{W^*}\right)\|w\|_W. \tag{A.15}$$

By the Riesz's representation theorem, the exists a unique $\widetilde{v} \in W$ such that

$$\langle G, w\rangle = -(\widetilde{v}, w), \quad \|\widetilde{v}\|_W = \|G\|_{W^*}. \tag{A.16}$$

Let $v \in W$ be $\widetilde{v} = S^{-1}v$.

Then, by (A.12),

$$B[v, w] + \langle G, w\rangle = 0 \quad (w \in W). \tag{A.17}$$

Therefore, v is the solution to (A.4). By the definition of v,

$$\begin{aligned} \|u\|_{\overline{H}^1(\Omega)} &\le \|v\|_{\overline{H}^1(\Omega)} + \|\widetilde{u}\|_{\overline{H}^1(\Omega)} \le C\Big(\|\widetilde{v}\|_{\overline{H}^1(\Omega)} + \|\widetilde{f}\|_{\overline{H}^{\frac{1}{2}}(\Gamma)}\Big) \\ &\le C'\Big(\|G\|_{W^*} + \|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)}\Big) \le C''\Big(\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} + \|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|F\|_{W^*}\Big) \end{aligned} \tag{A.18}$$

for some constants $C, C', C'' > 0$ independent of f, g, F. □

B The Green function

In this section we give the proof of the existence of the Green function which we used in Lemma 4.1. In [2], the existence is only proven for $n \ge 3$. So we have given here the proof of the existence including the case $n = 2$.

Let Ω be a bounded domain of $\boldsymbol{R}^n$ $(n \ge 2)$.

Definition B.1 *For a measurable set $A \subset \Omega$ and $u \in L^1(A)$, we define*

$$⨍_A u\,dx := \frac{1}{\mu(A)} \int_A u\,dx \tag{B.1}$$

where μ is Lebesgue measure in $\boldsymbol{R}^n$.

Definition B.2 *For $p > 0$, we define $L^p_*(\Omega)$ and $\|f\|_{L^p_*(\Omega)}$ by*

$$L^p_*(\Omega) := \big\{f : \Omega \to \boldsymbol{R} \cup \{\pm\infty\};\ \textit{measurable function such that } \|f\|_{L^p_*(\Omega)} < \infty\big\}, \tag{B.2}$$

$$\|f\|_{L^p_*(\Omega)} = \sup_{s>0}\Big\{s\,\mu(\{x \in \Omega;\ ; |f(x)| > s\})^{\frac{1}{p}}\Big\}. \tag{B.3}$$

Let $a_{ij} \in L^\infty(\Omega)$ $(1 \le i, j \le n)$ satisfy

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2 \quad (x \in \Omega,\ \xi = (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n) \tag{B.4}$$

and

$$\text{(B.5)} \qquad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\eta_j \le \Lambda|\xi||\eta| \quad (x \in \Omega,\ \xi = (\xi_1, \cdots, \xi_n),\ \eta = (\eta_1, \cdots, \eta_n) \in \boldsymbol{R}^n)$$

for some constants $0 < \lambda \le \Lambda < \infty$.

Theorem B.3 *There exists a nonnegative function* $K : \Omega \times \Omega \to \boldsymbol{R} \cup \{\infty\}$ *such that for each* $y \in \Omega$ *and any* $r > 0$

$$\text{(B.6)} \qquad K(\,\cdot\,, y) \in \overline{H}^1\big(\Omega \setminus \overline{B_r(y)}\big) \cap \dot{W}^{1,1}(\overline{\Omega})$$

and for all $\varphi \in C_0^\infty(\Omega)$

$$\text{(B.7)} \qquad a[K(\,\cdot\,, y), \varphi] = \varphi(y),$$

where $a[u,v] := \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x)\partial_i u \partial_j v\, dx$. *This is called Green function for* $A\cdot := \sum_{i,j=1}^{n} \partial_i\big(a_{ij}(x)\partial_j\,\cdot\big)$ *and it satisfies the following properties:*
For each fix $y \in \Omega$, *denote the function* $K(x) := K(x,y)$ *by* K. *Let* $\varepsilon \ge 4$ *and define* χ_i $(i = 1,2,3)$ *by*
(B.8)

$$\chi_1 = \begin{cases} \dfrac{\varepsilon}{2} & \text{for } n = 2 \\[2ex] \dfrac{n}{n-2} & \text{for } n \ge 3 \end{cases}, \quad \chi_2 = \begin{cases} \dfrac{2\varepsilon}{4+\varepsilon} & \text{for } n = 2 \\[2ex] \dfrac{n}{n-1} & \text{for } n \ge 3 \end{cases}, \quad \chi_3 = \begin{cases} \dfrac{\varepsilon}{4} & \text{for } n = 2 \\[2ex] n-2 & \text{for } n \ge 3. \end{cases}$$

Then, we have

$$\text{(B.9)} \qquad K \in L_*^{\chi_1}(\Omega) \quad \text{with } \|K\|_{L_*^{\chi_1}(\Omega)} \le C(n)\lambda^{-1}$$

for some constant $C(n) > 0$ *depending only on* n,

$$\text{(B.10)} \qquad \nabla K \in L_*^{\chi_2}(\Omega) \quad \text{with } \|\nabla K\|_{L_*^{\chi_2}(\Omega)} \le C(n, \lambda, \Lambda)$$

for some constant $C(n, \lambda, \Lambda) > 0$ *depending only on* n, λ, Λ,

$$\text{(B.11)} \qquad K \in \dot{W}^{1,p}(\overline{\Omega}) \quad \text{for each } 1 \le p \le \chi_2,$$

$$\text{(B.12)} \qquad K(x,y) \le C(n, \Lambda/\lambda)\lambda^{-1}|x-y|^{-\chi_3}.$$

Here, $C(n), C(n,\lambda,\Lambda)$ *and* $C(n,\Lambda/\lambda)$ *are positive constants which depend only on* n, $\{n, \lambda, \Lambda\}$ *and* $\{n, \Lambda/\lambda\}$, *respectively. Moreover,* $\dot{W}^{1,p}(\overline{\Omega})$ *is the Sobolev space with "$\dot{}$" having the same meaning as "$\dot{}$" of* $\dot{H}^{-\frac{1}{2}}(\overline{\Gamma_D})$.

Remark B.4 *For $n \geq 3$, the uniqueness of K is given in ([2]).*

Proof of Theorem B.3
Fix $y \in \Omega$ and $\rho > 0$. Write $B_\rho := B_\rho(y)$.

For the proof of Theorem B.3, we need the following Fact B.5 and Lemma B.6.

Fact B.5 *([9]) For $p > 1$,*

$$\|f\|_{L^p_*(\Omega)} \leq \|f\|_{L^p(\Omega)}. \tag{B.13}$$

$$\|f\|_{L^{p-q}(\Omega)} \leq \left(\frac{p}{q}\right)^{\frac{1}{p-q}} \mu(\Omega)^{\frac{q}{p(p-q)}} \|f\|_{L^p_*(\Omega)} \quad \textit{for } 0 < q \leq p-1. \tag{B.14}$$

Lemma B.6 *([2])*
Let $u \in \overline{H}^1(\Omega)$ satisfy $u \geq 0$ in Ω and

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x)\partial_i u \partial_j \varphi \leq 0 \quad \textit{for any } \varphi \in \dot{H}^1(\overline{\Omega}) \textit{ with } \varphi \geq 0 \textit{ in } \Omega. \tag{B.15}$$

Then, there exists a constant $C(n) > 0$ depending only on n, such that for $\alpha > 1$ and $B_\rho(x) \subset\subset \Omega$,

$$\sup_{B_{\frac{\rho}{2}}(x)} u^\alpha \leq C(n)\left(\frac{\alpha}{\alpha-1}\right)^2 \left(\frac{\Lambda}{\lambda}\right)^n \fint_{B_\rho(x)} u^\alpha \, dy. \tag{B.16}$$

We define T, which is bounded linear function on $\overline{H}^1(\Omega)$, by

$$T(\varphi) := \fint_{B_\rho} \varphi \, dx. \tag{B.17}$$

For any $u, v \in \dot{H}^1(\Omega)$,

$$|a[u,v]| \leq \Lambda \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq \Lambda \|u\|_{\overline{H}^1(\Omega)} \|v\|_{\overline{H}^1(\Omega)}, \tag{B.18}$$

$$a[u,u] \geq \lambda \|\nabla u\|^2_{L^2(\Omega)} \geq \lambda' \|u\|^2_{\overline{H}^1(\Omega)} \tag{B.19}$$

for some constant $\lambda' > 0$.

By the Lax-Milgram theorem and the Riesz representation theorem, there exists $G_\rho \in \dot{H}^1(\overline{\Omega})$ satisfying

$$a[G_\rho, \varphi] = \fint_{B_\rho} \varphi \, dx \tag{B.20}$$

for all $\varphi \in \dot{H}^1(\overline{\Omega})$. Taking $|G_\rho| \in \dot{H}^1(\overline{\Omega})$ as a test function,

$$a[G_\rho, G_\rho] = \fint_{B_\rho} G_\rho \, dx \leq \fint_{B_\rho} |G_\rho| \, dx = a[G_\rho, |G_\rho|]. \tag{B.21}$$

Put $M := \dfrac{a[G_\rho, |G_\rho|]}{a[G_\rho, G_\rho]} \geq 1$, then

$$a[G_\rho, G_\rho] = a\left[\frac{|G_\rho|}{M}, G_\rho\right] = a\left[G_\rho, \frac{|G_\rho|}{M}\right]. \tag{B.22}$$

From (B.22),

(B.23)

$$\begin{aligned}
a\left[\frac{|G_\rho|}{M}, \frac{|G_\rho|}{M}\right] &= \frac{1}{M^2} a[|G_\rho|, |G_\rho|] = \frac{1}{M^2} \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \partial_i |G_\rho| \partial_j |G_\rho| \, dx \\
&= \frac{1}{M^2} \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \partial_i G_\rho \partial_j G_\rho \, dx = \frac{1}{M^2} a[G_\rho, G_\rho] \leq a[G_\rho, G_\rho] = a\left[G_\rho, \frac{|G_\rho|}{M}\right].
\end{aligned}$$

Note that for $u \in \overline{H}^1(\Omega)$, $\nabla |u| \in L^2(\Omega)$ with $\nabla |u| = \begin{cases} \nabla u & \text{in } \{x \in \Omega;\, u > 0\} \\ 0 & \text{in } \{x \in \Omega;\, u = 0\} \\ -\nabla u & \text{in } \{x \in \Omega;\, u < 0\}. \end{cases}$

Then, we have

$$a\left[\frac{|G_\rho|}{M} - G_\rho, \frac{|G_\rho|}{M} - G_\rho\right] = a\left[\frac{|G_\rho|}{M} - G_\rho, \frac{|G_\rho|}{M}\right] - a\left[\frac{|G_\rho|}{M}, G_\rho\right] \leq 0. \tag{B.24}$$

Hence

$$G_\rho = \frac{|G_\rho|}{M} \geq 0. \tag{B.25}$$

At first, we prove

$$||G_\rho||_{L_*^{\chi_1}(\Omega)} \leq C(n) \lambda^{-1} \tag{B.26}$$

for some constant $C(n) > 0$ depending only on n.

Fixing $t > 0$, choose a test function $\varphi(x) = \left(\frac{1}{t} - \frac{1}{G_\rho(x)}\right)^+ \left(:= \max\left\{\frac{1}{t} - \frac{1}{G_\rho(x)}, 0\right\}\right)$. Then we have

(B.27)

$$\frac{1}{t} \geq \fint_{B_\rho} \varphi\, dx = \sum_{i,j=1}^n \int_\Omega a_{ij}(x)\partial_i G_\rho \partial_j \varphi\, dx = \sum_{i,j=1}^n \int_{\Omega_t} a_{ij}(x)\partial_i G_\rho \frac{\partial_j G_\rho}{{G_\rho}^2}\, dx \geq \lambda \int_{\Omega_t} \frac{|\nabla G_\rho|^2}{{G_\rho}^2}\, dx,$$

where $\Omega_t := \{x \in \Omega; G_\rho(x) > t\}$. By Sobolev's inequality,

$$\left(\int_{\Omega_t} \left|\log \frac{G_\rho}{t}\right|^{2\chi_1} dx\right)^{\frac{1}{\chi_1}} \leq C(n) \int_{\Omega_t} \left|\nabla \log \frac{G_\rho}{t}\right|^2 dx = C(n) \int_{\Omega_t} \frac{|\nabla G_\rho|^2}{{G_\rho}^2} \leq \frac{C(n)}{\lambda t} \tag{B.28}$$

for some constant $C(n) > 0$ depending only on n. Hence,

$$(\log 2)^2 \mu(\Omega_{2t})^{\frac{1}{\chi_1}} \leq \left(\int_{\Omega_{2t}} \left|\log \frac{G_\rho}{t}\right|^{2\chi_1} dx\right)^{\frac{1}{\chi_1}} \leq C(n)\lambda^{-1}t^{-1}. \tag{B.29}$$

Therefore,

$$2t\mu(\Omega_{2t})^{\frac{1}{\chi_1}} \leq \frac{2C(n)}{(\log 2)^2}\lambda^{-1} \tag{B.30}$$

and this gives (B.26).

Now, we take $G_\rho \in \overline{H}^1(\Omega)$ as a test function. Then we have

(B.31)

$$\begin{aligned}\lambda \int_\Omega |\nabla G_\rho|^2\, dx &\leq \sum_{i,j=1}^n a_{ij}(x)\partial_i G_\rho \partial_j G_\rho\, dx = \fint_{B_\rho} G_\rho\, dx = \frac{1}{\mu(B_\rho)} \int_{B_\rho} G_\rho\, dx \\ &\leq \frac{1}{\mu(B_\rho)} \|G_\rho\|_{L^{2\chi_1}(B_\rho)} \mu(B_\rho)^{1-\frac{1}{2\chi_1}} \leq C(n)\|\nabla G_\rho\|_{L^2(\Omega)} \mu(B_\rho)^{-\frac{1}{2\chi_1}} = C'(n)\|\nabla G_\rho\|_{L^2(\Omega)} \rho^{-\frac{n}{2\chi_1}}.\end{aligned}$$

for some constants $C(n), C'(n) > 0$ depending only on n.
Thus

$$\int_\Omega |\nabla G_\rho|^2\, dx \leq C'(n)\lambda^{-2}\rho^{-\frac{n}{\chi_1}}. \tag{B.32}$$

Next we will show

$$G_\rho(x) \leq C(n, \Lambda/\lambda)\lambda^{-1}|x-y|^{-\chi_3} \quad \text{if } |x-y| \geq 2\rho. \tag{B.33}$$

Let $R := |x-y| (\geq 2\rho)$.

First we consider the case: $\overline{B_{\frac{R}{2}}(x)} \subset \Omega$.
Since G_ρ is the solution of $Au = 0$ in $\Omega \setminus B_R$, we have

$$G_\rho(x)^\alpha \le C(\alpha, n, \Lambda/\lambda) \fint_{B_{\frac{R}{4}}(x)} G_\rho{}^\alpha \, dy \tag{B.34}$$

by using Lemma B.6. By (B.14) and (B.26), we have from $n \ge 3$,

$$\int_{B_{\frac{R}{4}}(x)} G_\rho{}^\alpha \, dx \le \frac{n}{n - \alpha(n-2)} \mu\Big(B_{\frac{R}{4}}\Big)^{1 - \frac{\alpha(n-2)}{n}} \|G_\rho\|_{L_*^{\frac{n}{n-2}}(\Omega)} \le C(n,\alpha)\lambda^\alpha R^{n-\alpha(n-2)} \tag{B.35}$$

for some constant $C(n, \alpha) > 0$ depending only on n, α. Hence, for (B.34) and (B.35),

$$G_\rho(x) \le C(n, \Lambda/\lambda)\lambda^{-1} R^{-(n-2)} \tag{B.36}$$

for some constant $C(n, \Lambda/\lambda) > 0$ depending only $n, \Lambda/\lambda$.
For $n = 2$, we have from (B.14) and (B.26),

$$\int_{B_{\frac{R}{4}}(x)} G_\rho{}^\alpha \, dx \le \frac{\varepsilon}{\varepsilon - 2\alpha} \mu\Big(B_{\frac{R}{4}}\Big)^{1 - \frac{2\alpha}{\varepsilon}} \|G_\rho\|_{L_*^{\frac{\varepsilon}{2}}(\Omega)} \le C(\alpha)\lambda^{-\alpha} R^{2 - \frac{4\alpha}{\varepsilon}} \tag{B.37}$$

for some constant $C(\alpha) > 0$ depending only on α.
Hence, for (B.34) and (B.37),

$$G_\rho(x) \le C(\Lambda/\lambda)\lambda^{-1} R^{-\frac{4}{\varepsilon}} \tag{B.38}$$

for some constant $C(\Lambda/\lambda) > 0$ depending only on Λ/λ.

Next we consider the case: $\overline{B_{\frac{R}{2}}(x)} \not\subset \Omega$. Consider a domain $\widetilde{\Omega}$ such that $\overline{B_{\frac{R}{2}}(x)} \subset \widetilde{\Omega}$ and extend operator A to $\widetilde{\Omega}$. Then, likewise G_ρ for A, we have $\widetilde{G}_\rho$ for this extended A. By restricting $\widetilde{G}_\rho$ to Ω, we have

$$A(G_\rho - \widetilde{G}_\rho) = 0 \text{ in } \Omega. \tag{B.39}$$

$G_\rho = 0 \le \widetilde{G}_\rho$ on $\partial\Omega$, therefore the maximal principle implies

$$G_\rho \le \widetilde{G}_\rho \quad \text{in } \Omega. \tag{B.40}$$

Since $\widetilde{G}_\rho$ satisfies (B.33), we have

$$\widetilde{G}_\rho(x) \le C(n, \Lambda/\lambda)\lambda^{-1} R^{-\chi_3}. \tag{B.41}$$

This completes the proof of (B.33).

Next we will show

$$\|\nabla G_\rho\|_{L_*^{\chi_2}(\Omega)} \le C(\lambda,\Lambda) \tag{B.42}$$

for some constant $C(\lambda,\Lambda)>0$ depending only on λ,Λ.

To show (B.42), we will show

$$\int_{\Omega\setminus B_R} |\nabla G_\rho|^2\,dx \le C(n,\lambda,\Lambda)R^{-\chi_4} \tag{B.43}$$

for some constant $C(n,\lambda,\Lambda)>0$ depending only on n,λ,Λ, where $\chi_4=\dfrac{8}{\varepsilon}$ for $n=2$, $\chi_4=n-2$ for $n\ge 3$.

Choose a test function $\eta\in C^\infty(\Omega)$ satisfying $\eta=1$ in $\Omega\setminus B_R$, $\eta=0$ in $B_{\frac{R}{2}}$ and $|\nabla\eta|\le\dfrac{C}{R}$ for some constant $C>0$.
Let $R\ge 4\rho$ and take $G_\rho\eta^2$ as a test function. Then, we have

(B.44)

$$\begin{aligned}0=\fint_{B_\rho} G_\rho\eta^2\,dx &= \sum_{i,j=1}^n\int_\Omega a_{ij}(x)\partial_iG_\rho\partial_j(G_\rho\eta^2)\,dx\\ &\ge \sum_{i,j=1}^n\int_{\Omega\setminus B_R} a_{ij}(x)\partial_iG_\rho\partial_jG_\rho\,dx + 2\sum_{i,j=1}^n\int_{\Omega\setminus B_{\frac{R}{2}}} a_{ij}(x)\partial_iG_\rho\partial_jG_\rho G_\rho\eta\,dx.\end{aligned}$$

This implies

(B.45)

$$\begin{aligned}\lambda\int_{\Omega\setminus B_R}|\nabla G_\rho|^2\,dx &\le \sum_{i,j=1}^n\int_{\Omega\setminus B_R} a_{ij}(x)\partial_iG_\rho\partial_jG_\rho\,dx \le 2\Lambda\int_{B_R\setminus B_{\frac{R}{2}}}|\nabla G_\rho|\frac{C}{R}G_\rho\eta\,dx\\ &\le \frac{2\Lambda C}{R}\int_{B_R\setminus B_{\frac{R}{2}}}|\nabla G_\rho|G_\rho\,dx \le \frac{\lambda}{2}\int_{B_R\setminus B_{\frac{R}{2}}}|\nabla G_\rho|^2\,dx + \frac{2\Lambda^2C^2}{\lambda R^2}\int_{B_R\setminus B_{\frac{R}{2}}} {G_\rho}^2\,dx.\end{aligned}$$

Hence

$$\int_{\Omega\setminus B_R}|\nabla G_\rho|^2\,dx \le 4\left(\frac{\Lambda}{\lambda}\right)^2\frac{C^2}{R^2}\int_{B_R\setminus B_{\frac{R}{2}}} {G_\rho}^2\,dx. \tag{B.46}$$

Combining this with (B.33), we have

$$\int_{\Omega\setminus B_R}|\nabla G_\rho|^2\,dx \le \begin{cases} C(\lambda,\Lambda)R^{-\frac{8}{\varepsilon}} & \text{for } n=2\\ C(n,\lambda,\Lambda)R^{-(n-2)} & \text{for } n\ge 3\end{cases} \tag{B.47}$$

for some constants $C(\lambda, \Lambda), C(n, \lambda, \Lambda) > 0$ depending only on $\{\lambda, \Lambda\}, \{n, \lambda, \Lambda\}$, respectively.

Next we consider the case $R < 4\rho$. From (B.32), we have

$$\int_{\Omega \setminus B_R} |\nabla G_\rho|^2 \, dx \le C(n)\lambda^{-2}\rho^{-\frac{n}{\chi_1}} = \begin{cases} C\lambda^{-2}\rho^{-\frac{4}{\varepsilon}} & \text{for } n = 2 \\ C(n)\lambda^{-2}\rho^{-(n-2)} & \text{for } n \ge 3 \end{cases} \tag{B.48}$$

for some constant $C > 0$ and some constant $C(n) > 0$ which depends on n. Observe that, for $n \ge 3$,

$$C(n)\lambda^{-2}\rho^{-(n-2)} \le C(n, \lambda)R^{-(n-2)} \tag{B.49}$$

for some constant $C(n, \lambda)$ depending only on n, λ and for $n = 2$,

$$C\lambda^{-2}\rho^{-\frac{4}{\varepsilon}} \le C(\lambda)R^{-\frac{4}{\varepsilon}} \le C(\lambda)R^{-\frac{8}{\varepsilon}} \tag{B.50}$$

for some constant $C(\lambda) > 0$ depending only on λ. Therefore we obtain (B.43).

Next we return to the proof of (B.42). For $n \ge 3$, we set $\Omega'_t := \{x \in \Omega;\ |\nabla G_\rho(x)| > t\}$ and $R = t^{-\frac{1}{n-1}}$ for fixed $t > 0$.
From (B.47) and (B.49),

$$t^2\mu(\Omega'_t \cap (\Omega \setminus B_R)) \le \int_{\Omega \setminus B_R} |\nabla G_\rho|^2 \, dx \le C(n, \lambda, \Lambda)t^{\frac{n-2}{n-1}} \tag{B.51}$$

for some constant $C(n, \lambda, \Lambda) > 0$ depending only on n, λ, Λ. That is

$$t\mu(\Omega'_t \cap (\Omega \setminus B_R))^{\frac{n-1}{n}} \le C(n, \lambda, \Lambda). \tag{B.52}$$

for some constant $C(n, \lambda, \Lambda) > 0$ depending only on n, λ, Λ. Combining this with

$$\mu(\Omega'_t \cap B_R) \le \mu(B_R) = C(n)R^n = (C'(n)t)^{\frac{n}{n-1}} \tag{B.53}$$

for some constants $C(n), C'(n) > 0$ depending only on n,

$$t\mu(\Omega'_t)^{\frac{n-1}{n}} \le C(n, \lambda, \Lambda). \tag{B.54}$$

for some constants $C(n, \lambda, \Lambda) > 0$ depending only on n, λ, Λ. Hence

$$||\nabla G_\rho||_{L_*^{\frac{n-1}{n}}(\Omega)} \le C(n, \lambda, \Lambda) \tag{B.55}$$

for some constants $C(n, \lambda, \Lambda) > 0$ depending only on n, λ, Λ.

For $n = 2$, we set $\Omega'_t = \{x \in \Omega;\ |\nabla G_\rho(x)| > t\}$ and $R = t^{-\frac{\varepsilon}{4+\varepsilon}}$ for fixed $t > 0$.
From (B.47) and (B.50),

$$t^2\mu(\Omega'_t \cap (\Omega \setminus B_R)) \le \int_{\Omega \setminus B_R} |\nabla G_\rho|^2 \, dx \le C(\lambda, \Lambda)t^{\frac{8}{4+\varepsilon}} \tag{B.56}$$

for some constant $C(\lambda, \Lambda) > 0$ depending only on λ, Λ. That is

$$t\mu(\Omega_t' \cap (\Omega \setminus B_R))^{\frac{4+\varepsilon}{2\varepsilon}} \le C'(\lambda, \Lambda) \tag{B.57}$$

for some constant $C'(\lambda, \Lambda) > 0$ depending only on λ, Λ. Combining this with

$$\mu(\Omega_t' \cap B_R) \le \mu(B_R) = \pi R^2 = \pi t^{-\frac{2\varepsilon}{4+\varepsilon}}, \tag{B.58}$$

$$t\mu(\Omega_t')^{\frac{4+\varepsilon}{2\varepsilon}} \le C'(\lambda, \Lambda) \tag{B.59}$$

for some constant $C'(\lambda, \Lambda) > 0$ depending only on λ, Λ. Hence

$$\|\nabla G_\rho\|_{L_*^{\frac{2\varepsilon}{4+\varepsilon}}(\Omega)} \le C'(\lambda, \Lambda) \tag{B.60}$$

for some constant $C'(\lambda, \Lambda) > 0$ depending only on λ, Λ.

Now by (B.13),

$$\|G_\rho\|_{L^{\chi_1}(\Omega)} \le C(n)\lambda^{-1} \quad \text{and} \quad \|\nabla G_\rho\|_{L^{\chi_2}(\Omega)} \le C(n, \lambda, \Lambda) \tag{B.61}$$

for some constants $C(n), C(n, \lambda, \Lambda) > 0$ depending only on n, $\{n, \lambda, \Lambda\}$, respectively. Note that $\chi_1 > \chi_2$ and $\chi_2 \ge 1$, because $\dfrac{2\varepsilon}{4+\varepsilon} < \dfrac{\varepsilon}{2}$, and $\dfrac{2\varepsilon}{4+\varepsilon} \ge 1$. Hence,

$$G_\rho \in \dot{W}^{1,\chi_2}(\overline{\Omega}). \tag{B.62}$$

Reminding Ω is bounded,

$$G_\rho \in \dot{W}^{1,s}(\overline{\Omega}) \quad \text{for } 1 \le s \le \chi_2. \tag{B.63}$$

Hence, fixing $s \in [1, \chi_2]$ and applying Rellich's compactness theorem, there exists $K \in \dot{W}^{1,s}(\overline{\Omega})$ such that

$$G_\rho \to K \quad \text{weakly in } \dot{W}^{1,s}(\overline{\Omega}) \quad (1 \le s \le \chi_2). \tag{B.64}$$

By (B.64) and,

$$\fint_{B_\rho} \varphi \, dx \to \varphi(y) \quad \text{as } \rho \to 0 \tag{B.65}$$

for any $\varphi \in C_0^\infty(\Omega)$, we have

$$a[K(\,\cdot\,, y), \varphi] = \varphi(y). \tag{B.66}$$

Furthermore, from (B.26) and (B.42), we get (B.9), (B.10). Also, from (B.47) and (B.48), we can prove (B.6).

Finally (B.12) is an easy consequence of (B.33), because $K(\,\cdot\,, y)$ is Hölder continuous in $\Omega \setminus \{y\}$. This follows from the famous De Giorgi-Nash-Moser regularity theorem, because $K(\,\cdot\,, y)$ is the solution of $Au = 0$ in $\Omega \setminus B_R(y)$. □

C Runge's theorem

In this Appendix two Runge's approximation theorems are given and they are applied to construct the two sequences of approximate functions $\{v_{1k}\}$ and $\{v_{2k}\}$ given in Section 2.

Let $\Omega, \Gamma, \Gamma_D, \Gamma_N, \gamma_0$ and L_{γ_0} be as in Section 1. Then, we have the first Runge's approximation theorem.

Theorem C.1 *Let U be an open subset of Ω such that $\overline{U} \subset \Omega$ and $\Omega \setminus \overline{U}$ is connected. Define*

$$\text{(C.1)} \qquad \begin{cases} X = \left\{u\big|_U ; u \in \overline{H}^1(\widetilde{U}), L_{\gamma_0} u = 0 \text{ in an open neighborhood } \widetilde{U} \text{ of } U\right\} \\ Y = \left\{v\big|_\Omega ; v \in \overline{H}^1(\widetilde{U}), L_{\gamma_0} v = 0 \text{ in } \Omega, \partial_{L_{\gamma_0}} v\big|_{\Gamma_N} = 0, \ \mathrm{supp}\big(v\big|_{\Gamma_D}\big) \subset \Gamma_0\right\}, \end{cases}$$

where $\widetilde{U}$ is an open subset of Ω depending on u such that

$$\text{(C.2)} \qquad \overline{U} \subset \widetilde{U} \subset \overline{\widetilde{U}} \subset \Omega$$

and Γ_0 is a fixed open subset of Γ_D. Then, Y is dense in X with respect to $\overline{H}^1(U)$ topology.

Proof By the Hahn-Banach theorem, it is enough to prove

$$\text{(C.3)} \qquad f \in \overline{H}^1(\widetilde{U})^*, f\big(v\big|_U\big) = 0 \ (v \in Y) \Longrightarrow f\big(u\big|_U\big) = 0 \quad (u \in X).$$

Suppose $f \in \overline{H}^1(\widetilde{U})^*, \ f\big(v\big|_U\big) = 0 \quad (v \in Y)$.
Let $y \in \Gamma_0$ and take a small open ball B centered at y and $\Omega_0 := \Omega \cup B$. We extend $\gamma_0 \in C^{0,1}(\overline{\Omega})$ to a neighborhood of $\overline{\Omega_0}$ preserving its regularity. Also, let

$$\text{(C.4)} \qquad T : \left\{\Psi \in \overline{H}^1(\widetilde{U}); \ \Psi\big|_{\Gamma_D} = 0\right\} \to \boldsymbol{R}, \quad T(\Psi) = f\big(\Psi\big|_U\big).$$

T has a bounded linear extension $\widetilde{T} \in \overline{H}^1(\Omega)^*$. Hence, by the unique solvability to variational problem, there exists $w \in \overline{H}^1(\Omega)$ such that $w = 0$ in Γ_D and

$$\text{(C.5)} \qquad -\int_\Omega \gamma_0 \nabla w \cdot \nabla \Psi \, dx = \widetilde{T}(\Psi) \quad \big(\Psi \in \overline{H}^1(\Omega), \ \Psi\big|_{\Gamma_D} = 0\big).$$

Therefore

$$\text{(C.6)} \qquad -\int_\Omega \gamma_0 \nabla w \cdot \nabla \Psi \ dx = f(\Psi\big|_U) \qquad \big(\Psi \in \overline{H}^1(\Omega), \ \Psi\big|_{\Gamma_D} = 0\big).$$

Define $\widetilde{w}$ by

$$\text{(C.7)} \qquad \widetilde{w} = \begin{cases} w & \text{in } \Omega \\ 0 & \text{in } \Omega_0 \setminus \Omega. \end{cases}$$

Since

$$w\big|_{\Gamma_D} = 0, \tag{C.8}$$

$$\widetilde{w} \in \overline{H}^1(\widetilde{\Omega}). \tag{C.9}$$

Now we the following.

Claim

$$\int_{\Omega_0} \gamma_0 \nabla \widetilde{w} \cdot \nabla \varphi = f\big(\varphi\big|_U\big) \quad (\varphi \in \dot{H}^1(\overline{\Omega})) \tag{C.10}$$

The proof of this claim will be given later.

From this claim,

$$L_{\gamma_0} \widetilde{w} = 0 \text{ in } \Omega_0 \setminus \overline{U}. \tag{C.11}$$

Note that

$$\begin{cases} \widetilde{w} = 0 \text{ in } \Omega_0 \setminus \overline{\Omega} \supset \Omega_0 \setminus \overline{U} \\ \Omega_0 \setminus \overline{U} \text{ is connected.} \end{cases} \tag{C.12}$$

Hence, by the weak unique continuation theorem for L_{γ_0} due to $\gamma_0 \in C^{0,1}(\overline{\Omega_0})$, we have

$$\widetilde{w} = 0 \text{ in } \Omega_0 \setminus \overline{U}. \tag{C.13}$$

Therefore

$$w = 0 \text{ in } \Omega \setminus \overline{U}. \tag{C.14}$$

Now let $v \in X$. Then, for some $\widetilde{U}$ which is an open neighborhood of $\overline{U}$, there exists $u \in \overline{H}^1(\widetilde{U})$ such that

$$L_{\gamma_0} u = 0 \text{ in } \widetilde{U},\ u\big|_U = v.$$

By taking a cut off function, for some $\widetilde{\widetilde{U}} \subset \widetilde{U}$ which is an open neighborhood of $\overline{U}$, there exists $\widetilde{u} \in \dot{H}^1(\overline{\Omega})$ such that

$$\widetilde{u}\big|_{\widetilde{\widetilde{U}}} = u\big|_{\widetilde{\widetilde{U}}}.$$

Hence, by reminding (C.6) and (C.14), $w \in \dot{H}^1(\overline{\widetilde{\widetilde{U}}})$ and $L_{\gamma_0} u = 0$ in $\widetilde{\widetilde{U}}$,

$$\begin{aligned} f(v) = f\big(u\big|_U\big) = f\big(\widetilde{u}\big|_U\big) &= \int_\Omega \gamma_0 \nabla w \cdot \nabla \widetilde{u}\, dx \\ &= \int_{\widetilde{\widetilde{U}}} \gamma_0 \nabla w \cdot \nabla \widetilde{u}\, dx = \int_{\widetilde{\widetilde{U}}} \gamma_0 \nabla w \cdot \nabla u\, dx = 0. \end{aligned} \tag{C.15}$$

Finally, we prove the Claim. For any $\varphi \in \dot{H}^1(\overline{\Omega}_0)$,

$$\int_{\Omega_0} \gamma_0 \nabla \widetilde{w} \cdot \nabla u\, dx = \int_{\Omega_0 \setminus \overline{\Omega}} \gamma_0 \nabla \widetilde{w} \cdot \nabla u\, dx + \int_{\Omega} \gamma_0 \nabla \widetilde{w} \cdot \nabla u\, dx = \int_{\Omega} \gamma_0 \nabla \widetilde{w} \cdot \nabla u\, dx. \tag{C.16}$$

Let $v \in \overline{H}^1(\Omega)$ be the solution to

$$\begin{cases} L_{\gamma_0} v = 0 \text{ in } \Omega, \\ \partial_{L_{\gamma_0}} v = 0 \text{ in } \Gamma_N, \ \ v = \varphi \text{ in } \Gamma_D. \end{cases} \tag{C.17}$$

Clearly,

$$v - \varphi \in \overline{H}^1(\Omega), \ \ (v - \varphi)\big|_{\Gamma_D} = 0. \tag{C.18}$$

By (C.6),

$$-\int_{\Omega} \gamma_0 \nabla w \cdot \nabla (v - \varphi)\, dx = f\big(v\big|_U - \varphi\big|_U\big). \tag{C.19}$$

Here note that $v\big|_U \in Y$ by $\operatorname{supp}\big(v\big|_{\Gamma_D}\big) \subset \Gamma_0$,

$$f\big(v\big|_U\big) = 0. \tag{C.20}$$

On the other hand, remind that

$$w \in \overline{H}^1(\Omega), w\big|_{\Gamma_D} = 0 \text{ and } L_{\gamma_0} v = 0 \text{ in } \Omega, \ \partial_{\gamma_0} v\big|_{\Gamma_N} = 0, \ v\big|_{\Gamma_D} = \varphi. \tag{C.21}$$

By the definition of weak solution,

$$\int_{\Omega} \gamma_0 \nabla w \cdot \nabla v\, dx = 0. \tag{C.22}$$

By (C.7), (C.19), (C.20) and (C.22),

$$-\int_{\Omega} \gamma_0 \nabla \widetilde{w} \cdot \nabla \varphi\, dx = f\big(\varphi\big|_U\big). \tag{C.23}$$

□

Likewise the proof given in [5] we have the second Runge's approximation theorem.

Theorem C.2 *Let U be an open subset of Ω such that $\overline{U} \subset \Omega$ and $\Omega \setminus \overline{U}$ is connected. Define the two spaces X, Y of functions by*

$$\begin{cases} X := \{u\big|_U; \ u \in \overline{H}^1(\widetilde{U}), L_{\gamma_0} \widetilde{u} = 0 \quad \text{in } \widetilde{U}\}, \\ Y := \{v\big|_U; \ v \in \overline{H}^1(\Omega), L_{\gamma_0} v = 0 \quad \text{in } \Omega, \ \operatorname{supp}\big(v\big|_{\Gamma}\big) \subset \Gamma_0\}, \end{cases} \tag{C.24}$$

where $\widetilde{U}$ is an open subset of Ω depending on u such that $\overline{U} \subset \widetilde{U} \subset \overline{\widetilde{U}} \subset \Omega$ and Γ_0 is a fixed open subset of Γ_N. Then, Y is dense in X with respect to $\overline{H}^1(U)$ norm.

Next we construct $\{v_{jk}\}$ $(j=1,2)$. By Theorems C.1, C.2, there exist $\{v''_{1k}\}, \{v''_{2k}\} \subset \overline{H}^1(\Omega)$ such that $v''_{jk} \to H\big(\,\cdot\,, c(t)\big)$ in $\overline{H}^1_{\rm loc}(\Omega \setminus \mathcal{C}_t)$ for each j $(j=1,2)$,

$$\text{(C.25)} \qquad \begin{cases} L_{\gamma_0} v''_{1k} = 0 & \text{in } \Omega \\ \partial_{L_{\gamma_0}} v''_{2k} = 0 & \text{on } \Gamma_N, \quad \operatorname{supp}\big(v''_{1k}\big|_\Gamma\big) \subset \Gamma_{10} \end{cases}$$

and

$$\text{(C.26)} \qquad \begin{cases} L_{\gamma_0} v''_{2k} = 0 \quad \text{in } \Omega \\ \operatorname{supp}\big(v''_{2k}\big|_\Gamma\big) \subset \Gamma_{20}, \end{cases}$$

where $\Gamma_{10} \subset \Gamma_D$, $\Gamma_{20} \subset \Gamma_N$ are open subsets.

Then, we only have to define each $\{v_{jk}\}$ $(j=1,2)$ by

$$\text{(C.27)} \qquad v_{jk} := v'_j + v''_{jk},$$

D Some preliminary estimates

In this Appendix we prove some estimates used in Section 3. Let $u \in \overline{H}^1(\Omega)$ be the solution to (1.6) and $v \in \overline{H}^1(\Omega)$ be the solution to (1.6) with $\gamma = \gamma_0$. Then, we have

Theorem D.1
(i)

$$\text{(D.1)} \qquad \langle (\Lambda_\gamma - \Lambda_{\gamma_0}) f, f \rangle_1 \le \int_D \gamma_1 \nabla v \cdot \nabla v \, dx - \int_\Omega F(u-v)\, dx + \langle g, u-v \rangle_2$$

and

$$\text{(D.2)} \quad \langle (\Lambda_\gamma - \Lambda_{\gamma_0}) f, f \rangle_1 \ge \int_D {\gamma_0}^{-1} \gamma_1 \gamma^{-1} (\gamma_0 \nabla) v \cdot (\gamma_0 \nabla) v \, dx - \int_\Omega F(u-v)\, dx + \langle g, u-v \rangle_2.$$

(ii)

$$\text{(D.3)} \qquad \langle g, (\Pi_\gamma - \Pi_{\gamma_0}) g \rangle_2 \le \int_D \gamma_1 \nabla v \cdot \nabla v \, dx - \int_\Omega F(u-v)\, dx - \langle \partial_{L_{\gamma_0}}(u-v), f \rangle_1$$

and

$$\text{(D.4)} \quad \langle g, (\Pi_\gamma - \Pi_{\gamma_0}) g \rangle_2 \ge \int_D {\gamma_0}^{-1} \gamma_1 \gamma^{-1} (\gamma_0 \nabla) v \cdot (\gamma_0 \nabla) v \, dx - \int_\Omega F(u-v)\, dx - \langle \partial_{L_{\gamma_0}}(u-v), f \rangle_1.$$

Proof We use the inequality given in [5]:

$$\text{(D.5)} \qquad \gamma_0 \nabla(v-u)\cdot\nabla(v-u) + (\gamma-\gamma_0)\nabla u\cdot\nabla u \ge {\gamma_0}^{-1}(\gamma-\gamma_0)\gamma^{-1}(\gamma_0\nabla v)\cdot(\gamma_0\nabla v).$$

We first prove (i). Observe that

$$\text{(D.6)} \quad \int_\Omega \{\gamma_0\nabla(u-v)\cdot\nabla(u-v) + (\gamma-\gamma_0)\nabla v\cdot\nabla v\}\,dx = \int_\Omega \big(\gamma\nabla u\cdot\nabla v - 2(\gamma\nabla u)\cdot\nabla v\big)\,dx + \int_\Omega \gamma_0\nabla v\cdot\nabla v\,dx$$

and

$$\text{(D.7)} \quad \int_\Omega \{\gamma\nabla(v-u)\cdot\nabla(v-u) + (\gamma_0-\gamma)\nabla u\cdot\nabla u\}\,dx = \int_\Omega \big(\gamma_0\nabla v\cdot\nabla v - 2(\gamma_0\nabla v)\cdot\nabla u\big)\,dx + \int_\Omega \gamma\nabla u\cdot\nabla u\,dx.$$

By the definitions of the Dirichlet to Neumann map and the Neumann to Dirichlet map, we have from (D.6),

$$\text{(D.8)} \quad \int_\Omega \{\gamma\nabla(u-v)\cdot\nabla(u-v) + (\gamma_0-\gamma)\nabla v\cdot\nabla v\}\,dx = \begin{cases} \langle(\Lambda_{\gamma_0}-\Lambda_\gamma)f, f\rangle_1 + \int_\Omega F(v-u)\,dx - \langle g, v-u\rangle_2 \\ \langle g, (\Pi_{\gamma_0}-\Pi_\gamma)g\rangle_2 + \int_\Omega F(v-u)\,dx + \langle \partial_{L_{\gamma_0}}(v-u), f\rangle_1, \end{cases}$$

where $d\sigma$ is the line segment for $n=2$ and the surface measure for $n\ge 3$. Also, we have from (D.7),

$$\text{(D.9)} \quad \int_\Omega \{\gamma_0\nabla(v-u)\cdot\nabla(v-u) + (\gamma-\gamma_0)\nabla u\cdot\nabla u\}\,dx = \begin{cases} \langle(\Lambda_\gamma-\Lambda_{\gamma_0})f, f\rangle_1 + \int_\Omega F(u-v)\,dx - \langle g, u-v\rangle_2 \\ \langle g, (\Pi_\gamma-\Pi_{\gamma_0})g\rangle_2 + \int_\Omega F(u-v)\,dx + \langle \partial_{L_{\gamma_0}}(u-v), f\rangle_1. \end{cases}$$

Reminding (1.2), we have (D.1) and (D.3) from (D.8). Also, by (D.5), we have (D.2) and (D.4) from (D.9).

References

[1] Y. Daido, M. Ikehata and G. Nakamura, Reconstruction of inclusion for the inverse boundary value problem with mixed type boundary condition (submitted).

[2] M. Grütuer and K-O. Widman, The Green function for uniformly elliptic equations, manuscripta math., 37 (1982).

[3] L. Hörmander, The Analysis of Linear Partial Differential Operators III, Springer, Berlin, 1985.

[4] M. Ikehata, Size estimation of inclusion, J. Inverse and Ill-Posed Problems, 6 (1998) 127-140.

[5] M. Ikehata, Reconstruction of inclusion from boundary measurements, J.Inverse and Ill-Posed Problems, 10(2002) 37- 65.

[6] M. Ikehata and G. Nakamura, Reconstruction procedure for identifying cracks (submitted).

[7] M. Ikehata, G. Nakamura and K. Tanuma, Identification of the shape of the inclusion in the anisotoropic elastic body, Applicable Analysis, 72(1999) 17-26.

[8] V. Isakov, On uniquness of recovery of a discontinuous conductivity coefficient, Comm. Pure Appl. Math. 41 (1988) 865-877.

[9] J. Moser, On Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math. 14 (1961) 577-591.

[10] R. Potthast, Point sources and multipoles in inverese scattering theory, CHAPMAN & HALL/CRC, London, 2001.