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Series #582. February 2003

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Reconstruction of Inclusions for the Inverse Boundary Value Problem with Mixed Type Boundary Condition

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We consider an inverse boundary value problem for identifying the inclusion inside a known anisotropic conductive medium. We give a reconstruction procedure for identifying the inclusion from the Dirichlet-Neumann map or the Neumann-Dirichlet map associated with the mixed type boundary condition.

Keywords: Inverse boundary value problem; Conductiviy equation; Probe method; Mixed type boundary condition

AMS Classifications: 35J25; 35Q72; 35R30

1 INTRODUCTION

Let $\Omega \subset \boldsymbol{R}^n$ $(n = 2$ or $3)$ be a bounded domain with C^2 boundary Γ. Ω is considered as a conductive medium with conductivity:

$$\gamma = \gamma_0 + \chi_D h \gamma_0 \tag{1.1}$$

with a matrix $\gamma_0 = (\gamma_{0ij}) \in C^{0,1}(\overline{\Omega})$ and a real valued scalar function $h \in L^\infty(\Omega)$. Here D is a bounded domain with Lipschitz boundary ∂D such that $\overline{D} \subset \Omega$, $\Omega \setminus \overline{D}$ is connected, χ_D is the characteristic function of D and $C^{0,1}(\overline{\Omega})$ is the space of functions which are Lipschitz continuous on $\overline{\Omega}$. We assume that $\gamma_0 = \big(\gamma_{0ij}(x)\big)$ is a symmetric matrix satisfying

$$\sum_{i,j=1}^{n} \gamma_{0ij}(x)\xi_i\xi_j \geq C_1|\xi|^2 \quad \big(\xi = (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n,\ \text{a.e. } x \in \overline{\Omega}\big) \tag{1.2}$$

for some constant $C_1 > 0$, and for any $a \in \partial D$, there exists a $\delta > 0$ such that either

$$h(x) \geq C_2 \quad \big(\text{ a.e. } x \in B_\delta(a)\big) \tag{1.3}$$

or

$$h(x) \leq -C_2 \quad \big(\text{ a.e. } x \in B_\delta(a)\big) \tag{1.4}$$

holds for some constant $C_2 > 0$, where $B_\delta(a) := \{x \in \boldsymbol{R}^n; |x-a| < \delta\}$. Moreover, we assume that $h \in C^{0,\alpha}$ near ∂D with $0 < \alpha < 1$ for $n = 2$ and $\frac{1}{2} < \alpha < 1$ for $n = 3$.

Let Γ consist of two parts. That is

$$\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}, \tag{1.5}$$

where Γ_D, Γ_N are open subsets of Γ such that $\Gamma_D \cap \Gamma_N = \phi$, $\Gamma_D \neq \phi$, $\Gamma_N \neq \phi$ and for $n = 3$, the boundaries $\partial\Gamma_D, \partial\Gamma_N$ of Γ_D, Γ_N are C^2, respectively.

Consider the mixed type boundary value problem:

$$\begin{cases} (L_\gamma u)(x) := \displaystyle\sum_{i,j=1}^{n} \partial_i\big(\gamma_{ij}(x)\partial_j u(x)\big) = F(x) \ \text{ in } \Omega \\ u = f \ \text{ on } \Gamma_D, \quad \partial_{L_\gamma} u = g \ \text{ on } \Gamma_N \end{cases} \tag{1.6}$$

for given $f \in \overline{H}^{\frac{1}{2}}(\Gamma_D), g \in \overline{H}^{-\frac{1}{2}}(\Gamma_N), F \in L^2(\Omega)$ where $x = (x_1, \cdots, x_n)$, $\partial_i := \partial/\partial x_i$ and

$$\big(\partial_{L_\gamma} u\big)(x) := \sum_{i,j=1}^{n} \nu_i \gamma_{ij}(x)\partial_j u(x) \tag{1.7}$$

with the unit outer normal vector $\nu = (\nu_1 \cdots \nu_n)$ of Γ. Here we have used the notations given in [2] to denote Sobolev spaces.

By Appendix A, there exists a unique solution $u = u(f, g, F) \in \overline{H}^1(\Omega)$ to (1.6) with the estimate:

$$\|u\|_{\overline{H}^1(\Omega)} \leq C\Big(\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} + \|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|F\|_{L^2(\Omega)}\Big), \tag{1.8}$$

where the constant $C > 0$ does not depend on f, g, F.

Moreover, even for $F \in W^*$ with $W := \big\{w \in \overline{H}^1(\Omega);\ w = 0 \quad \text{on } \Gamma_D\big\}$ and $\operatorname{supp} F \subset \Omega$, we have the similar result that $\|F\|_{L^2(\Omega)}$ in (1.8) has to be replaced by $\|F\|_{W^*}$. Hereafter, the norm $\|\cdot\|_W$ and inner product $(\,,\,)$ of W are those of $\overline{H}^1(\Omega)$, and the norm of the dual space W^* of W is denoted by $\|\cdot\|_{W^*}$.

Next, we define the Dirichlet to Neumann map Λ_γ and the Neumann to Dirichlet map Π_γ as follows.

DEFINITION 1.1 *Let $u(f, g, F)$ be the solution to (1.6).*

(i) Fixing g and F, define $\Lambda_\gamma : \overline{H}^{\frac{1}{2}}(\Gamma_D) \to \overline{H}^{-\frac{1}{2}}(\Gamma_D)$ by

$$\Lambda_\gamma f := \partial_{L_\gamma} u(f, g, F) \ \text{ on } \Gamma_D \tag{1.9}$$

(ii) Fixing f and F, define $\Pi_\gamma : \overline{H}^{-\frac{1}{2}}(\Gamma_N) \to \overline{H}^{\frac{1}{2}}(\Gamma_N)$ by

$$\Pi_\gamma g := u(f, g, F) \ \text{ on } \Gamma_N. \tag{1.10}$$

REMARK 1.2 *The trace of* $\partial_{L_\gamma} u(f,g,F) \in \overline{H}^{-\frac{1}{2}}(\Gamma_D)$ *exists, because* $F \in L^2(\Omega)$ *or* $F \in W^*$ *with* $\operatorname{supp} F \subset \Omega$.

Now, we consider two kinds of inverse problems *(IP1)* and *(IP2)*:

(IP1) Suppose γ_0 is known and h, D are unknown. Reconstruct D from Λ_γ.
(IP2) Suppose γ_0 is known and h, D are unknown. Reconstruct D from Π_γ.

THEOREM 1.3 *There are reconstruction procedures for the both inverse problems (IP1) and (IP2).*

REMARK 1.4
If it is possible to relate Λ_γ *or* Π_γ *to a Dirichlet to Neumann map* $\Lambda_{1,\gamma}$ *defined on the boundary of a subdomain* Ω_1 *of* Ω *such that* $\overline{\Omega_1} \subset \Omega, \Omega \setminus \overline{\Omega_1}$ *is connected and its boundary* $\partial\Omega_1$ *is Lipschitz smooth, then by a similar argument given in [3], D can be reconstructed from* $\Lambda_{1,\gamma}$*. However, we are interested in obtaining a reconstruction procedure which directly uses* Λ_γ *or* Π_γ*. For our reconstruction we need to analyze the behavior of the terms with* f, g, F *in the estimates of indicator functions given in (3.8) and (3.9). We have used the De Giorgi-Nash-Moser theorem for doing this.*

The probe method for the inverse boundary value problem with mixed type boundary condition was shown in [4] for identifying cracks. A new ingredient of this paper is an application of the De Giorgi-Nash-Moser theorem (see Appendix B) for estimating the indicator function given later when the conductivity of the inclusion is Hölder continuous near boundary. We also have to point out that the inverse boundary value problem identifying inclusions was initiated by Isakov [6] and there are closely related works done by Potthast and his collaborators ([7]) which use singular solutions for reconstructing an unknown scatter. Our reconstruction procedure without using any reduction to a known reconstruction procedure is totally new.

2 RECONSTRUCTION PROCEDURE

DEFINITION 2.1 (needle) *We call a nonselfintersecting piecewise* C^1 *curve* $\mathcal{C} := \{c(t); 0 \le t \le 1\}$ *joining* $c(0), c(1) \in \Gamma$ *needle if it satisfies* $\mathcal{C} \setminus \{c(0), c(1)\} \subset \Omega$.

DEFINITION 2.2 (singular solution)
(i) Fix $x^0 \in \Omega$ and let $G(x-x^0) \in \mathcal{D}'(\boldsymbol{R}^n)$ be a fundamental solution of

$$\nabla \cdot \big(\gamma_0(x^0)\nabla G(x-x^0)\big) + \delta(x-x^0) = 0 \ \textit{ in } \boldsymbol{R}^n. \tag{2.1}$$

(ii) Let $H_j(x,x^0) \in \mathcal{D}'(\boldsymbol{R}^n_x)$ $(j=1,2)$ be solutions of

$$L_{\gamma_0} H_j(x,x^0) + \delta(x-x^0) = 0 \ \textit{ in } \Omega \tag{2.2}$$

such that

$$H_j(x,x^0) - G(x-x^0) \in \overline{H}^1(\Omega) \tag{2.3}$$

and

$$\begin{cases} \partial_{L_{\gamma_0}} H_1(x,x^0) = 0 \quad \textit{on } \Gamma_N \\ H_2(x,x^0) = 0 \quad \textit{on } \Gamma_D. \end{cases} \tag{2.4}$$

We call each $H_j(x,x^0)$ singular solution.

REMARK 2.3 *The construction of singular solution can be done similarly to Lemma 3 in [5]*

Let $\mathcal{C} := \{c(t); 0 \le t \le 1\}$ be a needle. By the Runge approximation theorem given in Appendix C, there exist sequences of approximate functions $\{v_{1k}\}, \{v_{2k}\} \subset \overline{H}^1(\Omega)$ such that $v_{jk} \to v_j' + H_j\big(\,\cdot\,, c(t)\big)$ $(k \to \infty)$ in $\overline{H}^1_{\mathrm{loc}}(\Omega \setminus \mathcal{C}_t)$ for each j $(j=1,2)$,

$$\begin{cases} L_{\gamma_0} v_{1k} = F \ \text{ in } \Omega \\ \partial_{L_{\gamma_0}} v_{1k} = g \ \text{ on } \Gamma_N, \quad \mathrm{supp}\big(v_{1k}\big|_{\Gamma_N}\big) \subset \Gamma_D \end{cases} \tag{2.5}$$

and

$$\begin{cases} L_{\gamma_0} v_{2k} = F \ \text{ in } \Omega \\ v_{2k} = f \ \text{ on } \Gamma_D, \end{cases} \tag{2.6}$$

where $\mathcal{C}_t := \{c(s); 0 \le s \le t\}$ and $v_j' \in \overline{H}^1(\Omega)$ $(j=1,2)$ are the solutions to

$$\begin{cases} L_{\gamma_0} v_1' = F \ \text{ in } \Omega \\ v_1' = 0 \ \text{ on } \Gamma_D, \quad \partial_{L_{\gamma_0}} v_1' = g \ \text{ on } \Gamma_N \end{cases} \tag{2.7}$$

and

$$\begin{cases} L_{\gamma_0} v_2' = F \ \text{ in } \Omega \\ v_2' = f \ \text{ on } \Gamma_D, \quad \partial_{L_{\gamma_0}} v_2' = 0 \ \text{ on } \Gamma_N \end{cases} \tag{2.8}$$

(see Appendix C for the details).

DEFINITION 2.4 (indicator function) *Let* $\mathcal{C} = \{c(t); 0 \le t \le 1\}$ *be a needle,* t $(0 < t < 1)$ *satisfy* $\mathcal{C}_t \cap \overline{D} = \phi$ *and* $\{v_{jk}\} \subset \overline{H}^1(\Omega)$ $(j = 1, 2)$ *be the sequences of approximate functions given above. Then, we define two indicator functions* $I_1(t, \mathcal{C})$ *and* $I_2(t, \mathcal{C})$ *associated with (IP1) and (IP2):*

$$I_1(t, \mathcal{C}) := \lim_{k \to \infty} \langle (\Lambda_\gamma - \Lambda_{\gamma 0})\big(v_{1k}\big|_{\Gamma_D}\big), v_{1k}\big|_{\Gamma_D} \rangle_1 \tag{2.9}$$

and

$$I_2(t, \mathcal{C}) := \lim_{k \to \infty} \big\langle \big(\partial_{L_\gamma} v_{2k}\big)\big|_{\Gamma_N}, (\Pi_\gamma - \Pi_{\gamma 0})\big((\partial_{L_\gamma} v_{2k})\big|_{\Gamma_N}\big)\big\rangle_2 \tag{2.10}$$

where $\langle\,,\,\rangle_1$ *and* $\langle\,,\,\rangle_2$ *are the pairings for* $\{\dot{H}^{-\frac{1}{2}}(\overline{\Gamma_D}), \overline{H}^{\frac{1}{2}}(\Gamma_D)\}$ *and* $\{\overline{H}^{-\frac{1}{2}}(\Gamma_N), \dot{H}^{\frac{1}{2}}(\overline{\Gamma_N})\}$, *respectively.*

DEFINITION 2.5 (first hitting time) *Let* $\mathcal{C} = \{c(t); 0 \le t \le 1\}$ *be a needle such that* $\mathcal{C} \cap \overline{D} \ne \phi$. *We define* $T(\mathcal{C}, D)$ *by*

$$T(\mathcal{C}, D) := \sup\{\, t\,; 0 < t < 1,\ \ c(s) \notin \overline{D}\ \ (0 \le s < t)\}. \tag{2.11}$$

We call $T(\mathcal{C}, D)$ *the first hitting time of* $\mathcal{C}$ *to* D *.*

DEFINITION 2.6 (detecting time) *Let* $\mathcal{C}$ *be as in Definition 2.5. For the indicator functions* $I_j(t, \mathcal{C})$ $(j = 1, 2)$, *we define their detecting times* $t_j(t, \mathcal{C})$ $(j = 1, 2)$ *by*

$$t_j(\mathcal{C}, D) := \sup\Big\{0 < t < 1;\ \sup_{0<s<t} |I_j(s, \mathcal{C})| < \infty\Big\}. \tag{2.12}$$

Then, we have our main theorem

THEOREM 2.7 *For each* j $(j = 1, 2)$, *we have*

$$T(\mathcal{C}, D) = t_j(\mathcal{C}, D) \quad \textit{if } \mathcal{C} \cap \overline{D} \ne \phi. \tag{2.13}$$

Since we can reconstruct D by knowing $t_j(\mathcal{C}, D)$ for all possible $\mathcal{C}$, Theorem 2.7 implies Theorem 1.3.

3 ESTIMATES OF INDICATOR FUNCTIONS

In this Section we give some estimates for the indicator functions $I_j(t,\mathcal{C})$ $(j=1,2)$.

Let $u_{jk} \in \overline{H}^1(\Omega)$ $(j=1,2;\ k\in \boldsymbol{N})$ be

$$\begin{cases} u_{1k} := u\big(v_{1k}\big|_{\Gamma_D}, g, F\big) \\ u_{2k} := u\big(f, (\partial_{L_{\gamma_0}} v_{2k})\big|_{\Gamma_N}, F\big), \end{cases} \tag{3.1}$$

where $u = u(f,g,F)$ is the solution to (1.6). Also, let

$$w_{jk} := u_{jk} - v_{jk} \quad (j=1,2;\ k\in \boldsymbol{N}). \tag{3.2}$$

Then, we have

$$\begin{cases} L_\gamma w_{jk} = \nabla\cdot(\gamma_0-\gamma)\nabla v_{jk} \ \text{ in } \Omega \\ w_{jk} = 0 \ \text{ on } \Gamma_D, \ \ \partial_{L_\gamma} w_{jk} = 0 \ \text{ on } \Gamma_N. \end{cases} \tag{3.3}$$

More precisely, $w_{jk}\in W$ is the solution of the variational equation:

$$\int_\Omega \gamma\nabla w_{jk}\cdot\nabla\varphi\,dx = \int_\Omega (\gamma_0-\gamma)\nabla v_{jk}\cdot\nabla\varphi\,dx \quad (\varphi\in W) \tag{3.4}$$

Since

$$\sup_{\|\varphi\|_W\le 1}\left|\int_\Omega (\gamma_0-\gamma)\nabla(v_{jk}-v_{jl})\cdot\nabla\varphi\,dx\right| \le \|h\gamma_0\|_{L^\infty(D)}\|v_{jk}-v_{jl}\|_{\overline{H}^1(D)} \to 0 \tag{3.5}$$

as $k,l\to\infty$ by $\mathcal{C}_t\cap\overline{D}=\phi$ and $v_{jk}\to v'_j + H_j(\,\cdot\,,c(t))$ $\ (k\to\infty)$ in $\overline{H}^1_{\rm loc}(\Omega\setminus\mathcal{C}_t)$, we have from (1.8)

$$\|w_{jk}-w_{jl}\|_{\overline{H}^1(\Omega)} \to 0 \quad (k,l\to\infty). \tag{3.6}$$

Hence, there exist limits $w_j := \lim_{k\to\infty} w_{jk} \in \overline{H}^1(\Omega)$ $(j=1,2)$ and they satisfy

$$\begin{cases} L_\gamma w_j = \nabla\cdot(\gamma_0-\gamma)\nabla(v'_j + H_j(\,\cdot\,,c(t))) \ \text{ in } \Omega \\ w_j = 0 \ \text{ on } \Gamma_D, \quad \partial_{L_\gamma} w_j = 0 \ \text{ on } \Gamma_N. \end{cases} \tag{3.7}$$

Therefore, by Theorem D.1 in Appendix D, we have

$$\begin{aligned} \int_D h\gamma^{-1}(\gamma_0\nabla V_1)\cdot(\gamma_0\nabla V_1)\,dx - \int_\Omega F w_1\,dx + \langle g, w_1\rangle_2 &\le I_1(t,\mathcal{C}) \\ &\le \int_D h\gamma_0\nabla V_1\cdot\nabla V_1\,dx - \int_\Omega F w_1\,dx + \langle g, w_1\rangle_2 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \int_D h\gamma^{-1}(\gamma_0\nabla V_2)\cdot(\gamma_0\nabla V_2)\,dx - \int_\Omega F w_2\,dx + \langle \partial_{L_{\gamma_0}} w_2, f\rangle_1 &\le I_2(t,\mathcal{C}) \\ &\le \int_D h\gamma_0\nabla V_2\cdot\nabla V_2\,dx - \int_\Omega F w_2\,dx - \langle \partial_{L_{\gamma_0}} w_2, f\rangle_1, \end{aligned} \tag{3.9}$$

where V_j $(j=1,2)$ are defined by $V_j := v'_j + H_j(\,\cdot\,,c(t))$ $(j=1,2)$.

4 BEHAVIOR OF INDICATOR FUNCTIONS

In this Section we analyze the behavior of the indicator functions $I_j(t,\mathcal{C})$ as $t \uparrow T(\mathcal{C},D)$ when $\mathcal{C}\cap\overline{D}\neq\phi$. Hereafter, constants $C, C' > 0$ which will appear in the estimates are general constants.

Let $\mathcal{C}\cap\overline{D}\neq\phi$ and $0<t<1$ satisfy $\mathcal{C}_t\cap\overline{D}=\phi$.

LEMMA 4.1 *There exists a constant $M>0$ independent of t such that*

$$\|w_j\|_{L^2(\Omega)} \le M \quad (j=1,2) \quad \textit{as } t\uparrow T(\mathcal{C},D). \tag{4.1}$$

Proof For simplicity, put $t_0 := T(\mathcal{C},D)$ and $a = c(t_0)$. For $0<t<t_0$, let $z_j \in \overline{H}^1(\Omega)$ $(j=1,2)$ be the solutions to

$$\begin{cases} L_\gamma z_j = w_j \\ z_j = 0 \ \text{ on } \Gamma \end{cases} \tag{4.2}$$

Then, by (3.7) and $z_j \in W$, we have

$$\int_\Omega |w_j|^2\,dx = -\int_\Omega \gamma\nabla w_j\cdot\nabla z_j\,dx = \int_D h\gamma_0\nabla V_j\cdot\nabla z_j\,dx. \tag{4.3}$$

It is easy to see that

$$\left|\int_D h\gamma_0\nabla v_j'\cdot\nabla z_j\,dx\right|,\ \left|\int_{D\setminus B_\delta(a)} h\gamma_0\nabla H\big(x,c(t)\big)\cdot\nabla z_j\,dx\right| \le C\|w_j\|_{L^2(\Omega)} \tag{4.4}$$

as $t\uparrow t_0$ for some constant $C>0$ independent of t. Since $h\in C^{0,\alpha}$ near ∂D with $0<\alpha<1$ for $n=2$ and $\frac{1}{2}<\alpha<1$ for $n=3$, there exists $\eta>0$ such that

$$\big|(h\gamma_0)(x)-(h\gamma_0)\big(c(t)\big)\big| \le K|x-c(t)|^\alpha \quad \big(0<t_0-t\le\eta,\ x\in B_\delta(a)\big) \tag{4.5}$$

for some constant $K>0$ independent of t and x. Hence,

$$\left|\int_{D\cap B_\delta(a)}\Big((h\gamma_0)(x)-(h\gamma_0)\big(c(t)\big)\Big)\nabla H\big(x,c(t)\big)\cdot\beta\nabla z_j(x)\,dx\right|$$
$$\le CK\int_{D\cap B_\delta(a)} |x-c(t)|^\beta|\nabla z_j(x)|\,dx, \tag{4.6}$$

where $\beta=-1+\alpha$ for $n=2$, $-2+\alpha$ for $n=3$ and $C>0$ is some constant independent of t. By $0<\alpha<1$ for $n=2$ and $\frac{1}{2}<\alpha<1$ for $n=3$, there exists a constant $C>0$ independent of t such that

$$\int_{D\cap B_\delta(a)} |x-c(t)|^{2\beta}\,dx \le C \quad (0<t-t_0\le\eta). \tag{4.7}$$

Hence, we have from (4.6) and (4.7),

$$\left| \int_{D\cap B_\delta(a)} \Big((h\gamma_0)(x) - h\gamma_0\big(c(t)\big)\Big) \nabla H\big(x, c(t)\big) \cdot \beta \nabla z_j(x)\, dx \right| \le C\|w_j\|_{L^2(\Omega)} \qquad (0 < t - t_0 \le \eta) \tag{4.8}$$

for some constant $C > 0$ independent of t. Moreover, by (2.3)

$$\left| \int_{D\cap B_\delta(a)} (h\gamma_0)\big(c(t)\big) \nabla \Big(H\big(x, c(t)\big) - G\big(x - c(t)\big)\Big) \cdot \nabla z_j(x)\, dx \right| \le C\|w_j\|_{L^2(\Omega)} \qquad (0 < t - t_0 \le \eta) \tag{4.9}$$

for some constant $C > 0$ independent of t. Therefore, the proof is complete if we show

$$\left| \int_{D\cap B_\delta(a)} \gamma_0\big(c(t)\big) \nabla G\big(x - c(t)\big) \cdot \nabla z_j(x)\, dx \right| \le C\|w_j\|_{L^2(\Omega)} \quad (0 < t - t_0 \le \eta) \tag{4.10}$$

for some constant $C > 0$ independent of t.

Since $\nabla \cdot \gamma_0\big(c(t)\big)\nabla G\big(x - c(t)\big) = 0$ in $D \cap B_\delta(a)$,

$$\begin{aligned} \int_{D\cap B_\delta(a)} \gamma_0\big(c(t)\big) \nabla G\big(x - c(t)\big) \cdot \nabla z_j(x)\, dx &= \int_{\partial\left(D\cap B_\delta(a)\right)} \Big(\gamma_0\big(c(t)\big) \nabla G\big(x, -c(t)\big) \cdot \nu\Big) z_j\, d\sigma \\ &= \int_{\partial\left(D\cap B_\delta(a)\right)} \Big(\gamma_0\big(c(t)\big) \nabla G\big(x, -c(t)\big) \cdot \nu\Big) \Big(z_j - z_j\big(c(t)\big)\Big)\, d\sigma. \end{aligned} \tag{4.11}$$

By Theorem B.1 in Appendix B, there exist some σ $(0 < \sigma < 1)$ and a constant $C > 0$ depending only on n, C_1 and $\|\gamma\|_{L^\infty(\Omega)}$ such that

$$\left| z_j(x) - z_j\big(c(t)\big)\right| \le C|x - c(t)|^\sigma \|w_j\|_{L^2(\Omega)} \quad \big(x \in B_\delta(a),\ 0 < t - t_0 \le \eta\big). \tag{4.12}$$

Hence,

$$\begin{aligned} \left| \int_{D\cap B_\delta(a)} \gamma_0\big(c(t)\big) \nabla G\big(x - c(t)\big) \cdot \nabla z_j(x)\, dx \right| &\le C \int_{\partial\left(D\cap B_\delta(a)\right)} |x - c(t)|^{-n+1+\sigma}\, d\sigma \|w\|_{L^2(\Omega)} \\ &\le C'\|w\|_{L^2(\Omega)} \qquad (0 < t - t_0 \le \eta) \end{aligned} \tag{4.13}$$

for some constants $C, C' > 0$ independent of t. □

Let $\alpha \in C_0^\infty(\Omega)$ satisfy $\alpha = 1$ in an open neighborhood of $\overline{D}$ and $\widetilde{w}_j := w_j - \alpha w_j$ $(j = 1, 2)$. From (3.7) and $\mathrm{supp}(\gamma - \gamma_0) \subset \overline{D}$, we have

$$(1 - \alpha) L_\gamma w_j = 0 \quad \text{in } \Omega \tag{4.14}$$

and

$$L_\gamma \alpha = L_{\gamma_0} \alpha. \tag{4.15}$$

Then $\widetilde{w}_j$ satisfies

$$\begin{cases} L_\gamma \widetilde{w}_j = F_j \ \text{ in } \Omega \\ \widetilde{w}_j = 0 \ \text{ on } \Gamma_D, \ \ \partial_{L_\gamma} \widetilde{w}_j = 0 \ \text{ on } \Gamma_N, \end{cases} \tag{4.16}$$

where $F_j := -(L_\gamma \alpha) w_j - 2\gamma \nabla \alpha \cdot \nabla w_j$ satisfies $\mathrm{supp}\, F_j \subset \Omega$ and $\|F_j\|_{W^*}$ is uniformly bounded for any t $(0 < t - t_0 \leq \eta)$. Therefore, by the continuity of the trace and $\partial_{L_{\gamma_0}} w_2 = \partial_{L_\gamma} \widetilde{w}_2$,

$$\|w_1\|_{\overline{H}^{\frac{1}{2}}(\Gamma_N)} + \|\partial_{L_{\gamma_0}} w_2\|_{\dot{H}^{-\frac{1}{2}}(\overline{\Gamma_D})} \leq M \quad \text{as } t \uparrow T(\mathcal{C}, D) \tag{4.17}$$

for another constant $M > 0$ independent of t.

Now it is easy to see that the dominant parts of $\int_D h\gamma^{-1}(\gamma_0 \nabla V_j) \cdot (\gamma_0 \nabla V_j)\, dx$ and $\int_D h \nabla V_j \cdot \nabla V_j \, dx$ are $\int_{D \cap B_\delta(a)} h\gamma^{-1} \Big(\gamma_0 \nabla G_j\big(x - c(t)\big)\Big) \cdot \Big(\gamma_0 \nabla G_j\big(x - c(t)\big)\Big)\, dx$ and $\int_{D \cap B_\delta(a)} h\gamma_0 \nabla G_j\big(x - c(t)\big) \cdot \nabla G_j\big(x - c(t)\big)\, dx$ which are positive or negative according to (1.3) or (1.4) and blow up as $t \uparrow t_0$. Therefore, by (3.8), (3.9), (4.1) and (4.17),

$$|I_j(t, \mathcal{C})| \to \infty \quad (t \uparrow t_0). \tag{4.18}$$

Finally, (2.13) can be proven by the standard argument given in [3], so we omit its proof.

APPENDIX

A BOUNDARY VALUE PROBLEM FOR FORWARD PROBLEM

In this Appendix we give the proof of the well-posedness of the boundary value problem (1.6).

THEOREM A.1 *If $\gamma \in L^\infty(\Omega)$ satisfies $\gamma \geq \delta$ in Ω, there exists a unique solution of (1.6). Moreover u satisfies (1.8).*

Proof For $f \in \overline{H}^{\frac{1}{2}}(\Gamma_D)$, there exists $\widetilde{f} \in \overline{H}^{\frac{1}{2}}(\Gamma)$ which is the extension of f.

Let $\widetilde{u} \in \overline{H}^1(\Omega)$ be the solution to

$$\begin{cases} L_\gamma \widetilde{u} = 0 \text{ in } \Omega \\ \widetilde{u} = \widetilde{f} \text{ on } \Gamma. \end{cases} \tag{A.1}$$

Then, it is well known that

$$\|\widetilde{u}\|_{\overline{H}^1(\Omega)} \leq C\|\widetilde{f}\|_{\overline{H}^{\frac{1}{2}}(\Gamma)} \leq C'\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} \tag{A.2}$$

and

$$\|\partial_{L_\gamma}\widetilde{u}\|_{\overline{H}^{-\frac{1}{2}}(\Gamma)} \leq C\|\widetilde{u}\|_{\overline{H}^1(\Omega)} \leq C'\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} \tag{A.3}$$

for some constant $C, C' > 0$ independent of f.

Put $v := u - \widetilde{u} \in W$. This v has to satisfy

$$\begin{cases} L_\gamma v = F \quad \text{in } \Omega \\ v = 0 \quad \text{on } \Gamma_D, \quad \partial_{L_\gamma} v = \widetilde{g} \quad \text{on } \Gamma_N, \end{cases} \tag{A.4}$$

where $\widetilde{g} = g - \partial_{L_\gamma}\widetilde{u}$. Define

$$\langle G, w\rangle := \langle F, w\rangle - \int_{\Gamma_N} \widetilde{g} w \, d\Gamma \tag{A.5}$$

and

$$B[v, w] := \int_\Omega \gamma \nabla v \cdot \nabla w \, dx \tag{A.6}$$

for any $v, w \in W$.

By the Schwarz inequality,

$$|B[v, w]| \leq \int_\Omega |\gamma||\nabla v||\nabla w| \, dx \leq M\|v\|_{\overline{H}^1(\Omega)}\|w\|_{\overline{H}^1(\Omega)}. \tag{A.7}$$

By the Poincaré inequality

$$B[v, v] \geq \int_\Omega \gamma|\nabla v|^2 \, dx \geq \delta\|\nabla v\|_{L^2(\Omega)} \geq \delta'\|v\|_{\overline{H}^1(\Omega)} \tag{A.8}$$

for some constant $\delta' > 0$ independent of v, w.

Now we remind the Lax-Milgram theorem.

THEOREM A.2 (Lax-Milgram) *Let X be a real Hilbert space and $B : X \times X \to \boldsymbol{R}$ be a bilinear map satisfying*

$$|B[x, y]| \le \gamma \|x\| \|y\| \tag{A.9}$$

and

$$B[x, x] \ge \delta \|x\|^2, \tag{A.10}$$

then there exists a unique bounded linear bijective operator $S : X \to X$ such that

$$(x, y) = B[Sx, y] \quad \textit{with } \|S\| \le \delta^{-1}, \ \|S^{-1}\| \le \gamma. \tag{A.11}$$

By applying Theorem A.2, there exists a unique bounded linear bijective operator $S : W \to W$ such that

$$(S^{-1}v, w) = B[v, w] \quad (v, w \in W) \quad \text{with } \|S\| \le (\delta')^{-1}, \ \|S^{-1}\| \le M. \tag{A.12}$$

As immediate estimates, we have

$$|\langle F, w\rangle| \le \|F\|_{W^*} \|w\|_W \tag{A.13}$$

and

$$\begin{aligned} \left| \int_{\Gamma_N} \widetilde{g} w \, d\Gamma \right| &\le C\left(\|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|\partial_{L_\gamma} \widetilde{u}\|_{\overline{H}^{-\frac{1}{2}}(\Gamma)} \right) \|w\|_W \\ &\le C'\left(\|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} \right) \|w\|_W \end{aligned} \tag{A.14}$$

for some constants $C, C' > 0$ independent of f, g, F. Hence, by (A.13) and (A.14),

$$|\langle G, w\rangle| \le C\left(\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} + \|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|F\|_{W^*} \right) \|w\|_W. \tag{A.15}$$

By the Riesz representation theorem, the exists a unique $\widetilde{v} \in W$ such that

$$\langle G, w\rangle = -(\widetilde{v}, w), \quad \|\widetilde{v}\|_W = \|G\|_{W^*}. \tag{A.16}$$

Let $v \in W$ be $\widetilde{v} = S^{-1}v$.

Then, by (A.12),

$$B[v, w] + \langle G, w\rangle = 0 \quad (w \in W). \tag{A.17}$$

Therefore, v is the solution to (A.4). By the definition of v,

$$\begin{aligned} \|u\|_{\overline{H}^1(\Omega)} &\le \|v\|_{\overline{H}^1(\Omega)} + \|\widetilde{u}\|_{\overline{H}^1(\Omega)} \le C\left(\|\widetilde{v}\|_{\overline{H}^1(\Omega)} + \|\widetilde{f}\|_{\overline{H}^{\frac{1}{2}}(\Gamma)} \right) \\ &\le C'\left(\|G\|_{W^*} + \|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} \right) \le C''\left(\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} + \|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|F\|_{W^*} \right) \end{aligned} \tag{A.18}$$

for some constants $C, C', C'' > 0$ independent of f, g, F. □

B DE GIORGI-NASH-MOSER THEOREM

We provide the De Giorgi-Nash-Moser theorem for the readers' convenience. For its proof see [1].

THEOREM B.1 *Let Ω be as given in Section 1 and $\gamma = \big(\gamma_{ij}(x)\big) \in L^\infty(\Omega)$ be a symmetric matrix satisfying (1.2). For a given $f \in L^q(\Omega)$ for some $q > \dfrac{n}{2}$, let $u \in \overline{H}^1(\Omega)$ satisfy*

$$\int_\Omega \gamma_{ij}\partial_i u \partial_j \varphi \, dx = \int_\Omega f\varphi \, dx \quad (\varphi \in \dot{H}^1(\overline{\Omega})) \tag{B.1}$$

Then, $u \in C^{0,\sigma}(\Omega)$ for some σ $(0 < \sigma < 1)$ depending only n, q, C_1 and $\|\gamma\|_{L^\infty(\Omega)}$. Moreover, for any compact $K \subset \Omega$ and open ball B_R with radius R such that $\overline{B_R} \subset K$, we have the estimate:

$$|u(x) - u(y)| \le C\left(\frac{|x-y|}{R}\right)^\sigma (R^{-\frac{n}{2}}\|u\|_{L^2(B_R)} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}) \quad \big(x, y \in B_{\frac{R}{2}}\big) \tag{B.2}$$

where $B_{\frac{R}{2}}, B_R$ are equicentered and $C > 0$ is a constant depending only on $n, q, K, C_1, \|\gamma\|_{L^\infty(\Omega)}$.

REMARK B.2 *Ω can be any domain in $\boldsymbol{R}^n$ with $n \ge 2$; $n \in \boldsymbol{N}$.*

C RUNGE'S THEOREM

In this Appendix two Runge's approximation theorems are given and they are applied to construct the two sequences of approximate functions $\{v_{1k}\}$ and $\{v_{2k}\}$ given in Section 2.

Let $\Omega, \Gamma, \Gamma_D, \Gamma_N, \gamma_0$ and L_{γ_0} be as in Section 1. Then, we have the first Runge's approximation theorem.

THEOREM C.1 *Let U be an open subset of Ω such that $\overline{U} \subset \Omega$ and $\Omega \setminus \overline{U}$ is connected. Define*

$$\begin{cases} X = \Big\{u\big|_U; u \in \overline{H}^1(\widetilde{U}), L_{\gamma_0}u = 0 \ \textit{in an open neighborhood } \widetilde{U} \textit{ of } U\Big\} \\ Y = \Big\{v\big|_\Omega; v \in \overline{H}^1(\widetilde{U}), L_{\gamma_0}v = 0 \ \textit{in } \Omega, \partial_{L_{\gamma_0}}v\big|_{\Gamma_N} = 0, \ \mathrm{supp}\big(v\big|_{\Gamma_D}\big) \subset \Gamma_0\Big\}, \end{cases} \tag{C.1}$$

where $\widetilde{U}$ is an open subset of Ω depending on u such that

$$\overline{U} \subset \widetilde{U} \subset \overline{\widetilde{U}} \subset \Omega \tag{C.2}$$

and Γ_0 is a fixed open subset of Γ_D. Then, Y is dense in X with respect to $\overline{H}^1(U)$ topology.

Proof By the Hahn-Banach theorem, it is enough to prove

$$f \in \overline{H}^1(\widetilde{U})^*, f\big(v\big|_U\big) = 0 \ (v \in Y) \Longrightarrow f\big(u\big|_U\big) = 0 \quad (u \in X). \tag{C.3}$$

Suppose $f \in \overline{H}^1(\widetilde{U})^*$, $f\big(v\big|_U\big) = 0 \quad (v \in Y)$.
Let $y \in \Gamma_0$ and take a small open ball B centered at y and $\Omega_0 := \Omega \cup B$. We extend $\gamma_0 \in C^{0,1}(\overline{\Omega})$ to a neighborhood of $\overline{\Omega_0}$ preserving its regularity. Also, let

$$T : \big\{\Psi \in \overline{H}^1(\widetilde{U});\ \Psi\big|_{\Gamma_D} = 0\big\} \to \boldsymbol{R}, \quad T(\Psi) = f\big(\Psi\big|_U\big). \tag{C.4}$$

T has a bounded linear extension $\widetilde{T} \in \overline{H}^1(\Omega)^*$. Hence, by the unique solvability to variational problem, there exists $w \in \overline{H}^1(\Omega)$ such that $w = 0$ in Γ_D and

$$-\int_\Omega \gamma_0 \nabla w \cdot \nabla \Psi \, dx = \widetilde{T}(\Psi) \quad \big(\Psi \in \overline{H}^1(\Omega),\ \Psi\big|_{\Gamma_D} = 0\big). \tag{C.5}$$

Therefore

$$-\int_\Omega \gamma_0 \nabla w \cdot \nabla \Psi \ dx = f(\Psi\big|_U) \quad \big(\Psi \in \overline{H}^1(\Omega),\ \Psi\big|_{\Gamma_D} = 0\big). \tag{C.6}$$

Define $\widetilde{w}$ by

$$\widetilde{w} = \begin{cases} w & \text{in } \Omega \\ 0 & \text{in } \Omega_0 \setminus \Omega. \end{cases} \tag{C.7}$$

Since

$$w\big|_{\Gamma_D} = 0, \tag{C.8}$$

$$\widetilde{w} \in \overline{H}^1(\widetilde{\Omega}). \tag{C.9}$$

Now we claim the following.
Claim

$$\int_{\Omega_0} \gamma_0 \nabla \widetilde{w} \cdot \nabla \varphi = f\big(\varphi\big|_U\big) \quad (\varphi \in \dot{H}^1(\overline{\Omega})) \tag{C.10}$$

The proof of this claim will be given later.

From this claim,

$$L_{\gamma_0} \widetilde{w} = 0 \text{ in } \Omega_0 \setminus \overline{U}. \tag{C.11}$$

Note that

$$\begin{cases} \widetilde{w} = 0 \text{ in } \Omega_0 \setminus \overline{\Omega} \supset \Omega_0 \setminus \overline{U} \\ \Omega_0 \setminus \overline{U} \text{ is connected.} \end{cases} \tag{C.12}$$

Hence, by the weak unique continuation theorem for L_{γ_0} due to $\gamma_0 \in C^{0,1}(\overline{\Omega_0})$, we have

$$\widetilde{w} = 0 \text{ in } \Omega_0 \setminus \overline{U}. \tag{C.13}$$

Therefore

$$w = 0 \text{ in } \Omega \setminus \overline{U}. \tag{C.14}$$

Now let $v \in X$. Then, for some $\widetilde{U}$ which is an open neighborhood of $\overline{U}$, there exists $u \in \overline{H}^1(\widetilde{U})$ such that

$$L_{\gamma_0} u = 0 \text{ in } \widetilde{U}, \ \ u\big|_U = v.$$

By taking a cut off function, for some $\widetilde{\widetilde{U}} \subset \widetilde{U}$ which is an open neighborhood of $\overline{U}$, there exists $\widetilde{u} \in \dot{H}^1(\overline{\Omega})$ such that

$$\widetilde{u}\big|_{\widetilde{\widetilde{U}}} = u\big|_{\widetilde{\widetilde{U}}}.$$

Hence, by reminding (C.6) and (C.14), $w \in \dot{H}^1(\overline{\widetilde{\widetilde{U}}})$ and $L_{\gamma_0} u = 0$ in $\widetilde{\widetilde{U}}$,

$$\begin{aligned} f(v) = f\big(u\big|_U\big) = f\big(\widetilde{u}\big|_U\big) &= \int_\Omega \gamma_0 \nabla w \cdot \nabla \widetilde{u}\, dx \\ &= \int_{\widetilde{\widetilde{U}}} \gamma_0 \nabla w \cdot \nabla \widetilde{u}\, dx = \int_{\widetilde{\widetilde{U}}} \gamma_0 \nabla w \cdot \nabla u\, dx = 0. \end{aligned} \tag{C.15}$$

Finally, we prove the claim. For any $\varphi \in \dot{H}^1(\overline{\Omega}_0)$,

$$\int_{\Omega_0} \gamma_0 \nabla \widetilde{w} \cdot \nabla u\, dx = \int_{\Omega_0 \setminus \overline{\Omega}} \gamma_0 \nabla \widetilde{w} \cdot \nabla u\, dx + \int_\Omega \gamma_0 \nabla \widetilde{w} \cdot \nabla u\, dx = \int_\Omega \gamma_0 \nabla \widetilde{w} \cdot \nabla u\, dx. \tag{C.16}$$

Let $v \in \overline{H}^1(\Omega)$ be the solution to

$$\begin{cases} L_{\gamma_0} v = 0 \text{ in } \Omega, \\ \partial_{L_{\gamma_0}} v = 0 \text{ in } \Gamma_N, \ \ v = \varphi \text{ in } \Gamma_D. \end{cases} \tag{C.17}$$

Clearly,

$$v - \varphi \in \overline{H}^1(\Omega), \ \ (v - \varphi)\big|_{\Gamma_D} = 0. \tag{C.18}$$

By (C.6),

$$-\int_\Omega \gamma_0 \nabla w \cdot \nabla (v - \varphi)\, dx = f\big(v\big|_U - \varphi\big|_U\big). \tag{C.19}$$

Here note that $v\big|_U \in Y$ by $\operatorname{supp}\big(v\big|_{\Gamma_D}\big) \subset \Gamma_0$,

$$f\big(v\big|_U\big) = 0. \tag{C.20}$$

On the other hand, remind that

$$w \in \overline{H}^1(\Omega), w\big|_{\Gamma_D} = 0 \text{ and } L_{\gamma_0} v = 0 \text{ in } \Omega, \ \ \partial_{\gamma_0} v\big|_{\Gamma_N} = 0, \ \ v\big|_{\Gamma_D} = \varphi. \tag{C.21}$$

By the definition of weak solution,

$$\int_\Omega \gamma_0 \nabla w \cdot \nabla v \, dx = 0. \tag{C.22}$$

By (C.7), (C.19), (C.20) and (C.22),

$$-\int_\Omega \gamma_0 \nabla \widetilde{w} \cdot \nabla \varphi \, dx = f\big(\varphi\big|_U\big). \tag{C.23}$$

□

Likewise the proof given in [3] we have the second Runge's approximation theorem.

THEOREM C.2 *Let U be an open subset of Ω such that $\overline{U} \subset \Omega$ and $\Omega \setminus \overline{U}$ is connected. Define the two spaces X, Y of functions by*

$$\begin{cases} X := \{u\big|_U;\ u \in \overline{H}^1(\widetilde{U}), L_{\gamma_0}\widetilde{u} = 0 \quad in\ \widetilde{U}\}, \\ Y := \{v\big|_U;\ v \in \overline{H}^1(\Omega), L_{\gamma_0} v = 0 \quad in\ \Omega,\ \operatorname{supp}\big(v\big|_\Gamma\big) \subset \Gamma_0\}, \end{cases} \tag{C.24}$$

where $\widetilde{U}$ is an open subset of Ω depending on u such that $\overline{U} \subset \widetilde{U} \subset \overline{\widetilde{U}} \subset \Omega$ and Γ_0 is a fixed open subset of Γ_N. Then, Y is dense in X with respect to $\overline{H}^1(U)$ norm.

Next we construct $\{v_{jk}\}$ $(j = 1, 2)$. By Theorems C.1, C.2, there exist $\{v''_{1k}\}, \{v''_{2k}\} \subset \overline{H}^1(\Omega)$ such that $v''_{jk} \to H\big(\,\cdot\,, c(t)\big)$ in $\overline{H}^1_{\rm loc}(\Omega \setminus \mathcal{C}_t)$ for each j $(j = 1, 2)$,

$$\begin{cases} L_{\gamma_0} v''_{1k} = 0 \quad \text{in } \Omega \\ \partial_{L_{\gamma_0}} v''_{2k} = 0 \quad \text{on } \Gamma_N, \quad \operatorname{supp}\big(v''_{1k}\big|_\Gamma\big) \subset \Gamma_{10} \end{cases} \tag{C.25}$$

and

$$\begin{cases} L_{\gamma_0} v''_{2k} = 0 \quad \text{in } \Omega \\ \operatorname{supp}\big(v''_{2k}\big|_\Gamma\big) \subset \Gamma_{20}, \end{cases} \tag{C.26}$$

where $\Gamma_{10} \subset \Gamma_D,\ \Gamma_{20} \subset \Gamma_N$ are open subsets.

Then, we only have to define each $\{v_{jk}\}$ $(j = 1, 2)$ by

$$v_{jk} := v'_j + v''_{jk}, \tag{C.27}$$

D SOME PRELIMINARY ESTIMATES

In this Appendix we prove some estimates used in Section 3. Let $u \in \overline{H}^1(\Omega)$ be the solution to (1.6) and $v \in \overline{H}^1(\Omega)$ be the solution to (1.6) with $\gamma = \gamma_0$. Then, we have

THEOREM D.1

(i)

$$\langle(\Lambda_\gamma - \Lambda_{\gamma_0})f, f\rangle_1 \le \int_D h\gamma_0 \nabla v\cdot\nabla v\,dx - \int_\Omega F(u-v)\,dx + \langle g, u-v\rangle_2 \tag{D.1}$$

and

$$\langle(\Lambda_\gamma - \Lambda_{\gamma_0})f, f\rangle_1 \ge \int_D h\gamma^{-1}(\gamma_0\nabla)v\cdot(\gamma_0\nabla)v\,dx - \int_\Omega F(u-v)\,dx + \langle g, u-v\rangle_2. \tag{D.2}$$

(ii)

$$\langle g, (\Pi_\gamma - \Pi_{\gamma_0})g\rangle_2 \le \int_D h\gamma_0\nabla v\cdot\nabla v\,dx - \int_\Omega F(u-v)\,dx - \langle \partial_{L_{\gamma_0}}(u-v), f\rangle_1 \tag{D.3}$$

and

$$\langle g, (\Pi_\gamma - \Pi_{\gamma_0})g\rangle_2 \ge \int_D h\gamma^{-1}(\gamma_0\nabla)v\cdot(\gamma_0\nabla)v\,dx - \int_\Omega F(u-v)\,dx - \langle \partial_{L_{\gamma_0}}(u-v), f\rangle_1. \tag{D.4}$$

Proof We use the inequality given in [3]:

$$\gamma_0\nabla(v-u)\cdot\nabla(v-u) + (\gamma-\gamma_0)\nabla u\cdot\nabla u \ge {\gamma_0}^{-1}(\gamma-\gamma_0)\gamma^{-1}(\gamma_0\nabla v)\cdot(\gamma_0\nabla v). \tag{D.5}$$

We first prove (i). Observe that

$$\int_\Omega \{\gamma_0\nabla(u-v)\cdot\nabla(u-v) + (\gamma-\gamma_0)\nabla v\cdot\nabla v\}\,dx$$
$$= \int_\Omega \big(\gamma\nabla u\cdot\nabla v - 2(\gamma\nabla u)\cdot\nabla v\big)\,dx + \int_\Omega \gamma_0\nabla v\cdot\nabla v\,dx \tag{D.6}$$

and

$$\int_\Omega \{\gamma\nabla(v-u)\cdot\nabla(v-u) + (\gamma_0-\gamma)\nabla u\cdot\nabla u\}\,dx$$
$$= \int_\Omega \big(\gamma_0\nabla v\cdot\nabla v - 2(\gamma_0\nabla v)\cdot\nabla u\big)\,dx + \int_\Omega \gamma\nabla u\cdot\nabla u\,dx. \tag{D.7}$$

By the definitions of the Dirichlet to Neumann map and the Neumann to Dirichlet map, we have from (D.6),

$$\int_\Omega \{\gamma\nabla(u-v)\cdot\nabla(u-v) + (\gamma_0-\gamma)\nabla v\cdot\nabla v\}\,dx$$
$$= \begin{cases} \langle(\Lambda_{\gamma_0} - \Lambda_\gamma)f, f\rangle_1 + \int_\Omega F(v-u)\,dx - \langle g, v-u\rangle_2 \\ \langle g, (\Pi_{\gamma_0} - \Pi_\gamma)g\rangle_2 + \int_\Omega F(v-u)\,dx + \langle \partial_{L_{\gamma_0}}(v-u), f\rangle_1, \end{cases} \tag{D.8}$$

where $d\sigma$ is the line segment for $n = 2$ and the surface measure for $n = 3$. Also, we have from (D.7),

$$\int_\Omega \{\gamma_0 \nabla(v-u)\cdot\nabla(v-u) + (\gamma-\gamma_0)\nabla u\cdot\nabla u\}\,dx$$
$$= \begin{cases} \langle(\Lambda_\gamma - \Lambda_{\gamma_0})f, f\rangle_1 + \displaystyle\int_\Omega F(u-v)\,dx - \langle g, u-v\rangle_2 \\ \langle g, (\Pi_\gamma - \Pi_{\gamma_0})g\rangle_2 + \displaystyle\int_\Omega F(u-v)\,dx + \langle \partial_{L_{\gamma_0}}(u-v), f\rangle_1. \end{cases} \tag{D.9}$$

Reminding (1.2), we have (D.1) and (D.3) from (D.8). Also, by (D.5), we have (D.2) and (D.4) from (D.9).

Acknowledgements

The second author is partially supported by Grant-in-Aid for Scientific Research (C)(2) (No.13640152) of Japan Society for the Promotion of Science and the third author is partially supported by Grant-in-Aid for Scientific Research (B)(2) (No.14340038) of Japan Society for the Promotion of Science.

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