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Spectra Of Toeplitz Operators And Uniform Algebras

by

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Abstract. Let A be a uniform algebra on X and $\mathcal{P}$ a set of all probability measures on X. For each μ in $\mathcal{P}$, $H^2(\mu)$ is the closure of A in $L^2(\mu)$ and T^μ_ϕ is a Toeplitz operator on $H^2(\mu)$ for a continuous function ϕ on X. In this paper we study the invertibility and the spectrum of $\mathbf{T}_\phi = \sum_{\mu \in \mathcal{P}} \oplus T^\mu_\phi$. We show that if $\mathbf{T}_\phi$ is invertible then the index of ϕ is zero and if the converse is true for an arbitrary continuous function ϕ then A is a Dirichlet algebra on X. Moreover we study the spectrum of $\mathbf{T}_\phi$.

§1. Introduction

Let X be a compact Hausdorff space, $C = C(X)$ an algebra of all continuous complex-valued functions on X, and A a uniform algebra on X. $\mathcal{P}$ denotes the set of all positive Borel measures on X with total mass 1. For each μ, $H^2(\mu)$ is defined as the closure of A in $L^2(\mu)$. Let P^μ be the orthogonal projection from $L^2(\mu)$ onto $H^2(\mu)$. For ϕ in C and f in $H^2(\mu)$, put

$$T^\mu_\phi f = P^\mu(\phi f).$$

In this paper, we study Toeplitz operators $\mathbf{T}_\phi$ on the Hardy space $\mathbf{H}^2$ for ϕ in C, where

$$\mathbf{T}_\phi = \sum_{\mu\in\mathcal{P}} \oplus T^\mu_\phi \quad \text{and} \quad \mathbf{H}^2 = \sum_{\mu\in\mathcal{P}} \oplus H^2(\mu).$$

$\sigma(\mathbf{T}_\phi)$ denotes the spectrum of $\mathbf{T}_\phi$ and then $\sigma(\mathbf{T}_\phi) \supseteq \bigcup_{\mu\in\mathcal{P}} \sigma(T^\mu_\phi)$. For ϕ, ψ in C $T^\mu_\phi T^\mu_\psi = P^\mu \mathbf{T}_\phi \mathbf{T}_\psi | H^2(\mu)$ and so $\mathbf{T}_\phi$ is a power dilation of T^μ_ϕ. $\|\mathbf{T}_\phi\| = \sup_{x\in X}|\phi(x)|$ and $\|T^\mu_\phi\| = \mu - ess\sup_{x\in X}|\phi(x)|$. T^μ_ϕ is the local part of $\mathbf{T}_\phi$ and $\mathbf{T}_\phi$ is more strongly related with A than T^μ_ϕ. The local part of $\mathbf{T}_\phi$ has been studied in [7] for arbitrary uniform algebra.

When X is the unit circle ∂D, A is the disc algebra $\mathcal{A}$ and μ is the normalized Lebesgue measure $d\theta/2\pi$, $H^2 = H^2(\mu)$ is the classical Hardy space and $T_\phi = T^\mu_\phi$ is the usual Toeplitz operator on H^2. When μ is a finite positive Borel measure on ∂D, $H^2(\mu)$ is called a weighted Hardy space. In this classical case, our result shows that $\sigma(\mathbf{T}_\phi) = \sigma(T_\phi)$ for arbitrary ϕ in C. When X is the closed unit disc $\bar{D}$, A is the disc algebra $\mathcal{A}$ and μ is the normalized area measure rdr$d\theta/\pi$, $L^2_a = H^2(\mu)$ is the Bergman space and $T'_\phi = T^\mu_\phi$ is the usual Toeplitz operator on L^2_a. In this case, $\sigma(\mathbf{T}_\phi) \neq \sigma(T'_\phi)$. By definition, $\mathbf{T}_\phi$ is strongly related with the uniform algebra, that is, $\mathcal{A}|\partial D$ or $\mathcal{A}|\bar{D}$ but T_ϕ or T'_ϕ is related with $d\theta/2\pi$ or rdr$d\theta/\pi$. Suppose the classical Hardy space H^∞ is the weak $*$ closure of $\mathcal{A}$ in $L^\infty = L^\infty(\mu)$ with $\mu = d\theta/2\pi$. When X is the maximal ideal space of L^∞, A is the Gelfand transform of H^∞ on X and μ is a representing measure on X for the origin, $H^2(\mu)$ is also the classical Hardy space H^2 and T^μ_ϕ is the usual Toeplitz operator T_ϕ. Our result implies that $\sigma(\mathbf{T}_\phi) = \sigma(T_\phi)$ for arbitrary ϕ in $C =$ the Gelfand transform of L^∞.

Let A be an arbitrary uniform algebra on X and $M(A)$ the maximal ideal space of A. For a function ϕ in C, we say that the index of ϕ is zero if there exists a nonvanishing function g in $C(M(A))$ such that ϕg has a continuous logarithm on X. By the Arens-Royden theorem [3, p89], we can choose g as an element in A^{-1}. When A is a natural uniform algebra on the complex plane, our definition of the index of ϕ is same to the classical case (see [4, p281]).

In Section 2, we show that $\mathbf{T}_\phi$ is invertible if and only if there exist a positive constant δ and a function g in A^{-1} such that

$$\mathrm{Re}\phi g \geq \delta > 0 \quad \text{on} \quad X.$$

This result is similar as that of the classical Toeplitz operator by Widom and Devinatz (see [2]). That is, T^μ_ϕ is invertible if and only if there exist a positive constant δ and a function g in $H^\infty(\mu)^{-1}$ such that $\mathrm{Re}\phi g \geq \delta > 0$ *a.e.* on ∂D, when ϕ is in $L^\infty(\mu)$ and $\mu = d\theta/2\pi$. When C is a commutative C^*-subalgebra of $L^\infty(\mu)$, our theorem implies that if ϕ is a function in C and $\mathbf{T}_\phi$ is invertible then there exist a positive constant δ and a function g in $H^\infty(\mu)^{-1} \cap C$ such that $\mathrm{Re}\phi g \geq \delta > 0$ on ∂D. In the general setting, our result shows that if $\mathbf{T}_\phi$ is invertible then the index of ϕ is zero. When A is a Dirichlet algebra on X, if the index of ϕ is zero then $\mathbf{T}_\phi$ is invertible. We show that the converse is also true. If $\mathbf{T}_\phi$ is always invertible for an arbitrary function ϕ in C with index $\phi = 0$, then A is a Dirichlet algebra on X. In Section 3, we show that $\phi(X) \subseteq \{\lambda \in \mathbb{C}\ ;\ \text{index } (\phi - \lambda) \neq 0\} \subseteq \sigma(\mathbf{T}_\phi) \subseteq$ the convex hull of $\phi(X)$. Moreover when ϕ is in A or ϕ is real-valued in C, $\sigma(\mathbf{T}_\phi)$ is completely described.

In this paper, $A^\perp$ denotes the set of all annihilating measures on X and $M(A)$ denotes the maximal ideal space of A. $M(A)$ is called simply connected when the first Ćech cohomology group of $M(A)$ with integer coefficients is zero. $\mathcal{R}(\phi)$ is the range $\phi(X)$ of ϕ in X. $\langle f, g\rangle_\mu = \int_X f\bar{g}d\mu$ is the inner product in $L^2(\mu)$ and $\|f\|_\mu = (\langle f, f\rangle_\mu)^{1/2}$. $\|F\|$ is the norm of F in $\mathbf{H}^2$. $\|f\|_X = \sup_{x\in X}|f(x)|$ and $\|f + A\|_X = \inf_{g\in A}\|f + g\|_X$. $|||\mathbf{T}_\phi|||$ denotes the norm of the operator $\mathbf{T}_\phi$ on $\mathbf{H}^2$ and $|T^\mu_\phi|$ denotes the norm of the operator T^μ_ϕ on $H^2(\mu)$.

§2. Invertibility of $\mathbf{T}_\phi$

For each μ in $\mathcal{P}$, let I^μ be the identity operator on $L^2(\mu)$. For ϕ in C and f in $H^2(\mu)$, put $H^\mu_\phi f = (I^\mu - P^\mu)(\phi f)$ and $\mathbf{H}_\phi = \sum_{\mu\in\mathcal{P}} \oplus H^\mu_\phi$. Lemma 1 is similar to a theorem of Nehari [8]. Lemma 2 is similar to a result of Nakazi [6] which was proved by Widom and Devinatz [2] when ϕ is unimodular. Theorem 1 is an analogue of a theorem of Widom and Devinatz [2] in the classical case.

Lemma 1. *If ϕ is a function in C, then $|||\mathbf{H}_\phi||| = \|\phi + A\|_X$.*

Proof. It is clear that $|||\mathbf{H}_\phi||| \leq \|\phi + A\|_X$. Fix $\phi \in C$ with $\phi \notin A$. By the Hahn-Banach theorem and the Riesz representation theorem, there exists a finite Borel measure $\nu \in A^\perp$ with $\|\nu\| = 1$ such that

$$\|\phi + A\|_X = \int \phi d\nu.$$

Put $F = d|\nu|/d\nu$ and $\mu = |\nu|$, then $F \in L^2(\mu) \cap A^\perp$. Hence

$$\|\phi + A\|_X = \int \phi \cdot 1 \cdot \bar{F} d\mu = \langle H^\mu_\phi 1, F\rangle_\mu \leq |H^\mu_\phi| \leq |||\mathbf{H}_\phi|||.$$

Lemma 2. *Suppose ϕ is a function in C. $\mathbf{T}_\phi$ is left invertible if and only if there exists a positive constant ε and a function g in A such that*

$$|\phi + g|^2 \leq |\phi|^2 - \varepsilon \ \text{ on } \ X.$$

Proof. The 'if' part is easy. In fact,

$$\mathbf{H}_\phi^*\mathbf{H}_\phi + \mathbf{T}_\phi^*\mathbf{T}_\phi = \mathbf{T}_{|\phi|^2}$$

because $(H_\phi^\mu)^* H_\phi^\mu + (T_\phi^\mu)^* T_\phi^\mu = T_{|\phi|^2}^\mu$ for all $\mu \in \mathcal{P}$. Hence

$$\mathbf{H}_\phi^*\mathbf{H}_\phi = \mathbf{H}_{\phi+g}^*\mathbf{H}_{\phi+g} \le \mathbf{T}_{|\phi+g|^2} \le \mathbf{T}_{|\phi|^2-\varepsilon}$$

and so

$$\mathbf{T}_\phi^*\mathbf{T}_\phi = \mathbf{T}_{|\phi|^2} - \mathbf{H}_\phi^*\mathbf{H}_\phi \ge \mathbf{T}_\varepsilon.$$

Now we will show the 'only if' part. If $\mathbf{T}_\phi$ is left invertible, then there exists $\varepsilon > 0$ such that $\mathbf{T}_\phi^*\mathbf{T}_\phi \ge \mathbf{T}_{\sqrt{2\varepsilon}}$. Hence $(T_\phi^\mu)^* T_\phi^\mu \ge T_{\sqrt{2\varepsilon}}^\mu$ for all $\mu \in \mathcal{P}$ and so

$$\int |\phi f|^2 d\mu \ge 2\varepsilon \int |f|^2 d\mu \quad (f \in A)$$

for all $\mu \in \mathcal{P}$. Therefore $|\phi|^2 \ge 2\varepsilon > 0$. Hence $\mathbf{H}_\phi^*\mathbf{H}_\phi \le \mathbf{T}_{|\phi|^2-2\varepsilon} \le \mathbf{T}_{|\phi|^2-\varepsilon}$ and so $\|\mathbf{H}_\phi F\|^2 \le \|(\mathbf{T}_{|\phi|^2-\varepsilon})^{1/2} F\|^2$ for all $F \in \mathbf{H}^2$. Therefore for any $\mu \in \mathcal{P}$, $f \in A$ and $g \in L^2(d\mu) \cap A^\perp$,

$$\begin{aligned}\left|\int \phi f \bar{g} d\mu\right|^2 &= |\langle H_\phi^\mu f, g\rangle_\mu|^2 \le \|H_\phi^\mu f\|_\mu^2 \|g\|_\mu^2 \\ &\le \langle T_{|\phi|^2-\varepsilon}^\mu f, f\rangle \|g\|_\mu^2 \le \int (|\phi|^2 - \varepsilon)|f|^2 d\mu \int |g|^2 d\mu\end{aligned}$$

because $\|H_\phi^\mu f\|_\mu^2 \le \|(T_{|\phi|^2-\varepsilon}^\mu)^{1/2} f\|_\mu^2$ for all $f \in A$. If $v^2 d\mu \in \mathcal{P}$ and v is an invertible function in C, then for any $f \in A$ and $G \in L^2(v^2 d\mu) \cap A^\perp$

$$\left|\int \phi f \bar{G} v^2 d\mu\right|^2 \le \int (|\phi|^2 - \varepsilon)|f|^2 v^2 d\mu \int |G|^2 v^2 d\mu.$$

It is easy to see that $v^2(L^2(v^2 d\mu) \cap A^\perp) = L^2(d\mu) \cap A^\perp$. Hence for any $f \in A$ and $g \in L^2(d\mu) \cap A^\perp$

$$\left|\int \phi f \bar{g} d\mu\right|^2 \le \int (|\phi|^2 - \varepsilon)|f|^2 v^2 d\mu \int |g|^2 v^{-2} d\mu$$

because $G = g/v^2$ belongs to $L^2(v^2 d\mu) \cap A^\perp$. Put $a^{-1} = \int (|\phi|^2 - \varepsilon)^{-1/2} d\mu$ and $v^2 = a(|\phi|^2 - \varepsilon)^{-1/2}$, then $\int v^2 d\mu = 1$ and so $v^2 d\mu \in \mathcal{P}$ by the definition of $\mathcal{P}$ in Introduction. Then v^2 is an invertible function in C because $|\phi|^2 \ge 2\varepsilon > 0$. Hence for any $f \in A$ and $g \in L^2(d\mu) \cap A^\perp$

$$\left|\int \phi f \bar{g} d\mu\right|^2 \le \int (|\phi|^2 - \varepsilon)^{1/2} a |f|^2 d\mu \int (|\phi|^2 - \varepsilon)^{1/2} a^{-1} |g|^2 d\mu. \qquad (\star)$$

Put $u = (|\phi|^2 - \varepsilon)^{1/2}$, then $(A \cdot u^{-1})^\perp \cap (C \cdot u^{-1})^* = \{u d\lambda \ ; \ \lambda \in A^\perp\}$ where $*$ denotes the dual. By the Hahn-Banach theorem,

$$\inf\{\|(\phi + h)u^{-1}\|_X \ ; \ h \in A\} = \sup\left\{\left|\int \phi d\lambda\right| \ ; \ \lambda \in A^\perp \ \text{and} \ \int u d|\lambda| \le 1\right\}.$$

If $\phi \notin A$ then there exists a nonzero $\nu \in A^\perp$ with $\int u d|\nu| = 1$ such that

$$\inf \|(\phi + h)u^{-1}\|_X = \int \phi d\nu.$$

Put $F = d|\nu|/d\nu$, then $F \in L^2(d\nu) \cap A^\perp$ and $1 \in A$, hence by $(\star)$

$$\int \phi d\nu = \int \phi \cdot 1 \cdot \bar{F} d|\nu| \leq \int (|\phi|^2 - \varepsilon)^{1/2} d|\nu| = \int u d|\nu| = 1.$$

Thus $\inf \|(\phi + h)u^{-1}\|_X \leq 1$. If $\varepsilon_1 < \varepsilon$, then for some $\delta > 0$ $|\phi|^2 - \varepsilon_1 \geq (1+\delta)(|\phi|^2 - \varepsilon)$ and hence there exists a function g in A

$$|\phi|^2 - \varepsilon_1 \geq (1+\delta)(|\phi|^2 - \varepsilon) \geq |\phi + g|^2.$$

Lemma 3. *Suppose ϕ is a function in C. $\mathbf{T}_\phi$ is left invertible if and only if there exists a positive constant δ and a function g in A such that*

$$\mathrm{Re}\bar{\phi} g \geq \delta > 0 \quad \text{on} \quad X.$$

Proof. If $\mathbf{T}_\phi$ is left invertible, then by Lemma 2 there exists g in A such that $|\phi|^2 \geq \varepsilon^2 + |\phi - g|^2$ for some constant $\varepsilon > 0$. Then, $|\phi|^2 \geq \varepsilon^2$, and so $1 \geq \varepsilon^2/|\phi|^2 \geq \varepsilon_1 > 0$ for some constant ε_1. Hence

$$1 - \varepsilon_1 \geq 1 - \frac{\varepsilon^2}{|\phi|^2} \geq \left|1 - \frac{g}{\phi}\right|^2.$$

Therefore there exist constants $\varepsilon_2, \varepsilon_3$ such that $\mathrm{Re}\frac{g}{\phi} \geq \varepsilon_2 > 0$ and so $\mathrm{Re}\bar{\phi} g \geq \varepsilon_2 |\phi|^2 \geq \varepsilon_2 \varepsilon^2 > 0$. Conversely if $\mathrm{Re}\bar{\phi} g \geq \delta > 0$, then $0 < \varepsilon_3 \leq |\phi| \leq \gamma < \infty$ and so $\mathrm{Re}\frac{g}{\phi} \geq \frac{\delta}{\gamma} > 0$. Hence there exist two positive constant ε_4 and ε_5 such that $\left|\varepsilon_4 \frac{g}{\phi} - 1\right|^2 \leq 1 - \varepsilon_5$. Hence $|\varepsilon_4 g - \phi|^2 \leq (1 - \varepsilon_5)|\phi|^2$ and so

$$|\varepsilon_4 g - \phi|^2 + \varepsilon_5 \varepsilon^2 \leq |\varepsilon_4 g - \phi|^2 + \varepsilon_5 |\phi|^2 \leq |\phi|^2.$$

Lemma 4. *Suppose ϕ is a function in C. $\mathbf{T}_\phi$ is invertible if and only if $\mathbf{T}_{\frac{\phi}{|\phi|}}$ and $\mathbf{T}_{|\phi|}$ are invertible.*

Proof. If $\mathbf{T}_\phi$ is invertible, and $\mathbf{T}_\phi^* = \mathbf{T}_{\bar{\phi}}$ is also invertible and so by Lemma 3 there exists a constant δ_1 and a function g in A such that

$$\mathrm{Re}\phi g \geq \delta_1 > 0.$$

Then ϕ is invertible in C and so there exists a constant δ_2 such that

$$\mathrm{Re}\frac{\phi}{|\phi|}g \geq \delta_2 > 0.$$

Hence by Lemma 3 $\mathbf{T}_{\phi/|\phi|}$ and $\mathbf{T}_{|\phi|}$ are invertible. The converse can be proved similarly by Lemma 3.

Theorem 1. *Suppose ϕ is a function in C. Then, $\mathbf{T}_\phi$ is invertible if and only if there exist a constant δ and a function g in A^{-1} such that*

$$\mathrm{Re}\phi g \geq \delta > 0 \ \text{ on } \ X.$$

Proof. By Lemma 4, we may assume that ϕ is unimodular. If $\mathbf{T}_\phi$ is invertible, then by Lemma 2 there exists a function g in A such that $\|\phi + g\| < 1$ and so $\|1 + \bar{\phi}g\| < 1$. Hence $\mathbf{T}_{\bar{\phi}}\mathbf{T}_g$ is invertible and so $\mathbf{T}_g$ is invertible because $\mathbf{T}_\phi^* = \mathbf{T}_{\bar{\phi}}$ is invertible. Therefore $gH^2(\mu) = H^2(\mu)$ for any $\mu \in \mathcal{P}$ and so g^{-1} belongs to $H^2(\mu)$. Lemma 4 implies that $g^{-1} \in C$. We will show that g^{-1} belongs to A. Then the proof of Lemma 3 implies the theorem. If $g^{-1} \notin A$, there exists a finite measure λ in $A^\perp$ such that $\|\lambda\| = 1$ and $\int g^{-1} d\lambda \neq 0$. Let $F = d\lambda/d|\lambda|$, then $\bar{F}$ is orthogonal to $H^2(|\lambda|)$. Since $\int g^{-1} F d|\lambda| \neq 0$, $g^{-1} \notin H^2(|\lambda|)$. This contradiction implies that $g^{-1} \in A$. Conversely if there exist a constant δ and a function g in A^{-1} such that $\mathrm{Re}\phi g \geq \delta > 0$, then there exists a constant δ' such that $\mathrm{Re}\phi\frac{g}{|g|} \geq \delta' > 0$. Hence $\mathrm{Re}\phi\frac{\bar{f}}{|f|} \geq \delta' > 0$ where $f = g^{-1}$. By lemma 3, $\mathbf{T}_\phi$ is invertible.

Corollary 1. *Let ϕ be a function in C. If $\mathbf{T}_\phi$ is invertible, then there exists a function g in A^{-1} such that ϕg has a continuous logarithm on X, that is, the index of ϕ is zero.*

The converse of Corollary 1 is not true as Corollary 2 shows.

Corollary 2. *If $\mathbf{T}_\phi$ is always invertible for an arbitrary function ϕ in C with index $\phi = 0$, then A is a Dirichlet algebra on X.*

Proof. If $v \in C$ is real-valued and $\phi = e^{iv}$, then index $\phi = 0$. By hypothesis, $\mathbf{T}_\phi$ is invertible and so by Theorem 1 there exists a function g in A such that $\mathrm{Re}g\bar{\phi} > 0$ on X. Corollary 4.7 in [5] shows that A is a Dirichlet algebra.

§3. Spectrum of $\mathbf{T}_\phi$

Let B be a subset of C^{-1}. We define a generalization of the convex hull of the range $\mathcal{R}(\phi)$ of ϕ in C. That is, Hull $\{\mathcal{R}(\phi), B\}$ is a set of all complex numbers λ which satisfy the following : There do not exist a constant δ and a function g in B such that

$\mathrm{Re}(\phi-\lambda)g \geq \delta > 0$ on X. Hull $\{\mathcal{R}(\phi), B\}$ need not be determined by $\mathcal{R}(\phi)$ and B in general. Hull $\{\mathcal{R}(\phi), B\}$ contains $\mathcal{R}(\phi)$. If $B = \{\lambda \in \mathbb{C} \ ; \ \lambda \neq 0\}$, then Hull $\{\mathcal{R}(\phi), B\}$ is the convex hull of $\mathcal{R}(\phi)$. If $B = C^{-1}$, then Hull $\{\mathcal{R}(\phi), B\} = \mathcal{R}(\phi)$. If $B = \exp C$, then Hull $\{\mathcal{R}(\phi), B\} = \{\lambda \in \mathbb{C} \ ; \ \phi - \lambda$ does not have a continuous logarithm on $X\}$. In this section, we are interested in Hull $\{\mathcal{R}(\phi), \exp A\}$ and Hull $\{\mathcal{R}(\phi), A^{-1}\}$. If ϕ is in A then Hull $\{\mathcal{R}(\phi), A^{-1}\} = \hat{\phi}(M(A))$. This is proved in the proof of (4) of Theorem 2. (2) and (4) of Theorem 2, and Corollaries 3 and 4 are similar to theorems in [1, Chapter 7]. For example, (2) of Theorem 2 is an analogue of a theorem of Hartman-Wintner (cf.[1, 7.20 Theorem]).

Theorem 2. *Let ϕ be a function in C.*
(1) $\sigma(\mathbf{T}_\phi) = $ Hull $\{\mathcal{R}(\phi), A^{-1}\}$.
(2) $\mathcal{R}(\phi) \subseteq \sigma(\mathbf{T}_\phi) \subseteq$ the convex hull of $\mathcal{R}(\phi)$.
(3) $\sigma(\mathbf{T}_\phi) \supseteq \{\lambda \in \mathbb{C} \ ; \ \mathrm{index}(\phi - \lambda) \neq 0\}$.
(4) If ϕ is in A, then $\sigma(\mathbf{T}_\phi) = \hat{\phi}(M(A))$.
(5) If ϕ is real-valued, $a = \min \phi$ and $b = \max \phi$, then $\sigma(\mathbf{T}_\phi) \supseteq \mathcal{R}(\phi) \ni a, b$ and $\sigma(\mathbf{T}_\phi) \subseteq [a,b]$. Moreover $\lambda \in [a,b] \backslash \sigma(\mathbf{T}_\phi)$ if and only if χ_{E_λ} belongs to A where $E_\lambda = \{x \in X \ ; \ \phi(x) - \lambda > 0\}$.

Proof. (1) is clear by Theorem 1. (2) is a result of (1). (3) is a result of (1) and the definition of the index. (4) If $\lambda \notin \sigma(\mathbf{T}_\phi)$, then by Theorem 1 there exist a positive constant δ and $g \in A^{-1}$ such that $\mathrm{Re}(\phi - \lambda)g \geq \delta > 0$. Hence there exists a function $f \in A$ such that $(\phi - \lambda)g = e^f$ and so $\phi - \lambda \in A^{-1}$. Therefore $\lambda \notin \hat{\phi}(M(A))$. Conversely if $\lambda \notin \hat{\phi}(M(A))$, then $\phi - \lambda \in A^{-1}$. Put $g = (\phi - \lambda)^{-1}$ and $\delta = 1$, then $\mathrm{Re}(\phi - \lambda)g = \delta > 0$. Hence Theorem 1 implies that $\lambda \notin \sigma(\mathbf{T}_\phi)$. (5) (1) implies that $\sigma(\mathbf{T}_\phi) \supseteq \mathcal{R}(\phi) \ni a, b$ and $\sigma(\mathbf{T}_\phi) \subseteq [a,b]$. If $\lambda \in [a,b] \backslash \sigma(\mathbf{T}_\phi)$, then by (1) there exist δ and $g \in A^{-1}$ such that $\mathrm{Re}(\phi - \lambda)g \geq \delta > 0$. Since $(\phi - \lambda)/|\phi - \lambda| = 2\chi_{E_\lambda} - 1$, $\mathrm{Re}g \geq \delta/|\phi - \lambda|$ on E_λ and $\mathrm{Re}g \leq -\delta/|\phi - \lambda|$ off E_λ. A theorem of Runge implies that χ_{E_λ} belongs to A. Conversely if $\chi_{E_\lambda} \in A$, then $g = 2\chi_{E_\lambda} - 1$ belongs to A^{-1} and so $\mathrm{Re}(\phi - \lambda)g \geq \delta > 0$. (1) implies that $\lambda \in [a,b] \backslash \sigma(\mathbf{T}_\phi)$.

Corollary 3. *Suppose χ_E is a characteristic function of a subset E in X, and a and b are real numbers with $a < b$. If $\phi = a\chi_E + b\chi_{E^c}$ is in C, then $\sigma(\mathbf{T}_\phi) = \{a, b\}$ when χ_E is in A and $\sigma(\mathbf{T}_\phi) = [a,b]$ when χ_E is not in A.*

Corollary 4. *Suppose A is antisymetric. If ϕ is a real-valued function in C with $a = \min \phi$ and $b = \max \phi$, then $\sigma(\mathbf{T}_\phi) = [a,b]$.*

Corollary 5. *Suppose A is a Dirichlet algebra on X and ϕ is a function in C.*
(1) $\sigma(\mathbf{T}_\phi) \subseteq \{\lambda \in \mathbb{C} \ ; \ \phi - \lambda$ does not have a continuous logarithm on $X\}$.
(2) If $M(A)$ is simplly connected, then $\sigma(\mathbf{T}_\phi) = \{\lambda \in \mathbb{C} \ ; \ \phi - \lambda$ does not have the continuous logarithm on $X\}$.

Proof. (1) If $\log(\phi - \lambda) \in C$, then there exist two real-valued functions u, v

such that $\phi - \lambda = e^{u+iv}$. Since A is a Dirichlet algebra, there exists $f \in A$ such that $\|v - \mathrm{Im} f\| < \pi/2$. Put $g = e^{-f}$ then $\mathrm{Re}(\phi - \lambda)g \geq \delta > 0$ for some constant δ and $g \in A^{-1}$. By Theorem 2, λ belongs to $\sigma(\mathbf{T}_\phi)^c$. (2) Since $M(A)$ is simply connected, by the Arens-Royden theorem $A^{-1} = \exp A$. Hence (1) of Theorem 2 imply (2).

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