PHYSICAL MEASURES FOR PARTIALLY HYPERBOLIC SURFACE ENDOMORPHISMS

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PHYSICAL MEASURES FOR PARTIALLY HYPERBOLIC SURFACE ENDOMORPHISMS

MASATO TSUJII

ABSTRACT. We consider dynamical systems generated by partially hyperbolic surface endomorphisms of class C^r with one-dimensional strongly unstable subbundle. As the main result, we prove that such a dynamical system generically admits finitely many ergodic physical measures whose union of basins of attraction has total Lebesgue measure, provided that $r \geq 19$.

1. INTRODUCTION

In the study of smooth dynamical systems from the standpoint of ergodic theory, one of the most fundamental questions is whether the following preferable picture is true for almost all of them: The asymptotic distribution of the orbit for Lebesgue almost every initial point exists and coincides with one of the finitely many ergodic invariant measures that are given for the dynamical system. The answer is expected to be affirmative in general[12]. However it seems far beyond the scope of researches at present to answer the question in the general setting. The purpose of this paper is to provide an affirmative answer to the question in the case of partially hyperbolic surface endomorphisms with one-dimensional strongly unstable subbundle.

Let M be the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ or, more generally, a region on the torus whose boundary consists of finitely many C^2curves: e.g. an annulus $(\mathbb{R}/\mathbb{Z}) \times [-1/3, 1/3]$. We equip M with the standard Riemannian metric $\|\cdot\|$ and the Lebesgue measure $\mathbf{m}$ that are inherited from $\mathbb{R}^2$. In this paper, we call a C^1mapping $F : M \to M$ a *partially hyperbolic endomorphism* if there are positive constants λ and c and a continuous decomposition of the tangent bundle $TM = \mathbf{E}^c \oplus \mathbf{E}^u$ with $\dim \mathbf{E}^c = \dim \mathbf{E}^u = 1$ such that, for all $z \in M$ and $n \geq 0$,

1. $\|DF^n|_{\mathbf{E}^u(z)}\| > \exp(\lambda n - c)$ and
2. $\|DF^n|_{\mathbf{E}^c(z)}\| < \exp(-\lambda n + c)\|DF^n|_{\mathbf{E}^u(z)}\|$.

The subbundles $\mathbf{E}^c$ and $\mathbf{E}^u$ are called the central and strongly unstable subbundle respectively. Notice that we do not assume these subbundles to be invariant in the definition above. The set of partially hyperbolic C^rendomorphisms on M is an open subset in the space $C^r(M, M)$, provided $r \geq 1$.

A *physical measure* for a continuous mapping $F : M \to M$ is an F-invariant probability measure whose basin of attraction

$$\mathcal{B}(\mu) = \mathcal{B}(\mu; F) := \left\{ z \in M \;\middle|\; \frac{1}{n}\sum_{i=0}^{n-1} \delta_{F^i(z)} \to \mu \text{ weakly as } n \to \infty \right\},$$

has positive Lebesgue measure. One of the main results of this paper is

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Theorem 1.1. *A partially hyperbolic C^r endomorphism on M generically admits finitely many ergodic physical measures whose union of basins of attraction has total Lebesgue measure, provided that $r \geq 19$.*

More detailed versions of this theorem will be given in the next section. Here we explain the ideas behind the results of this paper. First of all, the readers should notice that we do *not* claim in the theorem above that the physical measures are hyperbolic. Instead, we will show that the physical measures for a generic partially hyperbolic endomorphism have nice properties even if they are not hyperbolic. This is the novelty of the argument in this paper.

For a partially hyperbolic endomorphism $F : M \to M$, Lyapunov exponent takes two distinct values at each point: The larger is positive and the smaller is indefinite. The latter is called the central Lyapunov exponent, as it is attained by the vectors in the central subbunle. An invariant measure is hyperbolic if the central Lyapunov exponent is non-zero almost everywhere with respect to it. In the former part of this paper, we will analyze hyperbolic invariant measures by using the techniques in Pesin theory or smooth ergodic theory. As conclusions, we will prove the following facts under generic conditions on F: Let $\chi > 0$ be a positive number, which may be arbitrarily small. Then there are only finitely many ergodic physical measures whose central Lyapunov exponent is larger than χ in absolute value. Further, if X is the complement of the union of their basins of attraction, and if μ is a weak limit point of the sequence $n^{-1}\sum_{i=0}^{n-1} \mathbf{m}|_X \circ F^{-i}$, then the absolute value of the central Lyapunov exponent is not larger than χ for μ-almost every point. These facts are far from trivial. But the argument in the proofs does not deviate far from the existing ones in the smooth ergodic theory.

The key claim in the argument of this paper is that, if the number χ is small enough, the measure μ as above for a generic partially hyperbolic endomorphism is absolutely continuous with respect to Lebesgue measure and the density satisfies some regularity conditions. The regularity conditions on the density enables us to show that μ is a convex combination of finitely many ergodic physical measures whose union of basins of attraction has total measure with respect to $\mathbf{m}|_X$. With these facts and those mentioned in the preceding paragraph, we conclude the theorem above.

The conclusion of the key claim may appear unusual, since the measure μ may have neutral or even negative central Lyapunov exponent while we usually meet absolutely continuous invariant measures as a consequence of expanding property of dynamical systems in all directions. Intuitively, we can explain the conclusion as follows: As a consequence of the dominating expansion in the strongly unstable directions, the limit measure μ should have some smoothness or uniformity in those directions. In fact, we can show that the natural extensions of μ and its ergodic components to the inverse limit are absolutely continuous along the strongly unstable manifolds. So, for each ergodic component μ' of μ, we can cut a curve γ out of a strongly unstable manifold so that μ' is attained as a weak limit point of the sequence $n^{-1}\sum_{i=0}^{n-1} \nu_\gamma \circ F^{-i}$ where ν_γ is a smooth measure on γ. Since F expands the curve γ uniformly, the image $F^n(\gamma)$ for large n should be a long curve which is transversal to the central subbundle $\mathbf{E}^c$. Imagine to look into a small neighborhood of a point in the support of μ'. Then the image $F^n(\gamma)$ should appear as a bunch of short pieces of curve in that neighborhood. The number of the pieces of curve should grow exponentially as n gets large. And they would not concentrate in the central direction strongly, as the central Lyapunov exponent is nearly neutral almost everywhere with respect to μ'. These suggest that the ergodic component μ' should have some smoothness or uniformity in the central direction as well as in the strongly unstable direction and, hence, the measure μ should be absolutely continuous with respect to Lebesgue measure.

On the technical side, an important idea in the proof of the key claim is that we look at the angles between the short pieces of curve mentioned in the preceding paragraph rather than their positions. As we perturb the mapping F, it turns out that we can control the angles between those pieces of curve to some extent, though we can not control their positions by the usual problem of interference. And we can show that the pieces of curve satisfy some transversality condition generically. In order to show the conclusion of the key claim, we relate the transversality condition to absolute continuity of the measure μ. To this end, we make use of an idea in the paper[13] by Peres and Solomyak, which treat a problem in fractal geometry posed by Erdös. Since we can not explain the idea in short, we will illustrate it in the beginning of section 6 by using a simple example. Actually we have used the same idea in our previous paper[19], which the reader can regard as a study for this work. Lastly, the author would like to note that the idea in [13] can traces back to the papers of Falconer[5] and Simon and Pollicot[16].

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2. Statement of the main results

Let $\mathcal{PH}^r$ be the set of partially hyperbolic C^rendomorphisms on M and $\mathcal{PH}_0^r$ the subset of those without critical points. We consider the subset $\mathcal{R}^r$ of mappings $F \in \mathcal{PH}^r$ that satisfy the following two conditions:

(1) F admits a finite collection of ergodic physical measures whose union of basins of attraction has total Lebesgue measure on M, and
(2) a physical measure for F is absolutely continuous with respect to Lebesgue measure if the sum of its Lyapunov exponents is positive.

In this paper, we claim that almost all partially hyperbolic C^rendomorphisms on M satisfy the conditions (1) and (2) above or, in other words, belong to the subset $\mathcal{R}^r$. The former part of our main result is stated as follows:

Theorem 2.1. (I) *The subset $\mathcal{R}^r$ is a residual subset in $\mathcal{PH}^r$, provided $r \geq 19$.*
(II) *The intersection $\mathcal{R}^r \cap \mathcal{PH}_0^r$ is a residual subset in $\mathcal{PH}_0^r$, provided $r \geq 2$.*

The conclusions of this theorem mean that the complement of the subset $\mathcal{R}^r$ is a meager subset in the sense of Baire's category argument. However the recent progress in dynamical system theory has thrown serious doubt that the notion of genericity based on Baire's category argument may not have its literal meaning. In fact, it can happen that the dynamical systems in some meager subset appear as subsets with positive Lebesgue measure in the parameter spaces of typical families. For example, compare Jakobson's theorem[21] and the density of Axiom A[11, 17] in one-dimensional dynamical systems. For this reason, we dare to state our claim also in a measure-theoretical framework, though no measure-theoretical definition that corresponds to the notion of genericity has been firmly established yet.

Let B be a Banach space. Let $\tau_v : B \to B$ be the translation by $v \in B$, so that $\tau_v(x) = x + v$. A Borel finite measure $\mathcal{M}$ on B is said to be *quasi-invariant* along a linear subspace $L \subset B$ if $\mathcal{M} \circ \tau_v^{-1}$ is equivalent to $\mathcal{M}$ for any $v \in L$. In the case where B is finite-dimensional, a Borel finite measure on B is equivalent to the Lebesgue measure if and only if it is quasi-invariant along the whole space B. But, unfortunately enough, it is known that no Borel finite measures on an infinite-dimensional Banach space are quasi-invariant along the whole space[22]. This is one of the reasons why we do not have obvious definitions for the concept like *Lebesgue almost everywhere* in the cases of infinite-dimensional Banach spaces or Banach manifolds such as the space $C^r(M, M)$. Nevertheless, there may be Borel finite measures on B that is quasi-invariant along dense subspaces. In fact, on the Banach space $C^r(M, \mathbb{R}^2)$, there exist Borel finite measures that are quasi-invariant along the dense subspace $C^{r+2}(M, \mathbb{R}^2)$. (Lemma 3.16.) For integers $s \geq r \geq 1$, we will denote by $\mathcal{Q}_s^r$ the set of Borel probability measures on $C^r(M, \mathbb{R}^2)$ that are quasi-invariant along the subspace $C^s(M, \mathbb{R}^2)$ and regard the measures in these sets as substitutions for the (non-existing) Lebesgue measure.

Let us consider the space $C^r(M, \mathbb{T})$ of C^rmappings from M to the torus $\mathbb{T}$, which contains the space $C^r(M, M)$ of C^rendomorphisms on M. For a mapping G in $C^r(M, \mathbb{T})$, we consider the mapping

$$\Phi_G : C^r(M, \mathbb{R}^2) \to C^r(M, \mathbb{T}), \quad F \mapsto G + F. \tag{1}$$

We say that a subset $\mathcal{X} \subset C^r(M, M)$ is *shy* with respect to a measure $\mathcal{M}$ on $C^r(M, \mathbb{R}^2)$ if $\Phi_G^{-1}(\mathcal{X})$ is a null subset with respect to $\mathcal{M}$ for any $G \in C^r(M, \mathbb{T})$. This is a slight modification of the notion of shyness introduced by Hunt, Sauer and Yorke [8, 9]. (See also [20].)

Put $\mathcal{S}^r := \mathcal{PH}^r \setminus \mathcal{R}^r$. The latter part of our main result is stated as follows.

Theorem 2.2. (I) *The subset $\mathcal{S}^r$ is shy with respect to a measure $\mathcal{M}_s$ in $\mathcal{Q}_{s-1}^r$ if the integers $r \geq 2$ and $s \geq r + 3$ satisfy*

$$(r-2)(r+1) < (r-\nu-2)\left(2(r-3) - \frac{(2s-r-\nu+1)}{\nu}\right) \tag{2}$$

for some integer $3 \leq \nu \leq r-2$. Moreover, $\mathcal{S}^r$ is shy with respect to any measure in $\mathcal{Q}_{s-1}^r$ if $r \geq 2$ and $s \geq r+3$ satisfy the condition (2) with s replaced by $s+2$ for some integer $3 \leq \nu \leq r-2$.
(II) *$\mathcal{S}^r \cap \mathcal{PH}_0^r$ is shy with respect to any measure in $\mathcal{Q}_s^r$ for $s \geq r \geq 2$.*

Remark. The inequality (2) holds for the combinations $(r, s, \nu) = (19, 22, 3)$ and $(21, 26, 3)$ for example. But it does not hold for any $s \geq r + 3$ and $3 \leq \nu \leq r-2$ unless $r \geq 19$.

As an advantage of the measure-theoretical notions introduced above, we can derive the following corollary on the families of mappings in $\mathcal{PH}^r$, whose proof is given in the appendix. Let us regard the space $C^r(M \times [-1, 1]^k, M)$ as that of C^rfamilies of endomorphisms on M with parameter space $[-1, 1]^k$. We can introduce the notion of shyness for the Borel subsets in this space in the same way as we did for those in $C^r(M, M)$. Let $\mathbf{m}_{\mathbb{R}^k}$ be the Lebesgue measure on $\mathbb{R}^k$.

Corollary 2.3. *The set of C^rfamilies $F(z, \mathbf{t})$ in $C^r(M \times [-1, 1]^k, M)$ such that*

$$\mathbf{m}_{\mathbb{R}^k}(\{\mathbf{t} \in [-1, 1]^k \mid F(\cdot, \mathbf{t}) \in \mathcal{S}^r\}) > 0$$

is shy with respect to any Borel finite measure on $C^r(M \times [-1, 1]^k, \mathbb{R}^2)$ that is quasi-invariant along the subspace $C^{s-1}(M \times [-1, 1]^k, \mathbb{R}^2)$, provided that the integers $r \geq 2$

and $s \geq r+3$ satisfy the condition (2) with s replaced by $s+2$ for some integer $3 \leq \nu \leq r-2$.

We give a few comments on the main result above. The restriction that the surface M is a region on the torus is actually not very essential. We could prove theorem 2.1 with M being a general compact surface by modifying the proof slightly. The main reason for this restriction is the difficulty in generalizing the notion of shyness to the spaces of endomorphisms on general compact surfaces. Since the definition depends heavily on the linear structure of the space $C^r(M, \mathbb{R}^2)$, we hardly know how we can modify this notion naturally so that it is consistent under the non-linear coordinate transformations. The generalization or modification of the notion of shyness should be an important issue in the future. Besides, the restriction on M simplifies the proof considerably and does not exclude the interesting examples such as the so-called Viana-Alves maps[1, 23].

The assumptions on differentiability in the main theorems are crucial in our argument especially in the part where we consider the influence of the critical points on the dynamics. We do not know whether they are technical ones or not.

As we called attention in the introduction, the main theorems tell nothing about hyperbolicity of the physical measures. Of course, it is natural to expect that the physical measures are hyperbolic generically. The author think it not too optimistic to expect that $\mathcal{R}^r$ contains an open dense subset of $\mathcal{PH}^r$ in which the physical measures for the mappings are hyperbolic and depend on the mapping continuously.

Generalization of the main theorems to partially hyperbolic diffeomorphisms on higher dimensional manifolds is an interesting subject to study. Our argument on physical measures with nearly neutral central Lyapunov exponent seems to be complementary to the recent works[2, 3] of Alves, Bonatti and Viana. However, as far as the author understand, there exist essential difficulties in the case where the dimension of the central subbundle is higher than one.

The plan of this paper is as follows: We give some preliminary arguments in section 3. We first define some basic notations and then introduce the notions of *admissible curve* and *admissible measure*, which play central rolls in our argument. The former is taken from the paper of Viana[23] with slight modification and the latter is a corresponding notion for measures. Next we introduce two conditions on partially hyperbolic endomorphisms, namely, *the transversality condition on unstable cones* and *the no flat contact condition.* At the end of section 3, we shall give a concrete plan of the proof of the main theorems using the terminology introduced in this section. In section 4, we study hyperbolic physical measures using Pesin theory. Section 5 is devoted to basic estimates on the distortion of the iterates of partially hyperbolic endomorphisms. Then we go into the main part of this paper, which consists of three mutually independent sections. In section 6, we prove that a partially hyperbolic endomorphism belongs to the subset $\mathcal{R}^r$ if it satisfies the two conditions above. In section 7 and 8 respectively, we prove that each of the two conditions holds for almost all partially hyperbolic endomorphisms.

3. Preliminaries

In this section, we prepare some notations, definitions and basic lemmas that we shall use frequently in the later sections.

3.1. **Notations.** Throughout this paper, we assume $r \geq 2$. Let $F : M \to M$ be a C^rmapping and $\mathcal{C}(F)$ the critical set of F. For a non-zero tangent vector $v \in T_zM$

at a point $z \in M$, we define

$$D_*F(z,v) = \|DF_z(v)\|/\|v\|$$

and

$$D^*F(z,v) = (\det DF_z)/D_*F(z,v) \qquad \text{if } DF(v) \neq \mathbf{0}.$$

Then we have $|D^*F(z,v)| = \|DF^*(v^*)\| / \|v^*\|$ for any cotangent vector $v^* \neq \mathbf{0}$ at $F(z)$ that is normal to $DF(v)$. We shall write $D_*F(v)$ and $D^*F(v)$ for $D_*F(z,v)$ and $D^*F(z,v)$ respectively in places where the point z is clear from the context.

For a C^rmapping $F : M \to \mathbb{R}^2$, the C^rnorm of F is defined by

$$\|F\|_{C^r} = \max_{z\in M} \max_{0\le a+b\le r} \left\| \frac{\partial^{a+b}F}{\partial^a x \partial^b y}(z) \right\|$$

where (x,y) is the coordinate on $\mathbb{T}$ that is induced by the standard one on $\mathbb{R}^2$. Similarly, for C^rmappings F and G in $C^r(M,\mathbb{T})$, the C^rdistance is defined by

$$d_{C^r}(F,G) = \max_{z\in M} \max \left\{ d(F(z),G(z)), \max_{1\le a+b\le r} \left\| \frac{\partial^{a+b}F}{\partial^a x \partial^b y} - \frac{\partial^{a+b}G}{\partial^a x \partial^b y} \right\| \right\}.$$

For a tangent vector $v \in TM$, we denote by $v^{\perp}$ the tangent vector that is obtained by rotating v by the right angle in the counter-clockwise direction. For two tangent vectors u and v, we denote by $\angle(u,v)$ the angle between them even if they belongs to the tangent spaces at different points. Let $\exp_z : T_z\mathbb{T} \to \mathbb{T}$ be the exponential mapping, which is defined simply by $\exp_z(v) = z + v$ in our case. For a point z of an metric space and a positive number δ, let $\mathbf{B}(z,\delta)$ be the open disk with center at z and radius δ. Likewise, for a subset X in a metric space, let $\mathbf{B}(X,\delta)$ be its open δ-neighborhood. For a positive number δ, we define a lattice $\mathbb{L}(\delta)$ as the subset of points (x,y) in $\mathbb{T}$ whose components, x and y, are multiples of $1/([1/\delta]+1)$, so that the disks $\mathbf{B}(z,\delta)$ for points $z \in \mathbb{L}(\delta)$ cover the torus $\mathbb{T}$.

3.2. **Some open subsets in $\mathcal{PH}^r$.** In this subsection, we introduce bounded open subsets in the space $\mathcal{PH}^r$ in which the mappings enjoy some uniform estimates. For the proof of the main theorems, we can restrict ourselves to such open subsets. This simplifies the argument considerably.

Let $\mathcal{S}_0^r$ be the subset of mappings F in $\mathcal{PH}^r$ that violate either of the conditions:

(A1) The image $F(M)$ is contained in the interior of M;
(A2) The function $z \mapsto \det DF_z$ has 0 as its regular value;
(A3) The restriction of F to the critical set $\mathcal{C}(F)$ is transversal to $\mathcal{C}(F)$.

Notice that the condition (A2) and (A3) are trivial if the mapping F has no critical points. To prove the following lemma, we have only to apply Thom's jet transversality theorem[6] and its measure-theoretical version[20, Theorem C].

Lemma 3.1. *The subset $\mathcal{S}_0^r$ is a closed nowhere dense subset in $\mathcal{PH}^r$ and shy with respect to any measure in $\mathcal{Q}_s^r$ for $s \ge r \ge 2$.*

Remark. The terminology in [20] is different from that in this paper. But we can put theorem C and other results in [20] into our terminology without difficulty.

Consider a C^rmapping $F_\sharp$ in $\mathcal{PH}^r$ and let $TM = \mathbf{E}^c \oplus \mathbf{E}^u$ be a decomposition of the tangent bundle which satisfies the conditions in the definition of partially hyperbolic endomorphism for $F = F_\sharp$. Notice that, though the central subbundle $\mathbf{E}^s$ is uniquely determined by the conditions in the definition, the strongly unstable subbundle $\mathbf{E}^u$ is *not*. Indeed any continuous subbundle transversal to $\mathbf{E}^s$ satisfies the conditions in the definition possibly with different constants λ and c. Making use of this arbitrariness, we can assume that $\mathbf{E}^u$ is a C^∞subbundle. Further, by taking $\mathbf{E}^u$ nearly orthogonal to $\mathbf{E}^c$ and by changing the constants λ and c, we can assume that there exist positive-valued C^∞functions θ^c and θ^u on M such that the cone fields

$$\mathbf{S}^u(z) = \{v \in T_zM \setminus \{0\} \mid \angle(v, \mathbf{E}^u(z)) \leq \theta^u(z)\} \quad \text{and}$$
$$\mathbf{S}^c(z) = \{v \in T_zM \setminus \{0\} \mid \angle(v^\perp, \mathbf{E}^u(z)) \leq \theta^c(z)\}$$

satisfy the following conditions at every point $z \in M$:

(B1) $\mathbf{S}^s(z) \cap \mathbf{S}^u(z) = \emptyset$;
(B2) $\mathbf{E}^c(z) \setminus \{0\}$ is contained in the interior of the cone $\mathbf{S}^c(z)$;
(B3) $DF_\sharp(\mathbf{S}^u(z))$ is contained in the interior of $\mathbf{S}^u(F_\sharp(z))$;
(B4) $(DF_\sharp)_z^{-1}(\mathbf{S}^c(F_\sharp(z)))$ is contained in the interior of $\mathbf{S}^c(z)$;
(B5) For any $v \in \mathbf{S}^u(z)$ and $n \geq 1$, we have
  (1) $\|D_*F_\sharp^n(z,v)\| > \exp(\lambda n - c)$ and
  (2) $\|D^*F_\sharp^n(z,v)\| < \exp(-\lambda n + c)\|D_*F_\sharp^n(z,v)\|$.

Suppose that the mapping $F_\sharp$ does *not* belong to $\mathcal{S}_0^r$. Then we can take a small number $\rho > 0$ and a large integer $\Lambda > 0$ so that the following conditions hold for any C^rmapping F satisfying $d_{C^r}(F, F_\sharp) < 2\rho$:

(C1) The conditions (B3), (B4) and (B5) with $F_\sharp$ replaced by F;
(C2) The parallel translation of $\mathbf{E}^c(F_\sharp(z))$ to $F(z)$ is contained in $\mathbf{S}^c(F(z)) \cup \{0\}$ for any $z \in M$;
(C3) $d(F(M), \partial M) > \rho$;
(C4) The function $z \mapsto \det DF_z$ has no critical points on $\mathbf{B}(\mathcal{C}(F), \rho)$ and it holds $|\det DF_z| > \rho \cdot d(z, \mathcal{C}(F))$ for $z \in \mathbf{B}(\mathcal{C}(F), \rho)$;
(C5) If a point $z \in M$ satisfies $d(z, w_1) < \rho$ and $d(F(z), w_2) < \rho$ for some points $w_1, w_2 \in \mathcal{C}(F)$ and if $v \in \mathbf{S}^u(z)$, the angle between $DF(v)$ and the tangent vector of $\mathcal{C}(F)$ at w_2 is larger than ρ;
(C6) $\# F^{-1}(z) < \Lambda$ and $\|DF_z\| < \Lambda$ for any $z \in M$.

We can choose countably many pairs of a C^rmapping $F_\sharp$ in $\mathcal{PH}^r \setminus \mathcal{S}_0^r$ and a positive number ρ as above so that the open subsets

$$\mathcal{U} = \{F \in C^r(M, M) \mid d_{C^r}(F_\sharp, F) < \rho\}$$

cover $\mathcal{PH}^r \setminus \mathcal{S}_0^r$. In order to prove the main theorems, theorem 2.1 and 2.2, it is enough to prove their claims by restricting ourselves to each of such open subsets $\mathcal{U}$. Therefore we henceforth fix a C^rmapping $F_\sharp$ in $\mathcal{PH}^r \setminus \mathcal{S}_0$, subbundles $\mathbf{E}^c$ and $\mathbf{E}^u$, C^∞functions θ^c and θ^u, cone fields $\mathbf{S}^s(\cdot)$ and $\mathbf{S}^u(\cdot)$ and positive numbers λ, c, ρ and Λ as above, and consider the mappings in the corresponding open subset $\mathcal{U}$.

3.3. **Remarks on the notation for constants.** In this paper, we shall introduce various constants that depend only on the integer $r \geq 2$ and the objects that we fixed at the end of the last subsection. In order to distinguish such kind of constants, we make it as a rule to denote them by symbols with subscript g. Obeying this rule, we shall denote λ_g, c_g, ρ_g and Λ_g for the constants λ, c, ρ and Λ_g hereafter. Notice that, once we denote a constant by a symbol with subscript g, we mean that it is a constant of this kind. In order to save symbols for constants, we shall frequently use a generic symbol C_g for large positive constants of this kind. Note that the value of the constants denoted by C_g may be different from place to place even in a single expression. For instance, ridiculous expressions like $2C_g < C_g$ can be true, though we shall not really meet such ones. Also note that we shall omit the phrases on the choice of the constants denoted by C_g in most cases.

Also we can and do introduce a constant $A_g > 0$ such that

$$A_g^{-1}\frac{|D^*F^n(z,w)|}{D_*F^n(z,w)} \le \frac{\angle(DF^n(u),DF^n(v))}{\angle(u,v)} \le A_g\frac{|D^*F^n(z,w)|}{D_*F^n(z,w)} \tag{3}$$

for any $z \in M$, $n \ge 1$ and tangent vectors $u, v, w \notin \mathbf{S}^s(z)$. We shall use the following relations frequently: For any $F \in \mathcal{U}$, $z \in M$, $v \in \mathbf{S}^u(z)$ and $n \ge 1$, we have

$$C_g^{-1}\cdot d(z,\mathcal{C}(F)) \le |\det DF_z| \le \exp(\Lambda_g)\|D^*F^n(z,v)\| \le C_g\cdot d(z,\mathcal{C}(F)), \tag{4}$$

$$C_g^{-1} < \|D_*F^n(z,v)\| \le \|DF_z^n\| \le C_g\|D_*F^n(z,v)\| \tag{5}$$

and, if $z \notin \mathcal{C}(F)$, also

$$C_g^{-1}\cdot\|D^*F^n(z,v)\| \le \|(DF_z^n)^{-1}\|^{-1} \le \|D^*F^n(z,v)\|. \tag{6}$$

3.4. **Admissible curves.** In this subsection, we introduce the notion of admissible curve. From the forward invariance of the unstable cones $\mathbf{S}^u$, the mappings in $\mathcal{U}$ preserve the class of C^1curves whose tangent vectors belong to $\mathbf{S}^u$. We shall investigate such class of curves and find a subclass which is uniformly bounded in C^{r-1}sense and essentially invariant under the iterates of mappings in $\mathcal{U}$. We shall call a curve in this subclass an admissible curve.

In this paper, we always assume that the curves are regular and parameterized by length. Let $\gamma : [0,a] \to M$ be a C^rcurve such that $\gamma'(t) \in \mathbf{S}^u(\gamma(t))$ for $t \in [0,a]$. As we assume $\|\gamma'(t)\| \equiv 1$, the second differential of γ is written in the form

$$\frac{d^2}{dt^2}\gamma(t) = d^2\gamma(t)\cdot(\gamma'(t))^{\perp}$$

where $d^2\gamma : [0,a] \to \mathbb{R}$ is a C^{r-2}function. We define $d^k\gamma(t)$ for $3 \le k \le r$ as the $(k-2)$-th differential of the function $d^2\gamma(t)$.

Let $F_*\gamma : [0,a'] \to M$ be the image of the curve γ under a mapping $F \in \mathcal{U}$. Notice that $F_*\gamma$ is not simply the composition $F\circ\gamma$, because we assume $F_*\gamma$ to be parameterized by length. The right relation between γ and $F_*\gamma$ is given by

$$F_*\gamma(p(t)) = F(\gamma(t)) \tag{7}$$

where $p : [0,a] \to [0,a']$ is the unique C^rdiffeomorphism satisfying $p(0) = 0$ and $\frac{d}{dt}p(t) = D_*F(\gamma(t),\gamma'(t))$. Differentiating the both sides of (7), we get the formula

$$D_*F(\gamma(t),\gamma'(t))\cdot(F_*\gamma)'(p(t)) = DF_{\gamma(t)}(\gamma'(t))$$

for $t \in [0,a]$. Differentiating the both sides again and considering the components normal to $(F\gamma)'(p(t))$, we get

$$d^2F_*\gamma(p(t)) = \frac{D^*F(\gamma(t),\gamma'(t))}{D_*F(\gamma(t),\gamma'(t))^2}\cdot d^2\gamma(t) + \frac{Q_2(\gamma(t),\gamma'(t);F)}{D_*F(\gamma(t),\gamma'(t))^3} \tag{8}$$

where $Q_2(a,b;F)$ is a polynomial of the components of the unit vector b whose coefficients are polynomials of the differentials of F at a up to the second order. Likewise, examining the differentials of the both sides of (8), we obtain

$$d^kF_*\gamma(p(t)) = \frac{D^*F(\gamma(t),\gamma'(t))}{D_*F(\gamma(t),\gamma'(t))^k}\cdot d^k\gamma(t) + \frac{Q_k(\gamma(t),\gamma'(t),\{d^i\gamma(t)\}_{i=2}^{k-1};F)}{D_*F(\gamma(t),\gamma'(t))^{3k-3}} \tag{9}$$

for $3 \le k \le r$, where $Q_k(a, b, \{c_i\}_{i=2}^{k-1}; F)$ is a polynomial of the components of the unit vector b and the scalars c_i whose coefficients are polynomials of the differentials of F at a up to the k-th order.

Remark. In addition, we can check that Q_k for $2 \le k \le r$ is written in the form

$$D_*F(\gamma(t), \gamma'(t))^{2k-3} \cdot v^*((D^kF)_a(b, b, \cdots, b)) + \tilde{Q}_k(a, b, \{c_i\}_{i=2}^{k-1}; F)$$

where v^* is a unit cotangent vector at the point $F(a)$ that is normal to $DF_a(b)$ and $\tilde{Q}_k(a, b, \{c_i\}_{i=2}^{k-1}; F)$ is a polynomial of the components of b and the scalars c_i whose coefficients are polynomials of the differentials of F at a up to the $(k-1)$-th order.

Fix an integer $n_g > 0$ such that $n_g\lambda_g - c_g > 0$. From the condition (C1) in subsection 3.2 and from the formulae (8) and (9) above, we can get

Lemma 3.2. *There exist constants $K_g^{(k)} > 1$ for $2 \le k \le r$ such that, if a curve $\gamma : [0, a] \to M$ of class C^{r-1} satisfies*

(a) *$\gamma'(t) \in \mathbf{S}^u(\gamma(t))$ for $t \in [0, a]$,*
(b) *$|d^k\gamma(t)| \le K_g^{(k)}$ for $2 \le k \le r-1$ and $t \in [0, a]$, and*
(c) *the function $d^{r-1}\gamma$ satisfies Lipschitz condition with constant $K_g^{(r)}$:*

$$\left|d^{r-1}\gamma(t) - d^{r-1}\gamma(s)\right| \le K_g^{(r)}|t - s| \qquad \text{for any } 0 \le s < t \le a.$$

then $F_*^n\gamma$ for $n \ge n_g$ satisfies the same conditions.

Henceforth we fix the constants $K_g^{(k)}$, $2 \le k \le r$, in lemma 3.2 and put

Definition. A C^{r-1}curve γ is called *an admissible curve* if γ satisfies the conditions (a), (b) and (c) above.

Corollary 3.3. *If a C^{r-1}curve γ is admissible, so is $F_*^n\gamma$ for $n \ge n_g$.*

For a positive number a, let $\mathcal{AC}'(a)$ be the set of C^1curves $\gamma : [0, a] \to M$ such that $\gamma'(t) \in \mathbf{S}^u(\gamma(t))$ for $t \in [0, a]$, and $\mathcal{AC}(a)$ the set of admissible curves in $\mathcal{AC}'(a)$. Note that $\mathcal{AC}(a)$ is a compact subset with respect to the C^{r-1}topology. For an interval $J \subset (0, \infty)$, we define the set $\mathbf{AC}(J)$ (resp. $\mathbf{AC}'(J)$) as the disjoint union of the product spaces $\mathcal{AC}(a) \times [0, a]$ (resp. $\mathcal{AC}'(a) \times [0, a]$) for $a \in J$:

$$\mathbf{AC}(J) := \coprod_{a \in J} \mathcal{AC}(a) \times [0, a] \qquad (\text{ resp. } \mathbf{AC}'(J) := \coprod_{a \in J} \mathcal{AC}'(a) \times [0, a] \text{ }).$$

We equip the space $\mathbf{AC}(J)$ with the distance $d_{\mathbf{AC}}$ defined by

$$d_{\mathbf{AC}}((\gamma_1, t_1), (\gamma_2, t_2)) = |t_2 - t_1| + \|\gamma_2 - \gamma_1\|_{C^{r-1}} + \left(\max_{2 \le k \le r} K_g^{(k)}\right) \cdot |a_2 - a_1|$$

for $(\gamma_i, t_i) \in \mathcal{AC}(a_i) \times [0, a_i]$, $i = 1, 2$ in $\mathbf{AC}(J)$, where $\|\gamma_2 - \gamma_1\|_{C^{r-1}}$ is

$$\max_{0 \le \theta \le \min\{a_1, a_2\}} \left\{ d(\gamma_2(\theta), \gamma_1(\theta)),\ \angle(\gamma_1'(\theta), \gamma_2'(\theta)),\ \max_{2 \le k \le r-1} \left|d^k\gamma_2(\theta) - d^k\gamma_1(\theta)\right| \right\}.$$

Then the space $\mathbf{AC}((0, \infty))$ with this distance is a complete separable metric space. The space $\mathbf{AC}(J)$ for an interval J is compact if and only if J is compact.

Each mapping $F \in \mathcal{U}$ naturally induces the continuous action

$$F_* : \mathbf{AC}'((0, \infty)) \to \mathbf{AC}'((0, \infty))$$

that maps a point $(\gamma, t) \in \mathcal{AC}'(a) \times [0, a]$ to $(F_*\gamma, p(t)) \in \mathcal{AC}'(a') \times [0, a']$, where a' is the length of the curve $F_*\gamma$ and $p : [0, a] \to [0, a']$ is the unique diffeomorphism that satisfies $p(0) = 0$ and $\frac{d}{dt}p(t) = D_*F(\gamma(t), \gamma'(t))$ for $t \in [0, a]$. Corollary 3.3 implies that the iterate F_*^n for $n \geq n_g$ maps the subset $\mathbf{AC}((0,\infty))$ into itself.

We define the mapping $\Pi : \mathbf{AC}'((0,\infty)) \to M$ by $\Pi(\gamma, t) = \gamma(t)$. Then we have the commutative relation:

$$\begin{array}{ccc} \mathbf{AC}'((0,\infty)) & \xrightarrow{F_*} & \mathbf{AC}'((0,\infty)) \\ \Pi\downarrow & & \Pi\downarrow \\ M & \xrightarrow{F} & M \end{array} \tag{10}$$

3.5. **Admissible measures.** In this subsection, we introduce the notion of admissible measure. Let $\Xi_{\mathbf{AC}}$ be the partition of the space $\mathbf{AC}((0,\infty))$ into the subsets $\{\gamma\} \times [0, a]$ for $a > 0$ and $\gamma \in \mathcal{AC}(a)$. On each element $\xi = \{\gamma\} \times [0, a]$ of $\Xi_{\mathbf{AC}}$, we consider the measure $\mathbf{m}_\xi$ that corresponds to Lebesgue measure on $[0, a]$ through the bijection $(\gamma, t) \mapsto t$. For a Borel finite measure $\tilde{\mu}$ on $\mathbf{AC}((0,\infty))$, let $\{\tilde{\mu}_\xi\}_{\xi\in\Xi_{\mathbf{AC}}}$ be the conditional measures with respect to the partition $\Xi_{\mathbf{AC}}$.

For a positive number $L > 0$, let us consider the following condition on a Borel finite measure $\tilde{\mu}$ on $\mathbf{AC}((0,\infty))$:

(∗) the conditional measure $\tilde{\mu}_\xi$ is absolutely continuous with respect to $\mathbf{m}_\xi$ and the density $d\tilde{\mu}_\xi/d\mathbf{m}_\xi$ has a version such that $\log(d\tilde{\mu}_\xi/d\mathbf{m}_\xi)$ satisfies Lipschitz condition with constant L, for $\tilde{\mu}$-almost every $\xi \in \Xi_{\mathbf{AC}}$.

Since the iteration of the mappings in $\mathcal{U}$ is expanding on the admissible curves uniformly, we can show the following lemma by using the standard argument on the iteration of expanding maps.

Lemma 3.4. *There exists a positive constant L_g such that, if a Borel finite measure $\tilde{\mu}$ on $\mathbf{AC}((0,\infty))$ satisfies the condition (∗) for $L = L_g$ and if $F \in \mathcal{U}$, then the image $\tilde{\mu} \circ F_*^{-n}$ for $n \geq n_g$ satisfies the same condition.*

Henceforth we fix the constant L_g in lemma 3.4 and put the following definition:

Definition. A Borel finite measure $\tilde{\mu}$ on $\mathbf{AC}((0,\infty))$ is called an admissible measure if it satisfies the condition (∗) for $L = L_g$.

For an interval $J \subset (0,\infty)$, let $\mathbf{AM}(J)$ be the set of admissible measures that is supported on $\mathbf{AC}(J)$. Then we can see

Lemma 3.5. *If a measure $\tilde{\mu}$ belongs to $\mathbf{AM}([a,\infty))$ for some $a \geq 0$ and if $F \in \mathcal{U}$, then $\tilde{\mu} \circ F_*^{-n}$ belongs to $\mathbf{AM}([a',\infty))$ for $n \geq n_g$ where $a' = a\exp(\lambda_g n - c_g) > a$.*

Lemma 3.6. *The subset $\mathbf{AM}(J)$ for a closed interval $J \subset (0,\infty)$ is closed in the space of Borel finite measures on $\mathbf{AC}((0,\infty))$.*

Definition. A Borel finite measure μ on M is said to have *an admissible lift* if there is an admissible measure $\tilde{\mu} \in \mathbf{AM}((0,\infty))$ such that $\tilde{\mu} \circ \Pi^{-1} = \mu$. The measure $\tilde{\mu}$ is called *an admissible lift* of μ.

For an interval $J \subset (0,\infty)$, let $\mathcal{AM}(J)$ be the sets of Borel finite measures on M that have admissible lifts in $\mathbf{AM}(J)$. Lemma 3.5 implies

Corollary 3.7. *If $\mu \in \mathcal{AM}([a,\infty))$ for some $a > 0$ and if $F \in \mathcal{U}$, then the measure $\mu \circ F^{-n}$ belongs to $\mathcal{AM}([a',\infty))$ for $n \geq n_g$ where $a' = a\exp(\lambda_g n - c_g) > a$. Especially, if an invariant measure for $F \in \mathcal{U}$ has an admissible lift, it belongs to $\mathcal{AM}([a,\infty))$ for any $a > 0$.*

For $a > 0$, let $\Delta_a : \mathbf{AC}([a, \infty)) \to \mathbf{AC}([a, 2a])$ be the mapping that brings an element $(\gamma, t) \in \mathcal{AC}(b) \times [0, b]$ to

$$\Delta_a((\gamma, t)) = (\gamma|_{[m(t), m(t)+(b/n)]}, t - m(t)) \in \mathcal{AC}(b/n) \times [0, b/n] \tag{11}$$

where $n = [b/a]$ and $m(t) = [tn/b](b/n)$. Then we have $\Pi \circ \Delta_a = \Pi$ and, for any $\tilde{\mu} \in \mathbf{AM}([a, \infty)$, the image $\tilde{\mu} \circ \Delta_a^{-1}$ belongs to $\mathbf{AM}([a, 2a])$. Thus we obtain

Lemma 3.8. *$\mathcal{AM}([a, \infty)) = \mathcal{AM}([a, 2a])$ for $a > 0$.*

From this lemma and lemma 3.6, it follows

Lemma 3.9. *The set $\mathcal{AM}([a, \infty))$ for $a > 0$ is a closed subset in the space of Borel finite measures on M. Especially, for a mapping $F \in \mathcal{U}$, the set of F-invariant Borel probability measures that have admissible lifts is compact.*

Suppose that P is a small parallelogram on the torus $\mathbb{T}$ whose center z belongs to M and two of whose sides are parallel to the unstable subspace $\mathbf{E}^u(z)$. Then the restriction of Lebesgue measure $\mathbf{m}$ to P has an admissible lift, provided that P is sufficiently small. Moreover the linear combinations of such measures have admissible lifts. Thus we obtain

Lemma 3.10. *For any Borel finite measure ν on M that is absolutely continuous with respect to Lebesgue measure, there exist a sequence $b_n \to +0$ and measures $\nu_n \in \mathcal{AM}([b_n, \infty))$ such that $|\nu - \nu_n| \to 0$ as $n \to \infty$. Further we can take the measures ν_n so that the densities $d\nu_n/d\mathbf{m}$ are square integrable.*

The following is a consequence of the last two lemmas and corollary 3.7.

Lemma 3.11. *Let F be a mapping in $\mathcal{U}$ and ν a probability measure on M that is absolutely continuous with respect to $\mathbf{m}$. Then any weak limit point of the sequence $n^{-1} \sum_{i=0}^{n-1} \nu \circ F^{-i}$ is contained in $\mathcal{AM}([a, \infty))$ for any $a > 0$. Especially, physical measures for F are contained in $\mathcal{AM}([a, \infty))$ for any $a > 0$.*

Finally we prove

Lemma 3.12. *Let F be a mapping in $\mathcal{U}$. If an F-invariant Borel probability measure has an admissible lift, so do its ergodic components.*

Proof. From lemma 3.9, it is enough to show the following claim: If an F-invariant measure μ that has an admissible lift splits into two non-trivial F-invariant measures μ_1 and μ_2 that are totally singular with respect to each other, then the measures μ_1 and μ_2 have admissible lifts. We are going to show this claim. From corollary 3.7, we can take an admissible lift $\tilde{\mu}$ of μ that is supported on $\mathbf{AC}([1, \infty))$. Consider the mapping $G = \Delta_1 \circ F_*^{n_g} : \mathbf{AC}([1, \infty)) \to \mathbf{AC}([1, 2])$, where Δ_1 is the mapping defined by (11). Replacing $\tilde{\mu}$ by $\tilde{\mu} \circ G^{-1}$, we assume that $\tilde{\mu}$ is supported on $\mathbf{AC}([1, 2])$. From the assumption, we can take an F-invariant Borel subset $X \subset M$ such that $\mu_1(M \backslash X) = \mu_2(X) = 0$. Then, by the relation $F^{n_g} \circ \Pi = \Pi \circ G$, the set $\tilde{X} := \pi^{-1}(X)$ is G-invariant. If we prove that $\tilde{X}$ is a $\Xi_{\mathbf{AC}}$-set, that is, a union of elements of the partition $\Xi_{\mathbf{AC}}$, modulo null subsets with respect to $\tilde{\mu}$, then the claim above follows because the restriction of the measure $\tilde{\mu}$ to $\tilde{X}$ is an admissible lift of μ_1.

Put $\Xi_1 = \Xi_{\mathbf{AC}}$ and define the sequence Ξ_n, $n = 1, 2, \ldots$ inductively by the relation $\Xi_{n+1} = G^{-1}(\Xi_n) \vee \Xi_1$. Then Ξ_n is increasing with respect to n and the limit $\bigvee_{n=1}^{\infty} \Xi_n$ is the partition into individual points. Thus the conditional expectation $E(\tilde{X}|\Xi_n)$ with respect to $\tilde{\mu}$ converges to the indicator function of $\tilde{X}$ as

$n \to \infty$, $\tilde{\mu}$-a.e. Note that the restriction of G^n to each element of the partition Ξ_n is a bijection onto an element of Ξ_1 and its distortion is uniformly bounded. Hence, using the assumption that $\tilde{\mu}$ is an admissible measure and the invariance of $\tilde{X}$, we can see that the conditional expectation $E(\tilde{X}|\Xi_1)$ equals to the indicator function of $\tilde{X}$, or $\tilde{X}$ is a $\Xi_{\mathbf{AC}}$-set modulo null subsets with respect to $\tilde{\mu}$. □

3.6. **The no flat contact condition.** In this subsection, we consider the influence of the critical points on ergodic behavior of partially hyperbolic endomorphisms. We first explain a problem that the critical points may cause. And then we give a mild condition on the mappings in $\mathcal{U}$, *the no flat contact condition*, which allows us to avoid that problem. In the last part of this paper, we will prove that this condition holds for almost all partially hyperbolic endomorphisms in $\mathcal{U}$.

Let $\chi_c(z;F) < \chi_u(z;F)$ be Lyapunov exponents of a mapping $F \in \mathcal{U}$ at $z \in M$. For a mapping $F \in \mathcal{U}$ and a Borel finite measure μ on M, we define

$$\chi_c(\mu;F) = \frac{1}{|\mu|}\int \log \|DF|_{E^c(z)}\|\, d\mu(z) \quad \text{and}$$
$$\chi_u(\mu;F) = \frac{1}{|\mu|}\int \log(|\det DF_z|/\|DF|_{E^c(z)}\|)\, d\mu(z).$$

These are called the central and unstable Lyapunov exponent of μ, respectively. If μ is an ergodic invariant measure for $F \in \mathcal{U}$, Lyapunov exponents $\chi_c(z;F)$ and $\chi_u(z;F)$ take constant values $\chi_c(\mu;F)$ and $\chi_u(\mu;F)$ respectively, μ-a.e.

Let F be a mapping in $\mathcal{U}$ and μ_n, $n = 1, 2, \cdots$, a sequence of ergodic invariant probability measures for F that have admissible lifts. Suppose that μ_n converges weakly to some measure μ_∞ as $n \to \infty$. Then μ_∞ has an admissible lift from lemma 3.9. It is not difficult to see that the Lyapunov exponent $\chi_u(\mu_n;F)$ always converges to $\chi_u(\mu_\infty;F)$. However, for the central Lyapunov exponent, we only have the inequality

$$\limsup_{i\to\infty} \chi_c(\mu_n;F) \le \chi_c(\mu_\infty;F) \tag{12}$$

when F has critical points, because the function $\log \|DF|_{E^c(z)}\|$ is not continuous at the critical points. Though the strict inequality in (12) is not likely to hold often, we can not avoid such cases in general. And, once the strict inequality holds, the ergodic behavior of F can be wild.

Remark. It is not easy to construct examples in which the strict inequality (12) holds. For example, consider the direct product of the quadratic mappings given in the paper[7] and an angle multiplying mapping $\theta \mapsto d\cdot\theta$ on the circle.

In order to avoid the irregularity described above, we introduce a mild condition:

Definition. We say that a mapping $F \in \mathcal{U}$ satisfies *the no flat contact condition* if there exist positive constants $C = C(F)$, $n_0 = n_0(F) \ge n_g$ and $\beta = \beta(F)$ such that, for any admissible curve $\gamma \in \mathcal{AC}(a)$, $n \ge n_0$ and $\epsilon > 0$, it holds

$$\mathbf{m}_{\mathbb{R}}\left(\{t \in [0,a]\ ;\ d(F^n(\gamma(t)), \mathcal{C}(F)) < \epsilon\}\right) < C\cdot\epsilon^\beta \max\{a,1\}$$

where $\mathbf{m}_{\mathbb{R}}$ is the Lebesgue measure on $\mathbb{R}$. If F has no critical points, we regard that $d(z,\mathcal{C}(F)) = +\infty$ for $z \in M$ and that F satisfies the no flat contact condition.

Remark. The definition above is motivated by the argument in a paper of Viana[23], in which the condition as above for $\beta = 1/2$ is considered.

Below we give simple consequences of the no flat contact condition. For $F \in \mathcal{U}$ and $z \in M$, we define

$$L(z; F) := \log\left(\min_{v \in \mathbf{S}^u(z)} |D^*F(z, v)|\right) \in \mathbb{R} \cup \{-\infty\}. \tag{13}$$

This function is continuous outside the critical set $\mathcal{C}(F)$ and satisfies

$$L(z; F) \geq \log d(z, \mathcal{C}(F)) - C_g$$

from (4), provided that $\mathcal{C}(F) \neq \emptyset$. Thus we can get the following lemma.

Lemma 3.13. *Suppose that $F \in \mathcal{U}$ satisfies the no flat contact condition and let $n_0 = n_0(F)$ be that in the condition. For any $\delta > 0$ and $a > 0$, we can choose a positive number $h = h(\delta, a; F)$ such that*

$$\int \min\{0, L(z; F) + h\}\, d(\mu \circ F^{-n})(z) \geq -\delta \cdot |\mu|$$

for any $\mu \in \mathcal{AM}([a, \infty))$ and $n \geq n_0$.

Using the inequality $\log \|DF|_{E^c(z)}\| \geq L(z; F) - C_g$, which follows from (4), together with lemma 3.13, corollary 3.7 and lemma 3.9, we can obtain

Corollary 3.14. *Suppose that $F \in \mathcal{U}$ satisfies the no flat contact condition. Then the central Lyapunov exponent $\chi_c(\mu; F)$ considered as a function on the space of F-invariant probability measures that have admissible lifts is continuous and, hence, uniformly bounded away from $-\infty$.*

This corollary implies that the irregularity of the central Lyapunov exponent mentioned above do not take place under the no flat contact condition.

3.7. **Multiplicity of tangencies between the images of the unstable cones.** By an iterate of a mapping $F \in \mathcal{U}$, the unstable cones $\mathbf{S}^u(z)$ at many points z will be brought to one point and some pairs of their images may tangent, that is, have a half-line in common. In this subsection, we introduce quantities that measure the multiplicity of such tangencies and then formulate a condition, *the transversality condition on unstable cones*, for mappings in $\mathcal{U}$.

First we introduce analogues of Pesin subsets. Let $\chi = \{\chi_c^-, \chi_c^+, \chi_u^-, \chi_u^+\}$ be a quadruple of real numbers that satisfy

$$\chi_c^- < \chi_c^+ < \chi_u^- < \chi_u^+. \tag{14}$$

Let ϵ be a positive number. For a mapping $F \in \mathcal{U}$, an integer $n > 0$ and a real number $k > 0$, we define a closed subset $\Lambda(\chi, \epsilon, k, n; F)$ of M as the set of all points $z \in M$ that satisfy

$$\chi_c^-(j-i) - \epsilon(n-j) - k \leq \log|D^*F^{j-i}(v)| \leq \chi_c^+(j-i) + \epsilon(n-j) + k$$

and

$$\chi_u^-(j-i) - \epsilon(n-j) - k \leq \log D_*F^{j-i}(v) \leq \chi_u^+(j-i) + \epsilon(n-j) + k$$

for any $0 \leq i < j \leq n$ and $v \in \mathbf{S}^u(F^i(z))$. Applying the standard argument in Pesin theory[14, 15] to the inverse limit, we can show

Lemma 3.15. *If μ is an invariant probability measure for $F \in \mathcal{U}$ and if*

$$\chi_c^- < \chi_c(z; F) < \chi_c^+ \quad \text{and} \quad \chi_u^- < \chi_u(z; F) < \chi_u^+ \quad \mu\text{-a.e.,}$$

we have $\lim_{k\to\infty} \liminf_{n\to\infty} \mu(\Lambda(\chi, \epsilon, k, n; F)) = 1$.

The subset $\Lambda(\chi, \epsilon, k, n; F)$ is increasing with respect to k and ϵ, and satisfies

$$F^i(\Lambda(\chi, \epsilon, k, n; F)) \subset \Lambda(\chi, \epsilon, k, n-i; F) \quad \text{and} \tag{15}$$

$$\Lambda(\chi, \epsilon, k, n; F) \subset \Lambda(\chi, \epsilon, k+\epsilon i, n-i; F) \quad \text{for } 0 \le i < n. \tag{16}$$

From (3), we can and do take a constant H_g such that

$$\angle(DF^n(u), DF^n(v)) < H_g \frac{|D^*F^n(z,w)|}{D_*F^n(z,w)} \le H_g \exp((\chi_c^+ - \chi_u^-)n + 2k) \tag{17}$$

for any $z \in \Lambda(\chi, \epsilon, k, n; F)$ and $u, v, w \in \mathbf{S}^u(z)$. For $z \in M$, let $\mathcal{E}(z; \chi, \epsilon, k, n; F)$ be the set of all pairs (w, w') of points in $F^{-n}(z) \cap \Lambda(\chi, \epsilon, k, n; F)$ such that

$$\angle(DF^n(\mathbf{E}^u(w')), DF^n(\mathbf{E}^u(w))) \le 5H_g \exp((\chi_c^+ - \chi_u^-)n + 2k). \tag{18}$$

Note that, if a pair (w, w') of points in $F^{-n}(z) \cap \Lambda(\chi, \epsilon, k, n; F)$ does *not* belong to $\mathcal{E}(z; \chi, \epsilon, k, n; F)$, we have

$$\angle(DF^n(u), DF^n(u')) > 3H_g \exp((\chi_c^+ - \chi_u^-)n + 2k) \tag{19}$$

for any $u \in \mathbf{S}(w)$ and $u' \in \mathbf{S}(w')$, from (17).

As a measure for the multiplicity of tangencies, we consider the number

$$\mathbf{N}(\chi, \epsilon, k, n; F) = \max_{z \in M} \max_{w \in F^{-n}(z) \cap \Lambda(\chi, \epsilon, k, n; F)} \#\{w' \mid (w, w') \in \mathcal{E}(z; \chi, \epsilon, k, n; F)\}.$$

This is increasing with respect to k and ϵ.

Definition. Let $\mathbf{X} = \{\chi(\ell)\}_{\ell=1}^{\ell_0}$ be a finite collection of quadruples of numbers $\chi(\ell) = \{\chi_c^-(\ell), \chi_c^+(\ell), \chi_u^-(\ell), \chi_u^+(\ell)\}$ that satisfy (14). We say that a mapping $F \in \mathcal{U}$ satisfies *the transversality condition on unstable cones for* $\mathbf{X}$ if

$$\lim_{\epsilon \to +0} \lim_{k \to \infty} \liminf_{n \to \infty} \max \left\{ \frac{\log(\mathbf{N}(\chi(\ell), \epsilon, k, n; F))}{n \cdot (\chi_c^-(\ell) + \chi_u^-(\ell) - \chi_c^\Delta(\ell) - \chi_u^\Delta(\ell))} \; ; \; 1 \le \ell \le \ell_0 \right\} < 1$$

where $\chi_c^\Delta(\ell) = \chi_c^+(\ell) - \chi_c^-(\ell)$ and $\chi_u^\Delta(\ell) = \chi_u^+(\ell) - \chi_u^-(\ell)$.

3.8. **Measures on the space of mappings.** In this subsection, we give some additional argument concerning measures on the space of mappings. Recall that we denote by $\tau_\psi : C^r(M, \mathbb{R}^2) \to C^r(M, \mathbb{R}^2)$ the translation by $\psi \in C^r(M, \mathbb{R}^2)$. For an integer $s \ge 0$ and a positive number $d > 0$, we put

$$\mathbf{D}^s(d) = \{G \in C^s(M, \mathbb{R}^2) \mid \|G\|_{C^s} \le d\}. \tag{20}$$

The following lemma gives measures on $C^r(M, \mathbb{R}^2)$ with nice properties.

Lemma 3.16. *For an integer $s \ge 3$, there exists a Borel probability measure $\mathcal{M}_s$ on $C^{s-3}(M, \mathbb{R}^2)$ such that*

1. *$\mathcal{M}_s$ is quasi-invariant along $C^{s-1}(M, \mathbb{R}^2)$ and*
2. *there exists a positive constant $\rho = \rho_s(d)$ for any $d > 0$ such that*

$$\frac{1}{2} \le \frac{d(\mathcal{M}_s \circ \tau_\psi^{-1})}{d\mathcal{M}_s} \le 2 \quad \mathcal{M}_s\text{-a.e. on } \mathbf{D}^{s-3}(d)$$

for any $\psi \in C^s(M, \mathbb{R}^2)$ with $\|\psi\|_{C^s} < \rho$.

We give the proof of lemma 3.16 in the appendix at the end of this paper, one because the lemma itself has nothing to do with dynamical systems and one because the proof is merely a combination of some results in probability theory.

Henceforth, we fix the measures $\mathcal{M}_s$ for $s \ge 3$ in lemma 3.16. Note that the measure $\mathcal{M}_s$ belongs to $\mathcal{Q}_{s-1}^r$ when $s \ge r + 3$.

Lemma 3.17. *Suppose $s \geq r+3$. If a Borel subset X in $C^r(M,M)$ is shy with respect to the measure $\mathcal{M}_{s+2}$, then X is shy with respect to any measure in $\mathcal{Q}^r_{s-1}$.*

Proof. Take an arbitrary measure $\mathcal{N}$ in $\mathcal{Q}^r_{s-1}$. The measure $\mathcal{M}_{s+2}$ is supported on the space $C^{s-1}(M,\mathbb{R}^2)$, along which $\mathcal{N}$ is quasi-invariant. Hence the convolution $\mathcal{N} * \mathcal{M}_{s+2}$ is equivalent to $\mathcal{N}$. From the assumption, we have

$$\begin{aligned}\mathcal{N} * \mathcal{M}_{s+2}(\Phi_G^{-1}(X)) &= \int \mathcal{M}_{s+2} \circ \tau_\psi^{-1}(\Phi_G^{-1}(X)) d\mathcal{N}(\psi)\\ &= \int \mathcal{M}_{s+2}(\Phi_{G+\psi}^{-1}(X)) d\mathcal{N}(\psi) = 0\end{aligned}$$

for any $G \in C^r(M,\mathbb{T})$. Thus X is shy with respect to $\mathcal{N}$. □

In order to evaluate subsets in $C^r(M,\mathbb{T})$ with respect to the measures $\mathcal{M}_s$, we will use the following lemma:

Lemma 3.18. *Let $s \geq r+3$ and $d > 0$. Suppose that mappings $\psi_i \in C^s(M,\mathbb{R}^2)$ and positive numbers T_i for $1 \leq i \leq m$ satisfy*

$$\sup_{|t_i| \leq T_i} \left\| \sum_{i=1}^m t_i \psi_i \right\|_{C^s} \leq \rho_s(d) \tag{21}$$

where $\rho_s(d)$ is that in lemma 3.16. If a Borel subset X in $C^r(M,\mathbb{T})$ satisfies, for some $\beta > 0$, that

$$\mathbf{m}_{\mathbb{R}^m}\left(\left\{ (t_i)_{i=1}^m \in \prod_{i=1}^m [-T_i, T_i] \;\middle|\; \varphi + \sum_{i=1}^m t_i \psi_i \in X \right\}\right) < \beta \prod_{i=1}^m 2T_i \tag{22}$$

for every $\varphi \in X$, then we have

$$\mathcal{M}_s(\Phi_G^{-1}(X) \cap \mathbf{D}^{s-3}(d)) \leq 2^{m+1}\beta \cdot \mathcal{M}_s(\Phi_G^{-1}(Y)) \leq 2^{m+1}\beta$$

for any $G \in C^r(M,\mathbb{T})$, where

$$Y = \left\{ \psi + \sum_{i=1}^m t_i \psi_i \;\middle|\; \psi \in X,\ \ |t_i| \leq T_i/2 \right\}.$$

Proof. Put $Z = \Phi_G^{-1}(X) \cap \mathbf{D}^{s-3}(d)$ and denote by $\mathbf{1}_Z$ the indicator function of Z. Using Fubini theorem and the properties of $\mathcal{M}_s$, we get

$$\begin{aligned}&\int \mathbf{m}_{\mathbb{R}^m}\left(\left\{ \mathbf{t} \in \mathbb{R}^m \;\middle|\; |t_i| \leq \frac{T_i}{2},\ \tilde{\psi} + \sum t_i \psi_i \in Z \right\}\right) d\mathcal{M}_s(\tilde{\psi})\\ &= \int_{\{\mathbf{t};|t_i|<T_i/2\}} \left(\int \mathbf{1}_Z \left(\tilde{\psi} + \sum t_i \psi_i\right) d\mathcal{M}_s(\tilde{\psi}) \right) d\mathbf{m}_{\mathbb{R}^m}(\mathbf{t})\\ &= \int_{\{\mathbf{t};|t_i|<T_i/2\}} \mathcal{M}_s\left(Z - \sum t_i \psi_i\right) d\mathbf{m}_{\mathbb{R}^m}(\mathbf{t}) \geq 2^{-1}\mathcal{M}_s(Z) \prod_{i=1}^m T_i.\end{aligned}$$

The integrand of the integral on the first line is positive only if $\tilde{\psi}$ belongs to $\Phi_G^{-1}(Y)$ and bounded by $\beta \prod_{i=1}^m 2T_i$ from the assumption (22). Thus we obtain the lemma. □

3.9. **The plan of the proof of the main theorems.** Now we can describe the plan of the proof of the main results, theorem 2.1 and 2.2, more concretely by using the terminology introduced in the preceding subsections. We split the proof into two parts. In the former part, which will be carried out in sections 4, 5 and 6, we study ergodic properties of partially hyperbolic endomorphisms in $\mathcal{U}$ that satisfy the no flat contact condition and the transversality condition on unstable cones for some finite collection of quadruples. The conclusion in this part is the following theorem: For a finite or countable collection $\mathbf{X} = \{\chi(\ell)\}_{\ell\in L}$ of quadruples $\chi(\ell) = \{\chi_c^-(\ell), \chi_c^+(\ell)), \chi_u^-(\ell), \chi_u^+(\ell)\}$ that satisfy the condition (14), we denote by $|\mathbf{X}|$ the union of the open rectangles $(\chi_c^-(\ell), \chi_c^+(\ell)) \times (\chi_u^-(\ell), \chi_u^+(\ell))$ over $\ell \in L$.

Theorem 3.19. *Let $\mathbf{X}$ be a finite collection of quadruples that satisfy (14),*

$$\chi_c^- + \chi_u^- > 0, \qquad \chi_c^- < 0 \tag{23}$$

and also

$$\{0\} \times [\lambda_g, \Lambda_g] \subset |\mathbf{X}| \subset (-2\Lambda_g, 2\Lambda_g) \times (0, 2\Lambda_g). \tag{24}$$

Suppose that a mapping F in $\mathcal{U}$ satisfies the no flat contact condition and the transversality condition on unstable cones for $\mathbf{X}$. Then F admits a finite collection of ergodic physical measures whose union of basins of attraction has total Lebesgue measure on M. In addition, if an ergodic physical measure μ for F satisfies either $(\chi_c(\mu; F), \chi_u(\mu; F)) \in |\mathbf{X}|$ or $\chi_c(\mu; F) > 0$, then μ is absolutely continuous with respect to Lebesgue measure.

In the latter part of the proof, which will be carried out in section 7 and 8, we show that the two conditions assumed on the mapping F in the theorem above hold for almost all partially hyperbolic endomorphisms in $\mathcal{U}$, provided that we choose the finite collection $\mathbf{X}$ of quadruple appropriately. On the one hand, we will prove the following theorem in section 7: For a finite collection $\mathbf{X}$ of quadruples that satisfy (14), we denote by $\mathcal{S}_1(\mathbf{X})$ the set of mappings $F \in \mathcal{U}$ that does *not* satisfy the transversality condition on unstable cones for $\mathbf{X}$.

Theorem 3.20. *There exists a countable collection $\mathbf{X} = \{\chi(\ell)\}_{\ell=1}^{\infty}$ of quadruples satisfying (14) and (23) such that*

(a) *$|\mathbf{X}|$ contains the subset $\{(x_c, x_u) \in \mathbb{R}^2 \mid x_c + x_u > 0, \lambda_g \le x_u \le \Lambda_g, x_c \le 0\}$,*
(b) *$|\mathbf{X}|$ is contained in $(-2\Lambda_g, 2\Lambda_g) \times (0, 2\Lambda_g)$, and*
(c) *the subset $\mathcal{S}_1(\mathbf{X}')$ for any finite sub-collection $\mathbf{X}' \subset \mathbf{X}$ is shy with respect to the measures $\mathcal{M}_s$ for $s \ge r+3$ and is a meager subset in $\mathcal{U}$ in the sense of Baire's category argument.*

On the other hand, we will show the following theorem in section 8. Let $\mathcal{S}_2$ be the set of mappings $F \in \mathcal{U}$ that does *not* satisfy the no flat contact condition.

Theorem 3.21. *If an integer $s \ge r+3$ satisfies the condition (2) for some integer $3 \le \nu \le r-2$, then the subset $\mathcal{S}_2$ is shy with respect to the measure $\mathcal{M}_s$. Moreover, $\mathcal{S}_2$ is contained in a closed nowhere dense subset in $\mathcal{U}$, provided that $r \ge 19$.*

It is easy to check that the three theorems above imply the main theorems: Consider a countable set of quadruples $\mathbf{X} = \{\chi(\ell)\}_{\ell=1}^{\infty}$ in theorem 3.20 and put $\mathbf{X}_m = \{\chi(\ell)\}_{\ell=1}^{m}$ for $m > 0$. Theorem 3.19 implies that the complement of $(\cup_{m=1}^{\infty}\mathcal{S}_1(\mathbf{X}_m)) \cup \mathcal{S}_2$ in $\mathcal{U}$ is contained in $\mathcal{R}^r$. Thus the main theorems, theorem 2.1 and 2.2, restricted to $\mathcal{U}$ follow from theorem 3.20, 3.21 and lemma 3.17. As we noted in subsection 3.2, this is enough for the proof of the main theorems.

4. Hyperbolic physical measures

In this section, we study hyperbolic physical measures for partially hyperbolic endomorphisms. Throughout this section, *we consider a mapping F in $\mathcal{U}$ that satisfies the no flat contact condition.*

4.1. **Physical measures with negative central exponent.** In this subsection, we study physical measures whose central Lyapunov exponent is negative.

Lemma 4.1. *If an ergodic probability measure μ with negative central Lyapunov exponent has an admissible lift, then it is a physical measure.*

Proof. The central Lyapunov exponent of the measure μ is bounded way from $-\infty$ by corollary 3.14. From Oceledec's theorem and the assumption that μ has an admissible lift, we can find an admissible curve γ such that almost all points with respect to the smooth measure on it are forward Lyapunov regular for μ. According to Pesin theory, the so-called Pesin's local stable manifold exists for each of such points on γ. These local stable manifolds are transversal to γ and contained in the basin $\mathcal{B}(\mu)$ of μ. Further, the union of them has positive Lebesgue measure from absolute continuity of Pesin's local stable manifolds[15, §4]. Therefore μ is a physical measure. □

From this lemma and lemma 3.12, we can get

Corollary 4.2. *If an F-invariant probability measure μ has an admissible lift, it has at most countably many ergodic components with negative central Lyapunov exponent, each of which is a physical measure and absolutely continuous w.r.t. μ.*

The basin of an ergodic physical measure with negative central Lyapunov exponent may not have interior even though we ignore null subsets with respect to Lebesgue measure. Nevertheless, we have

Lemma 4.3. *For an ergodic physical measure μ with negative central Lyapunov exponent, there is an open subset U with $\mu(U) = 1$ such that*

$$\nu(\mathcal{B}(\mu)) > 0 \quad \text{if and only if} \quad \limsup_{n\to\infty} \nu \circ F^{-n}(U) > 0 \tag{25}$$

for a Borel finite measure ν which has an admissible lift.

Proof. Recall the proof of lemma 4.1. From absolute continuity of Pesin's local stable manifolds, there exists an open neighborhood U_z for μ-almost every point z such that, if an admissible curve $\gamma : [0, a] \mapsto M$ with length $a > 2$ satisfies $\gamma([1, a-1]) \cap U_z \neq \emptyset$, the inverse image $\gamma^{-1}(\mathcal{B}(\mu))$ has positive Lebesgue measure. Let U be the union of such neighborhoods U_z. If $\limsup_{n\to\infty} \nu \circ F^{-n}(U) > 0$ for a Borel finite measure ν that has an admissible lift, we have $\nu(\mathcal{B}(\mu)) > 0$ from the choice of the neighborhoods U_z and corollary 3.7. The reverse implication is a consequence of the fact $\mu(U) = 1$. □

Lemma 4.4. *Let μ_i, $i = 1, 2, \dots$, be a sequence of mutually distinct F-invariant Borel probability measures each of which has an admissible lift. If μ_i converges to some measure μ_∞ as $i \to \infty$, we have $\chi_c(z; F) \geq 0$ for μ_∞-almost every $z \in M$.*

Proof. From lemma 3.9, μ_∞ has an admissible lift. If the conclusion of the lemma were not true, there should be an ergodic physical measure $\mu'_\infty \ll \mu_\infty$ with negative central Lyapunov exponent, from corollary 4.2. Take the open set U in lemma 4.3

for μ'_∞. On the one hand, $\mu'_\infty(U) = 1$ and hence $\mu_\infty(U) > 0$. On the other hand, since $\mu_i \neq \mu'_\infty$ except for one i at most, we should have $\mu_i(\mathcal{B}(\mu'_\infty)) = 0$ and hence $\mu_i(U) = 0$. These contradict the fact that μ_i converges to μ_∞. □

From this lemma and corollary 3.14, it follows

Corollary 4.5. *For any negative number $\chi < 0$, there exist at most finitely many ergodic physical measures for F that satisfies $\chi_c(\mu; F) \le \chi$.*

Finally we show

Lemma 4.6. *Let ν be a Borel finite measure that is absolutely continuous with respect to Lebesgue measure and μ a limit point of the sequence $n^{-1}\sum_{i=0}^{n-1} \nu \circ F^{-n}$, $n = 1, 2, \ldots$. Then we have either*

(a) *$\chi_c(z; F) \ge 0$ for μ-almost every point $z \in M$, or*
(b) *there is an ergodic physical measure $\mu' \ll \mu$ with negative central Lyapunov exponent and $\nu(\mathcal{B}(\mu')) > 0$.*

Especially, for a physical measure μ for F, we have either (a) *or*

(b′) *μ is ergodic and has negative central Lyapunov exponent.*

Proof. Suppose that (a) does not hold. Then, from corollary 4.2, there exists an ergodic physical measure $\mu' \ll \mu$ with negative central Lyapunov exponent. Take the open set U in lemma 4.3 for μ'. We have $\mu'(U) = 1$ and hence $\mu(U) > 0$. Thus

$$\limsup_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} \nu \circ F^{-n}(U) \ge \mu(U) > 0.$$

Though the measure ν may not have an admissible lift, we can use the approximation in lemma 3.10 to conclude $\nu(\mathcal{B}(\mu')) > 0$. □

4.2. **Physical measures with positive central exponent.** In this subsection, we investigate physical measures with positive central Lyapunov exponent. We shall prove the following three propositions.

Proposition 4.7. *Any physical measure μ with positive central Lyapunov exponent is ergodic and absolutely continuous with respect to Lebesgue measure. Moreover the basin $\mathcal{B}(\mu)$ is an open set modulo Lebesgue null subsets.*

Proposition 4.8. *For any positive number $\chi > 0$, there exist at most finitely many ergodic physical measures for F that satisfies $\chi_c(\mu; F) \ge \chi$.*

Let $\mathcal{B}^+(F)$ (resp. $\mathcal{B}^-(F)$) be the union of the basins of ergodic physical measures with positive (resp. negative) central Lyapunov exponent.

Proposition 4.9. *Suppose that a Borel probability measure ν on M is absolutely continuous with respect to Lebesgue measure and supported on the complement of $\mathcal{B}^-(F) \cup \mathcal{B}^+(F)$. If ν_∞ is a weak limit point of the sequence $n^{-1}\sum_{j=0}^{n-1} \nu \circ F^{-j}$, $n = 1, 2, \ldots$, then we have $\chi_c(z; F) = 0$ for ν_∞-almost every point z.*

We derive the propositions above from the following single proposition: Let $X(i)$, $i = 1, 2, \cdots$, be Borel subsets in M with positive Lebesgue measure. We denote by $\mathbf{m}_{X(i)}$ the normalization of the restriction of Lebesgue measure $\mathbf{m}$ to $X(i)$. For each $i \ge 1$, let $\mu_{i,\infty}$ be a weak limit point of the sequence $n^{-1}\sum_{j=0}^{n-1} \mathbf{m}_{X(i)} \circ F^{-j}$, $n = 1, 2, \cdots$. Assume that the sequence $\mu_{i,\infty}$ converges weakly to some measure μ_∞ as $i \to \infty$. Also assume that $\chi_c(\mu_\infty; F) > 0$ and that $\chi_c(z; F) \ge 0$, μ_∞-a.e.

Proposition 4.10. *In the situation as above, there exist an ergodic physical measure $\nu_{i,\infty}$ and a disk D_i in M for sufficiently large i such that*

(a) $\nu_{i,\infty} \ll \mu_{i,\infty}$ *and* $\nu_{i,\infty} \ll \mathbf{m}$,
(b) $\chi_c(\nu_{i,\infty}; F) > 0$,
(c) *the radius of* D_i *is independent of* i,
(d) $\nu_{i,\infty}(D_i) > 0$ *and* $D_i \subset \mathcal{B}(\nu_{i,\infty})$ *modulo Lebesgue null subsets.*

Below we prove proposition 4.7, 4.8 and 4.9 using proposition 4.10.

Proof of proposition 4.7. Let μ be a physical measure such that $\chi_c(\mu; F) > 0$. From lemma 4.6, we have $\chi_c(z; F) \geq 0$ for μ-almost every point z. Apply proposition 4.10 to the situation where $X(i) = \mathcal{B}(\mu)$ and $\mu_{i,\infty} = \mu_\infty = \mu$ for all $i \geq 1$. And let $\nu_{i,\infty}$ and D_i be those in the corresponding conclusion, which we can assume to be independent of i. Consider the open set $V = \bigcup_{n=0}^{\infty} F^{-n}(D_i)$. Then $\mathcal{B}(\nu_{i,\infty}) = V$ modulo Lebesgue null subsets. Since $\nu_{i,\infty}(V) \geq \nu_{i,\infty}(D_i) > 0$ and since $\nu_{i,\infty} \ll \mu$, we have $\mu(V) > 0$. Hence

$$\mathbf{m}_{\mathcal{B}(\mu)}(\mathcal{B}(\nu_{i,\infty})) = \lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} \mathbf{m}_{\mathcal{B}(\mu)} \circ F^{-i}(\mathcal{B}(\nu_{i,\infty})) \geq \mu(V) > 0.$$

This implies $\mu = \nu_{i,\infty}$. We have proved proposition 4.7. □

Proof of proposition 4.8. Suppose that there exist infinitely many ergodic physical measures μ_i, $i = 1, 2, \cdots$, that satisfy $\chi_c(\mu_i; F) \geq \chi > 0$. By taking a subsequence, we assume that μ_i converges to an invariant probability measure μ_∞ as $i \to \infty$. Then we have $\chi_c(\mu_\infty; F) \geq \chi > 0$ from corollary 3.14. From lemma 4.4, we have $\chi_c(z; F) \geq 0$ for μ_∞-almost every point z. Thus we can apply proposition 4.10 to the situation where $X(i) := \mathcal{B}(\mu_i)$ and $\mu_{i,\infty} = \mu_i$ for $i \geq 1$. Since μ_i are ergodic, the disks D_i in the corresponding conclusion are contained in $\mathcal{B}(\mu_i)$ modulo Lebesgue null subsets and hence mutually disjoint. But this is impossible because the radii of the disks D_i are positive and independent of i. □

Proof of proposition 4.9. Let $X = M \setminus (\mathcal{B}^-(F) \cup \mathcal{B}^+(F))$. For the proof of the proposition, it is enough to show the claim in the case $\mathbf{m}(X) > 0$ and $\nu = \mathbf{m}_X$. Let ν_∞ be a weak limit point of the sequence $n^{-1} \sum_{j=0}^{n-1} \nu \circ F^{-j}$. From lemma 4.6, it holds $\chi_c(z; F) \geq 0$ for ν_∞-almost every $z \in M$. Thus we have only to prove $\chi_c(\nu_\infty; F) \leq 0$. Suppose that we have $\chi_c(\nu_\infty; F) > 0$. Then we can apply proposition 4.10 to the situation where $X(i) := X$ for all $i \geq 1$. Let $\nu_{i,\infty} \ll \nu_\infty$ and D_i be those in the corresponding conclusion, which we can assume to be independent of i. We should have

$$\nu(\mathcal{B}(\nu_{i,\infty})) \geq \limsup_{n\to\infty} \nu(F^{-n}(D_i)) \geq \nu_\infty(D_i) > 0.$$

But this contradicts the definition of X because $\nu_{i,\infty}$ is an ergodic physical measure with positive central Lyapunov exponent. □

We proceed to the proof of proposition 4.10. For positive numbers χ, ϵ, k and a positive integer n, we define a closed subset $\Gamma(\chi, \epsilon, k, n; F)$ as the set of all points $z \in M$ such that, for any $0 \leq m < n$ and any $v \in \mathbf{S}^u(F^m(z))$,

(Γ1) $|D^*F^{n-m}(v)| \geq \exp(\chi(n-m) - k)$ and
(Γ2) $|D^*F(v)| \geq \exp(-\epsilon(n-m) - k)$.

For the points in $\Gamma(\chi, \epsilon, k, n; F)$, we have the following estimates on distortion:

Lemma 4.11. *For positive numbers $\chi > 0$, $0 < \epsilon < \chi/10$ and $k > 0$, there exists a positive constant $\alpha = \alpha(\chi, \epsilon, k)$, which depends only on χ, ϵ and k besides the objects that we fixed in subsection 3.2, such that, for any $n > 0$ and $z \in \Gamma(\chi, \epsilon, k, n; F)$, the restriction of F^n to some neighborhood V of z is a diffeomorphism onto the disk $\mathbf{B}(F^n(z), \alpha)$ and we have*

(1) $\|(DF^n_w)^{-1}\|^{-1} > C_g^{-1}\exp(\chi n - k)$ *for $w \in V$, and*

(2) $|\log|\det DF^n_w| - \log|\det DF^n_{w'}|| < 1$ *for $w, w' \in V$.*

Proof. Fix $v \in \mathbf{S}^u(z)$ and put $\delta(i) = |D^*F^{n-i}(DF^i(v))|^{-1}$ for $0 \le i < n$. Let D_n be the disk in the tangent space $T_{F^n(z)}M$ with center at the origin and radius α. We define the regions $D_i \subset T_{F^i(z)}M$ for $0 \le i < n$ so that $DF(D_i)$ is the $\delta(i)\alpha$-neighborhood of D_{i+1}. Then we have

$$\mathrm{diam} D_i \le \left\|(DF^{n-i}_{F^i(z)})^{-1}\right\| \cdot \alpha + \sum_{j=i}^{n-1} \left\|(DF^{j+1-i}_{F^i(z)})^{-1}\right\| \cdot \delta(j) \cdot \alpha$$

for $0 \le i < n$. Using the relation (6), we can check

$$\left\|(DF^{j+1-i}_{F^i(z)})^{-1}\right\| \cdot \delta(j) \le C_g \cdot |D^*F(DF^j(v))|^{-1} \cdot \delta(i).$$

Thus, from the conditions (Γ1) and (Γ2), it holds

$$\mathrm{diam} D_i \le C_g n \cdot \exp(\epsilon(n-i) + k)\delta(i) \cdot \alpha \le C_g n \cdot \exp(-(\chi - \epsilon)(n-i) + 2k) \cdot \alpha.$$

From the condition (Γ2) and the relation (6), we have

$$\|DF^{-1}_{F^i(z)}\|^{-1} \ge C_g^{-1}\exp(-\epsilon(n-i) - k).$$

Also we have the estimates

$$\begin{aligned}&\left\|\exp^{-1}_{F^{i+1}(z)} \circ F \circ \exp_{F^i(z)}(v) - DF_{F^i(z)}(v)\right\| \\ &\qquad \le C_g(\mathrm{diam} D_i)^2 \le C_g n^2 \exp(-(\chi - 2\epsilon)(n-i) + 3k)\delta(i)\alpha^2\end{aligned}$$

for $v \in D_i$ and

$$\begin{aligned}&\left\|D(\exp^{-1}_{F^{i+1}(z)} \circ F \circ \exp_{F^i(z)})_v - DF_{F^i(z)}\right\| \\ &\qquad \le C_g \mathrm{diam} D_i \le C_g n \cdot \exp(-(\chi - \epsilon)(n-i) + 2k) \cdot \alpha.\end{aligned}$$

for $v \in T_{F^i(z)}M$. Hence, if we take sufficiently small α depending only on χ, ϵ, k and C_g, the restriction of F to $\exp_{F^i(z)}(D_i)$ is a diffeomorphism onto a neighborhood of $\exp_{F^{i+1}(z)}(D_{i+1})$ for $0 \le i < n$. This implies the first claim of the lemma. We can check the other claims, (1) and (2), by straightforward estimates. □

From now to the end of this section, we consider the situation in proposition 4.10. For each i, we take a subsequence $n(j;i) \to \infty$ $(j \to \infty)$ such that the sequence of measures $n(j;i)^{-1}\sum_{m=0}^{n(j;i)-1} \mathbf{m}_{X(i)} \circ F^{-m}$ converges to $\mu_{i,\infty}$ as $j \to \infty$. The following is the key lemma in the proof of proposition 4.10.

Lemma 4.12. *There exist $\chi > 0$, $0 < \epsilon < \chi/10$ and $k > 0$ such that*

$$\liminf_{j\to\infty} \frac{1}{n(j;i)} \sum_{m=0}^{n(j;i)-1} \mathbf{m}_{X(i)}(\Gamma(\chi, \epsilon, k, m; F)) > 0 \quad \text{for sufficiently large } i. \tag{26}$$

The point of this lemma is that we can take χ, ϵ and k uniformly for sufficiently large i. Before proving this lemma, we finish the proof of proposition 4.10 assuming it.

Proof of proposition 4.10. Let χ, ϵ and k be those in lemma 4.12 and $\alpha = \alpha(\chi, \epsilon, k, F)$ that in lemma 4.11. We consider a large integer i for which (26) holds. Then we can take a compact subset $K \subset X(i)$ and a point $z_0 \in M$ such that

$$\liminf_{j\to\infty} \frac{1}{n(j;i)} \sum_{m=0}^{n(j;i)-1} (\mathbf{m}|_{K\cap\Gamma(\chi,\epsilon,k,m;F)} \circ F^{-m})(\mathbf{B}(z_0, \alpha/2)) > 0. \tag{27}$$

Let $\mathcal{D}_m$ be the union of the connected components of $F^{-m}(\mathbf{B}(z_0, \alpha/2))$ that meet $K \cap \Gamma(\chi, \epsilon, k, m; F)$. Then, on each of the connected components of $\mathcal{D}_m$, the mapping F^n is a diffeomorphism onto $\mathbf{B}(z_0, \alpha/2)$ and satisfies the estimates in lemma 4.11. Let ν be a limit point of the sequence $n(j;i)^{-1} \sum_{m=0}^{n(j;i)-1} (\mathbf{m}|_{\mathcal{D}_m}) \circ F^{-m}$. Then we have $\nu \le \mathbf{m}(X(i)) \cdot \mu_{i,\infty}$ and $\nu \ll \mathbf{m}$ and, further,

$$e^{-1} \cdot \frac{\nu(\mathbf{B}(z_0, \alpha/2))}{\mathbf{m}(\mathbf{B}(z_0, \alpha/2))} \le \frac{d\nu}{d\mathbf{m}} \le e \cdot \frac{\nu(\mathbf{B}(z_0, \alpha/2))}{\mathbf{m}(\mathbf{B}(z_0, \alpha/2))}.$$

We can check that ν is ergodic and $\chi_c(z; F) > 0$ for ν-almost every point z. (See the remark below.) Hence there is an ergodic component $\nu_{i,\infty}$ of $\mu_{i,\infty}$ such that $\nu \ll \nu_{i,\infty} \ll \mu_{i,\infty}$. The measure $\nu_{i,\infty}$ and disk $D_i = \mathbf{B}(z_0, \alpha/2)$ satisfy the conditions in proposition 4.10. □

Remark. Actually, it is not completely simple to prove that ν is ergodic and that $\chi_c(z; F) > 0$ for ν-almost every point z. But there are a few standard ways for it. For example, we can argue as follows: Consider the inverse limit space of F

$$\tilde{M}_F = \{(z_i)_{-\infty < i \le 0} \mid z_i \in M,\ F(z_i) = z_{i+1}\}$$

and the projection $\pi : \tilde{M}_F \to M$ defined by $\pi((z_i)_{-\infty<i\le 0}) = z_0$. Let $\tilde{\mu}_{i,\infty}$ be the natural extension of $\mu_{i,\infty}$. We can check that the part $\tilde{\nu}$ of $\tilde{\mu}_{i,\infty}$ that corresponds to ν is supported on a union of local unstable manifolds, each of which is projected onto the disk $\mathbf{B}(z_0, \alpha/2)$ by π. Further, the conditional measures on those local unstable manifolds given by $\tilde{\nu}$ are absolutely continuous with respect to the smooth measures on them. For any continuous function φ on M, the backward time-average of $\varphi \circ \pi$ is constant on each of the local unstable manifolds. From the ergodic theorem, the forward time-average coincides with the backward time-average almost everywhere with respect to $\tilde{\nu} \ll \tilde{\mu}_{i,\infty}$, and is the pull-back of a function on M by π. Thus it must be constant $\tilde{\nu}$-almost everywhere. This implies that ν is ergodic. The positivity of the central Lyapunov exponent is obtained by considering Lyapunov exponents with respect to the backward iteration.

In the remaining part of this subsection, we prove lemma 4.12. To begin with, we fix several constants: Fix $\chi_0 > 0$ and $0 < s_0 < 1$ such that

$$\mu_\infty(\{z \in M \mid \chi_c(z) > \chi_0\}) > s_0. \tag{28}$$

Also fix a positive number ϵ_0 such that $0 < \epsilon_0 < 10^{-4} s_0 \chi_0$. Recall that we are considering a mapping $F \in \mathcal{U}$ that satisfies the no flat contact condition. From lemma 3.13, we can take and fix a large positive constant $h_0 > \chi_0$ such that

$$\int \min\{0, L(z; F) + h_0\}\, d(\mu \circ F^{-n})(z) > -10^{-1} s_0 \epsilon_0$$

for any measure μ in $\mathcal{AM}([1,\infty))$ and $n \geq n_0(F)$, where $L(z;F)$ is the function defined by (13) and $n_0(F)$ is the constant in the definition of the no flat contact condition. From (28) and the assumption that $\chi_c(z;F) \geq 0$ for μ_∞-a.e. z, we can take and fix a constant $k_0 > h_0$ such that

$$\mu_\infty(\{z \in M; |D^*F^n(v)| \geq \exp(\chi_0 n - k_0), \forall v \in \mathbf{S}^u(z), \forall n \geq 0\}) > s_0$$

and

$$\mu_\infty(\{z \in M; |D^*F^n(v)| \geq \exp(-\epsilon_0 n - k_0), \forall v \in \mathbf{S}^u(z), \forall n \geq 0\}) > 1 - \frac{s_0\epsilon_0}{10h_0}.$$

Finally we fix a positive integer m_0 that satisfies $\epsilon_0 m_0 > 10k_0$.

Next we introduce the following subsets of M:

$$A = \{z \in M \ ; \ |D^*F^m(v)| > \exp(\chi_0 m - 2k_0), \forall v \in \mathbf{S}^u(z), 0 \leq \forall m \leq m_0\},$$
$$B = \{z \in M \ ; \ |D^*F^m(v)| > \exp(-\epsilon_0 m - 2k_0), \forall v \in \mathbf{S}^u(z), 0 \leq \forall m \leq m_0\} \supset A,$$
$$C = M \setminus B \quad \text{and} \quad D = \{z \in C \mid L(z;F) \leq -h_0\} \subset C.$$

Note that A and B are open subsets. From the assumption that the sequence $\mu_{i,\infty}$ converges to μ_∞ as $i \to \infty$, we have

$$\liminf_{j\to\infty} n(j;i)^{-1} \sum_{m=0}^{n(j;i)-1} \mathbf{m}_{X(i)}(F^{-m}(A)) > s_0, \qquad \text{and} \tag{29}$$

$$\liminf_{j\to\infty} n(j;i)^{-1} \sum_{m=0}^{n(j;i)-1} \mathbf{m}_{X(i)}(F^{-m}(B)) > 1 - \frac{s_0\epsilon_0}{10h_0} \tag{30}$$

for sufficiently large i.

Below we fix a large integer i for which (29) and (30) hold. Using lemma 3.10, we can find a small number $b_0 > 0$ and a probability measure μ_0 in $\mathcal{AM}([b_0,\infty))$ such that

$$|\mathbf{m}_{X(i)} - \mu_0| < s_0/10, \tag{31}$$

$$\liminf_{j\to\infty} n(j;i)^{-1} \sum_{m=0}^{n(j;i)-1} \mu_0(F^{-m}(A)) > s_0, \quad \text{and} \tag{32}$$

$$\liminf_{j\to\infty} n(j;i)^{-1} \sum_{m=0}^{n(j;i)-1} \mu_0(F^{-m}(B)) > 1 - \frac{s_0\epsilon_0}{10h_0}. \tag{33}$$

By modifying the measure μ_0 slightly if necessary, we can and do assume

$$\sum_{m=0}^{n_0(F)} \int \min\{0, L(F^m(z);F) + h_0\} d\mu_0 > -\infty$$

in addition. Then, from corollary 3.7 and the choice of h_0, we have also

$$\liminf_{j\to\infty} n(j;i)^{-1} \sum_{m=0}^{n(j;i)-1} \int \min\{0, L(F^m(z);F) + h_0\} d\mu_0 \geq -\frac{s_0\epsilon_0}{10}. \tag{34}$$

For $z \in M$ and integers $m < n$, we denote, by $A_z(m,n)$ $B_z(m,n)$, $C_z(m,n)$ and $D_z(m,n)$, the set of integers $m \leq q < n$ for which $F^q(z)$ belongs to A, B, C and D respectively. Then we have

Lemma 4.13. *A point $z \in M$ belongs to $\Gamma(s_0\chi_0/40, 4\epsilon_0, 6k_0, n; F)$ for $n > 0$ if*

(A) $\#A_z(m,n) \geq s_0(n-m)/10$ *for any* $0 \leq m < n$,

(C) $\#C_z(m,n) \leq \epsilon_0(n-m)/h_0$ *for any* $0 \leq m < n$, *and*

(D) $\displaystyle\sum_{q\in D_z(m,n)} \min\{0, L(F^q(z);F)+h_0\} \geq -\epsilon_0(n-m)$ *for any* $0 \leq m < n$.

Proof. Consider a point $z \in M$ and an integer n that satisfy the conditions (A), (C) and (D). Let $0 \leq m < n$ and $I = \{m, m+1, \dots, n-1\}$. We call a set of m_0 consecutive integers $\{q, q+1, \dots, q+m_0-1\}$ is an A-interval (resp. a B-interval) if the smallest element q belongs to $A_z(m,n)$ (resp. $B_z(m,n)$). If $\{q, q+1, \dots, q+m_0-1\}$ is an A-interval, we have

$$\sum_{j=0}^{m_0-1} \log|D^*F(DF^j(v))| \geq \chi_0 m_0 - 2k_0 > (\chi_0-\epsilon_0)m_0 + 2k_0 \tag{35}$$

for $v \in \mathbf{S}^u(F^q(z))$, where the second inequality follows from the choice of m_0. Similarly, if $\{q, q+1, \dots, q+m_0-1\}$ is a B-interval, we have

$$\sum_{j=0}^{m_0-1} \log|D^*F(DF^j(v))| \geq -\epsilon_0 m_0 - 2k_0 > -2\epsilon_0 m_0 \tag{36}$$

for $v \in \mathbf{S}^u(F^q(z))$.

Take mutually disjoint A-intervals that cover $A_z(m,n)$ and let I_A be the union of them. Then take mutually disjoint B-intervals that cover $B_z(m,n) \setminus I_A$ and let I_B be the union of them. We can and do take the B-intervals in I_B so that their smallest elements are *not* contained in I_A. Note that I_A and I_B are not necessarily contained in I.

Consider an arbitrary vector $v \in \mathbf{S}^u(F^m(z))$. Then $DF^{q-m}(v)$ belongs to $\mathbf{S}^u(F^q(z))$ for $q \geq m$. From (35) and the fact that all the A-intervals in I_A but one is contained in I, we have

$$\sum_{q\in I_A\cap I} \log|D^*F(DF^{q-m}(v))| \geq (\chi_0-\epsilon_0)\#(I_A\cap I) + 2k_0((\#I_A/m_0)-1) - 2k_0.$$

Each A-interval in I_A meets at most one B-interval in I_B. Thus the number of B-intervals in I_B whose intersection with $I \setminus I_A$ has cardinality less than m_0 is at most $(\#I_A/m_0)+1$. From this and (36), we obtain

$$\sum_{q\in I_B\cap(I\setminus I_A)} \log|D^*F(DF^{q-m}(v))| \geq -2\epsilon_0\#(I_B\cap(I\setminus I_A)) - 2k_0((\#I_A/m_0)+1).$$

Since the complement of $I_B \cup I_A$ in I is contained in $C_z(m,n)$, the condition (Γ1) in the definition of the set $\Gamma(s_0\chi_0/40, 4\epsilon_0, 6k_0, n; F)$ follows from the two inequalities above, the assumptions (A),(C) and (D) and the choice of ϵ_0. If m belongs to $B_z(m,n)$, the condition (Γ2) holds obviously. Otherwise, the condition (Γ2) follows from (D) because we have $\epsilon_0(n-m)/h_0 \geq \#C_z(m,n) \geq 1$ in that case from (C). □

In order to prove lemma 4.12, we see how often the assumptions (A), (C) and (D) in the lemma above hold. For this purpose, we prepare the following elementary lemma, which we shall use again in section 6.

Lemma 4.14. *Let μ be a measure on a measurable space X and ψ_m, $m = 0, 1, \dots,$ a sequence of non-negative integrable function on X. For a positive number $\alpha > 0$ and an integer $n \geq 0$, let $Y_n(\alpha)$ be the set of points $y \in X$ such that*

$$\sum_{\ell=m}^{n-1} \psi_\ell(y) \geq \alpha(n-m) \quad \text{for some } 0 \leq m < n.$$

Then it holds, for any $n > 0$,

$$\sum_{m=0}^{n-1} \mu(Y_m(\alpha)) \leq \sum_{m=0}^{n} \mu(Y_m(\alpha)) \leq \alpha^{-1} \sum_{m=0}^{n-1} \int \psi_m d\mu.$$

Proof. For each point $z \in M$, we define integers

$$n = q_0(z) \geq p_1(z) > q_1(z) \geq p_2(z) > q_2(z) \geq \cdots \geq p_{j(z)} > q_{j(z)} \geq 1$$

in the following inductive manner: Suppose that $q_j(z)$ has been defined. If there exist integers $p \leq q_j(z)$ such that $F^p(z) \in Y_p(\alpha)$, let $p_{j+1}(z)$ be the maximum of such integers and $q_{j+1}(z)$ the smallest integer $q < p_{j+1}(z)$ such that

$$\sum_{\ell=q}^{p_{j+1}(z)-1} \psi_\ell(y) \geq \alpha(p_{j+1}(z) - q). \tag{37}$$

Otherwise we put $j(z) = j$ and finish the definition. Consider the subsets

$$Z_m = \{z \in M \mid q_j(z) \leq m < p_j(z) \text{ for some } 1 \leq j \leq j(z)\}$$

for $0 \leq m < n$. Then we have $Y_n(\alpha) \subset Z_{n-1}$. From (37), we obtain

$$\sum_{m=0}^{n-1} \int \psi_m d\mu \geq \alpha \sum_{m=0}^{n-1} \mu(Z_m) \geq \alpha \sum_{m=0}^{n} \mu(Y_m(\alpha)). \quad \square$$

Now we can complete the proof of lemma 4.12.

Proof of lemma 4.12. For $n \geq 0$, let $\tilde{A}_n$, $\tilde{C}_n$ and $\tilde{D}_n$ be the set of points $z \in M$ for which the condition (A), (C) and (D) does NOT hold, respectively. First, apply lemma 4.14 to the case where $\alpha = 1 - s_0/10$, $n = n(j;i)$ and ψ_m is the indicator function of the complement of $F^{-m}(A)$. Then, from (32), we obtain

$$\begin{aligned} \frac{1}{n(j;i)} \sum_{m=0}^{n(j;i)-1} \mu_0(\tilde{A}_m(z)) &\leq \frac{1}{1 - \frac{s_0}{10}} \frac{1}{n(j;i)} \sum_{m=0}^{n(j;i)-1} \mu_0(M \setminus F^{-m}(A)) \\ &\leq \frac{1 - s_0}{1 - \frac{s_0}{10}} \leq 1 - \frac{9}{10} s_0 \end{aligned}$$

for sufficiently large j. Second, apply lemma 4.14 to the case where $\alpha = \epsilon_0/h_0$, $n = n(j;i)$ and ψ_m is the indicator function of the set $F^{-m}(C) = M \setminus F^{-m}(B)$. Then, from (33), we obtain

$$\frac{1}{n(j;i)} \sum_{m=0}^{n(j;i)-1} \mu_0(\tilde{C}_m(z)) \leq \frac{h_0}{\epsilon_0} \frac{1}{n(j;i)} \sum_{m=0}^{n(j;i)-1} \mu_0(F^{-m}(C)) \leq \frac{1}{10} s_0$$

for sufficiently large j. Third, apply lemma 4.14 to the case where $\alpha = \epsilon_0$, $n = n(j;i)$ and $\psi_m(z) = -\min\{0, L(F^m(z);F) + h_0\}$. Then, from (34), we obtain

$$\frac{1}{n(j;i)} \sum_{m=0}^{n(j;i)-1} \mu_0(\tilde{D}_m(z)) \leq \frac{-1}{\epsilon_0} \frac{1}{n(j;i)} \sum_{m=0}^{n(j;i)-1} \int \min\{0, L(F^m(z);F) + h_0\} d\mu_0(z) \leq \frac{1}{10} s_0$$

for sufficiently large j. From the three inequalities above and (31), we conclude

$$\frac{1}{n(j;i)} \sum_{m=0}^{n(j;i)-1} \left(\mathbf{m}_{X(i)}(\tilde{A}_m \cup \tilde{C}_m \cup \tilde{D}_m) \right) \leq 1 - \frac{6}{10} s_0$$

for sufficiently large j. Since the complement of $\tilde{A}_m \cup \tilde{C}_m \cup \tilde{D}_m$ is contained in the subset $\Gamma(s_0\chi_0/40, 4\epsilon_0, 6k_0, m; F)$ from lemma 4.13, this implies lemma 4.12. □

5. Some estimates on distortion

In this section, we give some basic estimates on distortion of the iterates of mappings in $\mathcal{U}$. The estimates are straightforward and may look rather tedious. But we need to check that some constants in the estimates can be taken uniformly for the mappings in $\mathcal{U}$. This is important especially in our argument in section 7, where we consider perturbations of mappings in $\mathcal{U}$.

Let $\chi = \{\chi_c^-, \chi_c^+, \chi_u^-, \chi_u^+\}$ be a quadruple satisfying (14) and (23), and $\epsilon > 0$ a small positive number satisfying

$$\epsilon < 10^{-3} \min\{\chi_c^- + \chi_u^-, \ |\chi_c^-|, \ \chi_u^+ - \chi_u^-, \ \lambda_g\}. \tag{38}$$

In the argument below, we will take several constants that depend only on χ and ϵ besides the integer r and the objects that we have already fixed in subsection 3.2. In order to distinguish such kind of constants, we denote them by symbols with subscript ϵ. Also we will use a generic symbol C_ϵ for large positive constants of this kind. The usage of these notations is the same as those introduced in subsection 3.3. The following lemma is the main ingredient of this section.

Lemma 5.1. *There exist positive constants $0 < \rho_\epsilon < 1$, $\kappa_\epsilon > 1$ and $\kappa_g > 1$ such that the following claim holds for any $F \in \mathcal{U}$, $k > 0$, $n \geq 1$, $z_0 \in \Lambda(\chi, \epsilon, k, n; F)$ and $0 < \rho \leq \rho_0$ where*

$$\rho_0 := \rho_\epsilon e^{-4\epsilon n - 2k} \min_{0 \leq i \leq j \leq n} \min_{v \in \mathbf{S}^u(F^i(z))} |D^* F^{j-i}(v)| \geq \rho_\epsilon \exp((\chi_c^- - 5\epsilon)n - 3k) : \tag{39}$$

For every mapping $G \in C^r(M, M)$ that satisfies $d_{C^1}(F, G) \leq \rho$, we can take a point $z(G)$ and its neighborhood $V_\rho(G) \ni z(G)$ in a unique manner so that

(i) *$z(G)$ depends on G continuously and $z(F) = z_0$,*
(ii) *$G^n(z(G)) \equiv F^n(z_0)$ and*
(iii) *the restriction of G^n to $V_\rho(G)$ is a diffeomorphism onto $\mathbf{B}(F^n(z_0), \rho)$.*

Further it holds

(iv) $\mathrm{diam}(V_\rho(G)) < \kappa_g \rho \exp(-\chi_c^- n + k)$,
(v) $\mathbf{B}(z(G), \kappa_g^{-1} \rho \exp(-\chi_u^+ n - k)) \subset V_\rho(G)$,
(vi) $V_\rho(G) \subset \Lambda(\chi, \epsilon, n, k+1; F)$,
(vii) $\angle(DG^n(\mathbf{E}^u(w)), DF^n(\mathbf{E}^u(z_0))) \leq \kappa_\epsilon e^{2k} \rho$ *for any point $w \in V_\rho(G)$, and*

(viii) *any admissible curve in* $\mathbf{B}(z_0, \kappa_g^{-1})$ *meets* $V_\rho(F)$ *in a single curve.*

Proof. First of all, check that the inequality in (39) follows from the assumption $z_0 \in \Lambda(\chi, \epsilon, k, n; F)$. We will give the conditions on the choice of the constants ρ_ϵ, κ_ϵ and κ_g in the course of the argument below. For $0 \le i \le n$, we put $\zeta(i) = F^i(z_0)$ and

$$\delta(i) = \frac{\rho \cdot \exp(\epsilon(n-i)+k)}{\min_{i\le \ell\le n}\min_{v\in \mathbf{S}^u(\zeta(i))}|D^*F^{\ell-i}(v)|}.$$

Then it holds

$$\rho < \rho\exp(\epsilon(n-i)+k) \le \delta(i) \le \rho_\epsilon \exp(-3\epsilon n - k) \quad \text{for } 0 \le i \le n. \tag{40}$$

Using the relation (6), we can see

$$\begin{aligned}\frac{\delta(j)}{\delta(i)} &\le \exp(-\epsilon(j-i))\frac{\min_{j\le \ell\le n}\min_{v\in \mathbf{S}^u(\zeta(i))}|D^*F^{\ell-i}(v)|}{\min_{j\le \ell\le n}\min_{v\in \mathbf{S}^u(\zeta(j))}|D^*F^{\ell-j}(v)|}\\ &\le C_g\exp(-\epsilon(j-i))\|(DF^{j-i}_{\zeta(i)})^{-1}\|^{-1}\end{aligned} \tag{41}$$

for $0 \le i \le j \le n$, and

$$\begin{aligned}\frac{\delta(i+1)}{\delta(i)} &= \exp(-\epsilon)\frac{\min\{1,\ \min_{i+1\le \ell\le n}\min_{v\in \mathbf{S}^u(\zeta(i))}|D^*F^{\ell-i}(v)|\}}{\min_{i+1\le \ell\le n}\min_{v\in \mathbf{S}^u(\zeta(i+1))}|D^*F^{\ell-i-1}(v)|}\\ &\ge C_g^{-1}\exp(-\epsilon)\min\{1, \|(DF_{\zeta(i)})^{-1}\|^{-1}\}\end{aligned} \tag{42}$$

for $0 \le i \le n$.

We put $D_n = \mathbf{B}(0,\rho) \subset T_{\zeta(n)}M$ and define the region $D_i \subset T_{\zeta(i)}M$ for $0 \le i < n$ inductively so that $DF_{\zeta(i)}(D_i)$ is the $2\delta(i+1)$-neighborhood of $D_{i+1} \subset T_{\zeta(i+1)}M$. Put $B_i = \exp_{\zeta(i)}(D_i)$. Then

$$\begin{aligned}\mathrm{diam}B_i = \mathrm{diam}D_i &\le 2\rho\cdot\|(DF^{n-i}_{\zeta(i)})^{-1}\| + \sum_{j=i+1}^{n} 4\delta(j)\cdot\|(DF^{j-i}_{\zeta(i)})^{-1}\|\\ &< C_\epsilon\delta(i) \le C_\epsilon\rho_\epsilon\exp(-3\epsilon n - k)\end{aligned} \tag{43}$$

for $0 \le i \le n$, where the second inequality follows from (41) and the third from (40). Since $\zeta(0) = z_0 \in \Lambda(\chi, \epsilon, k, n; F)$, we have

$$\|(DF_{\zeta(i)})^{-1}\|^{-1} \ge C_g^{-1}\exp(-\epsilon(n-i)-k) \quad \text{for } 0 \le i \le n \tag{44}$$

by (6). Therefore, if we take the constant ρ_ϵ sufficiently small, we can obtain

$$\|DG_w - DF_{\zeta(i)}\| \le \rho + C_g\cdot \mathrm{diam}B_i < \|(DF_{\zeta(i)})^{-1}\|^{-1}$$

and

$$d(G(w), \exp_{\zeta(i+1)}\circ DF_{\zeta(i)}\circ \exp^{-1}_{\zeta(i)}(w)) \le \rho + C_g\cdot(\mathrm{diam}B_i)^2 < 2\delta(i+1)$$

for $0 \le i < n$, $w \in \mathbf{B}(\zeta(i), \mathrm{diam}B_i)$ and $G \in \mathcal{U}$ satisfying $d_{C^1}(F,G) \le \rho \le \rho_0$, where we used the relation

$$(\mathrm{diam}B_i)^2 \le C_g\delta(i)^2 \le C_g\rho_\epsilon\exp(-2\epsilon n)\delta(i+1)$$

which follows from (40), (42) and (44). These two inequalities imply that the mapping G restricted to $\mathbf{B}(\zeta(i), \mathrm{diam}B_i) \supset B_i$ is a diffeomorphism and maps B_i onto a neighborhood of B_{i+1} for $0 \le i < n$. Put $V_\rho(G) = \cap_{i=0}^n G^{-i}(B_i)$. Then the restriction of G^n to $V_\rho(G)$ is a diffeomorphism onto $B_n = \mathbf{B}(F^n(z_0), \rho)$. Let $z(G)$ be the unique point in $V_\rho(G)$ that G^n brings to $F^n(z_0)$. Clearly $z(G)$ and $V_\rho(G)$ satisfy the conditions (i), (ii) and (iii).

We show the conditions (iv)-(viii). Using (5) and (6), we can check that (iv) and (v) follows from (vi). We prove (vi) and (vii). Let $G \in \mathcal{U}$ be a mapping that satisfies $d_{C^1}(F,G) \leq \rho \leq \rho_0$ and w a point in $V_\rho(G)$. We put $w(i) = G^i(w)$ for $0 \leq i \leq n$. Consider an integer $0 \leq i \leq n$ and tangent vectors $v \in \mathbf{S}^u(\zeta(i))$ and $u \in \mathbf{S}^u(w(i))$. For $0 \leq m \leq n-i$, we have

$$\angle(DG^m_{w(i)}(u), DF^m_{\zeta(i)}(v)) \leq \angle(DF^m_{\zeta(i)}(u), DF^m_{\zeta(i)}(v)) + \sum_{j=1}^{m} \angle(DF^{m-j+1}_{\zeta(i+j-1)}(DG^{j-1}_{w(i)}(u)), DF^{m-j}_{\zeta(i+j)}(DG^j_{w(i)}(u))).$$

Remark. In the expression above, we identified tangent vectors with their parallel translations and abused the notation slightly. Actually, $DF^{m-j}_{\zeta(i+j)}(DG^j_{w(i)}(u))$ should have been written $DF^{m-j}_{\zeta(i+j)}(\tau(DG^j_{w(i)}(u)))$ where τ is the parallel translation from $w(i+j)$ to $\zeta(i+j)$. We continue to use such identification below.

Since $w(i+j-1) \in B_{i+j-1}$ and $DG^{j-1}_{w(i)}(u) \in \mathbf{S}^u(w(i+j-1))$, the parallel translation of $DG^{j-1}_{w(i)}(u)$ to $\zeta(i+j-1)$ does not belongs to $\mathbf{S}^c(\zeta(i+j-1))$, provided that we take sufficiently small ρ_ϵ. Also we have

$$\angle(DF_{\zeta(i+j-1)}(DG^{j-1}_{w(i)}(u)), DG^j_{w(i)}(u)) \leq C_g\,(\mathrm{diam}(B_{i+j-1}) + d_{C^1}(F,G))$$

for $0 \leq j \leq n-i$. Using these and (3) in the inequality above, we obtain

$$\begin{aligned}(45)\quad &\angle(DG^m_{w(i)}(u), DF^m_{\zeta(i)}(v)) \\ &\leq A_g \frac{|D^*F^m(v)|}{D_*F^m(v)} \angle(u,v) + C_g \sum_{j=1}^{m} \frac{|D^*F^{m-j}(DF^j(v))|}{D_*F^{m-j}(DF^j(v))} (\mathrm{diam} B_{i+j-1} + \rho) \\ &\leq C_g \exp(-\lambda_g m)\angle(u,v) + C_g \sum_{j=1}^{m-1} \exp(-\lambda_g(m-j))(\mathrm{diam} B_{i+j-1} + \rho).\end{aligned}$$

In order to prove the condition (vii), we consider (45) in the case where $i = 0$, $m = n$ and v and u are unit tangent vectors in $\mathbf{E}^u(z_0)$ and $\mathbf{E}^u(w)$, respectively. In this case, we have

$$\begin{aligned}&\frac{|D^*F^{n-j}(DF^j(v))|}{D_*F^{n-j}(DF^j(v))} \cdot (\mathrm{diam} B_{j-1} + \rho) \leq \frac{|D^*F^{n-j}(DF^j(v))|}{D_*F^{n-j}(DF^j(v))} \cdot C_\epsilon \delta(j-1) \\ &\leq C_\epsilon \rho \exp(\epsilon(n-j) + k) \max_{j \leq \ell \leq n} \frac{|D^*F^{n-\ell}(DF^\ell(v))|}{D_*F^{n-\ell}(DF^\ell(v))} \frac{|D^*F(DF^{j-1}(v))|^{-1}}{D_*F^{\ell-j}(DF^j(v))} \\ &\leq C_\epsilon \rho \exp(-(\lambda_g - 2\epsilon)(n-j) + 2k)\end{aligned}$$

for $1 \leq j \leq n$, where we used (40) and (43) in the first inequality, (6) in the second, and the assumption $z_0 \in \Lambda(\chi, \epsilon, k, n; F)$ in the third. Likewise, using the estimate $\angle(v,u) \leq C_g d(z_0, w) \leq C_g \mathrm{diam} B_0$, we can show

$$\frac{|D^*F^n(v)|}{D_*F^n(v)} \angle(u,v) \leq \frac{|D^*F^n(v)|}{D_*F^n(v)} \cdot C_g \mathrm{diam} B_0 \leq C_\epsilon \rho \exp(-(\lambda_g - 2\epsilon)n + 2k).$$

Putting these inequalities in (45), we obtain the condition (vii).

Next we prove the condition (vi). Consider an integer $0 \leq i \leq n$ and a vector $u \in \mathbf{S}^u(w(i))$. Since $w(i)$ belongs to B_i, there is a vector $v \in \mathbf{S}^u(\zeta(i))$ such that

$\angle(u,v) < C_g \mathrm{diam} B_i$. From this, (43) and (45), we obtain

$$\begin{aligned}
&|D^*G(DG^\ell_{w(i)}(v)) - D^*F(DF^\ell_{\zeta(i)}(u))| \\
&< C_g(d_{C^1}(F,G) + \mathrm{diam} B_{i+\ell} + \angle(DG^\ell_{w(i)}(v), DF^\ell_{\zeta(i)}(u)) + \angle(DG^{\ell+1}_{w(i)}(v), DF^{\ell+1}_{\zeta(i)}(u))) \\
&< C_g \rho_\epsilon \exp(-3\epsilon n - k)
\end{aligned}$$

for $0 \le \ell \le n-i-1$. Thus, using (44), we can obtain

$$\log\left|\frac{D^*G^{j-i}(v)}{D^*F^{j-i}(u)}\right| < \sum_{\ell=0}^{j-i-1} \log\left|\frac{D^*G(DG^\ell_{w(i)}(v))}{D^*F(DF^\ell_{\zeta(i)}(u))}\right| < 1 \quad \text{for } 0 \le i \le j \le n, \tag{46}$$

provided that we take the constant ρ_ϵ sufficiently small. Likewise, we can get

$$\log\left|\frac{D_*G^{j-i}(v)}{D_*F^{j-i}(u)}\right| < 1 \quad \text{for } 0 \le i \le j \le n.$$

The condition (vi) follows from these two inequalities and the assumption that z_0 belongs to $\Lambda(\chi,\epsilon,n,k;F)$.

Finally we check the condition (viii). Let γ be an admissible curve in $\mathbf{B}(z_0, \kappa_g^{-1})$. From the argument in subsection 3.4, the curvature of $F^i_*\gamma$ for $0 \le i \le n$ is bounded by some constant C_g, even though $F^i_*\gamma$ for $0 \le i \le n_g$ may not be admissible. Thus we can take the constant κ_g so large that any arc in $F^i_*\gamma$ with length less than $4\Lambda_g\kappa_g^{-1}$ meets any ball with diameter not larger than $2\kappa_g^{-1}$ in a single sub-arc with length less than $4\kappa_g^{-1}$. The diameter of B_i is bounded by $2\kappa_g^{-1}$ provided that we take the constant ρ_ϵ sufficiently small. Thus, by induction on $0 \le j \le n$, we can check that $\gamma_j := \gamma \cap (\bigcap_{\ell=0}^{j} F^{-\ell}(\mathbf{B}(\zeta(\ell), \mathrm{diam} B_\ell)))$ consists of a single arc. We obtain the condition (viii) as the case $j = n$. □

Note that the claim of lemma 5.1 remains true even if we get the constant ρ_ϵ smaller and κ_ϵ and κ_g larger. By letting the constant ρ_ϵ smaller and κ_ϵ larger if necessary, we can show the following claim in addition:

Addendum to lemma 5.1 Suppose that $F \in \mathcal{U}$, $n \ge 1$ and $k > 0$. Then there exists a neighborhood $W(z)$ for each point $z \in \Lambda(\chi,\epsilon,k,n;F)$ such that

(ix) The restriction of F^n to $W(z)$ is a diffeomorphism onto the image. Further, if $W(z) \cap W(w) \ne \emptyset$ for some $w \in \Lambda(\chi,\epsilon,k,n;F)$, then F^n is injective on the union $W(z) \cup W(w)$.

(x) $\mathbf{m}(W(z)) > \kappa_\epsilon^{-1}\exp(-(\chi_u^+ + \max\{\chi_c^+, 0\} + 7\epsilon)n - 6k)$.

Proof. We consider a point $z_0 \in \Lambda(\chi,\epsilon,k,n;F)$ and continue to use the notations in lemma 5.1 and its proof. Let γ be the curve in $V_{\rho_0}(F)$ that F^n maps onto the segment $\{\zeta(n) + t\cdot \mathbf{e}^c(\zeta(n)) \mid |t| < \rho_0\} \subset \mathbf{B}(\zeta(n), \rho_0)$ where $\mathbf{e}^c(\cdot)$ is a unit vector in $\mathbf{E}^c(\cdot)$. From backward invariance of the central cones $\mathbf{S}^c(\cdot)$, the tangent vectors of γ is contained in the central cones, provided that we take sufficiently small ρ_ϵ. From (46) and (6), the length of $F^i_*\gamma$ satisfies

$$|F^i_*\gamma| < C_g\rho_0\|(DF^{n-i}_{\zeta(i)})^{-1}\| < C_g\rho_\epsilon \exp(-4\epsilon n - 2k)$$

and, for the case $i = 0$,

$$\begin{aligned}
|\gamma| &> C_g^{-1}\rho_0\|(DF^n_{\zeta(0)})^{-1}\| > C_g^{-1} \min_{0\le i\le j\le n} \rho_\epsilon e^{-4\epsilon n - 2k}\|(DF^{n-j}_{\zeta(j)})^{-1}\|\|(DF^i_{\zeta(0)})^{-1}\| \\
&\ge C_g^{-1}\rho_\epsilon \exp(-\max\{\chi_c^+, 0\}n - 5\epsilon n - 4k).
\end{aligned}$$

Next consider the family of parallel segments

$$\gamma_y(t) = y + t \cdot \mathbf{e}^u(z_0), \quad |t| < \rho_\epsilon \exp(-(\chi_u^+ + 2\epsilon)n - 2k)$$

parameterized by the points $y \in \gamma$, where $\mathbf{e}^u(z_0)$ is a unit vector in $\mathbf{E}^u(z_0)$. And we define $W(z_0)$ as the region that this family of segments sweeps. From the estimate on the length of γ above, we can see that $W(z_0)$ satisfies the condition (x), provided that we take sufficiently large constant κ_ϵ. Since the mapping F is uniformly expanding in the unstable directions, we can show

$$|F_*^i \gamma_y| \le C_g \rho_\epsilon \exp(-(\chi_u^+ + 2\epsilon)n - 2k) D_* F^i(\mathbf{e}^u(z_0)) < C_g \rho_\epsilon \exp(-\epsilon n - k)$$

for $0 \le i \le n$. Hence the diameter of $F^i(W(z_0))$ is bounded by

$$|F_*^i \gamma| + \max_{y \in \gamma} |F_*^i \gamma_y| \le 2C_g \rho_\epsilon \exp(-\epsilon n - k).$$

If $W(z_0) \cap W(w) \neq \emptyset$ for some point $w \in \Lambda(\chi, \epsilon, n, k; F)$, the diameter of the image $F^i(W(z_0) \cup W(w))$ is bounded by $4C_g \rho_\epsilon \exp(-\epsilon n - k)$. On the other hand, the distance from $\zeta(i)$ to the critical set $\mathcal{C}(F)$ is not less than $C_g^{-1} \exp(-\epsilon n - k)$ from (44). Thus, if we take sufficiently small constant ρ_ϵ, the restrictions of F to $F^i(W(z_0) \cup W(w))$ for $0 \le i < n$ are diffeomorphisms and hence (ix) holds. □

The condition (ix) implies that, if two points z and w in $\Lambda(\chi, \epsilon, k, n; F)$ satisfy $F^n(z) = F^n(w)$, the neighborhoods $W(z)$ and $W(w)$ are disjoint. Thus we obtain the following corollary from the condition (x).

Corollary 5.2. *For any $F \in \mathcal{U}$, $n \ge 1$, $k > 0$ and $\zeta \in M$, we have*

$$\#(\Lambda(\chi, \epsilon, k, n; F) \cap F^{-n}(\zeta)) \le \kappa_\epsilon \exp((\chi_u^+ + \max\{\chi_c^+, 0\} + 7\epsilon)n + 6k).$$

6. Physical measures with neutral central Lyapunov exponent

In this section, we study physical measures with nearly neutral central Lyapunov exponent. The goal is the proof of theorem 3.19, which will be carried out in the last three subsections.

6.1. An illustration of the idea of the proof.

The argument in this section is based on a new idea that relate the transversality condition on unstable cones to absolute continuity of physical measures with nearly neutral central Lyapunov exponent. In this first subsection, we illustrate the idea using a simple example, one because it is quite new in the study of dynamical systems, as far as the author understands, and one because the argument in the following subsections is rather involved in spite of the simplicity of the idea.

As a simplified model of partially hyperbolic endomorphism, we consider the skew product $F : [0, 1) \times \mathbb{R} \to [0, 1) \times \mathbb{R}$ defined by

$$F(x, y) = (d \cdot x, a_i x + b_i y + c_i) \quad \text{on } [i/d, (i+1)/d) \times \mathbb{R},\ i = 0, 1, 2, \cdots, d-1,$$

where $d \ge 2$ is an integer and a_i, b_i and c_i are real numbers. And we assume that

- $|b_i| < d$ for $0 \le i < d$, so that F is partially hyperbolic with $\mathbf{E}^c = \langle \partial/\partial y \rangle$,
- $|b_i| > d^{-1}$ for $0 \le i < d$, so that F is volume-expanding, and
- $\sum_{i=0}^{d-1} \log |b_i| < 0$, so that most of the orbits are bounded.

Put $\theta = \max_{1\le i\le d}|a_i|/(d-|b_i|)$ and $b_{\max} = \max_{1\le i\le d}|b_i|$. Then F brings a segment with slope less than θ in absolute value to a union of segments with the same property. Assume in addition that

$$|a_i - a_{i'}| > 3\theta \cdot b_{\max} \quad \text{for any } i \neq i'. \tag{47}$$

This is a much simplified analogue of the transversality condition on unstable cones. Indeed, if ℓ_σ is a segment in $[i_\sigma/d, (i_\sigma+1)/d) \times \mathbb{R}$ for $\sigma = 1, 2$, and if their slopes are bounded by θ in absolute value, then (47) implies that the difference between the slopes of their images under the mapping F is larger than $\theta \cdot b_{\max}/d$, provided $i_1 \neq i_2$.

We prove the existence of an absolutely continuous invariant measure for F with *negative* central Lyapunov exponent. First of all, observe the following fact: if Lebesgue integrable functions ψ_1 and ψ_2 on $[0,1]\times\mathbb{R}$ take constant values on lines with slopes k_1 and k_2 respectively or, in other words, satisfy $\psi_i(x,y) = \psi_i(0, y-k_i x)$ for $0 \le x \le 1$ and $y \in \mathbb{R}$, then we have

$$\begin{aligned}(\psi_1,\psi_2)_{L^2} &= \int \psi_1(x,y)\psi_2(x,y)dxdy\\ &= \int \psi_1(0,y')\cdot\psi_2(0,y'+(k_1-k_2)x)dxdy' \quad \text{where } y' = y - k_1 x\\ &\le |k_1-k_2|^{-1}\|\psi_1\|_{L^1}\|\psi_2\|_{L^1} \quad \text{provided } k_1 \neq k_2.\end{aligned}$$

Let $\psi(x,y)$ be an L^2 function on $[0,1]\times\mathbb{R}$ and suppose that it is the sum of non-negative functions $\psi_j(y)$, $j = 1, 2, \cdots, m$, that take constant values on lines with slopes k_j with $|k_j| < \theta$ respectively. Let $\mathcal{P}_F$ and $\mathcal{P}_i$, $0 \le i < d$, be the Perron-Frobenius operator associated to F and its restriction to $[i/d, (i+1)/d]\times\mathbb{R}$ respectively, so that $\mathcal{P}_F = \sum_{i=0}^{d-1}\mathcal{P}_i$. By using the transversality condition (47) and the fact we observed above, we can obtain

$$\begin{aligned}\|\mathcal{P}_F\psi\|_{L^2}^2 &= \sum_i \|\mathcal{P}_i\psi\|_{L^2}^2 + \sum_{i\neq i'}(\mathcal{P}_i\psi, \mathcal{P}_{i'}\psi)_{L^2}\\ &\le \frac{1}{d\cdot\min\{|b_i|\}}\|\psi\|_{L^2}^2 + \frac{d}{\theta\cdot b_{\max}}\|\psi\|_{L^1}^2.\end{aligned} \tag{48}$$

Note that the coefficient $1/(d\cdot\min\{|b_i|\})$ is smaller than 1 from the assumption. The Perron-Frobenius operator $\mathcal{P}_F$ preserves L^1 norm of non-negative functions and not dissipative because of the assumption $\sum_{i=0}^{d-1}\log|b_i| < 0$. Since the images $\mathcal{P}_F^n\psi$ again satisfy the condition that we assumed for ψ, we can apply the inequality (48) repeatedly and see that $\mathcal{P}_F^n(\psi)$, $n = 1, 2, \ldots$, are uniformly bounded with respect to the L^2norm. Thus we can find a non-trivial fixed point of $\mathcal{P}_F$ in $L^2([0,1]\times\mathbb{R})$ as a L^2-weak limit point of the sequence $n^{-1}\sum_{m=0}^{n-1}\mathcal{P}_F^m(\psi)$, $n = 1, 2, \ldots$. The measure μ having this fixed point as density is an absolutely continuous invariant measure for F, whose central Lyapunov exponent is $d^{-1}\sum_{i=1}^{d}\log|b_i| < 0$.

In the argument above, we used the assumption $\sum_{i=1}^{d}\log|b_i| < 0$ only to ensure that the Perron-Frobenius operator $\mathcal{P}$ is not dissipative. So, if we consider mappings on compact surfaces, the same argument should be valid in the case where the central Lyapunov exponent is neutral or even slightly positive. This is the key idea that we will develop in the following subsections.

6.2. **Semi-norms on the space of measures.** For a Borel finite measure μ on M and $0 < \delta < 1$, we define the function

$$J_\delta \mu : \mathbb{T} \to \mathbb{R}, \quad J_\delta \mu(w) := \frac{\mu(\mathbf{B}(w,\delta))}{\pi\delta^2} = \frac{1}{\pi\delta^2}\int \mathbf{1}_\delta(w,z)d\mu(z)$$

where

$$\mathbf{1}_\delta : \mathbb{T}\times\mathbb{T} \to \mathbb{R}, \qquad \mathbf{1}_\delta(w,z) = \begin{cases} 1, & \text{if } d(w,z) < \delta; \\ 0, & \text{otherwise.} \end{cases}$$

And we put, for Borel finite measures μ and ν on M,

$$(\mu,\nu)_\delta = (J_\delta\mu, J_\delta\nu)_{L^2(\mathbf{m})}, \quad \|\mu\|_\delta = \sqrt{(\mu,\mu)_\delta} = \|J_\delta \mu\|_{L^2(\mathbf{m})}.$$

Obviously $\|\cdot\|_\delta$ is a semi-norm and satisfies

$$\|\mu\|_\delta \le \frac{|\mu|}{\pi\delta^2}. \tag{49}$$

The semi-norm $\|\mu\|_\delta$ for a measure μ is essentially decreasing with respect to the auxiliary parameter δ. More precisely, we have

Lemma 6.1. *There is an absolute constant $C_0 > 1$ such that*

$$\|\mu\|_\delta \le C_0 \|\mu\|_\rho \tag{50}$$

for any $0 < \rho \le \delta < 1$ and any Borel finite measure μ.

Proof. There is an absolute constant C such that, for any $0 < \rho \le \delta < 1$, we can cover the disk $\mathbf{B}(0,\delta)$ in $\mathbb{R}^2$ by disks $\mathbf{B}(w_i,\rho)$, $1 \le i \le [C\delta^2/\rho^2]$, by choosing the points w_i appropriately. Using Schwarz inequality, we obtain

$$\|\mu\|_\delta^2 = \frac{1}{\pi^2\delta^4}\int \mu(\mathbf{B}(z,\delta))^2 d\mathbf{m}(z) \le \frac{1}{\pi^2\delta^4}\int \left(\sum_{i=1}^{[C\delta^2/\rho^2]} \mu(\mathbf{B}(z+w_i,\rho))\right)^2 d\mathbf{m}(z)$$

$$\le \frac{1}{\pi^2\delta^4}\cdot C\frac{\delta^2}{\rho^2}\cdot \sum_{i=1}^{[C\delta^2/\rho^2]} \int \mu(\mathbf{B}(z+w_i,\rho))^2 d\mathbf{m}(z) \le C^2\|\mu\|_\rho^2$$

for any Borel finite measure μ on M. □

We will make use of the following properties of the semi-norm $\|\cdot\|_\delta$.

Lemma 6.2. *If we have $\liminf_{\delta\to 0}\|\mu\|_\delta < \infty$ for a Borel finite measure μ, then the measure μ is absolutely continuous with respect to Lebesgue measure $\mathbf{m}$ and it holds $\lim_{\delta\to 0}\|\mu\|_\delta = \|d\mu/d\mathbf{m}\|_{L^2(\mathbf{m})}$.*

Proof. The assumption implies that there exists a sequence $\delta(i) \to +0$ such that $J_{\delta(i)}\mu$ is uniformly bounded in $L^2(\mathbf{m})$. Taking a subsequence, we can assume that $J_{\delta(i)}\mu$ converges weakly to some $\psi \in L^2(\mathbf{m})$ as $i \to \infty$. Since $(f,\psi)_{L^2(\mathbf{m})} = \lim_{i\to\infty}\int f\cdot J_{\delta(i)}\mu \, d\mathbf{m} = \int f d\mu$ for any continuous function f on M, we have $\mu = \psi\cdot\mathbf{m}$. Now the last equality is standard. □

Lemma 6.3. *If a sequence of Borel finite measures μ_i, $i \ge 1$, converges weakly to some Borel finite measure μ_∞, then we have $\|\mu_\infty\|_\delta = \lim_{i\to\infty}\|\mu_i\|_\delta$ for $\delta > 0$.*

Proof. We have $\mu_\infty(\partial B(z,\delta)) = 0$ for Lebesgue almost every point z, because

$$\int \mu_\infty(\partial B(z,\delta))d\mathbf{m}(z) = \int_{d(z,w)=\delta} d\mu_\infty(w)d\mathbf{m}(z) = \int \mathbf{m}(\partial B(w,\delta))d\mu_\infty(w) = 0.$$

This implies that $J_\delta\mu_i$ converges to $J_\delta\mu_\infty$ Lebesgue almost everywhere as $i \to \infty$. Since the semi-norms $\|J_\delta\mu_i\|_\delta$, $i \geq 1$, are uniformly bounded from (49), the lemma follows from Lebesgue's convergence theorem. □

6.3. **Two lemmas on the semi-norm $\|\cdot\|_\delta$.** Let $\chi = \{\chi_c^-, \chi_c^+, \chi_u^-, \chi_u^+\}$ be a quadruple satisfying the conditions (14) and (23), and ϵ a small positive constant satisfying (38). For simplicity, we put

$$\chi_c^\Delta = \chi_c^+ - \chi_c^-, \qquad \chi_u^\Delta = \chi_u^+ - \chi_u^-.$$

Let F be a mapping in $\mathcal{U}$, k a positive number, n a positive integer and μ a Borel finite measure on M that is supported on the subset $\Lambda(\chi,\epsilon,k,n;F)$. The aim of this subsection is to give two lemmas that estimate $\|\mu\circ F^{-n}\|_\delta$. Below we shall use the notation in section 5.

Suppose that the measure μ is absolutely continuous with respect to Lebesgue measure $\mathbf{m}$ and that the density $d\mu/d\mathbf{m}$ is square integrable. Then it holds

$$\|d(\mu\circ F^{-n})/d\mathbf{m}\|^2_{L^2(\mathbf{m})} \leq m\cdot\exp(-(\chi_c^- + \chi_u^-)n + 2k)\|d\mu/d\mathbf{m}\|^2_{L^2(\mathbf{m})}$$

where $m = \max\{\#(F^{-n}(w)\cap\Lambda(\chi,\epsilon,k,n;F)) \mid w\in M\}$, because

$$|\det DF^n| \geq \exp((\chi_c^- + \chi_u^-)n - 2k) \quad \text{on } \Lambda(\chi,\epsilon,k,n;F).$$

The following lemma is a counterpart of this simple fact for the semi-norm $\|\cdot\|_\rho$. Recall the constants $0<\rho_\epsilon<1$ and $\kappa_\epsilon,\kappa_g>1$ in lemma 5.1.

Lemma 6.4. *Let ρ be a positive number satisfying*

$$0<\rho<\rho_\epsilon\exp((\chi_c^- - 5\epsilon)n - 3(k+1))/(10\kappa_g^2)$$

and put

$$\delta = 10\kappa_g\rho\exp(-\chi_c^- n + k + 1).$$

Suppose that a measure μ in $\mathcal{AM}([\delta,\infty))$ is supported on a Borel subset X in $\Lambda(\chi,\epsilon,k,n;F)$. Then we have

$$\|\mu\circ F^{-n}\|^2_\rho \leq I_g\cdot m\cdot\exp((-\chi_c^- - \chi_u^- + \chi_c^\Delta + \chi_u^\Delta)n + 6k)\|\mu\|^2_\delta \tag{51}$$

for some constant $I_g>0$, where $m = \max\{\#(F^{-n}(w)\cap\mathbf{B}(X,\delta)) \mid w\in M\}$.

Remark. The point of the lemma above is that the auxiliary parameter of the semi-norm on the right hand side of (51), that is δ, is larger than that on the left hand side, that is ρ. If the auxiliary parameter on the right hand side were allowed to be much smaller than that on the left hand side, the inequality (51) would hold without the assumption that μ has an admissible lift.

Proof. For each point $y\in\Lambda(\chi,\epsilon,k+1,n;F)$, there is a unique neighborhood $V(y)$ such that F^n restricted to $V(y)$ is a diffeomorphism onto the disk $\mathbf{B}(F^n(y),\rho)$, according to lemma 5.1. Note that the diameter of $V(y)$ is smaller than $\delta/10$ from lemma 5.1(iv) and the definition of δ. Let U be the union of the neighborhoods $V(y)$ for $y\in X$. Then U is contained in $\mathbf{B}(X,\delta/10)$ and also in $\Lambda(\chi,\epsilon,k+1,n;F)$

from lemma 5.1(vi) because X is a subset of $\Lambda(\chi, \epsilon, k, n; F)$. From the definition of U and the assumption that μ is supported on X, it follows

$$J_\rho(\mu \circ F^{-n})(w) = \frac{1}{\pi\rho^2}\, \mu \circ F^{-n}(\mathbf{B}(w, \rho)) = \frac{1}{\pi\rho^2} \sum_{z \in F^{-n}(w) \cap U} \mu(V(z))$$

for $w \in M$. Suppose that we have proved

$$\mu(V(z)) \le C_g \exp(-(\chi_c^- + \chi_u^-)n + 2k) \left(\frac{\rho}{\delta}\right)^2 \mu(\mathbf{B}(z, \delta)) \tag{52}$$

for any $z \in \Lambda(\chi, \epsilon, k+1, n; F)$. Then it follows

$$J_\rho(\mu \circ F^{-n})(w) \le C_g \exp(-(\chi_c^- + \chi_u^-)n + 2k) \sum_{z \in F^{-n}(w) \cap U} J_\delta \mu(z) \tag{53}$$

for each $w \in M$. As we have

$$|\det DF^n| \le \exp((\chi_c^+ + \chi_u^+)n + 2k + 2) \quad \text{on } U \subset \Lambda(\chi, \epsilon, k+1, n; F),$$

we can obtain the inequality (51) from (53) by integrating the squares of the both sides and using Schwarz inequality. Therefore, in order to prove the lemma, it is enough to show the inequality (52). Since the both sides of (52) is linear with respect to μ, we may assume without loss of generality that μ has an admissible lift that is supported on a single element of the partition $\Xi_{\mathbf{AC}}$ in $\mathbf{AC}([\delta, \infty))$.

Let $\gamma : [0, a] \to M$ be an admissible curve with length $a \ge \delta$ and z a point in $\Lambda(\chi, \epsilon, k+1, n; F)$. Consider a connected component I of $\gamma^{-1}(V(z))$ and let J be the connected component of $\gamma^{-1}(\mathbf{B}(z, \delta)) \supset \gamma^{-1}(V(z))$ that contains I. As $\delta < \kappa_g^{-1}$, lemma 5.1(viii) tells that the interval I is the unique connected component of $\gamma^{-1}(V(z))$ in J. For the length of I, we have

$$\mathbf{m}_{\mathbb{R}}(I) = |\gamma|_I| \le |F_*^n(\gamma|_I)| \exp(-\chi_u^- n + k + 2) \le C_g \rho \exp(-\chi_u^- n + k + 2)$$

where the first inequality follows from the fact that $\gamma|_I$ is an admissible curve in $V(z) \subset \Lambda(\chi, \epsilon, k+2, n; F)$ and the second from the fact that $F_*^n(\gamma|_I)$ is a curve in $F^n(V(z)) = \mathbf{B}(F^n(z), \rho)$ whose tangent vectors are contained in the unstable cones $\mathbf{S}^u$. For the length of J, we have $\mathbf{m}_{\mathbb{R}}(J) \ge \delta/2$ because the curve $\gamma|_J$ meets $V(z) \subset \mathbf{B}(z, \delta/10)$ while the length of γ is not less than δ. These estimates hold for each connected component of $\gamma^{-1}(V(z))$. Thus we obtain

$$\frac{\mathbf{m}_{\mathbb{R}}(\gamma^{-1}(V(z)))}{\mathbf{m}_{\mathbb{R}}(\gamma^{-1}(\mathbf{B}(z, \delta)))} < C_g \frac{\rho \exp(-\chi_u^- n + k)}{\delta} < C_g \frac{\rho^2}{\delta^2} \exp(-(\chi_c^- + \chi_u^-)n + 2k),$$

where we used the definition of δ in the second inequality. From this and the definition of admissible measure, we can conclude (52) for any measure μ that has an admissible lift supported on $\{\gamma\} \times [0, a]$. □

The next lemma is a counterpart of the inequality (48). Recall the definition of $\mathbf{N}(\chi, \epsilon, k, n; F)$ in subsection 3.7.

Lemma 6.5. *Let ρ and δ be positive numbers that satisfy*

$$\rho \cdot \exp((-\chi_c^- + \epsilon)n) \le \delta \le \exp((\chi_c^- - 2\chi_u^+ - 3\epsilon)n).$$

Suppose that a measure μ in $\mathcal{AM}([\delta, \infty))$ is supported on $\Lambda(\chi, \epsilon, k, n; F)$. Then we have

$$\|\mu \circ F^{-n}\|_\rho^2 \le \frac{\mathbf{N}(\chi, \epsilon, k+1, n; F)\|\mu\|_\rho^2}{\exp((\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - 2\epsilon)n)} + \frac{\exp((-2\chi_c^+ + 2\epsilon)n)}{\delta^2} |\mu|^2,$$

provided that n is larger than some integer $n_* = n_*(\chi, \epsilon, k)$ which depends only on χ, ϵ and k besides the objects that we have fixed in subsection 3.2.

Proof. Put $\rho_1 := \exp((\chi_c^- - \chi_u^+ - \epsilon)n)$ and choose $n_* = n_*(\chi, \epsilon, k)$ so large that

$$\rho_1 < \rho_\epsilon \exp((\chi_c^- - 5\epsilon)n_* - 3(k+1))/(10\kappa_g^2).$$

We shall impose additional conditions on the choice of n_* in some places below. Consider an integer $n \geq n_*$ and the lattice $\mathbb{L}(\rho_1)$ which we defined at the end of subsection 3.1.

For $w \in \mathbb{L}(\rho_1)$, let $D_3(w,i)$, $1 \leq i \leq m(w)$, be the connected components of $F^{-n}(\mathbf{B}(w, 3\rho_1))$ that meet $\Lambda(\chi, \epsilon, k, n; F)$. By lemma 5.1 and the choice of n_* above, we can check that the restriction of F^n to $D_3(w,i)$ is a diffeomorphism onto $\mathbf{B}(w, 3\rho_1)$ and that $D_3(w,i)$ is contained in $\Lambda(\chi, \epsilon, k+1, n; F)$. Let $D_1(w,i)$ and $D_2(w,i)$ be the part of $D_3(w,i)$ that F^n maps onto $\mathbf{B}(w, \rho_1)$ and $\mathbf{B}(w, 2\rho_1)$ respectively. For $\sigma = 1, 2, 3$, let $D_\sigma(w)$ be the union of $D_\sigma(w,i)$ for $1 \leq i \leq m(w)$.

Since the disks $\mathbf{B}(w, \rho_1)$ for $w \in \mathbb{L}(\rho_1)$ cover the torus $\mathbb{T}$, we have

$$\mu \circ F^{-n} \leq \sum_{w \in \mathbb{L}(\rho_1)} (\mu \circ F^{-n})|_{\mathbf{B}(w,\rho_1)}.$$

The function $J_\rho((\mu \circ F^{-n})|_{\mathbf{B}(w,\rho_1)})$ is supported on the disk $\mathbf{B}(w, 2\rho_1)$ as $\rho < \rho_1$ from the assumption on ρ. And the intersection multiplicity of the disks $\mathbf{B}(w, 2\rho_1)$ for $w \in \mathbb{L}(\rho_1)$ is bounded by 10^2 at most. Thus we obtain, by Schwarz inequality,

$$\begin{aligned} \|\mu \circ F^{-n}\|_\rho^2 &\leq \int \left(\sum_{w \in \mathbb{L}(\rho_1)} J_\rho((\mu \circ F^{-n})|_{\mathbf{B}(w,\rho_1)})(z) \right)^2 d\mathbf{m}(z) \\ &\leq 10^2 \int \sum_{w \in \mathbb{L}(\rho_1)} \left(J_\rho((\mu \circ F^{-n})|_{\mathbf{B}(w,\rho_1)})(z) \right)^2 d\mathbf{m}(z) \\ &= 10^2 \sum_{w \in \mathbb{L}(\rho_1)} \|(\mu \circ F^{-n})|_{\mathbf{B}(w,\rho_1)}\|_\rho^2. \end{aligned}$$

Since the intersection multiplicity of the regions $D_2(w)$ for $w \in \mathbb{L}(\rho_1)$ is also bounded by 10^2, we have $\sum_{w \in \mathbb{L}(\rho_1)} \mu|_{D_2(w)} \leq 10^2 \mu$ and hence

$$\begin{aligned} \sum_{w \in \mathbb{L}(\rho_1)} \|\mu|_{D_2(w)}\|_\rho^2 &= \int \sum_{w \in \mathbb{L}(\rho_1)} (J_\rho(\mu|_{D_2(w)})(z))^2 d\mathbf{m}(z) \\ &\leq \int (10^2 \cdot J_\rho \mu(z))^2 d\mathbf{m}(z) \leq 10^4 \|\mu\|_\rho^2. \end{aligned}$$

Therefore we can deduce the inequality in the lemma from its localized version:

$$\begin{aligned} \|(\mu \circ F^{-n})|_{\mathbf{B}(w,\rho_1)}\|_\rho^2 \leq & \frac{\mathbf{N}(\chi, \epsilon, k+1, n; F) \|\mu|_{D_2(w)}\|_\rho^2}{\exp((\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - \epsilon)n)} \\ & + \frac{\exp((-2\chi_c^+ + \epsilon)n)}{\delta^2} (\mu(D_2(w)))^2 \end{aligned} \tag{54}$$

for $w \in \mathbb{L}(\rho_1)$, provided that we take the constant n_* so large that $\exp(\epsilon n_*) > 10^6$.

Below we fix $w \in \mathbb{L}(\rho_1)$ and prove the inequality (54). From the definition of $D_3(w,i)$ and the assumption that μ is supported on $\Lambda(\chi,\epsilon,k,n;F)$, we have

$$\left(\mu \circ F^{-n}\right)|_{\mathbf{B}(w,\rho_1)} = \sum_{i=1}^{m(w)} \left(\mu|_{D_1(w,i)}\right) \circ F^{-n}.$$

Hence the left hand side of the inequality (54) is written in the form

$$\sum_{1\le i,j\le m(w)} ((\mu|_{D_1(w,i)}) \circ F^{-n}), (\mu|_{D_1(w,j)}) \circ F^{-n})_\rho. \tag{55}$$

For $1 \le i \le m(w)$, let z_i be the unique point in $D_3(w,i)$ such that $F^n(z_i) = w$, which belongs to $\Lambda(\chi,\epsilon,k+1,n;F)$. For $1 \le i,j \le m(w)$, we write $i \pitchfork j$ if the pair (z_i, z_j) does not belong to the subset $\mathcal{E}(w;\chi,\epsilon,k+1,n;F)$, that is,

$$\angle(DF^n(\mathbf{E}^u(z_i)), DF^n(\mathbf{E}^u(z_j))) > 5H_g \exp((\chi_c^+ - \chi_u^-)n + 2(k+1)).$$

(See subsection 3.7 for the definition of the set $\mathcal{E}(\cdot)$.) We split the sum (55) into two parts according to the condition $i \pitchfork j$ and reduce the inequality (54) to the following two inequalities:

$$\sum_{i \not\pitchfork j} ((\mu|_{D_1(w,i)}) \circ F^{-n}, (\mu|_{D_1(w,j)}) \circ F^{-n})_\rho \le \frac{\mathbf{N}(\chi,\epsilon,k+1,n;F)\|\mu|_{D_2(w)}\|_\rho^2}{\exp((\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - \epsilon)n)}$$

and

$$\sum_{i \pitchfork j} ((\mu|_{D_1(w,i)}) \circ F^{-n}, (\mu|_{D_1(w,j)}) \circ F^{-n})_\rho \le \frac{\exp((-2\chi_c^+ + \epsilon)n)}{\delta^2}(\mu(D_2(w)))^2.$$

We denote the sums on the left hand sides of these two inequalities by $\sum_{\not\pitchfork}$ and $\sum_{\pitchfork}$ respectively.

We prove the first inequality. By Schwarz inequality, we have

$$\sum_{\not\pitchfork} \le \sum_{i \not\pitchfork j} \frac{\|(\mu|_{D_1(w,i)}) \circ F^{-n}\|_\rho^2 + \|(\mu|_{D_1(w,j)}) \circ F^{-n}\|_\rho^2}{2}.$$

Since each term $\|(\mu|_{D_1(w,i)}) \circ F^{-n}\|_\rho$ appears for at most $2 \cdot \mathbf{N}(\chi,\epsilon,k+1,n;F)$ times on the right hand side, this implies

$$\sum_{\not\pitchfork} \le \mathbf{N}(\chi,\epsilon,k+1,n;F) \sum_{i=1}^{m(w)} \|(\mu|_{D_1(w,i)}) \circ F^{-n}\|_\rho^2.$$

Besides, we have $\sum_{i=1}^{m(w)} \|\mu|_{D_2(w,i)}\|_\rho^2 \le \|\mu|_{D_2(w)}\|_\rho^2$. Therefore it is enough to show

$$\|(\mu|_{D_1(w,i)}) \circ F^{-n}\|_\rho^2 \le \frac{\|\mu|_{D_2(w,i)}\|_\rho^2}{\exp(\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - \epsilon)}. \tag{56}$$

We show this inequality by using lemma 6.4. Unfortunately, we can not apply lemma 6.4 directly to the measure $\mu|_{D_2(w,i)}$ because some part of its admissible lift may be supported on the part of $\mathbf{AC}((0,\infty))$ that corresponds to very short admissible curves, as a consequence of the restriction. We argue as follows: Observe that F^n brings any C^1curve with length less than δ in $D_3(w,i) \subset \Lambda(\chi,\epsilon,k+1,n;F)$ to a curve with length less than ρ_1 from the assumption on δ and (5), provided that n_* is larger than some constant which depends only on ϵ, k and the constant C_g in (5). Suppose that an admissible curve γ with length $a \ge \delta$ meets $D_2(w,i)$ and that a connected component I of $\gamma^{-1}(D_2(w,i))$ has length less than δ. Then the curve $\gamma|_I$ meets the boundary of $D_2(w,i)$ and hence $F_*^n(\gamma|_I)$ meets the boundary of $\mathbf{B}(w,2\rho_1)$. From the observation above, $F_*^n(\gamma|_I)$ does not meet $\mathbf{B}(w,\rho_1)$ and hence $\gamma|_I$ does not meet $D_1(w,i)$. Using this fact, we can construct a measure $\tilde{\mu}$ in $\mathcal{AM}([\delta,\infty))$ that satisfies $\mu|_{D_1(w,i)} \le \tilde{\mu} \le \mu|_{D_2(w,i)}$ by discarding the part of the admissible lift of $\mu|_{D_2(w,i)}$ that is supported on $\mathbf{AC}((0,\delta))$. Note that the observation above implies also that the δ-neighborhood of $D_2(w,i)$ is contained in $D_3(w,i)$, so that $\max\{\#(F^{-n}(z)\cap \mathbf{B}(D_2(w,i),\delta)) \mid z\in M\} = 1$. Now we apply proposition 6.4 to $\tilde{\mu}$ and $X = D_2(w,i)$. Then the corresponding conclusion and (50) imply (56), provided that n_* is larger than some constant which depends only on ϵ, k, ρ_ϵ, κ_g and I_g.

Next we prove the second inequality. It is enough to show

$$\begin{aligned}(57)\quad &\big((\mu|_{D_1(w,i)})\circ F^{-n}, (\mu|_{D_1(w,j)})\circ F^{-n}\big)_\rho \\ &\qquad\le \delta^{-2}\exp((-2\chi_c^+ + \epsilon)n)\cdot \mu(D_2(w,i))\cdot\mu(D_2(w,j))\end{aligned}$$

for $1\le i,j\le m(w)$ such that $i \pitchfork j$. The both sides of this inequality are linear with respect to $\mu|_{D_2(w,i)}$ and $\mu|_{D_2(w,j)}$. Hence, without loss of generality, we can assume that $\mu|_{D_2(w,i)}$ (resp. $\mu|_{D_2(w,j)}$) has an admissible lift supported on a single element $\{\gamma_i\}\times[0,a_i]$ (resp. $\{\gamma_j\}\times[0,a_j]$) of the partition $\Xi_{\mathbf{AC}}$ and that the curve γ_i (resp. γ_j) is a connected component of the intersection of an admissible curve with length $\ge\delta$ with $D_2(w,i)$ (resp. $D_2(w,j)$). From the argument in the proof of the first inequality above, if the length of the curve γ_i (resp. γ_j) is less than δ, it can not meet $D_1(w,i)$ (resp. $D_1(w,j)$) and hence the inequality (57) is trivial. Thereby, we can assume also that the lengths of γ_i and γ_j, that is, a_i and a_j, are not less than δ.

By the definition of admissible measure and that of the semi-norm $\|\cdot\|_\rho$, we have

$$\begin{aligned}&\frac{\big((\mu|_{D_1(w,i)})\circ F^{-n}, (\mu|_{D_1(w,j)})\circ F^{-n}\big)_\rho}{\mu(D_2(w,i))\cdot\mu(D_2(w,j))}\\ &\le \frac{C_g}{a_ia_j(\pi\rho^2)^2}\int_{\mathbb{T}\times[0,a_i]\times[0,a_j]} \mathbf{1}_\rho(F^n\circ\gamma_i(t),\ y)\cdot \mathbf{1}_\rho(F^n\circ\gamma_j(s),\ y)d\mathbf{m}(y)dtds\\ &\le C_g\delta^{-2}\rho^{-2}\int_{[0,a_i]\times[0,a_j]}\mathbf{1}_{2\rho}(F^n\circ\gamma_i(t),\ F^n\circ\gamma_j(s))dtds.\end{aligned}$$

We estimate the last term by using the assumption $i\pitchfork j$. From (17), it holds

$$\angle(DF^n(\mathbf{E}^u(\gamma_i(t))), DF^n(\gamma_i'(t))) \le H_g\exp((\chi_c^+ - \chi_u^-)n + 2(k+1))$$

for $t\in[0,a_i]$. From lemma 5.1(vii), it holds

$$\begin{aligned}\angle(DF^n(\mathbf{E}^u(z_i)), DF^n(\mathbf{E}^u(\gamma_i(t)))) &\le \kappa_\epsilon e^{2(k+1)}\cdot 2\rho_1\\ &\le H_g\exp((\chi_c^+ - \chi_u^-)n + 2(k+1))\end{aligned}$$

$t\in[0,a_i]$, where the second inequality follows from the definition of ρ_1 provided that n_* is larger than some constant which depends only on ϵ, κ_ϵ and H_g. Thus we have

$$\angle(DF^n(\mathbf{E}^u(z_i)), DF^n(\gamma_i'(t))) \le 2H_g\exp((\chi_c^+ - \chi_u^-)n + 2(k+1)) \quad \text{for } t\in[0,a_i]$$

and the same estimate with the index i replaced by j. Therefore the condition $i\pitchfork j$ implies that, for any $t\in[0,a_i]$ and $s\in[0,a_j]$,

$$\angle(DF^n(\gamma_i'(t)), DF^n(\gamma_j'(s))) > H_g\exp((\chi_c^+ - \chi_u^-)n + 2(k+1)).$$

By simple geometric consideration using this fact, we can see that the part of the curve $F_*^n\gamma_i$ that is within distance 2ρ from the curve $F_*^n\gamma_j$ has length less than $C_g\rho\exp(-(\chi_c^+ - \chi_u^-)n - 2(k+1))$. Since γ_i and γ_j are admissible curves in $\Lambda(\chi,\epsilon,k+1,n;F)$, we obtain

$$\begin{aligned}\mathbf{m}_{\mathbb{R}}\{t\in[0,a_i] \mid d(F^n(\gamma_i(t),F_*^n\gamma_j)\le 2\rho\} &\le \frac{C_g\rho\exp(-(\chi_c^+ - \chi_u^-)n - 2(k+1))}{\exp(\chi_u^- n-(k+1))}\\ &= C_g\rho\exp(-\chi_c^+ n-(k+1))\end{aligned}$$

and the same inequality with the indices i and j exchanged. These imply

$$\int_{[0,a_1]\times[0,a_2]} \mathbf{1}_{2\rho}(F(\gamma_i(t)),F(\gamma_j(s)))dtds \le C_g\rho^2\exp(-2\chi_c^+ n-2(k+1)).$$

Therefore we can conclude (57) by taking the constant n_* larger if necessary. □

6.4. **The proof of theorem 3.19: Part I.** We give the proof of theorem 3.19 in the following three subsection. From this point to the end of this section, we consider the situation assumed in the theorem: Let $\mathbf{X}$ be a finite collection of quadruples $\chi(\ell) = \{\chi_c^-(\ell),\chi_c^+(\ell),\chi_u^-(\ell),\chi_u^+(\ell)\}$, $1\le\ell\le\ell_0$, satisfying (14), (23) and (24); Let F be a mapping in $\mathcal{U}$ that satisfy the no flat contact condition and the transversality condition on unstable cones for $\mathbf{X}$. The aim of this subsection is to derive the conclusions of theorem 3.19 from the following proposition:

Proposition 6.6. *Let μ_i, $i\ge 1$, be a sequence of Borel probability measures on M. We assume either*

(A) *every μ_i is invariant and has an admissible lift, or*

(B) *$\mu_i = n(i)^{-1}\sum_{j=0}^{n(i)-1}\mathbf{m}_X\circ F^{-j}$ for some subsequence $n(i)\to\infty$, where $\mathbf{m}_X$ is the normalization of the restriction of Lebesgue measure to some Borel subset $X\subset M$ with positive Lebesgue measure.*

Further, we assume that μ_i converges weakly to a Borel probability measure μ_∞ as $i\to\infty$ and that the pair of Lyapunov exponents $(\chi_c(z;F),\chi_u(z;F))$ is contained in the region $|\mathbf{X}|$ for μ_∞-almost every point z. Then, for sufficiently large i, there exists a measure $\nu_i\le\mu_i$ such that

(a) *$|\nu_i|>1/3$ and*

(b) *ν_i is absolutely continuous with respect to Lebesgue measure and the L^2-norm of the density $d\nu_i/d\mathbf{m}$ is bounded by a constant that is independent of i.*

We assume proposition 6.6 and prove theorem 3.19.

Proof of theorem 3.19. First, note that, if an ergodic invariant measure μ has an admissible lift, and if the pair of Lyapunov exponent $(\chi_c(\mu;F),\chi_u(\mu;F))$ of μ is contained in $|\mathbf{X}|$, then μ is absolutely continuous with respect to Lebesgue measure and, hence, is a physical measure. This follows immediately from proposition 6.6 if we set $\mu_i=\mu_\infty=\mu$ in the assumption (A).

We show that there exist at most finitely many ergodic physical measures. Suppose that there exist infinitely many mutually distinct ergodic physical measures μ_i, $i=1,2,\cdots$. By taking a subsequence, we can assume that μ_i converges weakly to some measure μ_∞ as $i\to\infty$. We have $\chi_c(\mu_\infty;F)=0$ from corollary 4.5, proposition 4.8 and corollary 3.14. Moreover, we have $\chi_c(z;F)=0$ for μ_∞-almost every point z. In fact, otherwise, there should be an ergodic physical measure $\mu'_\infty \ll \mu_\infty$ with negative central Lyapunov exponent from lemma 4.6 and hence $\mu_i = \mu'_\infty$

for sufficiently large i from lemma 4.3, which contradicts the assumption that μ_i are mutually distinct. Since $\lambda_g \le \chi_u(z;F) \le \Lambda_g$ for any point $z \in M$ from the choice of the constants λ_g and Λ_g, the assumption (24) implies that the pair of Lyapunov exponents $(\chi_c(z;F), \chi_u(z;F))$ is contained in $|\mathbf{X}|$ for μ_∞-almost every point z. Therefore we can apply proposition 6.6 with assumption (A) to the sequence μ_i and conclude that there is a measure $\nu_i \le \mu_i$ for sufficiently large i such that $|\nu_i| > 1/3$ and $\|d\nu_i/d\mathbf{m}\|_{L^2(\mathbf{m})} < C$ for a constant C that is independent of i. For these measures ν_i, Schwarz inequality gives

$$(1/3)^2 < |\nu_i|^2 \le \mathbf{m}(\mathcal{B}(\mu_i))\|d\nu_i/d\mathbf{m}\|^2_{L^2(\mathbf{m})} < C^2\mathbf{m}(\mathcal{B}(\mu_i)).$$

Obviously this contradicts the fact that the basins $\mathcal{B}(\mu_i)$ are mutually disjoint.

Let $\mathcal{B}^0$ be the union of the basins of ergodic physical measures whose central Lyapunov exponent is neutral. Below we prove that the Lebesgue measure of the subset $X := M \setminus (\mathcal{B}^- \cup \mathcal{B}^0 \cup \mathcal{B}^+)$ is zero. Again the proof is by contradiction. Suppose that the subset X has positive Lebesgue measure. Then, by choosing a subsequence $n(i) \to \infty$ appropriately, we can assume that the sequence of measures $\mu_i = n(i)^{-1}\sum_{j=0}^{n(i)-1} \mathbf{m}_X \circ F^{-j}$ converges to some measure μ_∞ as $i \to \infty$. Note that the measures μ_i are supported on X, for $F(X) \subset X$. From proposition 4.9, we have $\chi_c(z;F) = 0$ for μ_∞-almost every point z. Thus the assumption (24) imply that the pair of Lyapunov exponents $(\chi_c(z;F), \chi_u(z;F))$ is contained in $|\mathbf{X}|$ for μ_∞-almost every point z. Each ergodic component of μ_∞ has an admissible lift from lemma 3.12 and hence it is a physical measure with neutral central Lyapunov exponent from the fact we noted in the beginning. Especially μ_∞ is supported on $\mathcal{B}^0$. Now apply proposition 6.6 with assumption (B) to the sequence μ_i and then let ν_i be those in the corresponding conclusion. Since the density $\psi_i := d\nu_i/d\mathbf{m}$ has uniformly bounded L^2norm for sufficiently large i, we can assume that ψ_i converges weakly to some $\psi_\infty \in L^2(\mathbf{m})$, by taking a subsequence of $n(i)$. Note that ψ_∞ is not trivial because

$$(\psi_\infty, 1)_{L^2(\mathbf{m})} = \lim_{i\to\infty}(\psi_i, 1)_{L^2(\mathbf{m})} = \lim_{i\to\infty}|\nu_i| \ge 1/3.$$

On the one hand, we have $\int \psi_i d\mu_\infty = 0$ since $\nu_i \le \mu_i$ is supported on $X \subset M \setminus \mathcal{B}^0$. On the other hand, we should have

$$\lim_{i\to\infty}\int \psi_i d\mu_\infty \ge \lim_{i\to\infty}\int \psi_i\psi_\infty d\mathbf{m} = \lim_{i\to\infty}(\psi_i, \psi_\infty)_{L^2} = \|\psi_\infty\|^2_{L^2(\mathbf{m})} > 0$$

because $\psi_\infty \cdot \mathbf{m} \le \mu_\infty$. We have arrived at a contradiction.

We have proved that there exists only finitely many ergodic physical measures for F and that the union of basins of them has total Lebesgue measure. The last statement of theorem 3.19 follows from proposition 4.7 and the fact we noted in the beginning. □

6.5. **The proof of theorem 3.19: Part II.** In this subsection, we give the proof of proposition 6.6, assuming a lemma, lemma 6.8, whose proof is left to the next subsection. Let μ_i and μ_∞ be those in proposition 6.6. We denote

$$\chi_c^\Delta(\ell) = \chi_c^+(\ell) - \chi_c^-(\ell) \quad \text{and} \quad \chi_u^\Delta(\ell) = \chi_u^+(\ell) - \chi_u^-(\ell) \qquad \text{for } 1 \le \ell \le \ell_0.$$

To begin with, we take and fix several constants in the following order:

(K1) Take $0 < \epsilon < 1$ so small that (38) hold for all the quadruples $\chi \in \mathbf{X}$ and that

$$\lim_{k\to\infty} \liminf_{n\to\infty} \max_{1\le\ell\le\ell_0} \frac{\log(\mathbf{N}(\chi(\ell), \epsilon, k, n; F))}{n \cdot (\chi_c^-(\ell) + \chi_u^-(\ell) - \chi_c^\Delta(\ell) - \chi_u^\Delta(\ell) - 100\epsilon)} < 1.$$

This is possible from the transversality condition on unstable cones for $\mathbf{X}$.

(K2) Take positive constants ρ_ϵ so small and κ_ϵ so large that lemma 5.1 and lemma 6.4 hold for all the quadruples $\chi \in \mathbf{X}$ and ϵ above.

(K3) Take a positive constant η so small that

$$10\Lambda_g \eta < \epsilon \quad \text{and} \quad \eta < 10^{-3}\epsilon < 10^{-3}.$$

(K4) Take positive constants h_0 and m_0 so large that $h_0 > \Lambda_g > 1$, $m_0 \ge n_g$ and

$$\int \min\{0,\ L(F^n(z); F) + h_0\}\, d\mu(z) > -\frac{\eta}{100} \cdot |\mu|$$

for any $\mu \in \mathcal{AM}([1,\infty))$ and $n \ge m_0$, where $L(\cdot)$ is the function defined in (13). This is possible from lemma 3.13.

(K5) Take a positive constant k_0 such that $k_0 > h_0$ and that

$$\mu_\infty\left(\bigcup_{\ell=1}^{\ell_0} \Lambda(\chi(\ell), \epsilon, k_0 - 1, n; F)\right) > 1 - \frac{\eta}{200h_0} \quad \text{for any } n > 0.$$

This is possible from lemma 3.15, and the assumption on μ_∞.

(K6) Take a large positive integer p_0 such that
   (a) $\mathbf{N}(\chi(\ell), \epsilon, k_0+2, p_0; F) \le \exp((\chi_c^-(\ell) + \chi_u^-(\ell) - \chi_c^\Delta(\ell) - \chi_u^\Delta(\ell) - 100\epsilon)p_0)$,
   (b) $p_0 > n_*(\chi(\ell), \epsilon, k_0 + 1)$

   for $1 \le \ell \le \ell_0$ where $n_*(\cdot)$ is that in lemma 6.5. This is possible from the choice of ϵ and the fact that $\mathbf{N}(\chi(\ell), \epsilon, k, p_0; F)$ is increasing with respect to k.

Hereafter we will never change the constants taken in (K1)-(K5). Note that we can choose the integer p_0 arbitrarily large in the condition (K6) above. In some places below, we shall put additional conditions that p_0 is larger than some numbers that depend only on $\mathbf{X}$, c_g, λ_g, Λ_g, κ_g, ℓ_0 and the constants taken in (K1)-(K5).

For a point $z \in M$, we define

$$\mathbf{k}(z) = \min\left\{ k \in \mathbb{Z} \;\middle|\; k \ge k_0 \text{ and } z \in \bigcup_{1\le\ell\le\ell_0} \Lambda(\chi(\ell), \epsilon, k, p_0; F) \right\} \ge k_0$$

and $\mathbf{k}(z) = \infty$ if the set $\{\cdot\}$ above is empty. Also we define

$$\mathbf{I}(z) = \begin{cases} 0, & \text{if } \mathbf{k}(z) = k_0; \\ 1, & \text{if } \mathbf{k}(z) > k_0. \end{cases}$$

This is the indicator function of the complement of $\bigcup_{1\le\ell\le\ell_0} \Lambda(\chi(\ell), \epsilon, k_0, p_0; F)$. Let m be a positive integer and write it in the form $m = q(m) \cdot p_0 + d(m)$ where $q(m) = [m/p_0]$, so that $0 \le d(m) < p_0$. We define the subset $\mathcal{R}(m)$ as the set of points $z \in M$ that satisfy

(R1) $\#\{1 \le j \le q \mid \mathbf{I}(F^{m-jp_0}(z)) = 1\} < \dfrac{\eta \cdot q}{10h_0}$ for $1 \le q \le q(m)$;

(R2) $\displaystyle\sum_{j=1}^{q} (\mathbf{k}(F^{m-jp_0}(z)) - k_0) < \eta \cdot qp_0$ for $1 \le q \le q(m)$; and

(R3) $\mathbf{k}(z) - k_0 < \eta m$.

The following lemma gives a sufficient condition in order that $\mathcal{R}(m)$, $m = 1, 2, \cdots$, are not very small with respect to a measure μ.

Lemma 6.7. *Let μ be a Borel probability measure μ on M and n a positive integer such that $n \geq 10p_0$. Assume that*

$$\sum_{j=0}^{n-1} \int |L(F^j(z); F)| \cdot \mathbf{I}(F^j(z))\, d\mu(z) < \frac{\eta n}{10} \tag{58}$$

and that

$$\sum_{j=0}^{n-1} \int \mathbf{I}(F^j(z))\, d\mu(z) < \frac{\eta n}{100 h_0}. \tag{59}$$

Then we have $n^{-1} \sum_{m=0}^{n-1} \mu(\mathcal{R}(m)) \geq 1/2$.

Proof. For $0 \leq m < n$, let $\mathcal{Q}_1(m)$, $\mathcal{Q}_2(m)$ and $\mathcal{Q}_3(m)$ be the sets of points z that *violate* the condition (R1), (R2) and (R3) respectively. We are going to estimate the measures of these subsets by using lemma 4.14. First we give the estimate on the subset $\mathcal{Q}_1(m)$ for $0 \leq m < n$. If $z \in \mathcal{Q}_1(m)$, we have

$$\sum_{j=1}^{q} \mathbf{I}(F^{m-jp_0}(z)) \geq \frac{\eta \cdot q}{10 h_0}$$

for some $1 \leq q < q(m)$. Using lemma 4.14 with the assumption (59), we obtain

$$\begin{aligned} \sum_{m=0}^{n-1} \mu(\mathcal{Q}_1(m)) &= \sum_{d=1}^{p_0} \sum_{j=0}^{[(n-d)/p_0]} \mu(\mathcal{Q}_1((n-d) - jp_0)) \\ &\leq \sum_{d=1}^{p_0} \left(\frac{10 h_0}{\eta} \sum_{j=0}^{[(n-d)/p_0]} \int \mathbf{I}\left(F^{(n-d)-jp_0}(z)\right) d\mu(z) \right) \leq \frac{n}{10}. \end{aligned}$$

Next we give the estimate on the union $\mathcal{Q}_2(m) \cup \mathcal{Q}_3(m)$. Let us put

$$\psi(z) = (|L(z; F)| + 5\Lambda_g) \cdot \mathbf{I}(z) \qquad \text{for } j \geq 1.$$

We claim that

$$\mathbf{k}(z) - k_0 \leq \sum_{j=0}^{p_0 - 1} \psi(F^j(z)) \qquad \text{for } z \in M. \tag{60}$$

For a point z, take the smallest integer $0 \leq p < p_0$ such that $\mathbf{k}(F^p(z)) = k_0$, and set $p = p_0$ if there are no such integers. If $p = 0$, the inequality (60) is trivial. So we assume $p > 0$. In the case $0 < p < p_0$, we choose an integer $1 \leq \ell \leq \ell_0$ so that $\Lambda(\chi(\ell), \epsilon, k_0, p_0; F)$ contains $F^p(z)$. In the case $p = p_0$, we choose $1 \leq \ell \leq \ell_0$ arbitrarily. For $0 \leq i < i' \leq p$ and $v \in \mathbf{S}^u(F^i(z))$, we have the following obvious estimates:

$$\begin{aligned} &\sum_{j=i}^{i'-1} L(F^j(z); F) \leq \log |D^* F^{i'-i}(v)| \leq \Lambda_g (i' - i), \quad \textit{and} \\ &- \Lambda_g \leq -c_g \leq \log |D_* F^{i'-i}(v)| \leq \Lambda_g (i' - i). \end{aligned}$$

Using these estimates and the fact that $F^p(z) \in \Lambda(\chi(\ell), \epsilon, k_0, p_0; F)$ in the case $p < p_0$, we can check that z belongs to $\Lambda(\chi(\ell), \epsilon, k, p_0; F)$ for

$$k = k_0 + \left[\sum_{j=0}^{p-1} (|L(F^j(z); F)| + 3\Lambda_g + \epsilon) \right] + 1.$$

This implies (60).

If a point z belongs to $\mathcal{Q}_2(m)$ or $\mathcal{Q}_3(m)$ for $p_0 \le m < n$, we have, from (60),

$$\sum_{j=m'}^{m-1} \psi(F^j(z)) \ge \eta(m - m') \quad \text{for some } 0 \le m' < m.$$

As $h_0 > \Lambda_g$, the assumptions (58) and (59) imply

$$\sum_{j=0}^{n-1} \int \psi(F^j(z)) d\mu(z) \le \frac{\eta n}{5}.$$

Therefore, by using lemma 4.14, we can obtain

$$\sum_{m=p_0}^{n-1} \mu(\mathcal{Q}_2(m) \cup \mathcal{Q}_3(m)) \le \frac{n}{5}.$$

Note that we have $\sum_{m=0}^{p_0} \mu(\mathcal{Q}_2(m) \cup \mathcal{Q}_3(m)) \le p_0 \le n/10$ from the assumption on n. Since $\mathcal{R}(m)$ is the complement of $\mathcal{Q}_1(m) \cup \mathcal{Q}_2(m) \cup \mathcal{Q}_3(m)$, we can conclude the lemma from the estimates above. □

The following lemma is the key step in the proof of proposition 6.6.

Lemma 6.8. *Let μ be a Borel finite measure on M and n a non-negative integer. If μ has an admissible lift $\tilde{\mu}$ such that $\tilde{\mu} \circ F_*^{-i}$ belongs to $\mathbf{AM}([\exp(-\eta n), \infty))$ for $0 \le i < n$, then we have*

$$\|(\mu|_{\mathcal{R}(n)}) \circ F^{-n}\|_\rho < C|\mu| + C\exp(-\epsilon n)\|\mu\|_{\rho \exp(-10\eta n)}$$

for $0 < \rho \le \exp(-10\Lambda_g p_0)$, where $C > 0$ is a constant that does not depend on the measure μ nor the integer n.

Remark. Actually, the constant $C > 0$ above depends only on ϵ, p_0, c_g and Λ_g.

We give the proof of this lemma in the next subsection. Below we assume this lemma and complete the proof of proposition 6.6.

Proof of proposition 6.6. First consider the case where the assumption (A) holds. From the choice of k_0, we have

$$\mu_i \left(\bigcup_{\ell=1}^{\ell_0} \Lambda(\chi(\ell), \epsilon, k_0, p_0; F) \right) > 1 - \frac{\eta}{100 h_0}$$

or, in other words,

$$\int \mathbf{I}(z) d\mu_i < \frac{\eta}{100 h_0}$$

for sufficiently large i, because $\Lambda(\chi(\ell), \epsilon, k_0, p_0; F)$ contains an open neighborhood of the compact subset $\Lambda(\chi(\ell), \epsilon, k_0 - 1, p_0; F)$. The measure μ_i belongs to $\mathcal{AM}([1, \infty))$ from corollary 3.7. Thus, it follows from the choice of h_0 that

$$\int \min\{0, L(z; F) + h_0\} d\mu_i(z) > -\frac{\eta}{100}.$$

Hence

$$\int |L(z; F)| \cdot \mathbf{I}(z) d\mu_i(z) < h_0 \cdot \frac{\eta}{100 h_0} + \frac{\eta}{100} < \frac{\eta}{10}.$$

Now we can apply lemma 6.7 to the measure μ_i for sufficiently large i, and obtain

$$n^{-1} \sum_{j=0}^{n-1} \mu_i(\mathcal{R}(j)) \geq \frac{1}{2} \qquad \text{for } n \geq 10 p_0.$$

We put

$$\nu_{i,n} = \frac{1}{n} \sum_{j=0}^{n-1} (\mu_i|_{\mathcal{R}(j)}) \circ F^{-j} \ \leq \mu_i \qquad \text{for } n \geq 1,$$

so that $|\nu_{i,n}| \geq 1/2$ for $n \geq 10 p_0$. Obviously the measure μ_i has an admissible lift that satisfies the assumption of lemma 6.8 for any $n \geq 0$. Thus it holds

$$\|\nu_{i,n}\|_\rho \leq \frac{1}{n} \sum_{j=0}^{n-1} \left\| (\mu_i|_{\mathcal{R}(j)}) \circ F^{-j} \right\|_\rho \leq C + \frac{C}{n} \sum_{j=0}^{n-1} \exp(-\epsilon j) \|\mu_i\|_{\rho \exp(-10\eta j)}$$

for $0 < \rho \leq \exp(-10 \Lambda_g p_0)$. This, together with (49) and the choice of η, implies $\limsup_{n \to \infty} \|\nu_{i,n}\|_\rho \leq C$. Let ν_i be a weak limit point of the sequence $\nu_{i,n}$, $n = 1, 2, \ldots$. Then it holds $\nu_i \leq \mu_i$ and $|\nu_i| \geq 1/2$. Also we have $\|\nu_i\|_\rho \leq C$ for $0 < \rho \leq \exp(-10 \Lambda_g p_0)$ from lemma 6.3. From lemma 6.2, this implies that ν_i is absolutely continuous with respect to Lebesgue measure and the density satisfies $\|d\nu_i / d\mathbf{m}\|_{L^2(\mathbf{m})} \leq C$. Thus the measures ν_i satisfy the conditions in proposition 6.6.

Next we consider the case where the assumption (B) holds. Let $n_0 = n_0(F) > n_g$ be that in the definition of the no flat contact condition. Let X and $\mathbf{m}_X$ be those in the assumption (B). Using lemma 3.10, we can find a small positive number $b > 0$ and a probability measure $\omega' \in \mathcal{AM}([b, \infty))$ such that

- $|\mathbf{m}_X - \omega'| < 10^{-3} \eta / h_0$,
- $\omega' \circ F^{-n_0}$ is absolute continuous with respect to Lebesgue measure, and
- the density $d(\omega' \circ F^{-n_0}) / d\mathbf{m}$ is square integrable.

We put $\omega = \omega' \circ F^{-n_0}$ and

$$\mu'_i = n(i)^{-1} \sum_{j=0}^{n(i)-1} \omega \circ F^{-j}, \qquad \text{for } i = 1, 2, \ldots.$$

Then, for sufficiently large i, we have $|\mu_i - \mu'_i| < 10^{-3} \eta / h_0$ and hence

$$\mu'_i \left(\bigcup_{\ell=1}^{\ell_0} \Lambda(\chi(\ell), \epsilon, k_0, p_0; F) \right) > 1 - \frac{\eta}{100 h_0}, \quad \text{that is,} \quad \int \mathbf{I}(z) d\mu'_i < \frac{\eta}{100 h_0}$$

from the choice of k_0. From corollary 3.7, $\omega \circ F^{-j}$ belongs to $\mathcal{AM}([1,\infty))$ for sufficiently large j. Thus we have

$$\int |L(z;F)| \cdot \mathbf{I}(z) d\mu_i'(z) < h_0 \cdot \frac{\eta}{100h_0} + \frac{\eta}{100} < \frac{\eta}{10}$$

for sufficiently large i, from the choice of h_0. Now we can apply lemma 6.7 to $\mu = \omega$ and $n = n(i)$ and obtain

$$n(i)^{-1} \sum_{m=0}^{n-1} \omega(\mathcal{R}(m)) \geq 1/2$$

for sufficiently large i. Let $\tilde{\omega}'$ be an admissible lift of ω' that belongs to $\mathbf{AM}([b,\infty))$ and put $\tilde{\omega} = \tilde{\omega}' \circ F_*^{-n_0}$. Then $\tilde{\omega}$ is an admissible lift of ω. Take a large positive integer n_1 that satisfies $\exp(-\eta \cdot n_1) < b\exp(-c_g)$. From lemma 3.5, the measures $\tilde{\omega} \circ F_*^{-i} = \tilde{\omega}' \circ F_*^{-i-n_0}$ for $i \geq 0$ belongs to $\mathbf{AM}([\exp(-\eta n), \infty))$, provided $n \geq n_1$. Thus we can apply lemma 6.8 to ω, and obtain

$$\|(\omega|_{\mathcal{R}(n)}) \circ F^{-n}\|_\rho < C|\omega| + C\exp(-\epsilon n)\|\omega\|_{\rho \exp(-10\eta n)}$$

for $0 < \rho \leq \exp(-10\Lambda_g p_0)$ and $n \geq n_1$. We put

$$\nu_i' = \frac{1}{n(i)} \sum_{j=n_1}^{n(i)-1} (\omega|_{\mathcal{R}(j)}) \circ F^{-j} \leq \mu_i' \quad i = 1, 2, \ldots.$$

Then, for sufficiently large i, we have $|\nu_i'| \geq (2/5)$ and

$$\|\nu_i'\|_\rho \leq C + \frac{C}{n(i)} \sum_{j=n_1}^{n(i)-1} \exp(-\epsilon j)\|\omega\|_{\rho\exp(-10\eta j)}$$

for $0 < \rho \leq \exp(-10\Lambda_g p_0)$. Letting $\rho \to +0$ in the last inequality, we obtain

$$\left\|\frac{d\nu_i'}{d\mathbf{m}}\right\|_{L^2} \leq C + \left(\frac{C}{n(i)} \sum_{j=n_1}^{n(i)-1} \exp(-\epsilon j)\right) \left\|\frac{d\omega}{d\mathbf{m}}\right\|_{L^2}$$

by lemma 6.2. Since we have $|\mu_i' - \mu_i| < 10^{-2}$ and $\nu_i' \leq \mu_i'$, we can find a Borel measure ν_i such that $\nu_i \leq \nu_i'$, $\nu_i \leq \mu_i$ and $|\nu_i| > 1/3$ for sufficiently large i. These measures ν_i satisfy the conditions in proposition 6.6. □

6.6. **The proof of theorem 3.19: Part III.** In this subsection, we give the proof of lemma 6.8 and complete the proof of theorem 3.19. Let n, μ and $\tilde{\mu}$ be those in lemma 6.8. Recall the mapping $\Pi : \mathbf{AC}'((0,\infty)) \to M$ and the commutative relation (10) in subsection 3.4. Below we divide the measure $\tilde{\mu}$ into many parts so that we can evaluate the semi-norms of their images under the mapping $\Pi \circ F_*^n$ by the two inequalities in subsection 6.3.

We write the integer n in the form $n = q(n)p_0 + d(n)$ where $q(n) = [n/p_0]$, so $0 \leq d(n) < p_0$. For integers $-1 \leq q \leq q(n)$, we put

$$\tau(q) = \begin{cases} qp_0 + d(n), & \text{in the case } 0 \leq q \leq q(n); \\ 0, & \text{in the case } q = -1, \end{cases}$$

so that $\tau(q(n)) = n$, and

$$\delta(q) = \begin{cases} \exp\left(-4\eta(n - \tau(q)) - 7\Lambda_g p_0 - c_g\right), & \text{in the case } 0 \leq q \leq q(n); \\ \exp\left(-4\eta n - 7\Lambda_g p_0\right), & \text{in the case } q = -1. \end{cases}$$

Take and fix a number $0 < \rho \le \exp(-10\Lambda_g p_0)$ arbitrarily and put

$$\rho(q) = \rho \exp(-10\eta(n - \tau(q))) \quad \text{for } -1 \le q \le q(n).$$

Also we put $W = \mathbf{AC}([\exp(-\eta n), \infty))$, so $\tilde{\mu} \circ F_*^{-i}$ for $0 \le i \le n$ are supported on W, from the assumption.

We begin with constructing measurable partitions $\xi(q)$, $-1 \le q \le q(n)$, of the space W such that

(Ξ1) $\xi(q)$ subdivides the partition $\Xi_{\mathbf{AC}}$ on W, which is defined in subsection 3.5. And $\xi(q)$ is increasing with respect to q, that is, $\xi(q+1)$ subdivides $\xi(q)$.

(Ξ2) Each element of the partition $\xi(q)$ is of the form $\{\gamma\} \times J$ where γ is an admissible curve in $\mathcal{AC}(a)$ with $a \ge \exp(-\eta n)$ and J is an interval in $[0, a]$ such that $\delta(q) \le |F_*^{\tau(q)}(\gamma|_J)| \le 2\delta(q)$.

The construction is done by induction on q easily. Since $\delta(-1) < \exp(-\eta n)$, we can construct a partition $\xi(-1)$ that satisfies (Ξ1) and (Ξ2) by subdividing the partition $\Xi_{\mathbf{AC}}$ on W. Let $0 \le q \le q(n)$ and suppose that we have constructed the partitions $\xi(j)$ for $-1 \le j < q$. For each element $\{\gamma\} \times J$ of $\xi(q-1)$, the length of the curve $F_*^{\tau(q)}(\gamma|_J)$ is not less than

$$\delta(q-1) \cdot \exp(\lambda_g(\tau(q) - \tau(q-1)) - c_g) > \delta(q),$$

provided that we take the constant p_0 so large that $(\lambda_g - 4\eta)p_0 > c_g$. (Recall the remark on the choice of the constant p_0 in the last subsection.) Hence we can construct the partition $\xi(q)$ satisfying (Ξ1) and (Ξ2) by subdividing $\xi(q-1)$.

A Borel measurable subset in W is said to be a $\xi(q)$-subset if it is a union of elements of $\xi(q)$. Note that, if Y is a $\xi(q)$-subset, the measure $(\tilde{\mu}|_Y) \circ F_*^{-\tau(q)} \circ \Pi^{-1}$ is contained in $\mathcal{AM}([\delta(q), 2\delta(q)])$ from the condition (Ξ2).

For $-1 \le q \le q(n)$ and an element $P = \{\gamma\} \times J$ of the partition $\xi(q)$, we define

$$\mathbf{k}_q(P) := \min\{\mathbf{k}(F^{\tau(q)}(\gamma(t))) \mid t \in J\} \ge k_0,$$

where $\mathbf{k}(\cdot)$ is that defined in the last subsection. For simplicity, we denote

$$\|\tilde{\nu}\|_\rho := \|\tilde{\nu} \circ \Pi^{-1}\|_\rho \quad \text{for a measure } \tilde{\nu} \text{ on } W.$$

The following is a consequence of the two inequalities in subsection 6.3.

Sublemma 6.9. *Let Y be a $\xi(q)$-subset in W for some $-1 \le q \le q(n)$ and k an integer such that*

$$k_0 \le k \le k_0 + \eta(n - \tau(q)). \tag{61}$$

If $\mathbf{k}_q(P) \le k$ for all $P \in \xi(q)$ in Y, we have

$$\left\|(\tilde{\mu}|_Y) \circ F_*^{-\tau(q+1)}\right\|_{\rho(q+1)} \le \exp(6\Lambda_g p_0 + 3(k - k_0)) \left\|(\tilde{\mu}|_Y) \circ F_*^{-\tau(q)}\right\|_{\rho(q)}.$$

Moreover, if $k = k_0$ and $q \ge 0$ in addition, we have either

$$\left\|(\tilde{\mu}|_Y) \circ F_*^{-\tau(q+1)}\right\|_{\rho(q+1)} \le \exp(-48\epsilon p_0) \left\|(\tilde{\mu}|_Y) \circ F_*^{-\tau(q)}\right\|_{\rho(q)}$$

or

$$\left\|(\tilde{\mu}|_Y) \circ F_*^{-\tau(q+1)}\right\|_{\rho(q+1)} \le \delta(q)^{-1} \exp(3\Lambda_g p_0) \cdot \tilde{\mu}(Y).$$

Proof. We put $p = \tau(q+1) - \tau(q) \le p_0$ so that p is smaller than p_0 only if $q = -1$. From the assumption, we can divide the subset Y into $\xi(q)$-subsets $Y(\ell)$, $1 \le \ell \le \ell_0$, so that $\Pi \circ F_*^{\tau(q)}(P) \cap \Lambda(\chi(\ell), \epsilon, k, p_0; F) \neq \emptyset$ for each $P \in \xi(q)$ in $Y(\ell)$. The measures $(\tilde{\mu}|_{Y(\ell)}) \circ F^{-\tau(q)} \circ \Pi^{-1}$ belong to $\mathcal{AM}([\delta(q), \infty))$, as we noted above.

We prove the first claim. By using (61) and (24), we can check

$$2\delta(q) \le \kappa_g^{-1} \rho_\epsilon \exp((\chi_c^-(\ell) - \chi_u^+(\ell) - 5\epsilon)p_0 - 4k),$$

provided that p_0 is larger than some constant that depends only on k_0, ρ_ϵ, κ_g and Λ_g. This and the claim (v) and (vi) of lemma 5.1 imply that the subset $\Pi \circ F_*^{\tau(q)}(Y(\ell))$ is contained in $\Lambda(\chi(\ell), \epsilon, k+1, p_0; F)$ and, hence, is contained in $\Lambda(\chi(\ell), \epsilon, k+1+\epsilon p_0, p; F)$ even in the case $p < p_0$, by (16).

We have $\delta(q) > \delta := 10\kappa_g \rho(q+1) \exp(-\chi_c^-(\ell)p + k + \epsilon p_0 + 2) > \rho(q)$ and

$$0 < \rho(q+1) \le \rho_\epsilon \exp((\chi_c^-(\ell) - 5\epsilon)p - 3(k+2+\epsilon p_0))/(10\kappa_g^2)$$

from (61) and (24), provided that p_0 is larger than some constant which depends only on k_0, ρ_ϵ, κ_g, c_g and Λ_g. Also we have

$$\max_{w \in M} \#(F^{-p}(w) \cap \mathbf{B}(\Pi \circ F_*^{\tau(q)}(Y(\ell)), \delta)) \le \max_{w \in M} \#(F^{-p}(w)) < \exp(\Lambda_g p_0)$$

from the choice of Λ_g. Therefore we can apply lemma 6.4 and obtain

$$\begin{aligned} &\left\| (\tilde{\mu}|_{Y(\ell)}) \circ F_*^{-\tau(q+1)} \right\|_{\rho(q+1)}^2 \\ &\qquad \le I_g \cdot \exp(11\Lambda_g p_0 + 6(k + \epsilon p_0 + 1)) \left\| (\tilde{\mu}|_{Y(\ell)}) \circ F_*^{-\tau(q)} \right\|_{\delta}^2 \\ &\qquad \le \ell_0^{-2} \exp(12\Lambda_g p_0 + 6(k - k_0)) \left\| (\tilde{\mu}|_{Y}) \circ F_*^{-\tau(q)} \right\|_{\rho(q)}^2 \end{aligned}$$

using (50), provided that p_0 is larger than some constant which depends only on I_g k_0, ℓ_0 and Λ_g. Summing up the square root of the both sides over $1 \le \ell \le \ell_0$, we obtain the first claim.

We prove the second claim by using lemma 6.5. Note that $\Pi \circ F_*^{\tau(q)}(Y(\ell))$ is contained in $\Lambda(\chi(\ell), \epsilon, k_0+1, p_0; F)$ in this case, from the argument above. We can check

$$\rho(q+1)\exp((-\chi_c^-(\ell) + \epsilon)p_0) < \delta(q) < \exp((\chi_c^-(\ell) - 2\chi_u^+(\ell) - 3\epsilon)p_0),$$

provided that p_0 is larger than some constant which depends only on c_g and Λ_g. Recall that we took p_0 so large that $p_0 \ge n_*(\chi(\ell), \epsilon, k_0+1)$ in the condition (K6). Hence we can apply lemma 6.5 and obtain

$$\begin{aligned} \left\| (\tilde{\mu}|_{Y(\ell)}) \circ F_*^{-\tau(q+1)} \right\|_{\rho(q+1)}^2 &\le \exp(-98\epsilon p_0) \left\| (\tilde{\mu}|_{Y(\ell)}) \circ F_*^{-\tau(q)} \right\|_{\rho(q+1)}^2 \\ &\quad + \delta(q)^{-2} \exp((-2\chi_c^+(\ell) + 2\epsilon)p_0) \cdot \tilde{\mu}(Y(\ell))^2, \end{aligned}$$

where we used the condition (K6)(a) in the choice of p_0. This implies

$$\begin{aligned} \left\| (\tilde{\mu}|_{Y(\ell)}) \circ F_*^{-\tau(q+1)} \right\|_{\rho(q+1)} &\le \exp(-49\epsilon p_0) \left\| (\tilde{\mu}|_{Y}) \circ F_*^{-\tau(q)} \right\|_{\rho(q+1)} \\ &\quad + \delta(q)^{-1} \exp((-\chi_c^+(\ell) + \epsilon)p_0) \cdot \tilde{\mu}(Y). \end{aligned}$$

Summing up the both sides for $1 \le \ell \le \ell_0$ and using (50), we conclude

$$\left\| (\tilde{\mu}|_Y) \circ F_*^{-\tau(q+1)} \right\|_{\rho(q+1)} \le C_0 \ell_0 \cdot \exp(-49\epsilon p_0) \left\| (\tilde{\mu}|_Y) \circ F_*^{-\tau(q)} \right\|_{\rho(q)} + \ell_0 \cdot \delta(q)^{-1} \exp((2\Lambda_g + \epsilon) p_0) \cdot \tilde{\mu}(Y).$$

The second claim follows from this inequality, provided that p_0 is larger than some constant that depends only on ℓ_0, Λ_g and ϵ. □

For integers $-1 \le q' \le q \le q(n)$, let $\mathcal{K}(q', q)$ be the set of sequences $\sigma = (\sigma_j)_{j=q'}^{q-1}$ of $(q - q')$ integers that satisfy

$$0 \le \sigma_j \le \eta(n - \tau(j)) \qquad \text{for } q' \le j < q. \tag{62}$$

In the case $q' = q$, we regard that $\mathcal{K}(q', q) = \mathcal{K}(q, q)$ consists of one empty sequence, which we denote by $\emptyset_q$. We put

$$\mathcal{K}(q) = \bigcup \{ \mathcal{K}(q', q) \mid -1 \le q' \le q \}$$

for $0 \le q \le q(n)$. Below we construct subsets $\mathcal{D}(\sigma)$ in W for $\sigma \in \bigcup_{q=-1}^{q(n)} \mathcal{K}(q)$ so that the following conditions hold:

(D1) $\mathcal{D}(\sigma)$ for $\sigma \in \mathcal{K}(q)$ are mutually disjoint $\xi(q-1)$-subsets.
(D2) The union of $\mathcal{D}(\sigma)$ for $\sigma \in \mathcal{K}(q)$ contains the subset $\Pi^{-1}(\mathcal{R}(n)) \cap W$.
(D3) For $-1 \le q' < q \le q(n)$ and $\sigma = (\sigma_j)_{j=q'}^{q-1} \in \mathcal{K}(q', q)$, we have

$$\left\| (\tilde{\mu}|_{\mathcal{D}(\sigma)}) \circ F_*^{-\tau(q)} \right\|_{\rho(q)} \le \exp(6\Lambda_g p_0 + 3\sigma_{q-1}) \left\| (\tilde{\mu}|_{\mathcal{D}(\sigma')}) \circ F_*^{-\tau(q-1)} \right\|_{\rho(q-1)}$$

where $\sigma' = (\sigma_j)_{j=q'}^{q-2} \in \mathcal{K}(q', q-1)$ (so $\sigma' = \emptyset_{q'}$ if $q' = q-1$). Further, we have

$$\left\| (\tilde{\mu}|_{\mathcal{D}(\sigma)}) \circ F_*^{-\tau(q)} \right\|_{\rho(q)} \le \exp(-48\epsilon p_0) \left\| (\tilde{\mu}|_{\mathcal{D}(\sigma')}) \circ F_*^{-\tau(q-1)} \right\|_{\rho(q-1)}$$

in the case where $q \ge 1$ and $\sigma_{q-1} = 0$.
(D4) For the empty sequence $\emptyset_q \in \mathcal{K}(q, q)$ for $q \ge 0$, we have

$$\left\| (\tilde{\mu}|_{\mathcal{D}(\emptyset_q)}) \circ F_*^{-\tau(q)} \right\|_{\rho(q)} \le \delta(q-1)^{-1} \exp(3\Lambda_g p_0) \cdot \tilde{\mu}(\mathcal{D}(\emptyset_q)).$$

The construction is done by induction on q. For the case $q = -1$, we just define $\mathcal{D}(\emptyset_{-1}) = W$. For the case $q = 0$, we have to define $\mathcal{D}(\sigma)$ for $\sigma = \emptyset_0 \in \mathcal{K}(0, 0)$ and $\sigma = (\sigma_{-1}) \in \mathcal{K}(-1, 0)$, where $0 \le \sigma_{-1} \le \eta n$ from (62). We put $\mathcal{D}(\emptyset_0) = \emptyset$ and

$$\mathcal{D}((\sigma_{-1})) = \bigcup \{ P \in \xi(-1) \mid \mathbf{k}_{(-1)}(P) = k_0 + \sigma_{-1} \} \qquad \text{for } 0 \le \sigma_{-1} \le \eta n.$$

Then the conditions (D1) and (D4) hold obviously. The condition (D2) follows from the condition (R3) in the definition of the subset $\mathcal{R}(n)$. The first claim of sublemma 6.9 implies that the condition (D3) holds also.

Next, let $q \ge 1$ and suppose that we have defined $D(\sigma)$ for $\sigma \in \mathcal{K}(q-1)$ so that the conditions (D1)-(D4) hold for them. Consider an element $\sigma = (\sigma_j)_{j=q'}^{q-1}$ in $\mathcal{K}(q', q)$ with $q' < q$ and put $\sigma' = (\sigma_j)_{j=q'}^{q-2}$. Let us set

$$\mathcal{D}_*(\sigma) = \bigcup \{ P \in \xi(q-1) \mid P \subset \mathcal{D}(\sigma') \text{ and } \mathbf{k}_{q-1}(P) = k_0 + \sigma_{q-1} \}. \tag{63}$$

In the case $\sigma_{q-1} > 0$, we define $\mathcal{D}(\sigma) = \mathcal{D}_*(\sigma)$. In the case $\sigma_{q-1} = 0$, we define $\mathcal{D}(\sigma)$ in the following manner: From the second claim of sublemma 6.9, we have either

$$\left\|(\tilde{\mu}|_{\mathcal{D}_*(\sigma)}) \circ F_*^{-\tau(q)}\right\|_{\rho(q)} \le \exp(-48\epsilon p_0) \left\|(\tilde{\mu}|_{\mathcal{D}_*(\sigma)}) \circ F_*^{-\tau(q-1)}\right\|_{\rho(q-1)} \tag{64}$$

or

$$\left\|(\tilde{\mu}|_{\mathcal{D}_*(\sigma)}) \circ F_*^{-\tau(q)}\right\|_{\rho(q)} \le \delta(q-1)^{-1} \exp(3\Lambda_g p_0) \cdot \tilde{\mu}(\mathcal{D}_*(\sigma));$$

We define $\mathcal{D}(\sigma) = \mathcal{D}_*(\sigma)$ in the case where (64) holds, and $\mathcal{D}(\sigma) = \emptyset$ otherwise. Finally we define $\mathcal{D}(\emptyset_q)$ as the union of $\mathcal{D}_*(\sigma)$ for the sequences $\sigma = (\sigma_j)_{j=q'}^{q-1}$ in $\bigcup_{-1<q'<q} \mathcal{K}(q', q) = \mathcal{K}(q) \setminus \{\emptyset_q\}$ such that $\sigma_{q-1} = 0$ and that (64) does *not* hold. As a consequence of this definition, the condition (D4) holds for the empty sequence $\emptyset_q$. The condition (D1) holds obviously. We can check the condition (D2) by using the condition (R2) in the definition of the subset $\mathcal{R}(n)$. The condition (D3) follows from the first claim of sublemma 6.9 and the construction above. We have finished the definition of the subsets $\mathcal{D}(\sigma)$.

For $-1 \le q' \le q(n)$, let $\mathcal{K}_*(q')$ be the set of sequences $\sigma = (\sigma_j)_{j=q'}^{q(n)-1}$ in $\mathcal{K}(q', q(n))$ that satisfy

- $|\sigma|_0 := \#\{q' \le j < q(n) \mid j \ge 0 \text{ and } \sigma_j > 0\} \le \eta(q(n) - q')$, and
- $|\sigma|_1 := \sum_{j=q'}^{q(n)-1} \sigma_j \le 2\eta(q(n) - q')p_0$.

Then, from the definition of the subsets $\mathcal{R}(n)$ and $\mathcal{D}(\sigma)$, we have

$$\Pi^{-1}(\mathcal{R}(n)) \cap W \subset \bigcup_{q'=-1}^{q(n)} \bigcup_{\sigma \in \mathcal{K}_*(q')} \mathcal{D}(\sigma)$$

and hence

$$(\tilde{\mu}|_{\Pi^{-1}(\mathcal{R}(n))}) \circ F_*^{-n} \le \sum_{q'=-1}^{q(n)} \sum_{\sigma \in \mathcal{K}_*(q')} (\tilde{\mu}|_{\mathcal{D}(\sigma)}) \circ F_*^{-n}.$$

For each $\sigma = (\sigma_j)_{j=q'}^{q(n)-1}$ in $\mathcal{K}_*(q')$ with $q' \ge 0$, we can obtain

$$\begin{aligned}\left\|(\tilde{\mu}|_{\mathcal{D}(\sigma)}) \circ F_*^{-n}\right\|_{\rho} &= \left\|(\tilde{\mu}|_{\mathcal{D}(\sigma)}) \circ F_*^{-\tau(q(n))}\right\|_{\rho(q(n))} \\ &\le \exp\left(6\Lambda_g p_0 |\sigma|_0 + 3|\sigma|_1 - 48\epsilon(q(n) - q' - |\sigma|_0)p_0\right) \cdot \left\|(\tilde{\mu}|_{\mathcal{D}(\emptyset_{q'})}) \circ F_*^{-\tau(q')}\right\|_{\rho(q')}\end{aligned}$$

from the conditions (D3) and, hence,

$$\left\|(\tilde{\mu}|_{\mathcal{D}(\sigma)}) \circ F_*^{-n}\right\|_{\rho} \le \exp(-47\epsilon(q(n) - q')p_0 + 11\Lambda_g p_0 + c_g) \cdot |\tilde{\mu}|$$

from the condition (D4) and the choice of η. Similarly, for $\sigma = (\sigma_j)_{j=-1}^{q(n)-1}$ in $\mathcal{K}_*(-1)$, we can obtain

$$\left\|(\tilde{\mu}|_{\mathcal{D}(\sigma)}) \circ F_*^{-n}\right\|_{\rho} \le \exp\left(6\Lambda_g p_0(|\sigma|_0 + 1) + 3|\sigma|_1 - 48\epsilon(q(n) - |\sigma|_0)p_0\right) \cdot \|\tilde{\mu}\|_{\rho(-1)}$$

and hence

$$\left\|(\tilde{\mu}|_{\mathcal{D}(\sigma)}) \circ F_*^{-n}\right\|_{\rho} \le \exp(-47\epsilon n + 8\Lambda_g p_0) \cdot \|\tilde{\mu}\|_{\rho(-1)}\,.$$

For the cardinality of the set $\mathcal{K}_*(q')$, we have

$$\#\mathcal{K}_*(q') \le \binom{q(n)-q'}{[\eta(q(n)-q')]} \cdot \binom{[2\eta p_0(q(n)-q')] + [\eta(q(n)-q')]}{[\eta(q(n)-q')]}$$

where the first factor on the right hand side is an upper bound for the number of possible arrangements of integers $j \ge 0$ for which σ_j may be positive and the second factor is an upper bound for the cardinality of $\sigma \in \mathcal{K}_*(q')$ when one of such arrangements is given. For positive numbers $\alpha, \beta > 0$ and an integer $m \ge 1$ such that $\alpha m \ge 1$ and $\beta m \ge 1$, we have

$$\log \binom{\alpha m + \beta m}{\beta m} \le \alpha m \log\left(1 + \frac{\beta}{\alpha}\right) + \beta m \log\left(1 + \frac{\alpha}{\beta}\right) + A_0$$

from Stirling's formula, where A_0 is an absolute constant. Hence we can obtain

$$\begin{aligned}\frac{\log \#\mathcal{K}_*(q')}{q(n)-q'} \le & -(1-\eta)\log(1-\eta) - \eta \log \eta \\ & + 2\eta p_0 \log\left(1 + \frac{1}{2p_0}\right) + \eta \log(1+2p_0) + 2A_0\end{aligned}$$

for $-1 \le q' < q(n)$. This implies

$$\#\mathcal{K}_*(q') \le \exp(\epsilon p_0(q(n)-q')) \quad \text{for } -1 \le q' < q(n),$$

provided that p_0 is larger than some constant which depends only on ϵ and η. Now we can conclude

$$\begin{aligned}&\left\|(\mu|_{\mathcal{R}(n)}) \circ F^{-n}\right\|_\rho = \left\|(\tilde{\mu}|_{\Pi^{-1}(\mathcal{R}(n))}) \circ F_*^{-n}\right\|_\rho \\ &\le \left(\sum_{q'=0}^{q(n)} \sum_{\sigma \in \mathcal{K}_*(q')} \left\|(\tilde{\mu}|_{\mathcal{D}(\sigma)}) \circ F_*^{-n}\right\|_\rho\right) + \left(\sum_{\sigma \in \mathcal{K}_*(-1)} \left\|(\tilde{\mu}|_{\mathcal{D}(\sigma)}) \circ F_*^{-n}\right\|_\rho\right) \\ &\le \sum_{q'=0}^{q(n)} \exp(-46\epsilon(q(n)-q')p_0 + 11\Lambda_g p_0 + c_g)|\mu| + \exp(-46\epsilon n + 8\Lambda_g p_0)\|\mu\|_{\rho(-1)}.\end{aligned}$$

This implies the inequality in lemma 6.8.

7. Genericity of the transversality condition on unstable cones

In this section, we consider multiplicity of tangencies between the images of the unstable cones under iterates of mappings in $\mathcal{U}$, and investigate to what extent we can resolve the tangencies by perturbation. The goal is the proof of theorem 3.20. The point of our argument in this section is that the dominating expansion in the unstable direction acts as uniform contraction on the angles between subspaces in the unstable cones. This enables us to control the images of the unstable cones in perturbations of mappings in $\mathcal{U}$. Notice that the content and the notation in this section is independent of those in the last section.

7.1. **Reduction of theorem 3.20: The first step.** In this subsection and the next, we reduce theorem 3.20 to more tractable propositions in two steps. For a quadruple $\chi = (\chi_c^-, \chi_c^+, \chi_u^-, \chi_u^+)$, we put

$$\chi_c^{++} := \max\{\chi_c^+, 0\}, \quad \chi_c^\Delta := \chi_c^+ - \chi_c^- \quad \text{and} \quad \chi_u^\Delta := \chi_u^+ - \chi_u^-.$$

For a quadruple χ satisfying (14) and a positive number ϵ, let $\mathcal{S}_1(\chi,\epsilon)$ be the set of mappings $F \in \mathcal{U}$ that satisfy

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{N}(\chi,\epsilon,\epsilon n, n; F) \geq \chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - \epsilon. \tag{65}$$

The first step of the reduction is simple. We show that we can deduce theorem 3.20 from the following proposition:

Proposition 7.1. *The subset $\mathcal{S}_1(\chi,\epsilon)$ is shy with respect to the measure $\mathcal{M}_s$ for $s \geq r+3$, if the quadruple $\chi = (\chi_c^-, \chi_c^+, \chi_u^-, \chi_u^+)$ satisfies the conditions (23),*

$$-2\Lambda_g < \chi_c^- < \chi_c^+ < \chi_u^- < \chi_u^+ < 2\Lambda_g, \tag{66}$$

$$\chi_c^\Delta + \chi_u^\Delta < \chi_c^- + \chi_u^- \tag{67}$$

and

$$\chi_u^- + \chi_c^- - \chi_c^{++} > \left(\frac{\chi_c^{++} + \chi_u^+}{\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta} + 1 \right) \cdot (\chi_c^\Delta + \chi_u^\Delta) \tag{68}$$

and if $\epsilon > 0$ is smaller than some constant which depends only on χ and s besides the integer $r \geq 2$ and the objects that we fixed in subsection 3.2.

Below we prove theorem 3.20 assuming this proposition.

Proof of theorem 3.20. For any point (χ_c, χ_u) in the subset given in the claim (a):

$$\left\{ (x_c, x_u) \in \mathbb{R}^2 \mid x_c + x_u > 0,\ \lambda_g \leq x_u \leq \Lambda_g,\ x_c \leq 0 \right\},$$

we can take a quadruple $\chi = (\chi_c^-, \chi_c^+, \chi_u^-, \chi_u^+)$ satisfying the conditions (23), (66), (67) and (68) so that the rectangle $(\chi_c^-, \chi_c^+) \times (\chi_u^-, \chi_u^+)$ contains the point (χ_c, χ_u). Thus we can choose a countable collection $\mathbf{X}$ of quadruples that satisfy (23), (66), (67) and (68) so that the conditions (a) and (b) in theorem 3.20 hold. We are going to show the condition (c) in theorem 3.20. We fix $s \geq r+3$. Let $\mathbf{X}'$ be an arbitrary finite subset of $\mathbf{X}$. Then we can take positive number $\epsilon > 0$ so small that the conclusion of proposition 7.1 holds for all the quadruples in $\mathbf{X}'$. For each $\chi \in \mathbf{X}'$ and $n \geq 1$, let $\mathcal{S}_1^*(\chi,\epsilon,n)$ be the closed subset of mappings $F \in \mathcal{U}$ that satisfy

$$\mathbf{N}(\chi,\epsilon,\epsilon n, n; F) \geq \exp\left((\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - \epsilon)\, n \right).$$

If a mapping $F \in \mathcal{U}$ belongs to $\mathcal{S}_1(\mathbf{X}')$, or F does not satisfy the transversality condition on unstable cones for $\mathbf{X}'$, then it holds

$$\liminf_{n\to\infty} \max \left\{ \frac{\log(\mathbf{N}(\chi,\epsilon,\epsilon n, n; F))}{n \cdot (\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta)} \ ;\ \chi = (\chi_c^-, \chi_c^+, \chi_u^-, \chi_u^+) \in \mathbf{X}' \right\} \geq 1$$

because $\mathbf{N}(\chi,\epsilon,k,n;F)$ is increasing with respect to ϵ and k. Hence we have

$$\mathcal{S}_1(\mathbf{X}') \subset \bigcup_{m>0} \bigcap_{n>m} \bigcup_{\chi\in\mathbf{X}'} \mathcal{S}_1^*(\chi,\epsilon,n) \subset \bigcup_{\chi\in\mathbf{X}'} \mathcal{S}_1(\chi,\epsilon).$$

From proposition 7.1, the subset $\bigcup_{\chi\in\mathbf{X}'} \mathcal{S}_1(\chi,\epsilon)$ is shy with respect to the measure $\mathcal{M}_s$ and, hence, so is $\mathcal{S}_1(\mathbf{X}')$. Further, the closed subset $\bigcap_{n>m} \bigcup_{\chi\in\mathbf{X}'} \mathcal{S}_1^*(\chi,\epsilon,n)$ is nowhere dense, because it is shy with respect to the measure $\mathcal{M}_s$. Thus $\mathcal{S}_1(\mathbf{X}')$ is a meager subset in $\mathcal{U}$ in the sense of Baire's category argument. □

7.2. **Reduction of theorem 3.20: The second step.** The second step of the reduction is rather involved. We reduce proposition 7.1 to yet another proposition, proposition 7.3, which will be proved in the remaining part of this section. Below we consider an integer $s \geq r+3$, a quadruple $\chi = \{\chi_c^-, \chi_c^+, \chi_u^-, \chi_u^+\}$ and a positive number ϵ. We assume that the quadruple χ satisfies the assumptions in proposition 7.1, that is, the conditions (23), (66), (67) and (68).

In this section, we will introduce several constants that depends only on the quadruple χ and the integers $s \geq r \geq 2$ besides the objects that we fixed in subsection 3.2. In order to distinguish such kind of constants, we will denote them by symbols with subscript χ. Also we will use a generic symbol C_χ for large positive constants of this kind. The usage of these notations is the same as those introduced in subsection 3.3 and section 5.

The choice of the number $\epsilon > 0$ is important for our argument not only in this subsection but also in the remaining part of this section. We claim that our argument in this section is true if ϵ is smaller than some constant ϵ_χ. Below we will assume $0 < \epsilon \leq \epsilon_\chi$ and give the conditions on the choice of ϵ_χ in the course of the argument.

From the condition (68), we can take and fix a positive constant h_χ such that

$$h_\chi + 1 > \frac{\chi_c^{++} + \chi_u^+}{\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta}$$

and that

$$\chi_u^- + \chi_c^- - \chi_c^{++} > (h_\chi + 2)(\chi_c^\Delta + \chi_u^\Delta).$$

Then we fix a positive integer q_χ such that

$$q_\chi > \frac{2(\chi_u^- - \chi_c^-) + \chi_c^{++} - \chi_c^- + \chi_c^\Delta + 2\chi_u^\Delta}{\chi_u^- + \chi_c^- - \chi_c^{++} - (h_\chi + 2)(\chi_c^\Delta + \chi_u^\Delta)}.$$

Also we put $r_\chi = 100(h_\chi + 1)^2 \Lambda_g^2 / \lambda_g$.

Definition. For integers $0 < p < n$ and a point $z \in M$, let $\mathcal{S}_1(\chi, \epsilon, n, p, z)$ be the set of mappings $F \in \mathcal{U}$ such that there exist a subset $\{w_i\}_{i=0}^{q_\chi}$ in $F^{-p}(z)$ and subsets E_i, $0 \leq i \leq q_\chi$, in $F^{-n+p}(w_i) \subset F^{-n}(z)$ that satisfy the following three conditions:

(S1) The subsets E_i for $0 \leq i \leq q_\chi$ are contained in $\Lambda(\chi, \epsilon, 2(h_\chi+1)\epsilon n, n; F)$, and

$$\# E_i = [\exp((\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - r_\chi \epsilon) n)] + 1.$$

(S2) For any points y and y' in the union $\bigcup_{i=0}^{q_\chi} E_i$, we have

$$\angle(DF^n(\mathbf{E}^u(y)), DF^n(\mathbf{E}^u(y'))) \leq \exp((\chi_c^+ - \chi_u^- + 6\epsilon + h_\chi(\chi_c^\Delta + \chi_u^\Delta + 4\epsilon))n).$$

(S3) For $0 \leq j \leq p$ and $0 \leq i, i' \leq q_\chi$, we have

$$F^j(\mathbf{B}(w_i, 10\exp(-r_\chi \epsilon n)) \cap \mathbf{B}(w_{i'}, 10\exp(-r_\chi \epsilon n)) = \emptyset$$

but for the case where both $i = i'$ and $j = 0$ hold.

For an integer $n \geq 1$, we consider the lattice

$$\mathbb{L}_n = \mathbb{L}(\exp((\chi_c^- - \chi_u^-)n))$$

where $\mathbb{L}(\cdot)$ is that defined at the end of subsection 3.1. The following lemma is the main ingredient of this subsection:

Lemma 7.2. *We have*

$$\mathcal{S}_1(\chi,\epsilon) \subset \limsup_{n\to\infty} \left(\bigcup_p \bigcup_{z\in\mathbb{L}_n} \mathcal{S}_1(\chi,\epsilon,n,p,z) \right), \tag{69}$$

where $\cup_p$ *indicates the union over integers* p *satisfying*

$$3h_\chi(\Lambda_g/\lambda_g)\epsilon n \le p \le 3h_\chi(h_\chi+1)(\Lambda_g/\lambda_g)\epsilon n + 1. \tag{70}$$

Proof. Let F be a mapping in $\mathcal{S}_1(\chi,\epsilon)$. We show that there are an arbitrarily large integer n, an integer p satisfying (70) and a point $z \in \mathbb{L}_n$ such that F belongs to $\mathcal{S}_1(\chi,\epsilon,n,p,z)$. From the definition of $\mathcal{S}_1(\chi,\epsilon)$, there are infinitely many integers m that satisfy

$$\mathbf{N}(\chi,\epsilon,\epsilon m, m; F) > \exp((\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - 2\epsilon)m).$$

In the argument below, we consider a large integer m that satisfy this condition. We shall replace it by larger one if necessary. From the definition of $\mathbf{N}(\cdot)$, there exist a point $\zeta \in M$ and a subset P in $\Lambda(\chi,\epsilon,\epsilon m, m; F)$ with cardinality

$$\#P > \exp((\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - 2\epsilon)m)$$

such that $F^m(P) = \{\zeta\}$ and that

$$\angle(DF^m(\mathbf{E}^u(w)), DF^m(\mathbf{E}^u(w'))) \le 10H_g \exp((\chi_c^+ - \chi_u^- + 2\epsilon)m)$$

for $w, w' \in P$. We put $p := [3h_\chi(\Lambda_g/\lambda_g)\epsilon m] + 1$ and consider the subsets of P,

$$P_\ell(w) = \left\{ w' \in P \mid F^{m-\ell p}(w') = F^{m-\ell p}(w) \right\}$$

for $0 \le \ell \le [m/p]$ and $w \in P$. Since the subset $P_\ell(w)$ is contained in the subset $\Lambda(\chi,\epsilon,(m+\ell p)\epsilon, m-\ell p; F)$ from (16), we have

$$\begin{aligned} \#P_\ell(w) &\le \kappa_\epsilon \exp((\chi_u^+ + \chi_c^{++} + 7\epsilon)(m-\ell p) + 6(m+\ell p)\epsilon)) \\ &\le \exp((\chi_u^+ + \chi_c^{++} + 7\epsilon)(m-\ell p) + 7(m+\ell p)\epsilon)) \end{aligned} \tag{71}$$

from corollary 5.2, where the second inequality holds when m is sufficiently large. Especially, for the case $\ell = [m/p]$, we have

$$\#P_{[m/p]}(w) \le \exp((\chi_u^+ + \chi_c^{++} + 7\epsilon)p + 14\epsilon m)) < \exp(-[m/p]\cdot \epsilon p)\cdot \#P$$

where the second inequality holds if ϵ_χ is smaller than some constant that depends only on χ, h_χ, Λ_g and λ_g and if m is sufficiently large. Thus there exist integers $0 \le \ell < [m/p]$ such that

$$\max_{w\in P} \#P_{\ell+1}(w) < \exp(-\epsilon p) \max_{w\in P} \#P_\ell(w). \tag{72}$$

Let ℓ_0 be the smallest integer $0 \le \ell < [m/p]$ such that (72) holds. Then we have

$$\max_{w\in P} \#P_{\ell_0}(w) \ge \exp(-\epsilon \ell_0 p)\cdot \#P.$$

Take a point $w_0 \in P$ such that $\#P_{\ell_0}(w_0) = \max_{w\in P} \#P_{\ell_0}(w)$, and put $n = m - \ell_0 p$, $z = F^n(w_0)$, $E = P_{\ell_0}(w_0)$. It holds

$$\#E = \#P_{\ell_0}(w_0) \ge \exp(-\epsilon(m-n))\#P \ge \exp((\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - 3\epsilon)m).$$

Comparing this with (71) for $\ell = \ell_0$, we obtain

$$m < \frac{\chi_c^{++} + \chi_u^+}{\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - 17\epsilon} \cdot n < (h_\chi + 1)n$$

where the second inequality follows from the choice of h_χ provided that ϵ_χ is smaller than some constant that depends only on χ and h_χ. Hence n and p satisfy the condition (70) and it holds

$$E \subset \Lambda(\chi, \epsilon, \epsilon m, m; F) \subset \Lambda(\chi, \epsilon, (m + \ell_0 p)\epsilon, m - \ell_0 p; F) \subset \Lambda(\chi, \epsilon, (2h_\chi + 1)\epsilon n, n; F).$$

From (3), we can obtain, for any points w and w' in E,

$$\begin{aligned}
&\angle(DF^n(\mathbf{E}^u(w)),\ DF^n(\mathbf{E}^u(w'))) \\
&\quad \le A_g \frac{D_* F^{m-n}(\mathbf{e}^u(F^n(w)))}{|D^* F^{m-n}(\mathbf{e}^u(F^n(w)))|} \angle(DF^m(\mathbf{E}^u(w)), DF^m(\mathbf{E}^u(w'))) \\
&\quad \le A_g \exp((-\chi_c^- + \chi_u^+)(m-n) + 2\epsilon m) \cdot 10 H_g \exp((\chi_c^+ - \chi_u^- + 2\epsilon)m) \\
&\quad = 10 H_g A_g \exp((\chi_c^+ - \chi_u^- + 4\epsilon)n + (\Delta\chi_c + \Delta\chi_u + 4\epsilon)(m-n)) \\
&\quad \le \exp((\chi_c^+ - \chi_u^- + 5\epsilon + h_\chi(\chi_c^\Delta + \chi_u^\Delta + 4\epsilon))n)
\end{aligned}$$

provided that m is sufficiently large.

Denote the points in the subset $F^{-p}(z)$ by w_i, $1 \le i \le i_0$, and divide the set E into the subsets $E_i = \{y \in E \mid F^{n-p}(y) = w_i\}$, $1 \le i \le i_0$. Then $i_0 = \#F^{-p}(z)$ is bounded by $\exp(\Lambda_g p)$ from the choice of the constant Λ_g. By changing the index i, we assume that the cardinality of the subset E_i is decreasing with respect to i. Let i_1 be the smallest positive integer such that

$$\sum_{i=1}^{i_1} \#E_i > \frac{1}{2} \sum_{i=1}^{i_0} \#E_i = \frac{\#E}{2}.$$

Then we have $\#E_{i_1} \cdot (i_0 - i_1 + 1) \ge \sum_{i=i_1}^{i_0} \#E_i \ge \#E/2$ and hence, for $1 \le i \le i_1$,

$$\#E_i \ge \#E_{i_1} \ge \frac{\#E}{2i_0} \ge \frac{\#E}{2\exp(\Lambda_g p)} > \exp((\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - r_\chi \epsilon)n),$$

where the last inequality follows from the definitions of p and r_χ provided that m is sufficiently large. Also we have

$$i_1 \ge \frac{\sum_{i=1}^{i_1} \#E_i}{\#E_1} \ge \frac{\#E}{2\#E_1} \ge \frac{\exp(\epsilon p)}{2}$$

from the condition (72) for $\ell = \ell_0$.

Notice that the point z that we took above may not be contained in $\mathbb{L}_n$, while it have to. So we want to shift it to the closest point in $\mathbb{L}_n$. The distance from the point z to the closest point in $\mathbb{L}_n$ is bounded by $\exp((\chi_c^- - \chi_u^-)n)$ and hence by $\rho_\epsilon \exp((\chi_c^- - 5\epsilon - 3(2h_\chi + 1)\epsilon))n)$, provided that ϵ_χ is smaller than some constant which depends only on χ and that we took sufficiently large m in the beginning. Thereby, by virtue of lemma 5.1, we can move the points w_i and those in E_i accordingly so that the relations $F^p(w_i) = z$ and $F^{n-p}(E_i) = \{w_i\}$ are preserved. Henceforth, we consider the points $z \in \mathbb{L}_n$, w_i and the subsets E_i thus obtained. Lemma 5.1 guarantees that the subsets E_i are contained in $\Lambda(\chi, \epsilon, 2(h_\chi + 1)\epsilon n, n; F)$ and that

$$\begin{aligned}
\angle(DF^n(\mathbf{E}^u(w)),\ DF^n(\mathbf{E}^u(w'))) &\le \exp((\chi_c^+ - \chi_u^- + 5\epsilon + h_\chi(\chi_c^\Delta + \chi_u^\Delta + 4\epsilon))n) \\
&\quad + 2\kappa_\epsilon \exp((\chi_c^- - \chi_u^- + (4h_\chi + 2)\epsilon)n) \\
&< \exp((\chi_c^+ - \chi_u^- + 6\epsilon + h_\chi(\chi_c^\Delta + \chi_u^\Delta + 4\epsilon))n)
\end{aligned}$$

for any points $w, w' \in \bigcup_{i=1}^{i_1} E_i$, provided that m is sufficiently large. Up to this point, we have found arbitrarily large integer n, an integer p, points z, w_i, $1 \leq i \leq i_1$, and subsets E_i, $1 \leq i \leq i_1$, that satisfy the conditions (70), (S1) and (S2). It remains to choose $(q_\chi + 1)$ points among w_i, $1 \leq i \leq i_1$, so that the condition (S3) holds.

Put $W = \{w_i; 1 \leq i \leq i_1\}$ and $\delta = 40p \cdot \exp(2\Lambda_g p - r_\chi \epsilon n)$. Note that the points w_i belong to $\Lambda(\chi, \epsilon, 2(h_\chi + 1)n, p; F)$ by (15). We can check

$$2\delta < \kappa_g^{-1} \rho_\epsilon \exp((-\chi_u^+ + \chi_c^- - 5\epsilon)p - 8(h_\chi + 1)\epsilon n)$$

by using the definition of p and r_χ and the condition (70), provided that m is large enough. Thus F^p is a diffeomorphism on the 2δ-neighborhood of each point in W from lemma 5.1(v). This implies that the distances between the points in $W \subset F^{-p}(z)$ are not less than 2δ. Let $L \subset W$ be the set of points in W that are within distance δ to either of the points $F^j(z)$, $0 \leq j < p$. Then we have $\#L \leq p$ obviously.

Consider a sequence $J = (j_\nu)_{\nu=0}^{\nu_0}$ of integers such that $1 \leq j_\nu \leq p$ for $0 \leq \nu \leq \nu_0$. We denote the sum of the integers in J by $|J| := \sum_{\nu=0}^{\nu_0} j_\nu$. For $x, x' \in W \setminus L$, we denote $x \succ_J x'$ if there is a sequence of points $x_0 = x, x_1, \cdots, x_{\nu_0+1} = x'$ in $W \setminus L$ such that

$$F^{j_\nu}(\mathbf{B}(x_\nu, 10\exp(-r_\chi \epsilon n)) \cap \mathbf{B}(x_{\nu+1}, 10\exp(-r_\chi \epsilon n)) \neq \emptyset \qquad \text{for } 0 \leq \nu \leq \nu_0.$$

From the definition of δ above, it is easy to see that we have $d(F^{|J|}(x), x') < \delta$ if $x \succ_J x'$ for some J with $|J| \leq 2p$. Hence, given a point $x \in W \setminus L$ and an integer $1 \leq i \leq 2p$, there is at most one point x' in $W \setminus L$ that satisfies $x \succ_J x'$ for some sequence J with $|J| = i$.

The relation $x \succ_J x'$ holds for some points x, x' in $W \setminus L$ only if $|J| < p$. In fact, otherwise, there should be a sequence J with $p \leq |J| < 2p$ and points x, x' in $W \setminus L$ such that $x \succ_J x'$ and hence $d(F^{|J|-p}(z), x') = d(F^{|J|}(x), x') < \delta$. But, since $0 \leq |J| - p < p$, this contradicts the definition of L.

The relation $x \succ_J x'$ never holds if $x = x'$. In fact, if $x \succ_J x$ for some J, the relation $x \succ_{J^n} x$ should hold for any $n \geq 1$ where J^n is the iteration of J for n times. But this obviously contradicts the fact proved in the preceding paragraph.

Let us denote $x \succ x'$ for $x, x' \in W \setminus L$ if either $x = x'$ or $x \succ_J x'$ for some sequence $J = (j_\nu)_{\nu=0}^{\nu_0}$ satisfying $1 \leq j_\nu \leq p$. Then, from the argument above, this relation is a partial order on the set $W \setminus L$ and, for each $x \in W \setminus L$, there exist at most p points x' in $W \setminus L$ such that $x \succ x'$. Let $W_{\max}$ be the set of the maximal elements in $W \setminus L$ with respect to the partial order $\succ$. Then we have

$$\#W_{\max} \geq \frac{\#(W \setminus L)}{p} \geq \frac{([\exp(\epsilon p)/2] - p)}{p} \geq (q_\chi + 1)$$

provided that m is large enough. Take $(q_\chi + 1)$ points $\{w_i\}_{i=0}^{q_\chi}$ from $W_{\max}$, then the condition (S3) holds for them. We have completed the proof of lemma 7.2. □

Using lemma 7.2, we can deduce proposition 7.1 from the following proposition:

Proposition 7.3. *Let $s \geq r+3$. Suppose that a quadruple χ satisfies the conditions (23), (66), (67) and (68) and that a positive number ϵ satisfies $0 < \epsilon \leq \epsilon_\chi$. Then, for any $d > 0$ and any mapping G in $C^r(M, \mathbb{T})$, there exists an integer n_0 such that*

$$\mathcal{M}_s(\Phi_G^{-1}(\mathcal{S}_1(\chi, \epsilon, n, p, z)) \cap \mathbf{D}^{s-3}(d)) < \exp((2\chi_c^- - 2\chi_u^- - \epsilon)n) \tag{73}$$

for $n \geq n_0$, $z \in \mathbb{L}_n$ and $0 < p < n$ that satisfies the condition (70).

Remark. Φ_G and $\mathbf{D}^{s-3}(d)$ above are those defined by (1) and (20) respectively.

In fact, because we have $\#L_n = ([\exp((-\chi_c^- + \chi_u^-)n)] + 1)^2$, it follows from proposition 7.3 and (69) that

$$\mathcal{M}_s\left(\Phi_G^{-1}(\mathcal{S}_1(\chi,\epsilon)) \cap \mathbf{D}^{s-3}(d)\right) = 0 \quad \text{for any } d > 0 \text{ and } G \in C^r(M,\mathbb{T}).$$

Since the measure $\mathcal{M}_s$ is supported on $C^{s-3}(M,\mathbb{R}^2) = \bigcup_{d>0} \mathbf{D}^{s-3}(d)$, this implies that the subset $\mathcal{S}_1(\chi,\epsilon)$ is shy with respect to the measure $\mathcal{M}_s$.

7.3. **Perturbations.** In this subsection, we introduce some families of mappings and give estimates on the variations of the images of the unstable subspaces $\mathbf{E}^u(z)$ under iterates of the mappings in the families. Henceforth, in this subsection and the next, we consider the situation in proposition 7.3: Let $s \geq r+3$; Let χ a quadruple that satisfies the conditions (23), (66), (67) and (68) and ϵ a positive number ϵ that satisfies $0 < \epsilon \leq \epsilon_\chi$.

Take and fix a C^∞ function $\psi : \mathbb{R}^2 \to \mathbb{R}$ such that $\|\psi\|_{C^1} \leq 1$ and that

$$\psi(w) = \begin{cases} x, & \text{if } \|w\| \leq 1/10; \\ 0, & \text{if } \|w\| \geq 1 \end{cases} \qquad \text{for } w = (x,y) \in \mathbb{R}^2.$$

For each point $z \in M$, we consider an isometric embedding

$$\varphi_z : \{w \in \mathbb{R}^2 \mid \|w\| < 1\} \to \mathbb{T}$$

that carries the origin to z and the x-axis $\mathbb{R} \times \{0\}$ to $\mathbf{E}^u(z)$. For an integer $n \geq 1$, we put $\delta_n = \exp(-r_\chi \epsilon n)$.

Recall that we took the subset $\mathcal{U}$ of mappings as a neighborhood of a C^rmapping $F_\sharp$ in subsection 3.2. For an integer $n \geq 1$ and a point $z \in M$, we define the C^∞mapping $\psi_{n,z} : M \to \mathbb{R}^2$ by

$$\psi_{n,z}(w) := \begin{cases} \delta_n^{s+3} \cdot \psi\left(\varphi_z^{-1}(w)/\delta_n\right) \cdot \mathbf{e}^c(F_\sharp(z)), & \text{if } d(w,z) < \delta_n; \\ 0, & \text{otherwise} \end{cases}$$

where $\mathbf{e}^c(\cdot)$ is either of the two unit vectors in the central subspace $\mathbf{E}^c(\cdot)$. Note that, for any mapping $F \in \mathcal{U}$, the parallel translation of the vector $\mathbf{e}^c(F_\sharp(z))$ to $F(z)$ is contained in $\mathbf{S}^c(F(z))$ from the choice of the constant ρ_g in subsection 3.2.

Let n and p be positive integers that satisfy the condition (70), $S = \{x_i\}_{i=0}^{q_\chi}$ an ordered subset of the lattice $\mathbb{L}(\delta_n/40)$ and F a mapping in $\mathcal{U}$. The family of mappings that we are going to consider is

$$F_{\mathbf{t}}(w) = F(w) + \sum_{i=1}^{q_\chi} t_i \cdot \psi_{n,x_i}(w) \ : \ M \to \mathbb{T}$$

where $\mathbf{t} = (t_i) \in \mathbb{R}^{q_\chi}$ is the parameter that ranges over the region

$$R = \{\mathbf{t} = (t_i) \in \mathbb{R}^{q_\chi} \mid |t_i| \leq \exp(\chi_c^- n)\}.$$

For this family, we have

$$d_{C^\ell}(F_{\mathbf{t}}, F) \leq C_g \cdot q_\chi \delta_n^{s-\ell+3} \|\mathbf{t}\| \cdot \|\psi\|_{C^\ell} \qquad \text{for } \mathbf{t} \in R \text{ and } 0 \leq \ell \leq s. \tag{74}$$

From this inequality in the case $\ell = 0$, we obtain

$$d_{C^0}(F_{\mathbf{t}}^j, F^j) \leq p\exp(\Lambda_g p) \cdot C_g \cdot q_\chi \delta_n^{s+3} \exp(\chi_c^- n) < \delta_n \tag{75}$$

for $0 \le j \le p$ and $\mathbf{t} \in R$, where the second inequality follows from the condition (70) and the definition of r_χ provided that n is larger than some constant N_ϵ. (Recall the notation introduced in section 5.)

Denote by ∂_i the partial differentiation with respect to the parameter t_i. Then

$$\|\partial_i F_{\mathbf{t}}(w)\| \le \delta_n^{s+3} \tag{76}$$

and

$$\|\partial_i (DF_{\mathbf{t}})(\mathbf{v})\| \le C_g \cdot \delta_n^{s+2} \|\mathbf{v}\| \tag{77}$$

for any $w \in M$, $\mathbf{v} \in \mathbf{S}^u(w)$ and $\mathbf{t} \in R$. If $d(w, x_i) < \delta_n/10$ in addition, we also have

$$|\mathbf{v}^*(\partial_i (DF_{\mathbf{t}}(\mathbf{v})))| \ge C_g^{-1} \cdot \delta_n^{s+2} \|\mathbf{v}\| \tag{78}$$

where $\mathbf{v}^*$ is the unit cotangent vector at $F_{\mathbf{t}}(w)$ that is normal to $DF_{\mathbf{t}}(\mathbf{v})$.

In the following argument, we assume that

$$F^j(\mathbf{B}(x_i, 2\delta_n)) \cap \mathbf{B}(x_{i'}, 2\delta_n) = \emptyset \tag{79}$$

for $0 \le i, i' \le q_\chi$ and $0 \le j \le p$ but for the case where both $i = i'$ and $j = 0$ hold. Note that (79) and the estimate (75) imply

$$F_{\mathbf{t}}^j(\mathbf{B}(x_i, \delta_n)) \cap \mathbf{B}(x_{i'}, \delta_n) = \emptyset \qquad \text{for } \mathbf{t} \in R. \tag{80}$$

Consider a point $z \in M$ and families of points $y_i(\mathbf{t}) \in M$, $0 \le i \le q_\chi$, parameterized by $\mathbf{t} \in R$ continuously. Suppose that it holds

(Y1) $F_{\mathbf{t}}^n(y_i(\mathbf{t})) = z$,

(Y2) $y_i(\mathbf{t}) \in \Lambda(\chi, \epsilon, (2h_\chi + 3)\epsilon n, n; F_{\mathbf{t}})$, and

(Y3) $d(F_{\mathbf{t}}^{n-p}(y_i(\mathbf{t})), x_i) < \delta_n/10$

for $0 \le i \le q_\chi$ and $\mathbf{t} \in R$. Let us put

$$A_i(\mathbf{t}) = \delta_n^{s+2} \frac{|D^* F_{\mathbf{t}}^{p-1}(DF_{\mathbf{t}}^{n-p+1}(\mathbf{e}^u(y_i(\mathbf{t}))))|}{D_* F_{\mathbf{t}}^{p-1}(DF_{\mathbf{t}}^{n-p+1}(\mathbf{e}^u(y_i(\mathbf{t}))))}$$

for $1 \le i \le q_\chi$, where $\mathbf{e}^u(z)$ is either of the two unit tangent vectors in $\mathbf{E}^u(z)$. Then we can show the following estimate, in which we take the constant N_ϵ larger if necessary:

Lemma 7.4. *If $n \ge N_\epsilon$, we have*

$$C_g^{-1} A_i(\mathbf{t}) \le |\partial_i \angle(DF_{\mathbf{t}}^n(\mathbf{E}^u(y_i(\mathbf{t}))), \mathbf{E}^u(z))| \le C_g A_i(\mathbf{t})$$

for $1 \le i \le q_\chi$, and also

$$|\partial_j \angle(DF_{\mathbf{t}}^n(\mathbf{E}^u(y_i(\mathbf{t}))), \mathbf{E}^u(z))| \le C_g \exp(-\lambda_g p) A_i(\mathbf{t})$$

for $0 \le i \le q_\chi$ and $1 \le j \le q_\chi$ provided $i \ne j$.

Proof. Let $1 \le i \le q_\chi$ and $0 \le j \le q_\chi$. For $0 \le m \le n$, we denote by $\mathbf{e}_m$ the unit tangent vector in the direction of $DF_{\mathbf{t}}^m(\mathbf{e}^u(y_i(\mathbf{t})))$ and by $\mathbf{e}_m^*$ the unit cotangent vector that is normal to $\mathbf{e}_m$. We can and do choose the orientation of the cotangent vectors $\mathbf{e}_m^*$ so that $(DF^{n-m})^*(\mathbf{e}_n^*) = D^* F^{n-m}(\mathbf{e}_m) \cdot \mathbf{e}_m^*$. Also we denote $z_m = F_{\mathbf{t}}^m(y_i(\mathbf{t}))$ for simplicity. Notice that $\mathbf{e}_m$, $\mathbf{e}_m^*$ and z_m depend on the parameter $\mathbf{t}$.

We first give simple consequences of the conditions (Y1) and (Y3). From (80) and the condition (Y3), the point z_m is not contained in $\mathbf{B}(x_j, \delta_n)$ for $n - 2p < m < n$ but for the case where both $m = n - p$ and $j = i$ hold. Since $z_m \notin \bigcup_{\ell=0}^{q_\chi} \mathbf{B}(x_\ell, \delta_n)$ for $n - p < m < n$ especially, the condition (Y1) implies that the point z_m does not depend on the parameter $\mathbf{t}$. For $0 \le m \le n-p$, differentiation of the both sides of the identity $F_{\mathbf{t}}^{n-p+1-m}(z_m) \equiv z_{n-p+1}$ gives

$$\left(DF_{\mathbf{t}}^{n-p+1-m}\right)_{z_m}(\partial_j z_m) + \sum_{\ell=m+1}^{n-p+1}\left(DF_{\mathbf{t}}^{n-p+1-\ell}\right)_{z_\ell}((\partial_j F_{\mathbf{t}})(z_{\ell-1})) = 0.$$

Applying $(DF_{\mathbf{t}}^{n-p+1-m})_{z_m}^{-1}$ to the both sides of this identity and using (76) and (6), we obtain

$$|\partial_j z_m| \le \sum_{\ell=m+1}^{n-p+1} C_g\left\|((DF_{\mathbf{t}}^{\ell-m})_{z_m})^{-1}\right\| \cdot \delta_n^{s+3} \le \sum_{\ell=m+1}^{n-p+1}\frac{C_g\delta_n^{s+3}}{|D^*F_{\mathbf{t}}^{\ell-m}(\mathbf{e}_m)|}. \tag{81}$$

Now we are going to estimate

$$\partial_j\angle(DF_{\mathbf{t}}^n(\mathbf{E}^u(y_i(\mathbf{t}))), \mathbf{E}^u(z)) = \partial_j\angle(\mathbf{e}_n, \mathbf{E}^u(z)) = \frac{\mathbf{e}_n^*(\partial_j(DF_{\mathbf{t}}^n(\mathbf{e}_0)))}{D_*F_{\mathbf{t}}^n(\mathbf{e}_0)}.$$

Differentiating the both sides of

$$DF_{\mathbf{t}}^n(\mathbf{e}_0) = (DF_{\mathbf{t}})_{z_{n-1}} \circ (DF_{\mathbf{t}})_{z_{n-2}} \circ \cdots \circ (DF_{\mathbf{t}})_{z_0}(\mathbf{e}_0)$$

and using the relation $DF_{\mathbf{t}}^m(\mathbf{e}_0) = D_*F_{\mathbf{t}}^m(\mathbf{e}_0)\cdot\mathbf{e}_m$, we can obtain

$$\begin{aligned}\partial_j(DF_{\mathbf{t}}^n(\mathbf{e}_0)) = &\sum_{m=0}^{n-1}(DF_{\mathbf{t}}^{n-m-1})_{z_{m+1}}((\partial_j(DF_{\mathbf{t}})_{z_m})(\mathbf{e}_m))\cdot D_*F_{\mathbf{t}}^m(\mathbf{e}_0)\\ &+\sum_{m=0}^{n-1}(DF_{\mathbf{t}}^{n-m-1})_{z_{m+1}}(D^2F_{\mathbf{t}}(\mathbf{e}_m,\partial_j z_m))\cdot D_*F_{\mathbf{t}}^m(\mathbf{e}_0)\\ &+(DF_{\mathbf{t}}^n)_{z_0}(D\mathbf{e}^u(\partial_j z_0)).\end{aligned}$$

From this and the relation $(DF^{n-m})^*(\mathbf{e}_n^*) = D^*F^{n-m}(\mathbf{e}_m)\mathbf{e}_m^*$, it follows

$$\begin{aligned}\mathbf{e}_n^*(\partial_j(DF_{\mathbf{t}}^n(\mathbf{e}_0))) = &\sum_{m=0}^{n-1}D^*F_{\mathbf{t}}^{n-m-1}(\mathbf{e}_{m+1})\cdot\mathbf{e}_{m+1}^*((\partial_j(DF_{\mathbf{t}})_{z_m})(\mathbf{e}_m))D_*F_{\mathbf{t}}^m(\mathbf{e}_0)\\ &+\sum_{m=0}^{n-1}D^*F_{\mathbf{t}}^{n-m-1}(\mathbf{e}_{m+1})\cdot\mathbf{e}_{m+1}^*(D^2F_{\mathbf{t}}(\mathbf{e}_m,\partial_j z_m))D_*F_{\mathbf{t}}^m(\mathbf{e}_0)\\ &+D^*F_{\mathbf{t}}^n(\mathbf{e}_0)\cdot\mathbf{e}_0^*(D\mathbf{e}^u(\partial_j z_0)).\end{aligned}$$

Note that we have $(\partial_j(DF_{\mathbf{t}}))_{z_m} = 0$ for $n-2p < m < n$ but for the case $m = n-p$, and $\partial_j z_m = 0$ for $n-p < m \le n$ as we noted above. Thus we obtain

$$\begin{aligned}&\frac{\mathbf{e}_n^*(\partial_j(DF_{\mathbf{t}}^n(\mathbf{e}_0)))}{D_*F_{\mathbf{t}}^n(\mathbf{e}_0)} - \frac{D^*F_{\mathbf{t}}^{p-1}(\mathbf{e}_{n-p+1})}{D_*F_{\mathbf{t}}^{p-1}(\mathbf{e}_{n-p+1})}\cdot\frac{\mathbf{e}_{n-p+1}^*((\partial_j(DF_{\mathbf{t}})_{z_{n-p}})(\mathbf{e}_{n-p}))}{D_*F_{\mathbf{t}}(\mathbf{e}_{n-p})}\\ &\quad=\sum_{m=0}^{n-2p}\frac{D^*F_{\mathbf{t}}^{n-m-1}(\mathbf{e}_{m+1})}{D_*F_{\mathbf{t}}^{n-m-1}(\mathbf{e}_{m+1})}\cdot\frac{\mathbf{e}_{m+1}^*((\partial_j(DF_{\mathbf{t}})_{z_m})(\mathbf{e}_m)))}{D_*F_{\mathbf{t}}(\mathbf{e}_m)}\\ &\qquad+\sum_{m=0}^{n-p}\frac{D^*F_{\mathbf{t}}^{n-m-1}(\mathbf{e}_{m+1})}{D_*F_{\mathbf{t}}^{n-m-1}(\mathbf{e}_{m+1})}\cdot\frac{\mathbf{e}_{m+1}^*(D^2F_{\mathbf{t}}(\mathbf{e}_m,\partial_j z_m))}{D_*F_{\mathbf{t}}(\mathbf{e}_m)}\\ &\qquad+\frac{D^*F_{\mathbf{t}}^n(\mathbf{e}_0)}{D_*F_{\mathbf{t}}^n(\mathbf{e}_0)}\cdot\mathbf{e}_0^*(D\mathbf{e}^u(\partial_j z_0)).\end{aligned} \tag{82}$$

From (77), the first sum on the right hand side is bounded in absolute value by

$$C_g\delta_n^{s+2}\cdot\frac{|D^*F_{\mathbf{t}}^{p-1}(\mathbf{e}^u_{n-p+1})|}{D_*F_{\mathbf{t}}^{p-1}(\mathbf{e}^u_{n-p+1})}\sum_{m=0}^{n-2p}\exp(-\lambda_g(n-p+1-m+2c_g))$$
$$\le C_g\cdot A_i(\mathbf{t})\exp(-\lambda_g p).$$

By the estimate (81) on $\partial_j z_m$ and the condition (Y2), the second sum on the right hand side is bounded in absolute value by

$$C_g\sum_{m=0}^{n-p}\sum_{\ell=m+1}^{n-p+1}\frac{|D^*F_{\mathbf{t}}^{n-m-1}(\mathbf{e}_{m+1})|}{D_*F_{\mathbf{t}}^{n-m}(\mathbf{e}_m)}\frac{\delta_n^{s+3}}{|D^*F_{\mathbf{t}}^{\ell-m}(\mathbf{e}_m)|}$$
$$=C_g\sum_{m=0}^{n-p}\sum_{\ell=m+1}^{n-p+1}\frac{|D^*F_{\mathbf{t}}^{n-\ell}(\mathbf{e}_\ell)|}{D_*F_{\mathbf{t}}^{n-\ell}(\mathbf{e}_\ell)}\frac{\delta_n^{s+3}}{D_*F_{\mathbf{t}}^{\ell-m}(\mathbf{e}_m)|D^*F_{\mathbf{t}}(\mathbf{e}_m)|}$$
$$<C_g\delta_n^{s+3}\frac{|D^*F_{\mathbf{t}}^{p-1}(\mathbf{e}_{n-p+1})|}{D_*F_{\mathbf{t}}^{p-1}(\mathbf{e}_{n-p+1})}\sum_{m=0}^{n-p}\sum_{\ell=m+1}^{n-p+1}\frac{\exp(-\lambda_g(n-p+1-m+2c_g))}{\exp(-(2h_\chi+4)\epsilon n)}$$
$$<C_gA_i(\mathbf{t})\cdot\delta_n\exp((2h_\chi+4)\epsilon n)<C_gA_i(\mathbf{t})\exp(-\lambda_g p),$$

where the last inequality follows from the definition of the constant r_χ and the condition (70) on p. Similarly, we can show that the last term on the right hand side is bounded by

$$C_g\sum_{\ell=1}^{n-p+1}\frac{|D^*F_{\mathbf{t}}^{n}(\mathbf{e}_0)|}{D_*F_{\mathbf{t}}^{n}(\mathbf{e}_0)}\frac{\delta_n^{s+3}}{|D^*F_{\mathbf{t}}^{\ell}(\mathbf{e}_0)|}<C_gA_i(\mathbf{t})\exp(-\lambda_g p).$$

From (77) and (78), we have

$$C_g^{-1}\delta_n^{s+2}<|\mathbf{e}^*_{n-p+1}(\partial_j DF_{\mathbf{t}}(\mathbf{e}_{n-p}))|<C_g\delta_n^{s+2}\qquad\text{if } j=i$$

and $\partial_i DF_{\mathbf{t}}(\mathbf{e}_{n-p})\equiv 0$ otherwise. Using these estimates in (82), we can conclude the lemma, by taking the constant N_ϵ larger if necessary. □

Consider the mapping $\Psi: R\to\mathbb{R}^{q_\chi}$ defined by

$$\Psi(\mathbf{t})=\Big(\angle(DF_{\mathbf{t}}^n(\mathbf{E}^u(y_i(\mathbf{t}))),DF_{\mathbf{t}}^n(\mathbf{E}^u(y_0(\mathbf{t})))\Big)_{i=1}^{q_\chi}. \tag{83}$$

As a consequence of the lemma above, we have the following corollary, where we take the constant N_ϵ still larger if necessary.

Corollary 7.5. *The mapping Ψ is injective and there is a constant B_χ such that*

$$|\det D\Psi(\mathbf{t})|>\exp(-B_\chi\epsilon n)\qquad\textit{for } \mathbf{t}\in R,$$

provided that $n\ge N_\epsilon$.

Proof. Let us denote by $D\Psi(\mathbf{t})_{ij}$ the (i,j)-entry of the representation matrix of $D\Psi(\mathbf{t})$ with respect to the standard coordinate on $\mathbb{R}^{q_\chi}$. Lemma 7.4 tells that the diagonal entries satisfy

$$C_g^{-1}A_i(\mathbf{t})<|D\Psi(\mathbf{t})_{ii}|<C_gA_i(\mathbf{t})$$

while the off-diagonal entries satisfy

$$|D\Psi(\mathbf{t})_{ij}|<C_g\exp(-\lambda_g p)A_j(\mathbf{t}),\qquad j\ne i.$$

These imply that Ψ is injective on R and $|\det D\Psi(\mathbf{t})|$ is bounded from below by $\prod_{i=1}^{q_\chi} C_g A_i(\mathbf{t})$, provided that n is larger than some constant C_χ. Therefore we have

$$|\det D\Psi(\mathbf{t})| > \Big(C_g \exp((\chi_c^- - \chi_u^+)p - (4h_\chi + 6 + (s+2)r_\chi)\epsilon n)\Big)^{q_\chi}$$

from the condition (Y2). Using the condition (70), we can obtain the corollary. □

7.4. **The proof of proposition 7.3.** In this subsection, we complete the proof of theorem 3.20 by proving proposition 7.3. Let G be a mapping in $C^r(M, \mathbb{T})$ and $d > 0$ a positive number. We consider a large integer $n > N_\epsilon$, an integer p satisfying the condition (70), and a point z in the lattice $\mathbb{L}_n$. We put $\delta_n = \exp(-r_\chi \epsilon n)$ as in the last subsection. Let $S = \{x_i\}_{i=0}^{q_\chi}$ be an ordered subset in the lattice $\mathbb{L}(\delta_n/40)$. We denote by $\mathcal{S}_1(\chi, \epsilon, n, p, z; S)$ the set of mappings F in $\mathcal{S}_1(\chi, \epsilon, n, p, z)$ such that the subset $\{w_i\}_{i=0}^{q_\chi}$ in the definition can be taken so that

(S4) $d(w_i, x_i) < \delta_n/20$ for $0 \le i \le q_\chi$.

The subset $\mathcal{S}_1(\chi, \epsilon, n, p, z)$ is contained in the union of $\mathcal{S}_1(\chi, \epsilon, n, p, z; S)$ over all ordered subsets $S = \{x_i\}_{i=0}^{q_\chi}$ of the lattice $\mathbb{L}(\delta_n/40)$. And the number of such ordered sets S is bounded by $(40\delta_n^{-1} + 1)^{2(q_\chi+1)}$. Therefore, in order to prove the inequality in proposition 7.3, it is enough to show

(84)
$$\mathcal{M}_s(\Phi_G^{-1}(\mathcal{S}_1(\chi, \epsilon, n, p, z; S)) \cap \mathbf{D}^{s-3}(d)) < \exp((2(\chi_c^- - \chi_u^-) - 2r_\chi(q_\chi + 2)\epsilon)n)$$

when n is sufficiently large.

Take a mapping F in $\mathcal{S}_1(\chi, \epsilon, n, p, z; S)$ arbitrarily and consider the family of mappings $F_\mathbf{t}$ defined for the ordered subset S in the last subsection. Note that the conditions (79) and (80) follows from the conditions (S3) and (S4) for F. Let $\mathcal{Y}$ be the set of continuous mappings

$$\mathbf{y} : R \to M \times M \times \cdots \times M, \quad \mathbf{y}(\mathbf{t}) = (y_i(\mathbf{t}))_{i=0}^{q_\chi}$$

that satisfy the conditions (Y1), (Y2) and (Y3) in the last subsection. A family $\mathbf{y}(\mathbf{t})$ in $\mathcal{Y}$ is uniquely determined once $\mathbf{y}(0)$ is given because of the conditions (Y1) and (Y2). Thus we have

$$\begin{aligned}\#\mathcal{Y} &\le (\#(\Lambda(\chi, \epsilon, (2h_\chi + 3)\epsilon n, n; F) \cap F^{-n}(z)))^{q_\chi+1}\\ &\le \kappa_\epsilon \exp((\chi_u^+ + \chi_c^{++} + 7\epsilon + 6(2h_\chi + 3)\epsilon)(q_\chi + 1)n)\\ &\le \exp((\chi_u^+ + \chi_c^{++})(q_\chi + 1)n + C_\chi \epsilon n)\end{aligned}$$

for sufficiently large n, from corollary 5.2 and the condition (Y2).

For a family $\mathbf{y} \in \mathcal{Y}$, we denote by $Z(\mathbf{y})$ the set of parameters $\mathbf{t} \in R$ such that

$$\begin{aligned}&\angle(DF_\mathbf{t}^n(\mathbf{E}^u(y_i(\mathbf{t}))), DF_\mathbf{t}^n(\mathbf{E}^u(y_0(\mathbf{t}))))\\ &\qquad\qquad \le \exp((\chi_c^+ - \chi_u^- + 6\epsilon + h_\chi(\chi_c^\Delta + \chi_u^\Delta + 4\epsilon))n)\end{aligned}$$

for all $1 \le i \le q_\chi$. Then lemma 7.5 implies that we have

$$\mathbf{m}(Z(\mathbf{y})) \le \exp((\chi_c^+ - \chi_u^- + h_\chi(\chi_c^\Delta + \chi_u^\Delta))q_\chi n + C_\chi \epsilon n)$$

provided that $n \ge N_\epsilon$.

Suppose that $F_\mathbf{s}$ belongs to $\mathcal{S}_1(\chi, \epsilon, n, p, z; S)$ for a parameter $\mathbf{s} \in R$. Then there are points $w_i \in F_\mathbf{s}^{-p}(z)$ and subsets $E_i \subset F_\mathbf{s}^{-(n-p)}(w_i)$, $0 \le i \le q_\chi$, which satisfy the conditions (S1)-(S4) with F replaced by $F_{\mathbf{s}}$. Consider a combination $(y_i)_{i=0}^{q_\chi}$ of points such that $y_i \in E_i$ for $0 \le i \le q_\chi$. From (74), we can check that

$$d_{C^1}(F_{\mathbf{t}}, F_{\mathbf{s}}) < \rho_\epsilon \exp((\chi_c^- - 5\epsilon)n - 3 \cdot 2(h_\chi + 1)\epsilon n) \qquad \text{for any } \mathbf{t} \in R$$

provided n is sufficiently large. Thus, the condition (S1) and lemma 5.1, we can check that there exists a unique element $\mathbf{y}(\mathbf{t}) = (y_i(\mathbf{y}))_{i=0}^{q_\chi}$ in $\mathcal{Y}$ such that $y_i(\mathbf{s}) = y_i$ for $0 \le i \le q_\chi$. The condition (S2) implies that $\mathbf{s}$ belongs to the subset $Z(\mathbf{y})$. Therefore, if $F_{\mathbf{s}}$ belongs to $\mathcal{S}_1(\chi, \epsilon, n, p, z; S)$, the parameter $\mathbf{s}$ belongs to the subset $Z(\mathbf{y})$ for at least

$$\prod_{i=0}^{q_\chi} \# E_i \ge \exp((\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - r_\chi \epsilon)(q_\chi + 1)n)$$

elements $\mathbf{y}$ in $\mathcal{Y}$. Now we arrive at the estimate

$$\begin{aligned}
\mathbf{m}(\{\mathbf{t} \in R \mid F_{\mathbf{s}} \in \mathcal{S}_1(\chi, \epsilon, n, p, z; S)\}) &\le \frac{\sum_{Y \in \mathcal{Y}} \mathbf{m}(Z(\mathbf{y}))}{\prod_{i=0}^{q_\chi} \# E_i} \\
\le \frac{\exp(((\chi_c^+ - \chi_u^- + h_\chi(\chi_c^\Delta + \chi_u^\Delta))q_\chi + (\chi_u^+ + \chi_c^{++})(q_\chi + 1))n + C_\chi \epsilon n)}{\exp((\chi_c^- + \chi_u^- - \chi_c^\Delta - \chi_u^\Delta - r_\chi \epsilon)(q_\chi + 1)n)}&.
\end{aligned}$$

Note that we have this estimate uniformly for the mappings F in $\mathcal{S}_1(\chi, \epsilon, n, p, z; S)$. Put $m = q_\chi$, $T_i = \exp(\chi_c^- n)$ and $\psi_i = \psi_{n, x_i}$ for $1 \le i \le q_\chi$ in lemma 3.18. Then the assumption (21) holds provided that n is sufficiently large. The conclusion is

$$\begin{aligned}
&\mathcal{M}_s(\Phi_G^{-1}(\mathcal{S}_1(\chi, \epsilon, n, p, z; S)) \cap \mathbf{D}^{s-3}(d)) \\
&\quad < 2^{q_\chi + 1} \exp\left((\chi_c^{++} - \chi_c^- - \chi_u^- + (h_\chi + 2)(\chi_c^\Delta + \chi_u^\Delta))q_\chi n\right) \\
&\qquad \times \exp((\chi_c^{++} - \chi_c^- + \chi_c^\Delta + 2\chi_u^\Delta + C_\chi \epsilon)n).
\end{aligned}$$

Using the condition in the choice of q_χ, we obtain (84) for sufficiently large n, provided that we take sufficiently small ϵ_χ.

8. Genericity of the no flat contact condition

In this section, we consider the situation where the images of admissible curves under an iterate of a mapping $F \in \mathcal{U}$ have flat contacts with the curves in the critical set $\mathcal{C}(F)$, and investigate whether we can resolve all of such flat contacts by perturbations. Our goal is the proof of theorem 3.21, which will be carried out in the last subsection. The key idea in the proof is that the non-flatness of contacts between curves is easier to establish if we assume higher differentiability. The reader should notice that the content and the notation in this section is independent of those in the last two sections.

8.1. **The jet spaces of curves.** We begin with formulating a sufficient condition for the no flat contact condition in terms of *jet*. For an integer $1 \le q \le r$ and a point $z \in M$, let Γ_z^q be the set of germs of C^qcurves $\gamma : (\mathbb{R}, 0) \to (M, z)$ at z. Recall that we always assume the curves to be parameterized by length. Two germs γ_1 and γ_2 in Γ_z^q are said to have contact of order q if $d(\gamma_1(t), \gamma_2(t))/|t|^q \to 0$ as $t \to 0$. This is an equivalence relation on the space Γ_z^q. The equivalence classes are called *q-jets of curve* and the quotient space is denoted by $\mathbb{J}^q\Gamma_z$. Suppose that a q-jet $\mathbf{j}$ of curve at $z \in M$ is represented by $\gamma \in \Gamma_z^q$. Then the the tangent vector $\frac{d}{dt}\gamma(0) \in T_z M$ at z does not depend on the choice of the representative γ, and neither do the differentials $d^i\gamma(0)$, $2 \le i \le q$, which are defined in subsection 3.4. Thus we put

$$\mathbf{j}^{(0)} = z, \quad \mathbf{j}^{(1)} = \frac{d}{dt}\gamma(0) \quad \text{and} \quad \mathbf{j}^{(i)} = d^i\gamma(0) \qquad \text{for } 2 \le i \le q.$$

The jet space of curves of order q is the disjoint union $\mathbb{J}^q\Gamma := \amalg_{z\in M}\mathbb{J}^q\Gamma_z$, which is equipped with the distance defined by

$$d_{\mathbb{J}}(\mathbf{j}_1, \mathbf{j}_2) = \max\left\{d(\mathbf{j}_1^{(0)}, \mathbf{j}_1^{(0)}), \angle(\mathbf{j}_1^{(1)}, \mathbf{j}_2^{(1)}), \max\left\{|\mathbf{j}_1^{(i)} - \mathbf{j}_2^{(i)}|; 2 \le i \le q\right\}\right\}.$$

Then the following mapping is a homeomorphism:

$$\mathbf{j} \in \mathbb{J}^q\Gamma \mapsto \left(\mathbf{j}^{(1)}, (\mathbf{j}^{(i)})_{i=2}^q\right) \in T^1M \times \mathbb{R}^{q-1}$$

where T^1M is the unit tangent bundle of M. Each mapping $F \in \mathcal{U}$ acts naturally on the space $\mathbb{J}^q\Gamma$. We denote this action simply by

$$F : \mathbb{J}^q\Gamma \to \mathbb{J}^q\Gamma, \qquad [\gamma] \mapsto [F_*\gamma].$$

Let $\mathbb{J}^q\mathcal{AC} \subset \mathbb{J}^q\Gamma$ be the compact subset of q-jets that are represented by germs of admissible curves. Lemma 3.2 tells that $F^n(\mathbb{J}^q\mathcal{AC}) \subset \mathbb{J}^q\mathcal{AC}$ for $n \ge n_g$.

For a C^qcurve $\gamma : I \to M$ defined on an interval I, its q-jet extension is the mapping $\mathbb{J}^q\gamma : I \to \mathbb{J}^q\Gamma$ that carries a parameter $t \in I$ to the jet in $\mathbb{J}^q\Gamma_{\gamma(t)}$ that is represented by the germ of γ at t. Recall that the critical set $\mathcal{C}(F)$ for any mapping F in $\mathcal{U}$ consists of finitely many C^{r-1}curves. Let $\mathbf{C}(F) \subset \mathbb{J}^{r-2}\Gamma$ be the union of the images of their $(r-2)$-jet extensions:

$$\mathbf{C}(F) = \{\mathbb{J}^{r-2}\gamma(I) \mid \gamma : I \to M \text{ is a } C^{r-1}\text{curve contained in } \mathcal{C}(F).\}$$

Lemma 8.1. *If a mapping $F \in \mathcal{U}$ satisfies*

$$F^n(\mathbb{J}^{r-2}\mathcal{AC}) \cap \mathbf{C}(F) = \emptyset \qquad \text{for some } n \ge 1, \tag{85}$$

then F satisfies the no flat contact condition.

Proof. For each point in $\mathcal{C}(F)$, we can find a small C^{r-1}coordinate neighborhood $(U, \psi : U \to \mathbb{R}^2)$ such that $\psi(\mathcal{C}(F) \cap U)$ is an interval in the x-axis $\mathbb{R} \times \{0\}$ and that it holds either

(a) $D\psi(\mathbf{S}^c(z))$ contains the x-axis $\mathbb{R} \times \{0\}$ for every $z \in U$, or
(b) $D\psi(\mathbf{S}^c(z))$ contains the y-axis $\{0\} \times \mathbb{R}$ for every $z \in U$.

Since the critical set $\mathcal{C}(F)$ is compact, we can cover it by finitely many coordinate neighborhood with these properties. So, for the purpose of proving the lemma, it is enough to show the following claim for each coordinate neighborhood (U, ψ) as above: For a constant $C > 0$ and $n_0 > 0$, it holds

$$\mathbf{m}_{\mathbb{R}}\left(\{t \in [0, a] \mid F^n(\gamma(t)) \in U \text{ and } d(F^n(\gamma(t)), \mathcal{C}(F)) < \epsilon\}\right) < C \cdot \epsilon^{1/r-2} \max\{a, 1\}$$

for any $a > 0$, $\gamma \in \mathcal{AC}(a)$, $n \ge n_0$, $\epsilon > 0$. If the condition (a) above holds, this claim is clear because the images of the admissible curves in it under ψ are curves whose slope is uniformly bounded away from 0. Thus it remains to check the claim above in the case where the condition (b) holds. To this end, it is enough to show the following:

Claim 8.2. *If a C^{r-1}function φ on a compact interval $I \subset \mathbb{R}$ satisfies,*

$$\max_{x\in I} \max\{|d^q\varphi/dx^q(x)|\ ;\ 1 \le q \le r-1\} \le K \quad \text{and}$$
$$\min_{x\in I} \max\{|d^q\varphi/dx^q(x)|\ ;\ 1 \le q \le r-2\} > \rho$$

for some positive constants K and ρ, then we have

$$\mathbf{m}_{\mathbb{R}}\{x \in \mathbb{R} \mid |\varphi(x)| < \epsilon\} < C(r,\rho,K,I)\cdot \epsilon^{1/(r-2)} \quad \text{for any } \epsilon > 0,$$

where $C(r,\rho,K,I)$ is a constant that depends only on r, ρ, K and the length of I.

We show this claim by using the following lemma[4, Lemma 5.3].

Lemma 8.3. *If a C^qfunction h on an interval J satisfies $|d^qh/dx^q(x)| \ge \rho > 0$ for all $x \in J$. Then $\mathbf{m}_{\mathbb{R}}(\{x \in J|\ |h(x)| \le \varepsilon\}) \le 2^{q+1}(\varepsilon/\rho)^{1/q}$ for any $\epsilon > 0$.*

Proof of claim 8.2. Let $X \subset I$ be the set of points $x \in I$ such that $|\varphi(x)| \le \rho/2$. For each point $x \in X$, there is an integer $1 \le m \le r-2$ such that $|d^m\varphi/dx^m(x)| > \rho$ and hence that $|d^m\varphi/dx^m| \ge \rho/2$ on the interval $J(x) := (x-\rho/(2K), x+\rho/(2K))$. We can take points $x_i \in X$, $i = 1,2,\dots,i_0$, so that the intervals $J(x_i)$ cover the subset X and that the intersection multiplicity is 2, so $i_0 \le (2\mathbf{m}_{\mathbb{R}}(I)/(\rho/K)) + 1$. Applying lemma 8.3 to each interval $J(x_i)$, we can see that $\mathbf{m}_{\mathbb{R}}\{x \in \mathbb{R} \mid |\varphi(x)| < \epsilon\}$ is bounded by $i_0 \cdot 2^{r-1}(\epsilon/(\rho/2))^{1/(r-2)}$ provided $\epsilon < \rho$. This implies claim 8.2. □

We finished the proof of lemma 8.1. □

8.2. **Lattices in the jet space.** In this subsection, we consider lattices in the space of admissible jets $\mathbb{J}^{r-2}\mathcal{AC}$ and formulate a sufficient condition for the no flat contact condition by using them. Henceforth, we fix integers $2 < \nu < r \le s$ satisfying the condition (2). Note that the condition (2) can be written in the form

$$(r-2)\left(r-1-\frac{r-3}{2}\right) < (r-\nu-2)\left(r-3-\frac{2s-r-\nu+1}{2\nu}\right).$$

Thus we can and do cover the interval $[\lambda_g/2, 2\Lambda_g]$ by finitely many intervals $I(\ell) = (\lambda^-(\ell), \lambda^+(\ell))$, $1 \le \ell \le \ell_0$, such that $\lambda^-(\ell) > \lambda_g/4$ and that

$$(r-2)\left(r-1-\frac{r-3}{2}\frac{\lambda^-(\ell)}{\lambda^+(\ell)}\right) < (r-\nu-2)\left(r-3-\frac{2s-r-\nu+1}{2\nu}\right).$$

For $n \ge 1$ and $1 \le \ell \le \ell_0$, let $\mathbf{Q}(n,\ell)$ be the set of jets $\mathbf{j}$ in $\mathbb{J}^{r-2}\mathcal{AC}$ that satisfy

(Q1) the point $\mathbf{j}^{(0)}$ is contained in the lattice $\mathbb{L}(\exp(-\lambda^+(\ell)(r-2)n))$,
(Q2) the angle $\angle(\mathbf{j}^{(1)}, \mathbf{e}^u(\mathbf{j}^{(0)}))$ is a multiple of $\exp(-\lambda^+(\ell)(r-3)n)$, and
(Q3) $\mathbf{j}^{(q)}$ is a multiple of $\exp((-\lambda^+(\ell)(r-3) + \lambda^-(\ell)(q-1))n)$ for $2 \le q \le r-2$.

We have

$$\#\mathbf{Q}(n,\ell) \le C_g \exp\left((r-2)\left((r-1)\lambda^+(\ell) - \frac{r-3}{2}\lambda^-(\ell)\right)n\right). \tag{86}$$

For integers $n \ge 1$, $1 \le \ell \le \ell_0$, a mapping $F \in \mathcal{U}$ and $\sigma = 0, 1$, we define $V_\sigma(n,\ell;F)$ as the set of jets $\mathbf{j}$ in $\mathbb{J}^{r-2}\mathcal{AC}$ that satisfy

$$\exp(\lambda^-(\ell)n - \sigma) \le |D_*F^n(\mathbf{j}^{(1)})| \le \exp(\lambda^+(\ell)n + \sigma).$$

Then, from the choice of the numbers $\lambda^\pm(\ell)$, the subsets $V_0(n,\ell;F)$ for $1 \le \ell \le \ell_0$ cover $\mathbb{J}^{r-2}\mathcal{AC}$ provided that n is larger than some constant C_g.

Lemma 8.4. *There is a constant $B_g > 1$ such that, for any jet $\mathbf{j}$ in $V_0(n,\ell;F)$ with $n \geq B_g$ and $1 \leq \ell \leq \ell_0$, there exists a jet $\mathbf{i} \in \mathbf{Q}(n,\ell) \cap V_1(n,\ell;F)$ such that*

$$d_{\mathbb{J}}(F^n(\mathbf{j}), F^n(\mathbf{i})) < B_g \exp(-\lambda^+(\ell)(r-3)n). \tag{87}$$

Proof. Let us take a jet $\mathbf{j} \in V_0(n,\ell;F)$ arbitrarily. Let w be the point in the lattice $\mathbb{L}(\exp(-\lambda^+(\ell)(r-2)n))$ that is closest to $\mathbf{j}^{(0)}$. As $\mathbf{j}^{(1)}$ belongs to $\mathbf{S}^u(\mathbf{j}^{(0)})$, the minimum angle between $\mathbf{j}^{(1)}$ and the vectors in $\mathbf{S}^u(w)$ is bounded by $C_g \cdot d(\mathbf{j}^{(0)}, w)$. Hence we can choose a jet $\mathbf{i} \in \mathbf{Q}(n,\ell)$ such that

(I1) $\mathbf{i}^{(0)} = w$ and hence $d(\mathbf{j}^{(0)}, \mathbf{i}^{(0)}) < \exp(-\lambda^+(\ell)(r-2)n)$,
(I2) $\angle(\mathbf{j}^{(1)}, \mathbf{i}^{(1)}) < \exp(-\lambda^+(\ell)(r-3)n) + C_g \exp(-\lambda^+(\ell)(r-2)n)$ and
(I3) $|\mathbf{j}^{(q)} - \mathbf{i}^{(q)}| < \exp((-\lambda^+(\ell)(r-3) + \lambda^-(\ell)(q-1))n)$ for $2 \leq q \leq r-2$.

For $0 \leq m \leq n$, we put $z(m) = F^m(\mathbf{j})^{(0)} = F^m(\mathbf{j}^{(0)})$, $w(m) = F^m(\mathbf{i})^{(0)} = F^m(\mathbf{i}^{(0)})$ and

$$\Delta_m^q = \begin{cases} d(F^m(\mathbf{j})^{(0)}, F^m(\mathbf{i})^{(0)}) = d(z(m), w(m)), & \text{if } q = 0; \\ \angle(F^m(\mathbf{j})^{(1)}, F^m(\mathbf{i})^{(1)}) = \angle(DF^m(\mathbf{j}^{(1)}), DF^m(\mathbf{i}^{(1)})), & \text{if } q = 1; \\ |F^m(\mathbf{j})^{(q)} - F^m(\mathbf{i})^{(q)}|, & \text{if } 2 \leq q \leq r-2. \end{cases}$$

In order to prove the inequality (87), it is enough to show

$$\Delta_n^q \leq C_g \exp(-\lambda^+(\ell)(r-3)n) \qquad \text{for } 0 \leq q \leq r-2.$$

First we prove

$$\Delta_m^0 \leq 2\|DF^m_{z(0)}\| \cdot \Delta_0^0 \leq C_g \exp(-\lambda^+(\ell)(r-3)n) \tag{88}$$

for $1 \leq m < n$. As $\mathbf{j} \in V_0(n,\ell;F)$, we have

$$\begin{aligned} \|DF^{m-k}_{z(k)}\| &\leq C_g \cdot D_* F^{m-k}(DF^k(\mathbf{j}^{(1)})) \leq \frac{C_g \cdot D_* F^n(\mathbf{j}^{(1)})}{D_* F^{n-m}(DF^m(\mathbf{j}^{(1)})) \cdot D_* F^k(\mathbf{j}^{(1)})} \\ &\leq C_g \exp(\lambda^+(\ell)n - \lambda_g(n-m+k)) \end{aligned} \tag{89}$$

for $0 \leq k \leq m \leq n$. So the second inequality in (88) follows from the condition (I1). We prove the first inequality in (88) by induction on $1 \leq m < n$. Using the simple estimate

$$\left\| \exp^{-1}_{z(m)}(w(m)) - DF_{z(m-1)}(\exp^{-1}_{z(m-1)}(w(m-1))) \right\| \leq C_g (\Delta^0_{m-1})^2$$

repeatedly, we can get the following inequality for $\Delta_m^0 = \|\exp^{-1}_{z(m)}(w(m))\|$:

$$\Delta_m^0 \leq \|DF^m_{z(0)}\| \Delta_0^0 + C_g \sum_{k=0}^{m-1} \|DF^{m-k-1}_{z(k+1)}\| (\Delta_k^0)^2 \qquad \text{for } 0 \leq m \leq n. \tag{90}$$

Note that we have, from (5),

$$\begin{aligned} \|DF^{m-k-1}_{z(k+1)}\| \|DF^k_{z(0)}\| &\leq C_g D_* F^{m-k-1}(DF^{k+1}(\mathbf{e}^u(z_0))) \cdot D_* F^k(\mathbf{e}^u(z_0)) \\ &\leq C_g \frac{D_* F^m(\mathbf{e}^u(z_0))}{D_* F(DF^k(\mathbf{e}^u(z_0)))} \leq C_g \|DF^m_{z(0)}\| \end{aligned} \tag{91}$$

for $0 \leq k \leq m-1$. Consider an integer $0 \leq m_0 \leq n$ and suppose that (88) holds for $0 \leq m < m_0$. Then, using them and the estimates (89) and (91) in (90), we can obtain

$$\begin{aligned}\Delta^0_{m_0} &\leq \|DF^{m_0}_{z(0)}\|\Delta^0_0 + C_g\sum_{k=0}^{m_0-1}\|DF^{m_0-k-1}_{z(k+1)}\|\cdot 2\|DF^k_{z(0)}\|\Delta^0_0\cdot\Delta^0_k\\ &\leq \|DF^{m_0}_{z(0)}\|\Delta^0_0\left(1 + C_g\cdot n\cdot\exp(-\lambda^+(\ell)(r-3)n)\right).\end{aligned}$$

This implies the first inequality in (88) for $m = m_0$, provided that n is larger than some constant C_g. Thus we can obtain (88) for $1 \leq m \leq n$ by induction.

Next we estimate Δ^1_m for $0 \leq m \leq n$. We have

$$\begin{aligned}\Delta^1_m \leq\ & \angle(DF^m_{z(0)}(\mathbf{j}^{(1)}), DF^m_{z(0)}(\mathbf{i}^{(1)}))\\ &+ \sum_{k=0}^{m-1}\angle(DF^{m-k}_{z(k)}(DF^k_{w(0)}(\mathbf{i}^{(1)})), DF^{m-k-1}_{z(k+1)}(DF^{k+1}_{w(0)}(\mathbf{i}^{(1)})))\end{aligned}$$

where we omit the operations of parallel translation. (Recall the remark given in the proof of lemma 5.1.) For $0 \leq k < n$, we have $DF^k_{w(0)}(\mathbf{i}^{(1)}) \in \mathbf{S}^u(w(k))$ and $d(z(k), w(k)) = \Delta^0_k \leq C_g\exp(-\lambda^+(\ell)(r-3)n)$. Hence the parallel translation of $DF^k_{w(0)}(\mathbf{i}^{(1)})$ to $z(k)$ does not belong to the central cone $\mathbf{S}^c(z(k))$ provided that n is larger than some constant C_g. Using (3), we can obtain

$$\begin{aligned}&\angle(DF^{m-k}_{z(k)}(DF^k_{w(0)}(\mathbf{i}^{(1)})), DF^{m-k-1}_{z(k+1)}(DF^{k+1}_{w(0)}(\mathbf{i}^{(1)})))\\ &\leq A_g\frac{|D^*F^{m-k-1}(DF^{k+1}(\mathbf{j}^{(1)}))|}{D_*F^{m-k-1}(DF^{k+1}(\mathbf{j}^{(1)}))}\angle(DF_{z(k)}(DF^k_{w(0)}(\mathbf{i}^{(1)})), DF_{w(k)}(DF^k_{w(0)}(\mathbf{i}^{(1)})))\\ &\leq C_g\exp(-\lambda_g(m-k-1))\Delta^0_k < C_g\exp(-\lambda_g(m-k-1) - \lambda^+(\ell)(r-3)n).\end{aligned}$$

Likewise we can obtain $\angle(DF^m_{z(0)}(\mathbf{j}^{(1)}), DF^m_{z(0)}(\mathbf{i}^{(1)})) \leq C_g\exp(-\lambda_g m)\Delta^1_0$. Therefore, by condition (I2), we conclude

$$\begin{aligned}\Delta^1_m &\leq C_g\exp(-\lambda_g m)\Delta^1_0 + \sum_{k=0}^{m-1}C_g\exp(-\lambda_g(m-k-1) - \lambda^+(\ell)(r-3)n)\\ &\leq C_g\exp(-\lambda^+(\ell)(r-3)n).\end{aligned}$$

Finally, we estimate Δ^q_n for $2 \leq q \leq r$. From the formula (9), we can see

$$\Delta^q_m \leq \frac{|D^*F(DF^{m-1}(\mathbf{j}^{(1)}))|}{D_*F(DF^{m-1}(\mathbf{j}^{(1)}))^q}\Delta^q_{m-1} + C_g\sum_{0\leq d<q}\Delta^d_{m-1}. \tag{92}$$

Consider this inequality (92) for $m = n$ and estimate the right hand side by using (92) recurrently as long as there exist terms Δ^q_m with $q > 1$ or $m > 0$ on the right hand side. Then we see that Δ^q_n is bounded by

$$\begin{aligned}&\frac{|D^*F^n(\mathbf{j}^{(1)})|}{D_*F^n(\mathbf{j}^{(1)})^q}\Delta^q_0\\ &+ C_g\sum_{1<d<q}\ \sum_{\substack{0=n_d\leq n_{d+1}\leq\cdots\\ \cdots\leq n_q<n_{q+1}=n+1}}\ \prod_{\ell=d}^{q}\ \prod_{n_\ell\leq j<n_{\ell+1}-1}\frac{|D^*F(F^j(\mathbf{j})^{(1)})|}{D_*F(F^j(\mathbf{j})^{(1)})^\ell}\Delta^d_0\\ &+ C_g\sum_{\substack{d=0,1\\ 0\leq m<n}}\ \sum_{\substack{m=n_d\leq n_{d+1}\leq\cdots\\ \cdots\leq n_q<n_{q+1}=n+1}}\ \prod_{\ell=d}^{q}\ \prod_{n_\ell\leq j<n_{\ell+1}-1}\frac{|D^*F(F^j(\mathbf{j})^{(1)})|}{D_*F(F^j(\mathbf{j})^{(1)})^\ell}\Delta^d_0.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\Delta^q_n \leq &\frac{\exp(-\lambda_g n + c_g)}{D_* F^n(\mathbf{j}^{(1)})^{q-1}}\Delta^q_0 + C_g n^q \sum_{1<d<q} \frac{\exp(-\lambda_g n + c_g)}{D_* F^n(\mathbf{j}^{(1)})^{d-1}}\Delta^d_0 \\ &+ C_g \sum_{0\leq m<n} (n-m)^q \exp(-\lambda_g(n-m) + c_g)(\Delta^0_m + \Delta^1_m) \\ \leq &C_g \max_{0\leq m<n}(\Delta^0_m + \Delta^1_m) + C_g \sum_{1<d\leq q} \exp(-(d-1)\lambda^-(\ell)n)\Delta^d_0,\end{aligned}$$

where the second inequality follows from the fact that the jet $\mathbf{j}$ belongs to $V_0(n,\ell;F)$. Using the estimates on Δ^0_m and Δ^1_m above and the condition (I3) in this inequality, we can conclude

$$\Delta^q_n < C_g \exp(-\lambda^+(\ell)(r-3)n) \quad \text{for } 2 \leq q \leq r-2.$$

We have proved the inequality (87). The jet $\mathbf{i}$ belongs to $V_1(n,\ell;F)$ because

$$\log \frac{D_* F^n(\mathbf{e}^u(\mathbf{i}^{(0)}))}{D_* F^n(\mathbf{e}^u(\mathbf{j}^{(0)}))} \leq C_g \sum_{m=0}^{n-1} (\Delta^0_m + \Delta^1_m) \leq C_g n \exp(-\lambda^+(\ell)(r-3)n) < 1,$$

provided that n is larger than some constant C_g. □

For integers $n \geq 1$, $1 \leq \ell \leq \ell_0$ and a jet $\mathbf{j} \in \mathbf{Q}(n,\ell)$, let $\mathcal{S}_2(n,\ell,\mathbf{j})$ be the set of mappings $F \in \mathcal{U}$ such that $\mathbf{j} \in V_1(n,\ell;F)$ and that

$$d_{\mathbb{J}}(F^n(\mathbf{j}), \mathbf{C}(F)) \leq 2B_g \exp(-\lambda^+(\ell)n(r-3)).$$

Then the last lemma implies

Corollary 8.5. *If there exists $n \geq B_g$ such that $F \notin \mathcal{S}_2(n,\ell,\mathbf{j})$ for all $1 \leq \ell \leq \ell_0$ and $\mathbf{j} \in \mathbf{Q}(n,\ell)$, then F satisfies the no flat contact condition.*

In the remaining part of this section, we shall estimate the measure of the subsets $\mathcal{S}_2(n,\ell,\mathbf{j})$ for $\mathbf{j} \in \mathbf{Q}(n,\ell)$ by using lemma 3.18.

8.3. **Perturbations.** In this subsection, we introduce some families of mappings and give a few estimates on the variation of the images of jets under the iterates of mappings in the families. In the argument below, we fix $1 \leq \ell \leq \ell_0$ and put $\delta_n = \exp(-\lambda^+(\ell)n/\nu)$ for $n \geq 1$.

For $1 \leq q \leq r-2$, we take and fix a C^∞function $\psi_q : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\psi_q(x,y) = \begin{cases} x^q/q!, & \text{for } (x,y) \in \mathbf{B}(0,1/10); \\ 0, & \text{for } (x,y) \notin \mathbf{B}(0,1). \end{cases}$$

For each point $\zeta \in M$, we consider an isometric embedding

$$\varphi_\zeta : \{w \in \mathbb{R}^2 \mid \|w\| < 1\} \to \mathbb{T}$$

that carries the origin to ζ and the x-axis $\mathbb{R} \times \{0\}$ to $\mathbf{E}^u(\zeta)$.

Recall that we took the subset $\mathcal{U}$ of mappings as a neighborhood of a C^rmapping $F_\sharp$ in subsection 3.2. For positive integers n, $1 \leq q \leq r-2$ and a point ζ in M, we define the C^∞ mapping $\psi_{q,n,\zeta} : M \to \mathbb{R}^2$ by

$$\psi_{q,n,\zeta}(z) = \begin{cases} \delta^s_n \cdot \psi_q(\varphi^{-1}_\zeta(z)/\delta_n) \cdot \mathbf{e}^c(F_\sharp(\zeta)) & \text{if } d(z,\zeta) < \delta_n; \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbf{e}^c(\cdot)$ is either of the two unit tangent vectors in the central subspace $\mathbf{E}^c(\cdot)$. Note that, for any mapping $F \in \mathcal{U}$, the parallel translation of the vector $\mathbf{e}^c(F_\sharp(z))$ to $F(z)$ is contained in $\mathbf{S}^c(F(z))$ from the choice of $\mathcal{U}$ in subsection 3.2.

For a positive integer n, a mapping $F \in \mathcal{U}$ and a point ζ in M, we define

$$F_{\mathbf{t}}(z) = F(z) + \sum_{q=\nu+1}^{r-2} t_q \psi_{q,n,\zeta}(z) : M \to \mathbb{T} \tag{93}$$

where $\mathbf{t} = (t_{\nu+1}, t_{\nu+2}, \cdots, t_{r-2})$ is the parameter that ranges over $R = [-1,1]^{r-2-\nu}$. This is the family of mappings that we are going to consider. Obviously we have

$$d_{C^q}(F_{\mathbf{t}}, F) \le C_g \delta_n^{s-q} \qquad \text{and} \qquad \|\partial_{\mathbf{t}} F_{\mathbf{t}}\|_{C^q} \le C_g \delta_n^{s-q} \tag{94}$$

for $0 \le q \le r$ and $\mathbf{t} \in R$. Especially, $F_{\mathbf{t}}(M) \subset M$ if n is sufficiently large.

We consider a jet $\mathbf{j} \in \mathbf{Q}(n,\ell) \cap V_1(n,\ell;F)$ and give some estimates on the variation of the image $F_{\mathbf{t}}^n(\mathbf{j})$. We begin with the estimate on the position $F_{\mathbf{t}}^n(\mathbf{j})^{(0)}$.

Lemma 8.6. *We have, for $0 \le m \le n$ and $\mathbf{t} \in R$,*

$$d(F_{\mathbf{t}}^m(\mathbf{j}^{(0)}), F^m(\mathbf{j}^{(0)})) < C_g \|DF_{\mathbf{j}^{(0)}}^m\| \delta_n^s \le C_g \delta_n^{s-\nu}$$

and

$$\|\partial_{\mathbf{t}} F_{\mathbf{t}}^m(\mathbf{j}^{(0)})\| < C_g \|DF_{\mathbf{j}^{(0)}}^m\| \delta_n^s \le C_g \delta_n^{s-\nu}$$

provided that n is larger than some constant C_g.

Proof. The following argument is a modification of that in the proof of lemma 8.4. We denote $z(m) = F^m(\mathbf{j}^{(0)})$, $w(m) = F_{\mathbf{t}}^m(\mathbf{j}^{(0)})$ and $\Delta_m = d(z(m), w(m))$ for $0 \le m \le n$, so $\Delta_0 = 0$. Using the simple estimate

$$\left\| \exp_{z(m)}^{-1}(w(m)) - (DF)_{z(m-1)}(\exp_{z(m-1)}^{-1}(w(m-1))) \right\| \le C_g(\delta_n^s + (\Delta_{m-1})^2)$$

repeatedly, we can obtain

$$\Delta_m \le \sum_{k=0}^{m-1} \|(DF^{m-k-1})_{z(k+1)}\| \cdot C_g(\delta_n^s + (\Delta_k)^2) \tag{95}$$

for $0 \le m \le n$. Consider an integer $0 \le m_0 \le n$ and a positive number $K > 0$ and suppose that we have

$$\Delta_m < K \|(DF^m)_{z(0)}\| \cdot \delta_n^s \tag{96}$$

for $0 \le m < m_0$. Note that we have, for $0 \le k \le m \le n$,

$$\|DF_{z(k+1)}^{m-k-1}\| \le C_g \frac{D_* F^m(\mathbf{j}^{(1)})}{D_* F^{k+1}(\mathbf{j}^{(1)})} \le C_g \|DF_{z(0)}^m\| \exp(-\lambda_g k)$$

and also $\|DF_{z(0)}^k\| \le C_g \|DF_{z(0)}^n\| \le C_g \delta_n^{-\nu}$. Thus, using (96) together with these estimates and (91) in the right hand side of (95) for $m = m_0$, we see

$$\Delta_{m_0} \le C_g \|(DF^{m_0})_{z(0)}\| \cdot \delta_n^s \cdot \left(\sum_{k=0}^{m-1} \left(\exp(-\lambda_g k) + K^2 \delta_n^{s-\nu} \right) \right).$$

This gives (96) for $m = m_0$, provided K and n are larger than some constant C_g. Thus we can obtain the first claim of the lemma by induction on m.

Put $\Delta'_m = \partial_{\mathbf{t}} F_{\mathbf{t}}^m(\mathbf{j}^{(0)})$ for $0 \le m \le n$. Using the simple inequality

$$\|\Delta'_m - (DF)_{z(m-1)} \Delta'_{m-1}\| \le C_g(\delta_n^s + \Delta_{m-1} \|\Delta'_{m-1}\|)$$

repeatedly, we can obtain

$$\Delta'_m \leq \sum_{k=0}^{m-1} \|(DF^{m-k-1})_{z(k+1)}\| \cdot C_g(\delta_n^s + \Delta_k \|\Delta'_k\|).$$

From this and the estimates on Δ_m we proved above, we can obtain the second claim of the lemma by induction on m, in a similar manner as above. □

Next we give the estimates on $\partial_{\mathbf{t}} F_{\mathbf{t}}^n(\mathbf{j})^{(q)}$ for $1 \leq q \leq r-2$. We denote by ∂_p the differentiation by the parameter t_p. For integers p and q satisfying $\nu+1 \leq p \leq r-2$ and $1 \leq q \leq r-2$, and for a jet $\mathbf{i} \in \mathbb{J}^{r-2}\mathcal{AC}$ and $\mathbf{t} \in R$, we define

$$\beta_p^{(q)}(\mathbf{i}, \mathbf{t}) = \pm \frac{\sin(\angle(\mathbf{e}^c(F_\sharp(z)), F_{\mathbf{t}}(\mathbf{i}^{(1)})) \cdot \|\partial_p((D^q F_{\mathbf{t}})_{\mathbf{i}^{(0)}}(\mathbf{i}^{(1)}, \mathbf{i}^{(1)}, \cdots, \mathbf{i}^{(1)}))\|}{D_* F_{\mathbf{t}}(\mathbf{i}^{(1)})^q}$$

where $(D^q F_{\mathbf{t}})_z : \otimes^q T_z M \to T_{F(z)} M$ is the q-th differential of $F_{\mathbf{t}}$ at z and the sign on the right hand side will be chosen appropriately in the argument below. Then

$$|\beta_p^{(q)}(\mathbf{i}, \mathbf{t})| \leq C_g \delta_n^{s-q}. \tag{97}$$

Lemma 8.7. *There exists a positive constant C_g such that, if $n \geq C_g$, it holds*

$$\left| \partial_p(F_{\mathbf{t}}^m(\mathbf{j})^{(q)}) - \sum_{k=0}^{m-1} \frac{D^* F_{\mathbf{t}}^{m-k-1}(F_{\mathbf{t}}^{k+1}(\mathbf{j})^{(1)})}{D_* F_{\mathbf{t}}^{m-k-1}(F_{\mathbf{t}}^{k+1}(\mathbf{j})^{(1)})^q} \cdot \beta_p^{(q)}(F_{\mathbf{t}}^k(\mathbf{j}), \mathbf{t}) \right| < C_g \delta_n^{s-q+1}$$

for any $\nu+1 \leq q \leq r-2$, $\nu+1 \leq p \leq r-2$, $\mathbf{t} \in R$ and $0 \leq m \leq n$.

Proof. Let $\nu+1 \leq p \leq r-2$. For $0 \leq q \leq r-2$ and $0 \leq m \leq n$, we put

$$\Delta_m^{(q)} = \begin{cases} \|\partial_p F_{\mathbf{t}}^m(\mathbf{j})^{(0)}\|, & \text{if } q = 0; \\ \partial_p(\angle(F_{\mathbf{t}}^m(\mathbf{j})^{(1)}, v_0)), & \text{if } q = 1; \\ \partial_p(F_{\mathbf{t}}^m(\mathbf{j})^{(q)}), & \text{if } q \geq 2 \end{cases}$$

where v_0 is some fixed vector. For $\Delta_m^{(1)}$, it holds

$$\left| \Delta_m^{(1)} - \frac{D^* F_{\mathbf{t}}(F_{\mathbf{t}}^{m-1}(\mathbf{j})^{(1)})}{D_* F_{\mathbf{t}}(F_{\mathbf{t}}^{m-1}(\mathbf{j})^{(1)})} \Delta_{m-1}^{(1)} - \beta_p^{(1)}(F_{\mathbf{t}}^{m-1}(\mathbf{j}), \mathbf{t}) \right| \leq C_g \Delta_{m-1}^{(0)} \leq C_g \delta_n^{s-\nu}$$

where the second inequality follows from lemma 8.6. From this inequality and the estimate (97) for $q = 1$, we can see

$$|\Delta_m^{(1)}| \leq C_g \sum_{k=0}^{m-1} \frac{|D^* F_{\mathbf{t}}^{m-k}(F_{\mathbf{t}}^k(\mathbf{j})^{(1)})|}{D_* F_{\mathbf{t}}^{m-k}(F_{\mathbf{t}}^k(\mathbf{j})^{(1)})} \left(\beta_p^{(1)}(F_{\mathbf{t}}^k(\mathbf{j}), \mathbf{t}) + \delta_n^{s-\nu} \right) < C_g \delta_n^{s-\nu}$$

for $0 \leq m \leq n$. Recall the formula (9) and the remark after it. By differentiating the both sides of the formula (9) with F replaced by $F_{\mathbf{t}}$ and using (94), we can get

(98)

$$\left| \Delta_m^{(q)} - \frac{D^* F_{\mathbf{t}}(F_{\mathbf{t}}^{m-1}(\mathbf{j})^{(1)})}{D_* F_{\mathbf{t}}(F_{\mathbf{t}}^{m-1}(\mathbf{j})^{(1)})^q} \Delta_{m-1}^{(q)} - \beta_p^{(q)}(F_{\mathbf{t}}^m(\mathbf{j}), \mathbf{t})) \right| \leq C_g \delta_n^{s-q+1} + C_g \sum_{0 \leq d < q} \Delta_{m-1}^{(d)}$$

for $2 \le q \le r-2$ and $0 \le m \le n$. This and (97) imply

$$|\Delta_m^{(q)}| \le C_g \sum_{k=1}^{m} \left(\frac{|D^* F_{\mathbf{t}}^{m-k}(F_{\mathbf{t}}^k(\mathbf{j})^{(1)})|}{D_* F_{\mathbf{t}}^{m-k}(F_{\mathbf{t}}^k(\mathbf{j})^{(1)})^q} \left(\delta_n^{s-q} + \sum_{0\le d<q} \Delta_{k-1}^{(d)} \right) \right)$$
$$\le C_g \left(\delta_n^{s-q} + \max_{0\le d<q} \max_{0<k<m} \Delta_k^{(d)} \right).$$

Hence, we can show the estimate $|\Delta_m^{(q)}| \le C_g \delta_n^{s-\nu}$ for $2 \le q \le \nu$ and $0 \le m \le n$ by induction on q, using lemma 8.6 and the estimate on $|\Delta_m^{(1)}|$ above. Using the inequality (98) repeatedly, we can see that the left hand side of the inequality in the lemma is bounded by

$$C_g \sum_{k=1}^{m} \frac{|D^* F_{\mathbf{t}}^{m-k}(F_{\mathbf{t}}^k(\mathbf{j})^{(1)})|}{D_* F_{\mathbf{t}}^{m-k}(F_{\mathbf{t}}^k(\mathbf{j})^{(1)})^q} \left(\delta_n^{s-q+1} + \sum_{0\le d<q} \Delta_{k-1}^{(d)} \right).$$

Hence, by induction on $\nu+1 \le q \le r-2$, we obtain the inequality in the lemma. □

Note that, for any $F \in \mathcal{U}$, the level curves of the function $\det F : z \mapsto \det DF_z$ are regular in the neighborhood $\mathbf{B}(\mathcal{C}(F), \rho_g)$ of the critical set $\mathcal{C}(F)$, from the choice of the constant ρ_g in subsection 3.2. For a point $w \in \mathbf{B}(\mathcal{C}(F), \rho_g)$, we denote by $\mathbf{c}(w; F)$ the $(r-2)$-jet at w that is given by the level curve passing through w.

Suppose that a jet $\mathbf{j} \in \mathbb{J}^{r-2}\mathcal{AC}$ satisfies, for all $\mathbf{t} \in R$,

(V1) $d(F_{\mathbf{t}}^{n-1}(\mathbf{j})^{(0)}, \zeta) < \delta_n/10$,
(V2) $d(F_{\mathbf{t}}^{n-1}(\mathbf{j})^{(0)}, \mathcal{C}(F_{\mathbf{t}})) > 3\delta_n$, and
(V3) $d(F_{\mathbf{t}}^{n}(\mathbf{j})^{(0)}, \mathcal{C}(F_{\mathbf{t}})) < \delta_n$.

Then, from the condition (V3), we can define the mapping $\Psi : R \to \mathbb{R}^{r-\nu-2}$ by

$$\Psi(\mathbf{t}) = \left(\frac{F_{\mathbf{t}}^n(\mathbf{j})^{(q)} - \mathbf{c}(F_{\mathbf{t}}^n(\mathbf{j})^{(0)}; F_{\mathbf{t}})^{(q)}}{\delta_n^{s-q}} \right)_{q=\nu+1}^{r-2}$$

provided that n is so large that $\delta_n < \rho_g$. The following is the goal of this subsection.

Lemma 8.8. *If the conditions (V1),(V2) and (V3) hold for all $\mathbf{t} \in R$, the mapping Ψ is a diffeomorphism and $|\det D\Psi(\mathbf{t})|$ is bounded from below by a constant C_g^{-1}, provided that n is larger than some constant C_g.*

Proof. From the condition (V1) and the definition of the family $F_{\mathbf{t}}$, we have

$$\beta_p^{(q)}(F_{\mathbf{t}}^{n-1}(\mathbf{j}), \mathbf{t}) = 0 \qquad \text{for } q > p, \text{ and}$$
$$|\beta_p^{(q)}(F_{\mathbf{t}}^{n-1}(\mathbf{j}), \mathbf{t})| \ge C_g^{-1} \delta_n^{s-q} \qquad \text{for } q = p,$$

in addition to (97). We show

$$\left| \sum_{k=0}^{n-2} \frac{D^* F_{\mathbf{t}}^{n-k-1}(F_{\mathbf{t}}^{k+1}(\mathbf{j})^{(1)})}{D_* F_{\mathbf{t}}^{n-k-1}(F_{\mathbf{t}}^{k+1}(\mathbf{j})^{(1)})^q} \beta_p^{(q)}(F_{\mathbf{t}}^k(\mathbf{j}), \mathbf{t}) \right| < C_g \delta_n^{s-q+1}. \tag{99}$$

Suppose that $\beta_p^{(q)}(F_{\mathbf{t}}^k(\mathbf{j}), \mathbf{t}) \ne 0$ for some integer $0 \le k \le m-2$. Then we have $d(F_{\mathbf{t}}^k(\mathbf{j})^{(0)}, \zeta) < \delta_n$ and

$$d(F_{\mathbf{t}}^{k+1}(\mathbf{j})^{(0)}, \mathcal{C}(F_{\mathbf{t}}))$$
$$\le d(F_{\mathbf{t}}^{k+1}(\mathbf{j})^{(0)}, F_{\mathbf{t}}(\zeta)) + d(F_{\mathbf{t}}(\zeta), F_{\mathbf{t}}^n(\mathbf{j})^{(0)}) + d(F_{\mathbf{t}}^n(\mathbf{j})^{(0)}, \mathcal{C}(F_{\mathbf{t}})) < C_g \delta_n$$

from (V1) and (V3). So we have $|D^*F_{\mathbf{t}}(F_{\mathbf{t}}^{k+1}(\mathbf{j})^{(1)})| < C_g\delta_n$ from (4). Hence

$$\left|\frac{D^*F_{\mathbf{t}}^{m-k-1}(F_{\mathbf{t}}^{k+1}(\mathbf{j})^{(1)})}{D_*F_{\mathbf{t}}^{m-k-1}(F_{\mathbf{t}}^{k+1}(\mathbf{j})^{(1)})^q}\beta_p^{(q)}(F_{\mathbf{t}}^k(\mathbf{j}),\mathbf{t})\right| \le C_g\delta_n^{s-q+1}\exp(-\lambda_g(m-k-1)+2c_g).$$

This implies (99).

The jet $\mathbf{c}(w;F_{\mathbf{t}})$ for $w \in \mathbf{B}(\mathcal{C}(F),\delta_n)$ does not depend on the parameter $\mathbf{t} \in R$ because $\mathbf{B}(\zeta,\delta_n) \cap \mathbf{B}(\mathcal{C}(F),\delta_n) = \emptyset$ from (V1) and (V2). So we have

$$\|\partial_p(\mathbf{c}(F_{\mathbf{t}}^n(\mathbf{j})^{(0)};F_{\mathbf{t}})^{(q)})\| < C_g\|\partial_p(F_{\mathbf{t}}^n(\mathbf{j})^{(0)})\| < C_g\delta_n^{s-\nu} \quad \text{for } \nu+1 \le p,q \le r-2$$

from lemma 8.6. From (99) and lemma 8.7, it follows

$$\left|\partial_p\left(F_{\mathbf{t}}^m(\mathbf{j})^{(q)} - \mathbf{c}(F_{\mathbf{t}}^n(\mathbf{j})^{(0)};F_{\mathbf{t}})^{(q)}\right) - \beta_p^{(q)}(F_{\mathbf{t}}^{m-1}(\mathbf{j}),\mathbf{t})\right| < C_g\delta_n^{s-q+1}.$$

Denote by $D\Psi(\mathbf{t})_{q,p}$ the (q,p)-entree of the representation matrix of $D\Psi(\mathbf{t})$ with respect to the standard basis of $\mathbb{R}^{r-2-\nu}$. Then, from the estimates above, we have

$$\begin{aligned} |D\Psi(\mathbf{t})_{q,p}| &< C_g\delta_n && \text{if } q > p, \\ |D\Psi(\mathbf{t})_{q,p}| &< C_g && \text{if } q \le p, \text{ and} \\ |D\Psi(\mathbf{t})_{q,p}| &> C_g^{-1} && \text{if } q = p. \end{aligned}$$

Now we can conclude the lemma by an elementary argument. □

8.4. **Resolution of the flat contacts.** In this subsection, we prove theorem 3.21. Until the last part of the proof, we fix $1 \le \ell \le \ell_0$ and put $\delta_n = \exp(-\lambda^+(\ell)n/\nu)$ for $n \ge 1$ as in the last subsection. Let n be a large integer, ζ a point in the lattice $\mathbb{L}(\delta_n/20)$ and $\mathbf{j}$ a jet in $\mathbf{Q}(n,\ell)$. We denote, by $Y_0(n,\ell,\mathbf{j},\zeta)$ (resp. $Y_1(n,\ell,\mathbf{j},\zeta)$), the set of mappings $F \in C^r(M,M)$ that satisfy

$$F^{n-1}(\mathbf{j})^{(0)} \in \mathbf{B}(\zeta,\delta_n/20) \qquad (\text{ resp. } F^{n-1}(\mathbf{j})^{(0)} \in \mathbf{B}(\zeta,\delta_n/5)). \tag{100}$$

Below we estimate

$$\mathcal{M}_s\left(\Phi_G^{-1}(\mathcal{S}_2(n,\ell,\mathbf{j}) \cap Y_0(n,\ell,\mathbf{j},\zeta)) \cap \mathbf{D}^r(d)\right) \qquad \text{for } G \in C^r(M,\mathbb{T}) \text{ and } d > 0,$$

where Φ_G and $\mathbf{D}^r(d)$ are those defined by (1) and (20) respectively.

Take a mapping F in $\mathcal{S}_2(n,\ell,\mathbf{j}) \cap Y_0(n,\ell,\mathbf{j},\zeta)$ arbitrarily and consider the family $F_{\mathbf{t}}$ defined by (93) in the last subsection. Note that the jet $\mathbf{j}$ belongs to $V_1(n,\ell;F)$ from the definition of $\mathcal{S}_2(n,\ell,\mathbf{j})$. We check that the conditions (V1), (V2) and (V3) hold for $\mathbf{t} \in R$ provided that n is larger than some constant C_g. Since F belongs to $\mathcal{S}_2(n,\ell,\mathbf{j})$, there exists a point $w_0 \in \mathcal{C}(F)$ such that

$$d_{\mathbb{J}}(F^n(\mathbf{j}),\mathbf{c}(w_0;F)) \le 2B_g\delta_n^{(r-3)\nu}. \tag{101}$$

Especially we have $d(F^n(\mathbf{j})^{(0)},w_0) < \rho_g$ and $\angle(F^n(\mathbf{j})^{(1)},\mathbf{c}(w_0;F)^{(1)}) < \rho_g$, provided that n is larger than some constant C_g. It follows from the condition (C5) in the choice of the constant ρ_g in subsection 3.2 that

$$d(F^{n-1}(\mathbf{j})^{(0)},\mathcal{C}(F)) > \rho_g. \tag{102}$$

Using (100), we can see

$$d(\zeta,\mathcal{C}(F)) \ge d(F^{n-1}(\mathbf{j})^{(0)},\mathcal{C}(F)) - d(F^{n-1}(\mathbf{j})^{(0)},\zeta) > \rho_g - 2B_g\delta_n^{(r-3)\nu} > 4\delta_n$$

provided that n is larger than some constant C_g. This implies that the critical set $\mathcal{C}(F_{\mathbf{t}})$ does not depend on $\mathbf{t} \in R$. Hence (V1), (V2) and (V3) follow from (100), (101), (102) and lemma 8.6, provided that n is larger than some constant C_g.

Let $\Psi : R \to \mathbb{R}^{r-\nu-2}$ be the mapping that we defined in the last subsection. Note that the conclusion of lemma 8.8 holds for this Ψ. Suppose that $F_{\mathbf{s}}$ belongs to $\mathcal{S}_2(n,\ell,\mathbf{j}) \cap Y_0(n,\ell,\mathbf{j},\zeta)$ for a parameter $\mathbf{s} \in R$. Then, by definition, there exists a point $w_1 \in \mathcal{C}(F)$ such that $d_{\mathbb{J}}(F_{\mathbf{s}}^n(\mathbf{j}), \mathbf{c}(w_1; F_{\mathbf{s}})) < 2B_g \exp(-\lambda^+(\ell)n(r-3))$. Since $\mathbf{c}(\cdot; F_{\mathbf{s}}) = \mathbf{c}(\cdot; F_{\mathbf{s}}) : \mathbf{B}(\mathcal{C}(F), \rho_g) \to \mathbb{J}^{r-2}\Gamma$ is a C^1mapping whose first-order differentials are bounded by some constant C_g, it follows

$$d_{\mathbb{J}}(F_{\mathbf{s}}^n(\mathbf{j}), \mathbf{c}(F_{\mathbf{s}}^n \mathbf{j}^{(0)}; F_{\mathbf{s}})) \le d_{\mathbb{J}}(F_{\mathbf{s}}^n(\mathbf{j}), \mathbf{c}(w_1; F_{\mathbf{s}})) + d_{\mathbb{J}}(\mathbf{c}(w_1; F_{\mathbf{s}}), \mathbf{c}(F_{\mathbf{s}}^n(\mathbf{j})^{(0)}; F_{\mathbf{s}}))$$
$$< (1+C_g) d_{\mathbb{J}}(F_{\mathbf{s}}^n(\mathbf{j}), \mathbf{c}(w_1; F_{\mathbf{s}})) < C_g \delta_n^{\nu(r-3)}.$$

Hence the image $\Psi(\mathbf{s})$ is contained in

$$\prod_{q=\nu+1}^{r-2} \left[-C_g \delta_n^{\nu(r-3)-(s-q)}, C_g \delta_n^{\nu(r-3)-(s-q)} \right] \subset \mathbb{R}^{r-\nu-2}.$$

We arrive at the estimate

$$\mathbf{m}_{\mathbb{R}^{r-\nu-2}} \{ \mathbf{t} \in R \mid F_{\mathbf{t}} \in \mathcal{S}_2(n,\ell,\mathbf{j}) \cap Y_0(n,\ell,\mathbf{j},\zeta) \} \le C_g \prod_{q=\nu+1}^{r-2} \delta_n^{\nu(r-3)-(s-q)},$$

which holds uniformly for $F \in \mathcal{S}_2(n,\ell,\mathbf{j}) \cap Y_0(n,\ell,\mathbf{j},\zeta)$, provided that n is larger than some constant C_g.

Now we apply lemma 3.18. Fix a small number $0 < T < 1$ such that

$$\max_{|t_q| \le T} \left\| \sum_{q=\nu+1}^{r-2} t_q \psi_{q,n,\zeta} \right\|_{C^s} \le r \cdot \max_{\nu \le q \le r-2} \|\psi_q\|_{C^s} \cdot T < \rho_s(d)$$

where $\rho_s(d)$ is that in lemma 3.16. Note that we can take T independent of n. Put $X = \mathcal{S}_2(n,\ell,\mathbf{j}) \cap Y_0(n,\ell,\mathbf{j},\zeta)$ and $T_i = T$ in lemma 3.18. Then the assumption (21) holds from the choice of T, and the subset Y in the statement of lemma 3.18 is contained in $Y_1(n,\ell,\mathbf{j},\zeta)$ from the condition (V1) which we have proved above. Therefore we can obtain, as the conclusion,

$$\frac{\mathcal{M}_s(\Phi_G^{-1}(\mathcal{S}_2(n,\ell,\mathbf{j}) \cap Y_0(n,\ell,\mathbf{j},\zeta)) \cap \mathbf{D}^r(d)))}{\mathcal{M}_s(\Phi_G^{-1}(Y_1(n,\ell,\mathbf{j},\zeta)))} \le C_g T^{-r+\nu+2} \prod_{q=\nu+1}^{r-2} \delta_n^{\nu(r-3)-(s-q)}$$

provided that n is larger than some constant C_g. The the subsets $Y_0(n,\ell,\mathbf{j},\zeta)$ for $\zeta \in \mathbb{L}(\delta_n/20)$ cover $C^r(M,M)$ while the intersection multiplicity of the subsets $Y_1(n,\ell,\mathbf{j},\zeta)$ for $\zeta \in \mathbb{L}(\delta_n/20)$ is bounded by some absolute constant. Hence we can conclude that the measure $\mathcal{M}_s(\Phi_G^{-1}(\mathcal{S}_2(n,\ell,\mathbf{j})) \cap \mathbf{D}^r(d))$ is bounded by

$$C_g T^{-r+\nu+2} \prod_{q=\nu+1}^{r-2} \delta_n^{\nu(r-3)-(s-q)}$$
$$= C_g T^{-r+\nu+2} \exp\left((r-\nu-2) \left(-(r-3) + \frac{2s-r-\nu+1}{2\nu} \right) \lambda^+(\ell) n \right).$$

The subset $\mathcal{S}_2$ is contained in the closed subset

$$\mathcal{S}_2' := \bigcap_{n \ge B_g} \bigcup_{\ell=1}^{\ell_0} \bigcup_{\mathbf{j} \in \mathbf{Q}(n,\ell)} \mathcal{S}_2(n,\ell,\mathbf{j})$$

from corollary 8.5. Hence the measure $\mathcal{M}_s(\Phi_G^{-1}(\mathcal{S}_2') \cap \mathbf{D}^r(d))$ is bounded by

$$C_g T^{-r+\nu+2} \sum_{\ell=1}^{\ell_0} \#\mathbf{Q}(n,\ell) \cdot \exp\left((r-\nu-2)\left(-(r-3) + \frac{2s-r-\nu+1}{2\nu}\right)\lambda^+(\ell)n\right)$$

for sufficiently large n. From the estimate (86) on the cardinality of $\mathbf{Q}(n,\ell)$ and the condition in the choice of $\lambda^{\pm}(\ell)$, this converges to 0 exponentially fast as $n \to \infty$. Thus we conclude $\mathcal{M}_s(\Phi_G^{-1}(\mathcal{S}_2) \cap \mathbf{D}^r(d)) = \mathcal{M}_s(\Phi_G^{-1}(\mathcal{S}_2') \cap \mathbf{D}^r(d)) = 0$. As d is an arbitrary positive number, $\mathcal{M}_s(\Phi_G^{-1}(\mathcal{S}_2)) = 0$ or $\mathcal{S}_2$ is shy with respect to $\mathcal{M}_s$.

Suppose that $r \geq 19$. Then the inequality (2) holds for $s = r+3$ and $\nu = 3$ and so $\mathcal{M}_{r+3}(\Phi_G^{-1}(\mathcal{S}_2')) = 0$ for any $G \in C^r(M, \mathbb{T})$. This implies that $\mathcal{U} \setminus \mathcal{S}_2'$ is dense. Therefore $\mathcal{S}_2$ is contained in the closed nowhere dense subset $\mathcal{S}_2'$.

Appendix A. Proof of corollary 2.3

To see that corollary 2.3 follows from theorem 2.2, it is enough to show

Lemma A.1. *If X is a Borel subset in $C^r(M, \mathbb{T}^2)$ that is shy with respect to any measure in $\mathcal{Q}_s^r$ for some $s > r$, then the subset*

$$Y = \left\{F(z,t) \in C^r(M \times [-1,1]^k, \mathbb{T}) \mid \mathbf{m}_{\mathbb{R}^k}(\{t \in [-1,1]^k \mid F(\cdot,t) \in X\}) > 0\right\},$$

is shy with respect to any Borel finite measure on $C^r(M \times [-1,1]^k, \mathbb{R}^2)$ that is quasi-invariant along $C^s(M \times [-1,1]^k, \mathbb{R}^2)$.

Proof. Take a mapping $G \in C^r(M \times [-1,1]^k, \mathbb{T})$ and put $G_0(z) = G(z,0)$. We define the mapping

$$P(f,\mathbf{t}) := G(\cdot,\mathbf{t}) - G_0(\cdot) + f(\cdot,\mathbf{t}) : C^r(M \times [-1,1]^k, \mathbb{R}^2) \times [-1,1]^k \to C^r(M, \mathbb{R}^2),$$

so that $\Phi_{G_0} \circ P(f,\mathbf{t}) = G(\cdot,\mathbf{t}) + f(\cdot,\mathbf{t})$. Let $\mathcal{N}$ be a Borel finite measure on $C^r(M \times [-1,1]^k, \mathbb{R}^2)$ that is quasi-invariant along $C^s(M \times [-1,1]^k, \mathbb{R}^2)$. Then the measure $(\mathcal{N} \times \mathbf{m}_{\mathbb{R}^k}|_{[-1,1]^k}) \circ P^{-1}$ on $C^r(M, \mathbb{R}^2)$ belongs to $\mathcal{Q}_s^r$ and so we have $(\mathcal{N} \times \mathbf{m}_{\mathbb{R}^k})((\Phi_{G_0} \circ P)^{-1}(X)) = 0$ from the assumption. This and Fubini theorem imply $\mathcal{N} \circ \Phi_G^{-1}(Y) = 0$ and hence the claim of the lemma. □

Appendix B. Proof of lemma 3.16

We use the definitions and results in the book[18] by Skorohod. We consider the functions $e_{nm}(x,y) = \exp\left(2\pi\sqrt{-1}(nx+my)\right)$ for $n, m \in \mathbb{Z}$, as a complete orthonormal basis of the space $L^2(\mathbb{T}, \mathbf{m})$. Let $A : L^2(\mathbb{T}, \mathbf{m}) \to L^2(\mathbb{T}, \mathbf{m})$ be the operator defined by

$$A\left(\sum_{(n,m)\in\mathbb{Z}^2} a_{nm} e_{nm}\right) = \sum_{(n,m)\in\mathbb{Z}^2} (n^2+m^2+1)^{-1/2} a_{nm} e_{nm}.$$

Let $\mathcal{N}$ be the Gaussian measure[18, §5] on $L^2(\mathbb{T}, \mathbf{m})$ whose characteristic function is $\Theta(\psi) = \exp(-(1/2)(A^{2s-3}\psi, \psi)_{L^2})$. Then $\mathcal{N}$ is supported on the Sobolev space $W^{s-3} := A^{s-3}(L^2(\mathbb{T}, \mathbf{m}))$. We can see, from [18, §16 theorem 2], that $\mathcal{N}$ is quasi-invariant along $W^{s-(3/2)} \supset C^{s-1}(\mathbb{T}, \mathbb{R})$ and it holds

$$\begin{aligned}\frac{d(\mathcal{N} \circ \tau_\psi^{-1})}{d\mathcal{N}}(\varphi) &= \exp\left((A^{-s}\psi, A^{-s+3}\varphi)_{L^2} - (1/2)\left\|A^{-s+(3/2)}\psi\right\|_{L^2}^2\right) \\ &\leq \exp(\|\psi\|_{W^s} \cdot \|\varphi\|_{W^{s-3}})\end{aligned}$$

for $\psi \in W^s$ and $\mathcal{N}$-a.e. $\varphi \in W^{s-3}$.

We show that the measure $\mathcal{N}$ is actually supported on $C^{s-3}(\mathbb{T}, \mathbb{R})$. We follow the argument in the proof of the fact that the measure corresponding to Brownian motion is supported on the space of continuous paths[10]. Let $\varphi^{(s-3)}$ be one of the $(s-3)$-th partial differentials of φ. Denoting the expectation with respect to the measure $\mathcal{N}$ by $E(\cdot)$, we have

$$E(|\varphi^{(s-3)}(z) - \varphi^{(s-3)}(w)|^5) \leq const \cdot d(w, z)^{5/2}$$

because the distribution of $\varphi^{(s-3)}(z) - \varphi^{(s-3)}(w)$ is a Gaussian distribution with average 0 and variance bounded by

$$\sum_{(n,m)\in\mathbb{Z}^2} \left(\min\{2, (n^2+m^2+1)^{1/2} d(z,w)\}(n^2+m^2+1)^{-3/4}\right)^2 \leq const. \cdot d(z,w).$$

By Borel-Cantelli lemma, there is a constant $i_0 > 0$ for $\mathcal{N}$-almost every φ such that

$$\left|\varphi^{(s-3)}(2^{-i}p, 2^{-i}q) - \varphi^{(s-3)}(2^{-i}p', 2^{-i}q')\right|^5 \leq 2^{-i/3}$$

for $i > i_0$ and $p, q, p', q' \in \mathbb{Z}$ such that $|p-p'| \leq 1$ and $|q-q'| \leq 1$. This implies that $\varphi^{(s-3)}$ is continuous for $\mathcal{N}$-almost every φ and hence $\mathcal{N}$ is supported on $C^{s-3}(\mathbb{T}, \mathbb{R})$.

As $C^{s-3}(\mathbb{T}, \mathbb{R}^2)$ is naturally identified with $C^{s-3}(\mathbb{T}, \mathbb{R}) \times C^{s-3}(\mathbb{T}, \mathbb{R})$, we regard the product $\mathcal{N} \times \mathcal{N}$ as a measure on $C^{s-3}(\mathbb{T}, \mathbb{R}^2)$. Put $\mathcal{M}_s = (\mathcal{N} \times \mathcal{N}) \circ \pi^{-1}$ where $\pi : C^{s-3}(\mathbb{T}, \mathbb{R}^2) \to C^{s-3}(M, \mathbb{R}^2)$ is the mapping that corresponds to the restriction to M. Then $\mathcal{M}_s$ satisfies the conditions in the lemma.

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