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Series #578. December 2002

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Blow Up Rate for Semilinear Heat Equation with Subcritical Nonlinearity

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1 Introduction

We consider the Cauchy problem for the semilinear heat equation of the form

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & \text{in} \quad \mathbf{R}^n \times (0,T), \\ u(x,0) = u_0(x) & \text{in} \quad \mathbf{R}^n, \end{cases} \tag{1.1}$$

with a subcritical nonlinearity, i.e.,

$$\begin{array}{c} 1 < p < (n+2)/(n-2) \quad \text{for} \quad n \geq 3 \quad \text{or} \\ 1 < p < \infty \quad \text{for} \quad n \leq 2. \end{array} \tag{1.2}$$

We are interested in a blow up rate estimate of the form

$$\|u(t); L^\infty(\mathbf{R}^n)\| \leq C(T-t)^{-1/(p-1)} \tag{1.3}$$

for a blowup solution $u(t) = u(\cdot, t)$ of (1.1) (with a blowup time T) where C is a some positive constant independent of time t. Here we denote by $\|f; X\|$ the norm of f in a Banach space X. Our goal is to prove (1.3) for all subcritical power p satisfying (1.2) without assuming nonnegativity for initial data u_0. We should note that the right hand side of (1.3) is constant multiple a spatially homogeous solution for (1.1). So the power appeared in (1.3) cannot be improved. The estimate (1.3) is crucial in studying the asymptotic behavior of blow up solution whose analysis has been started by Y. Giga and R. V. Kohn [8]. In [9] Y. Giga and R. V. Kohn established (1.3) for more restricted range of $p : 1 < p < (3n+8)/(3n-4)$ or for nonnegative initial data (so that the solution is positive everywhere) with subcritical p. It had been a long open problem whether or not (1.3) holds all subcritical p. Our result solves this important open problem. The range of p is expected not to be improved since for $p=(n+2)/(n-2)$ the estimate (1.3) does not hold when $n=3,4,5,6$ as announced in [3]. We refer Remark 2.2 for a list of related results.

Let us briefly sketch the idea of the proof. As in [9] we convert our problem to a uniform bound for a global-in-time solution w of the rescaled equation

$$w_s-\Delta w+\frac{y\cdot\nabla}{2}w+\beta w-|w|^{p-1}w=0, \qquad \beta=\frac{1}{p-1}$$

with

$$w(y,s)=(T-t)^{1/(p-1)}u(a+y\sqrt{T-t},t),$$

where $a\in\mathbf{R}^n$ is a center of the rescaling. A key step is to establish an integral estimate

$$\sup_{s\geq s_1}\int_s^{s+1}\|w(\tau);L^{p+1}(B_R)\|^{(p+1)q}d\tau\leq C_{q,s_1} \text{ for } s_1>s_0=-\log T \tag{1.4}$$

with C_{q,s_1} independent of a for $q\geq 2$. In [9] the case $q=2$ with the ball B_R replaced by $\mathbf{R}^n$ has been proved. This estimate with $q=2$ already yields (1.3) for $p<(3n+8)/(3n-4)$ as in [9]. To show this integral estimate we adjust the idea of P. Quittner [18] who obtained a uniform bound for global solution of $u_t=\Delta u+|u|^{p-1}u$ in a bounded domain with zero Dirichlet data for all subcritical p without assuming positivity of solutions. However, it is nontrivial to localize his argument to drive the above integral estimate. For this purpose we introduce two kinds of localized weighted energies of the form

$$\begin{aligned}E_\varphi[w](s) &= \frac{1}{2}\int_{\mathbf{R}^n}(|\nabla(\varphi w)|^2+(\beta\varphi^2-|\nabla\varphi|^2)|w|^2)\rho dy-\frac{1}{p+1}\int_{\mathbf{R}^n}\varphi^2|w|^{p+1}\rho dy,\\ \mathcal{E}_\varphi[w](s) &= \frac{1}{2}\int_{\mathbf{R}^n}\varphi^2(|\nabla w|^2+\beta|w|^2)\rho dy-\frac{1}{p+1}\int_{\mathbf{R}^n}\varphi^2|w|^{p+1}\rho dy\end{aligned}$$

with $\rho(y)=\exp(-|y|^2/4)$,where $\varphi=\varphi(y)$ is a cutoff function of a ball. (Here and hereafter we suppress the s-dependence of w in integrands. For example,

$$\int_{\mathbf{R}^n}\varphi^2|w|^{p+1}\rho dy=\int_{\mathbf{R}^n}(\varphi(y))^2|w(y,s)|^{p+1}\rho(y)dy.)$$

If $\varphi\equiv 1$, these energies agree with global energy $E[w]$ in [9]. The key observation is to derive upper and lower bounds for $\mathcal{E}_\varphi$ i.e.

$$-C_2\leq\mathcal{E}_\varphi[w]\leq C_1 \quad s\geq s_0 \quad \text{independent of} \quad a.$$

For the global energy $E[w](s)$ we know

$$0 \leq E[w](s) \leq E[w](s_0)$$

since $E[w](s)$ is monotone decreasing as a function of s. Unfortunately, such a monotonicity property is not expected for localized energies. We need a more sophisticated argument.

An upper bound for $\mathcal{E}_\varphi$ is obtained by bounds for

$$\sup_{s \geq s_0} \int_s^{s+1} \left(\frac{d\mathcal{E}_\varphi}{ds} \right)_+ ds \quad \text{and} \quad \sup_{s \geq s_0} \int_s^{s+1} \mathcal{E}_\varphi ds,$$

where $a_+ = \max\{a, 0\}$. A lower bound for $\mathcal{E}_\varphi$ is obtained by an integral identity of E_φ involving $d/ds \int_{\mathbf{R}^n} |\varphi w|^2 \rho ds$ and an upper bound for $(d\mathcal{E}_\varphi/ds)_+$ if one notices that w is a global solution. Note that the same argument with $\varphi \equiv 1$ yields $E[w] \geq 0$.

Applying bounds for $\mathcal{E}_\varphi$ we prove (1.4) by a bootstrap argument similar to that of P. Quittner [18]. We assume (1.4) with a fixed $q \geq 2$ and prove (1.4) for a bigger q denoted $\tilde{q} = \tilde{q}(p, q)$ by shrinking R. In [18] such a localization of shrinking R is unnecessary since the problem is global. Fortunately, we are able to prove $\tilde{q} - q \geq 1/(p+1)$ so the estimate (1.4) is obtained for every q in a finitely many steps by starting with $q = 2$ (obtained in [9]). We explicitly calculate $\tilde{q}$ to get the estimate $\tilde{q} - q \geq 1/(p+1)$ which was implicit in [18]. We present the detail of this bootstrap argument as well as calculation of exponents for the reader's convenience.

Once we get (1.4) for all $q \geq 2$ we obtain an upper bound for w which yields (1.3). Here is the only place that the subcriticality of p is invoked. From our proof it turns out that the constant C in (1.3) can be taken so that it depends only on n, p and a bound for $T^{1/(p-1)} \| u_0; L^\infty(\mathbf{R}^n) \|$ (Theorem 2.1). This estimate improves the uniform bound in [17], where $p < (3n+8)/(3n-4)$ is assumed and C in (1.3) depends only on bounds for T and C^2-norm of u_0 (other than p and n). Such a type of uniform bounds is crucial to obtain a uniform O. D. E. behavior of solutions [17]. From our estimates their result [17] can be extended for all subcritical p.

Our results extend to the Dirichlet problem on a convex domain. We shall discuss this topic in detail in a forthcoming paper.

Blowup rate estimates are important to study refined behavior of a blowup solution ([8] [9] [10] [14] [17] [20] [21]...). We do not exhaust all related references here. The reader is referred to a review paper [11] and papers cited there.

2 Main Theorem

We say that a function $u : \mathbf{R}^n \times [0, T) \to \mathbf{R}$ is a solution of (1.1) if u solves (1.1) and satisfies

$$\begin{cases} u, \ u_t, \ \nabla u \text{ and } \nabla^2 u \quad \text{are bounded and continuous} \\ \text{on } \mathbf{R}^n \times [0, \tau] \quad \text{for every } \tau < T. \end{cases} \tag{2.1}$$

If $u(t)$ satisfies that

$$\|u(t); L^\infty(\mathbf{R}^n)\| = \sup_{x \in \mathbf{R}^n} |u(x,t)| \to \infty \quad \text{as} \quad t \to T < \infty \tag{2.2}$$

for some $T > 0$, the time T is called the *blowup time* of u. Such a solution u is called a *blowup solution* of (1.1) with the blowup time T. Our main result is :

Theorem 2.1. *Assume that either $p > 1$ for $n \leq 2$ or $1 < p < (n+2)/(n-2)$ for $n \geq 3$. Let u be a blowup solution of (1.1) with a blowup time T. Then there exists a constant C depending only on n, p and a bound for $T^{1/(p-1)}\|u_0; L^\infty(\mathbf{R}^n)\|$ such that u satisfies (1.3) for all $t \in (0, T)$, i.e.*

$$\|u(t); L^\infty(\mathbf{R}^n)\| \leq C(T-t)^{-1/(p-1)}.$$

Remark 2.1. To prove this theorem we may assume that $T = 1$ by rescaling u, x, t as

$$z(x,t) = T^{1/(p-1)} u(\sqrt{T}x, Tt).$$

Indeed, z solves (1.1) with initial data

$$z_0 = T^{1/(p-1)} u_0(\sqrt{T}x),$$

and the blowup time of z equals 1. The estimate (1.3) for z is

$$\|z(s); L^\infty(\mathbf{R}^n)\| \leq C(1-s)^{-1/(p-1)}$$

with C depending only on n, p and a bound for $\|z_0; L^\infty(\mathbf{R}^n)\|$. The estimate (1.3) for u easily follows from this estimate with the same constant C.

Remark 2.2. We list known results of blowup rate estimate.

1. *General exponent (i.e. $p > 1$) for Dirichlet problem in a bounded, convex domain with zero boundary data.* In [4] it was proved that (1.3) holds when $\Delta u_0 + |u_0|^{n-1}u_0 \geq 0$ so that $u_t \geq 0$. The dependence of C with respect to u_0 and T is not explicit.

2. *Subcritical exponent (i.e.* $1 < p < (n+2)/(n-2)$*).*

   (a) In [9] it was shown that (1.3) holds when $1 < p < (3n+8)/(3n-4)$ or $u_0 \geq 0$.

   (b) *Dirichlet problem in a domain* Ω *with uniformly* $C^{2,\alpha}$*-smooth* $\partial\Omega$ *with zero boundary data.* In [5] it was shown that (1.3) holds when $1 < p < 1 + 2/(n+1)$ and $u_0 \geq 0$.

   (c) *Neumann problem in a bounded domain with zero Neumann data.* In [13] it was shown that (1.3) holds when $1 < p < 1 + 2/n$ and $u_0 \geq 0$.

   In all above results the dependence of C with respect to T and u_0 is implicit.

3. *Crtical exponent (i.e.* $p = (n+2)/(n-2)$*).* In [3] it was formally shown that there exists a sign-changing blowup solution such that

$$\lim_{t \to T} (T-t)^{-1/(p-1)} \|u(t); L^\infty(\mathbf{R}^n)\| = \infty. \tag{2.3}$$

4. *Supercrtical exponent (i.e.* $p > (n+2)/(n-2)$*).* In [12] it was shown that there exists a blowup solution satisfying (2.3) when $n \geq 11$ and $p > (n - 2\sqrt{n-1})/(n-4-2\sqrt{n-1})$.

5. *Universal bound, i.e. (1.3) for* $t \geq \tau > 0$ *with* C *depending only on* τ *and independent of the initial data.* (The type of estimate is expected only for positive solutions. So we assume that u is a positive solution in the following (a)-(c).)

   (a) In [6] it was shown that a universal bound exists when $1 < p < (n+1)/(n-1)$ for the Dirichlet problem in a bounded domain Ω with zero boundary data.

   (b) In [19] it was shown that a universal bound exists when $1 < p < (n+2)/(n-2)$ and $n \leq 3$ for the Dirichlet problem in a bounded domain Ω with zero boundary data.

   (c) In [16] it was shown that a universal bound exists when $1 < p < n/(n-2)$ and $n > 3$ with initial data u_0 which is radially symmetric and nonincreasing as a function of $r = |x|$.

Remark 2.3. The assumption (2.1) is actually unnecessary because of the regularizing effect (cf. §3.3). Indeed, if $u_0 \in L^\infty(\mathbf{R}^n)$, then (2.1) is fulfilled with $[0, \tau]$ replaced by $[\delta, \tau]$ for each $\delta > 0$ and $\tau < T$. We apply Theorem 2.1 in $[\delta, \tau)$ and letting $\delta \to 0$ yields our desired estimate since

$$\|u(\delta); L^\infty(\mathbf{R}^n)\| \leq 2\|u(0); L^\infty(\mathbf{R}^n)\|$$

for small $\delta > 0$ (Lemma 3.1).

Our main result removes the extra restrictions for subcritical exponent p and positivity of initial data in the results obtained in [9]. The result of [3] suggests that our restriction $p < (n+2)/(n-2)$ is optimal although the Dirichlet problem is studied in [3].

3 Preliminary

In this section we give conventions of notations and recall several estimates obtained in [9]. In this paper we always assume that $p > 1$, but we do not assume that p is subcritical in this section.

3.1 Rescaling variables

To study u near the blowup time T at a point $a \in \mathbf{R}^n$, as in [8] we introduce the rescaled function

$$w^a(y, s) = (T - t)^\beta u(a + y\sqrt{T - t}, t) \tag{3.1}$$

with

$$s = -\log(T - t), \quad \beta = \frac{1}{p - 1}.$$

If u solves (1.1), then w^a solves

$$w_s^a - \Delta w^a + \frac{y \cdot \nabla}{2} w^a + \beta w^a - |w^a|^{p-1} w^a = 0 \quad \text{in} \quad \mathbf{R}^n \times (s_0, \infty), \tag{3.2}$$

where

$$s_0 = -\log T.$$

By Remark 2.1 to prove Theorem 2.1 we may assume $T = 1$ so here and hereafter we assume $s_0 = 0$. We shall write a solution of (3.2) by w instead of w_a. The rescaled function $w^a(y,s)$ clearly exists for all $s > 0$, and it inherits bounds from those of u. In fact, (2.1) says that $w = w_a$ satisfies the property:

$$\begin{cases} w,\ w_s,\ \nabla w \text{ and } \nabla^2 w \quad \text{are bounded and continuous} \\ \text{on } \mathbf{R}^n \times [0,S] \quad \text{for all} \quad S < \infty. \end{cases} \tag{3.3}$$

We rewrite the equation (3.2) in a divergence form:

$$\rho w_s - \nabla \cdot (\rho \nabla w) + \beta \rho w - \rho |w|^{p-1} w = 0 \quad \text{in} \quad \mathbf{R}^n \times (0,\infty), \tag{3.4}$$

where $\rho = \rho(y)$ is a weight function:

$$\rho(y) = \exp\left(-\frac{|y|^2}{4}\right).$$

3.2 Global energy

We shall study a global solution w of (3.4) in $\mathbf{R}^n \times (0,\infty)$ which satisfies (3.3). We do not require that w is related to u and a. We recall the *energy* of w of the form

$$E[w](s) = \frac{1}{2}\int_{\mathbf{R}^n} (|\nabla w|^2 + \beta |w|^2)\rho dy - \frac{1}{p+1}\int_{\mathbf{R}^n} |w|^{p+1}\rho dy.$$

In this paper we call this functional the *global energy* to distinguish from other localized energies which will be defined later. This global energy satisfies the following two estimates which were obtained in [9].

Proposition 3.1. *Let w be a global solution of (3.4) satisfying (3.3). Then $E[w](s)$ is a nonnegative, monotone decreasing in s. In particular*

$$0 \le E[w](s) \le E[w](0). \tag{3.5}$$

Proposition 3.2. *Let w be a solution of (3.4) satisfying (3.3). Then w satisfies*

$$\int_0^\infty \|w_s; L^2_\rho(\mathbf{R}^n)\|^2 ds \le E[w](0). \tag{3.6}$$

Moreover, there exists a positive constant K depending only n, p and a bound for $E[w](0)$ such that

$$\|w; L^2_\rho(\mathbf{R}^n)\|^2 \le K^2 \tag{3.7}$$

and

$$\int_s^{s+1} \|w; L^{p+1}_\rho(\mathbf{R}^n)\|^{2(p+1)} ds \le K \qquad \textit{for all} \quad s \ge 0. \tag{3.8}$$

Here $L^p_\rho(\mathbf{R}^n)$ denotes a weighted L^p space:

$$L^p_\rho(\mathbf{R}^n) = \{u \in L^1_{loc}(\mathbf{R}^n) | \int_{\mathbf{R}^n} |u|^p \rho dx < \infty\}.$$

3.3 Regularizing effect

It is well known that the solution of (1.1) is smooth for $t > 0$ even if initial data is merely bounded, since the equation is parabolic. We here present a quantified version of this regularizing property.

Lemma 3.1. *For $u_0 \in L^\infty(\mathbf{R}^n)$ there exists a unique solution $u \in C(\mathbf{R}^n \times [0, T_0])$ satisfying (1.1) in $\mathbf{R}^n \times (0, T_0)$ for $T_0 = 1/2^p \|u_0\|_\infty^{p-1}$ such that*

$$\|u(t)\|_\infty \leq 2\|u_0\|_\infty \qquad for \quad t \in [0, T_0],$$

where $\|f\|_\infty = \|f; L^\infty(\mathbf{R}^n)\|$. Moreover, there exists $c_0 = c_0(n)$ and $c_1 = c_1(n, p)$ such that

$$\begin{aligned} t^{1/2}\|\nabla u(t)\|_\infty &\leq c_0\|u_0\|_\infty \\ t\|\nabla^2 u(t)\|_\infty &\leq c_1\|u_0\|_\infty \qquad \textit{for all} \quad t \in (0, T_0]. \end{aligned}$$

Sketch of the proof. The results are essentially well known, e.g. [7]. The local solution is odtained by a well known successive apporoximation

$$u^{(j+1)} = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}|u^{(j)}|^{p-1}u^{(j)}d\tau, \qquad u^{(1)} = e^{t\Delta}u_0 \quad (j \geq 1),$$

where $e^{t\Delta}$ denotes the solution operator of the heat equation. The estimate of derivatives is obtained by estimating the integral equation

$$u = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}|u|^{p-1}u d\tau.$$

For example, since $\|\nabla e^{t\Delta}u_0\|_\infty \leq c_0' t^{-1/2}\|u_0\|_\infty$ for $t > 0$ with c_0' depending only on n, we have

$$\begin{aligned} t^{1/2}\|\nabla u(t)\|_\infty &\leq c_0'\|u_0\|_\infty + 2c_0' t \sup_{t\in[0,T_0]} \||u(t)|^{p-1}u(t)\|_\infty \\ &\leq 3c_0'\|u_0\|_\infty. \square \end{aligned}$$

4 Uniform bound

In this section we give a local uniform bound for a global solution of (3.4) with (3.3) admitting our key integral estimate. Let B_R be an open ball of radius R centered at the origin in $\mathbf{R}^n$.

Theorem 4.1. *Assume that* $1 < p$ *for* $n \leq 2$ *or* $1 < p < (n+2)/(n-2)$ *for* $n \geq 3$. *Let* w *be a global solution of (3.4) satisfying (3.3) with the initial data* w_0. *Then for* $R > 0$ *there exists a positive constant* C_R *depending only on* n, p, R *and a bound for* $\|w_0\|_\infty$ *such that*

$$\|w(y,s); L^\infty(B_R)\| \leq C_R \quad \text{for} \quad s > 0. \tag{4.1}$$

Proof of Theorem 2.1 admitting Theorem 4.1. By Remark 2.2 we may assume that $T = 1$. By (4.1) we have

$$|w^a(0,s)| \leq C \quad \text{for} \quad s > 0$$

with C independent of $a \in \mathbf{R}^n$ where w^a is defined by (3.1). Here C depends only on n, p and a bound for $\|u_0\|_\infty$ since $\|u_0\|_\infty = \|w_0^a\|_\infty$. We thus obtain

$$|u(a,t)| \leq C(1-t)^{-1/(p-1)}, \quad t \in (0,1)$$

for all $a \in \mathbf{R}^n$. This yields the blowup rate estimate (1.3). □

To prove (4.1) in Theorem 4.1 we admit the following key integal estimate whose proof is postponed in Sections 5 and 6. Let $BC^m(\mathbf{R}^n)$ denote the space of all C^m functions on $\mathbf{R}^n$ such that all derivatives up to m-th order belong to $BC(\mathbf{R}^n)$. Here, $BC(\mathbf{R}^n)$ denotes the space of bounded continuous functions in $\mathbf{R}^n$.

Lemma 4.1. (Key integral estimate) *Let* w *be a global solution of (3.4) satisfying (3.3) with initial data* w_0. *For all* $q \geq 2$ *and* $R > 0$ *there exists a positive constant* C_q *depending only* p, q, R, n *and a bound for* $E[w](0)$ *and for* $\|w_0; BC^2(\mathbf{R}^n)\|$ *such that*

$$(\mathrm{A}_q) \qquad \sup_{s\geq 0} \int_s^{s+1} \|w; L^{p+1}(B_R)\|^{(p+1)q} d\tau \leq C_q. \tag{4.2}$$

Here we note that for the proof of Theorem 4.1 we require subcriticality of p while we do not require the subcriticality of p for the proof of Lemma 4.1. The way to derive uniform bound from integal estimate is essentially known by [9]. They use an *interpolation* theorem due to T. Cazenave and P. -L. Lions [2] and an *interior regularity* theorem for a linear parabolic equation [15], a covenient from for our purpose is stated in [9]. However, their integral bound is weaker than ours, so they are forced to impose a stronger restriction on the exponent p to derive a uniform bound. Actually, their integral bound corresponds to the case $q = 2$ in Lemma 4.1. For completeness we recall the interpolation theorem and the interior regularity theorem.

Lemma 4.2. (Interpolation theorem) *Assume that* $v \in L^\alpha((0,\infty); L^\beta(B_R))$, $v_t \in L^\gamma((0,\infty); L^\delta(B_R))$ *for some* $1 < \alpha, \beta, \gamma, \delta < \infty$. *Then*

$$v \in C([0,\infty); L^\lambda(B_R))$$

for all $\lambda < \lambda_0 = (\alpha + \gamma')\beta\delta/(\gamma'\beta + \alpha\delta)$ *with* $\gamma' = \gamma/(\gamma - 1)$, *and satisfies*

$$\sup_{t\geq 0} \|v(t); L^\lambda(B_R)\| \leq C \int_0^\infty \left(\|v; L^\beta(B_R)\|^\alpha + \|v_t; L^\delta(B_R)\|^\gamma\right) d\tau$$

for $\lambda < \lambda_0$. *The positive constant* C *depends only on* $\alpha, \beta, \gamma, \delta, n$ *and* R.

Lemma 4.3. (Interior regularity) *Suppose that* $v(x,t)$ *solves*

$$v_t - \nabla \cdot (A\nabla v) + B \cdot \nabla v + gv = 0 \tag{4.3}$$

in a cylinder $\Omega_R = B_R \times (0,\infty) \subset \mathbf{R}^n \times \mathbf{R}$ *under the following assumptions:*

1. *An* $n \times n$ *martrix* $A(x,t)$ *satisfies*

$$\mu_0|\xi|^2 \leq A(x,t)\xi \cdot \xi \leq \mu_0^{-1}|\xi|^2$$

   for all $\xi \in \mathbf{R}^n$ *and some positive constant* μ_0.

2. *An* n*-vector* $B(x,t)$ *satisfies*

$$|B(x,t)| \leq \mu_1 \qquad in \quad \Omega_R$$

   for some positive constant μ_1.

3. *A coefficient* $g(x,t) \in L^\beta((t_0, t_0+1); L^\alpha(B_R))$ *with* $1/\beta + n/2\alpha < 1$ *and* $\alpha \geq 1$ *satisfies*

$$\int_{t_0}^{t_0+1} \|g; L^\alpha(B_R)\|^\beta dt \leq \mu_2$$

   for some positive constant $\mu_2 = \mu_2(t_0)$.

If

$$\int_{t_0}^{t_0+1} \|v; L^2(B_R)\|^2 dt \le \mu_3$$

for some positive constant $\mu_3 = \mu_3(t_0)$, then there exists a positive constant C depending only on μ_j $(0 \le j \le 3)$, α, β, n, $\eta \in (0,1)$ and R such that

$$|v| \le C \quad on \quad B_{\eta R} \times \left(t_0 + \eta^2, t_0 + 1\right).$$

Remark 4.1. *If v is a solution of (4.3) in $B_R \times (0, \infty)$ and the constants $\mu_2(t_0)$ and $\mu_3(t_0)$ are uniformly bounded in $t_0 \ge 0$, then Lemma 4.3 implies*

$$|v| \le C \quad on \quad B_{\eta R} \times \left(\eta^2, \infty\right)$$

with C independent of $t_0 \ge 0$.

Proof of Theorem 4.1 admitting Lemma 4.1. By the regularizing effect Lemma 3.1 we observe that

$$\begin{array}{lll} \|w(s)\|_\infty \le 2\|w_0\|_\infty & \text{for} & s \in [0, s_1],\ s_1 = -\log(1 - T_0) \quad \text{and} \\ \|\nabla w(s)\|_\infty \le c_3 \|w_0\|_\infty & \text{and} & \|\nabla^2 w(s)\|_\infty \le c_3 \|w_0\|_\infty \\ & \text{for} & s \in [s_2, s_1],\ s_2 = s_1/2. \end{array} \tag{4.4}$$

with c_3 depending only on n, p, s_1. In particular $w(s_1) \in BC^2(\mathbf{R}^n)$. From (3.6) and (4.2) it follows that

$$\sup_{s \ge s_2} \int_s^{s+1} (\|w; L^{p+1}(B_R)\|^{(p+1)q} + \|w_s; L^2(B_R)\|^2) d\tau \le C_q + E[w](s_2) \tag{4.5}$$

for all $q \ge 2$ and $R > 0$ with C_q depending only on n, p, q, R and a bound for $E[w](s_2)$ and for $\|w(s_2); BC^2(\mathbf{R}^n)\|$. Since $E[w](s_2) \le E[w](0)$ by Proposition 3.1 and since $\|w(s_2); BC^2(\mathbf{R}^n)\|$ is bounded by (4.4), we conclude that the right hand side of (4.5) is dominated by a constant C_q'' depending only on n, p, q, R and a bound for $E[w](0)$ and for $\|w_0\|_\infty$. If we set

$$\lambda_1(q) = \frac{(p+1)q + 2}{q+1} = p + 1 - \frac{p-1}{q+1},$$

then Lemma 4.1 implies that

$$\sup_{s \ge s_2} \|w; L^\lambda(B_R)\| \le C_q' \quad \text{for all} \quad \lambda < \lambda_1(q) \quad \text{and all} \quad q \ge 2$$

with some positive constant C_q' depending only on p, q, n, R and a bound for $E[w](0)$ and for C_q''. We take q large so that $n/2\lambda_1(q) < 1$. This is always possible since $1 < p < (n+2)/(n-2)$. We fix such q and apply Lemma 4.2 (and Remark 4.1) with $g = |w|^{p-1}$. We thus conclude that

$$|w(y,s)| \leq C_{\eta R} \quad \text{for} \quad (y,s) \in B_{\eta R} \times (\eta^2 + s_2, \infty) \tag{4.6}$$

with some constant $C_{\eta R}$ depending only on p, n, R, η and a bound for $E[w](0)$ and for $\|w_0\|_\infty$. We now take η small so that $\eta^2 + s_2 < s_1$, and by (4.5) we observe that

$$\|w(s)\|_\infty \leq 2\|w_0\|_\infty, \quad s < s_1. \tag{4.7}$$

Since s_1 depends only on a bounded for $\|w_0\|_\infty$, η can be taken independent of R. Since R can be taken arbitrary, the estimates (4.6) and (4.7) yield (4.1) with C depending only on n, p, R and a bound for $E[w](0)$ and for $\|w_0\|_\infty$. Since $E[w](0)$ is bounded by a constant depending only on a bounded for $\|w_0\|_\infty$, we obtain (4.1) the desired dependance of C in (4.1).□

5 Local energies

Let w be a solution of (3.4) with (3.3). Hereafter we should suppress the word 'global'. Let ψ be a bounded C^2 function on $\mathbf{R}^n$. Then ψw satisfies

$$\rho(\psi w)_s - \nabla \cdot (\rho \nabla(\psi w)) + \nabla \cdot (\rho w \nabla \psi) + \rho \nabla \psi \cdot \nabla w + \beta \psi w \rho - \psi |w|^{p-1} w \rho = 0. \quad \text{in} \quad \mathbf{R}^n \times (0,\infty) \tag{5.1}$$

We introduce two types of *local energies.*

$$\begin{aligned} E_\psi[w] &= \frac{1}{2}\int_{\mathbf{R}^n} (|\nabla(\psi w)|^2 + (\beta\psi^2 - |\nabla\psi|^2)|w|^2)\rho dy \\ &\qquad - \frac{1}{p+1}\int_{\mathbf{R}^n} \psi^2 |w|^{p+1}\rho dy, \qquad (5.2) \\ \mathcal{E}_\psi[w] &= \frac{1}{2}\int_{\mathbf{R}^n} \psi^2(|\nabla w|^2 + \beta|w|^2)\rho dy - \frac{1}{p+1}\int_{\mathbf{R}^n} \psi^2|w|^{p+1}\rho dy. \qquad (5.3) \end{aligned}$$

In particular, if $\psi \equiv 1$ on $\mathbf{R}^n$, then the both of local energies equal the global energy $E[w]$. The aim of this section is to show a lower and an upper bound for $\mathcal{E}_\psi[w]$.

5.1 Integral identities

In this section we prepare two integral identities involving local energies. These identities will lead us to estimates of a local energy $\mathcal{E}_\psi$.

Proposition 5.1. *Let w be a solution of (3.4) with (3.3). Assume that $\psi \in BC^1(\mathbf{R}^n) \cap C^2(\mathbf{R}^n)$. Then $\mathcal{E}_\psi[w]$ satisfies*

$$\begin{aligned}
&\int_{\mathbf{R}^n} \psi^2 w w_s \rho dy + (p+1)\mathcal{E}_\psi[w](s) \\
= & -\int_{\mathbf{R}^n} 2\psi w \nabla\psi \cdot \nabla w \rho dy + \frac{p-1}{2}\int_{\mathbf{R}^n} \psi^2(|\nabla w|^2 + \beta|w|^2)\rho dy
\end{aligned} \tag{5.4}$$

for all $s > 0$. In particular, if $\psi \equiv 1$ in $\mathbf{R}^n$, then

$$\int_{\mathbf{R}^n} w w_s \rho dy + (p+1)E[w](s) = \frac{p-1}{2}\int_{\mathbf{R}^n} (|\nabla w|^2 + \beta|w|^2)\rho dy \tag{5.5}$$

for all $s > 0$.

Proof. Multiplying (3.4) by $\psi^2 w$ and integrating on $\mathbf{R}^n$, we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^n} \psi^2 w w_s \rho dy - \int_{\mathbf{R}^n} \psi^2 w \nabla \cdot (\rho \nabla w) dy \\
&\qquad + \int_{\mathbf{R}^n} \beta\psi^2|w|^2\rho dy + \int_{\mathbf{R}^n} \psi^2 |w|^{p+1}\rho dy = 0.
\end{aligned}$$

We integrate the second term by parts to get

$$\begin{aligned}
-\int_{\mathbf{R}^n} \psi^2 w \nabla \cdot (\rho\nabla w) dy &= \int_{\mathbf{R}^n} \nabla(\psi^2 w) \cdot \nabla w \rho dy \\
&= \int_{\mathbf{R}^n} 2\psi w \nabla\psi \cdot \nabla w \rho dy + \int_{\mathbf{R}^n} \psi^2 |\nabla w|^2 \rho dy
\end{aligned}$$

By the definition of $\mathcal{E}_\psi[w]$ we observe that

$$\int_{\mathbf{R}^n} \psi^2 |w|^{p+1}\rho dy = (p+1)\mathcal{E}_\psi[w] - \frac{p+1}{2}\int_{\mathbf{R}^n} \psi^2(|\nabla w|^2 + \beta|w|^2)\rho dy.$$

Combining the above three identities, we have (5.4). The identity (5.5) is nothing but (5.4) when $\psi \equiv 1$.□

Proposition 5.2. *Let w be a solution of (3.4) with (3.3). Assume that $\psi \in BC^1(\mathbf{R}^n) \cap C^2(\mathbf{R}^n)$. Then $E_\psi[w]$ satisfies*

$$\frac{1}{2}\frac{d}{ds}\int_{\mathbf{R}^n} |\psi w|^2 \rho dy = -2E_\psi[w] + \frac{p-1}{p+1}\int_{\mathbf{R}^n} \psi^2 |w|^{p+1}\rho dy \quad \textit{for all} \quad s > 0. \tag{5.6}$$

Proof. By (5.1) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int_{\mathbf{R}^n}|\psi w|^2\rho dy &= \int_{\mathbf{R}^n}\psi w(\psi w)_s\rho dy\\ &= \int_{\mathbf{R}^n}\psi w\nabla\cdot(\rho\nabla(\psi w))dy-\int_{\mathbf{R}^n}\psi w\nabla\cdot(\rho w\nabla\psi)dy\\ &\quad -\int_{\mathbf{R}^n}\psi w\rho\nabla w\cdot\nabla\psi dy-\int_{\mathbf{R}^n}\beta|\psi w|^2\rho dy+\int_{\mathbf{R}^n}\psi^2|w|^{p+1}\rho dy.\end{aligned}$$

We integrate the first and the second terms of the rightest hand side by parts. Then we get

$$\int_{\mathbf{R}^n}\psi w\nabla\cdot(\rho\nabla(\psi w))dy=-\int_{\mathbf{R}^n}|\nabla(\psi w)|^2\rho dy,$$

and

$$-\int_{\mathbf{R}^n}\psi w\nabla\cdot(\rho w\nabla\psi)dy=\int_{\mathbf{R}^n}\psi w\nabla w\cdot\nabla\psi\rho dy+\int_{\mathbf{R}^n}|w\nabla\psi|^2\rho dy,$$

since the weight function ρ is rapidly decreasing in y. By the definition of $E_\psi[w]$ we have

$$\int_{\mathbf{R}^n}(|\nabla(\psi w)|^2+(\beta\psi^2-|\nabla\psi|^2)|w|^2)\rho dy=2E_\psi[w]+\frac{2}{p+1}\int_{\mathbf{R}^n}\psi^2|w|^{p+1}\rho dy.$$

Combining these identities, we easily get (5.6).□

5.2 Upper bound for local energy

We shall prove an upper bound for the local energy $\mathcal{E}_\psi$. For the global energy such a bound is clear since $E[w](s)$ is nonincreasing in s (Proposition 3.1). Such a monotonicity property is not expected for $\mathcal{E}_\psi$ for general ψ. Instead, we estimate $d\mathcal{E}_\psi[w]/ds$ from above by $L^2(0,\infty)$ function plus a constant and control $\int_s^{s+1}\mathcal{E}_\psi[w]d\tau$ uniformly in $s\geq 0$. Fortunately, this is enough to obtain an upper bound for $\mathcal{E}_\psi[w](s)$. We first control $W^{1,2}_\rho$-norm. (The definition of this weighted Sobolev norm is found in the proof of the next proposition.)

Proposition 5.3. *Let w be a solution of (3.4) with (3.3). Then there exists a positive constant K_1 depending only on p, n and a bound for $E[w](0)$ such that*

$$\|w(s);W^{1,2}_\rho(\mathbf{R}^n)\|^2\leq K_1(1+\|w_s(s);L^2_\rho(\mathbf{R}^n)\|)\quad \text{for all}\quad s\geq 0. \tag{5.7}$$

Proof. We set

$$\|w(s);W^{1,2}_\rho(\mathbf{R}^n)\|^2=\|w(s);L^2_\rho(\mathbf{R}^n)\|^2+\beta\|\nabla w(s);L^2_\rho(\mathbf{R}^n)\|^2.$$

By (5.5) we observe that

$$\|w(s); W^{1,2}_\rho(\mathbf{R}^n)\|^2 \le \frac{2}{p-1}((p+1)E[w](s) + \|w(s)w_s(s); L^1_\rho(\mathbf{R}^n)\|).$$

Since $E[w](s) \le E[w](0)$ by (3.5) and $\|w(s); L^2_\rho(\mathbf{R}^n)\|^2 \le K$ by (3.7), we apply Schwarz's inequality and obtain (5.7) with $K_1 = \beta \max\{2(p+1)E[w](0),\ 2K^{1/2}\}$.□

Proposition 5.4. (Quasi-monotonicity of $\mathcal{E}_\psi$) *Let w be a solution of (3.4) with (3.3). For $\psi \in BC^1(\mathbf{R}^n) \cap C^2(\mathbf{R}^n)$ there exists a positive constant L_1 depending only on p, n, $\|\nabla\psi\|_\infty$ and a bound for $E[w](0)$ such that $\mathcal{E}_\psi[w]$ satisfies*

$$\frac{d}{ds}\mathcal{E}_\psi[w](s) \le L_1(1 + \|w_s(s); L^2_\rho(\mathbf{R}^n)\|) \quad \textit{for all} \quad s > 0. \tag{5.8}$$

Proof. Multiplying (3.4) by ψw_s and integrating in $\mathbf{R}^n$, we get

$$\int_{\mathbf{R}^n} \psi^2 |w_s|^2 \rho dy - \int_{\mathbf{R}^n} \psi^2 w_s \nabla\cdot(\rho\nabla w) dy + \int_{\mathbf{R}^n} \beta\psi^2 w w_s \rho dy - \int_{\mathbf{R}^n} \psi^2 |w|^{p-1} w w_s \rho dy = 0.$$

Integrating the second term by parts, we have

$$\begin{aligned} -\int_{\mathbf{R}^n} \psi^2 w_s \nabla\cdot(\rho\nabla w) dy &= \int_{\mathbf{R}^n} \nabla(\psi^2 w_s)\cdot(\nabla w)\rho dy \\ &= \int_{\mathbf{R}^n} \psi^2 \nabla w\cdot\nabla w_s \rho dy + 2\int_{\mathbf{R}^n} \psi w_s \nabla\psi\cdot\nabla w \rho dy. \end{aligned}$$

We thus obtain

$$\begin{aligned} &\int_{\mathbf{R}^n} \psi^2(\nabla w\cdot\nabla w_s + \beta w w_s)\rho dy - \int_{\mathbf{R}^n} \psi^2 |w|^{p-1} w w_s \rho dy \\ &= -\int_{\mathbf{R}^n} \psi^2 |w_s|^2 \rho dy - 2\int_{\mathbf{R}^n} \psi w_s \nabla\psi\cdot\nabla w \rho dy. \end{aligned}$$

We differentiate $\mathcal{E}_\psi[w]$ with respect to s and use this identity to get

$$\begin{aligned} \frac{d}{ds}\mathcal{E}_\psi[w] &= \int_{\mathbf{R}^n} \psi^2(\nabla w\cdot\nabla w_s + \beta w w_s)\rho dy - \int_{\mathbf{R}^n} |w|^{p-1} w w_s \psi^2 \rho dy \\ &= -\int_{\mathbf{R}^n} \psi^2 |w_s|^2 \rho dy - 2\int_{\mathbf{R}^n} \psi w_s \nabla\psi\cdot\nabla w \rho dy. \end{aligned}$$

We now apply Cauchy's inequality in estimating the last term and obtain

$$\begin{aligned} \frac{d}{ds}\mathcal{E}_\psi[w] &\le -\int_{\mathbf{R}^n} \psi^2 |w_s|^2 \rho dy + \frac{1}{2}\int_{\mathbf{R}^n} \psi^2 |w_s|^2 \rho dy + 2\int_{\mathbf{R}^n} |\nabla\psi|^2 |\nabla w|^2 \rho dy \\ &\le 2\int_{\mathbf{R}^n} |\nabla\psi|^2 |\nabla w|^2 \rho dy. \end{aligned}$$

The estimate (5.8) follows from (5.7) by setting $L_1 = 2K_1\|\nabla\psi\|^2_\infty$.□

Proposition 5.5. *Let w be a solution of (3.4) with (3.3). For $\psi \in BC(\mathbf{R}^n) \cap C^2(\mathbf{R}^n)$ there exists a positive constant K_2 depending only on n, p, $\|\psi\|_\infty$ and a bound for $E[w](0)$ such that*

$$\int_s^{s+1} \mathcal{E}_\psi[w](\tau)d\tau \leq K_2 \qquad \textit{for all} \quad s \geq 0 \tag{5.9}$$

Proof. For $s \geq 0$ we integrate $\mathcal{E}_\psi[w]$ on $[s, s+1]$, and use (3.5), Schwarz' inequality and (3.8) to get

$$\begin{aligned}
\int_s^{s+1} \mathcal{E}_\psi[w]d\tau &= \frac{1}{2}\int_s^{s+1}\int_{\mathbf{R}^n} \psi^2(|\nabla w|^2 + \beta|w|^2)\rho dyd\tau \\
&\qquad\qquad - \frac{1}{p+1}\int_s^{s+1}\int_{\mathbf{R}^n} \psi^2|w|^{p+1}\rho dyd\tau \\
&\leq \frac{M_0^2}{2}\int_s^{s+1}\int_{\mathbf{R}^n}(|\nabla w|^2 + \beta|w|^2)\rho dyd\tau \\
&= M_0^2\int_s^{s+1}\left(E[w](s) + \frac{1}{p+1}\|w; L_\rho^{p+1}(\mathbf{R}^n)\|^{p+1}\right)d\tau \\
&\leq M_0^2\left(E[w](0) + \frac{1}{p+1}\left(\int_s^{s+1}\|w; L_\rho^{p+1}(\mathbf{R}^n)\|^{2(p+1)}d\tau\right)^{1/2}\right) \\
&\leq M_0^2\left(E[w](0) + \frac{\sqrt{K}}{p+1}\right),
\end{aligned}$$

where $M_0^2 = \|\psi\|_\infty^2$. Setting

$$K_2 = M_0^2(E[w](0) + \sqrt{K}/(p+1)),$$

we now obtain (5.9).□

Lemma 5.1. *Assume that $f \in C^1(0,\infty) \cap C[0,\infty)$, $m \in L^1_{loc}(0,\infty)$ and $m \geq 0$. Let F and I be*

$$F = \sup_{s>0}\int_s^{s+1} f(\tau)d\tau < \infty,$$

$$I = \sup_{s>0}\int_s^{s+1} m(\tau)d\tau < \infty.$$

If $f'(s) \leq m(s)$ for all $s > 0$, then

$$\begin{aligned}
f(s) &\leq F + I \qquad \textit{for all} \quad s \geq 1, \\
f(s) &\leq I + f(0) \qquad \textit{for all} \quad s \in [0,1).
\end{aligned}$$

Proof. Let $s \geq 1$. By the mean value theorem there exists $t_0 \in (s-1, s)$ such that

$$f(t_0) = \int_{s-1}^{s} f(\tau)d\tau.$$

Since $t_0 \in (s-1, s)$, the assumption of f' and m implies

$$f(s) - f(t_0) = \int_{t_0}^{s} f'(\tau)d\tau \leq \int_{t_0}^{s} m(\tau)d\tau \leq \int_{s-1}^{s} m(\tau)d\tau \leq I.$$

Then we obtain

$$f(s) \leq f(t_0) + I \leq \int_{s-1}^{s} f(\tau)d\tau + I \leq F + I$$

for $s \geq 1$. The remaining inequality easily follows from

$$f(s) - f(0) \leq \int_{0}^{s} m(\tau)d\tau,$$

so that proof is now complete.□

Combining (5.8) and (3.6), we now apply Lemma 5.1 with (5.9) to get an upper bound for $\mathcal{E}_\psi[w]$.

Theorem 5.1. *Let w be a solution of (3.4) with (3.3). For $\psi \in BC^1(\mathbf{R}^n) \cap C^2(\mathbf{R}^n)$*

$$\mathcal{E}_\psi[w](s) \leq \max\{K_2, \mathcal{E}_\psi[w](0)\} + L_1(1+K) \quad \textit{for all} \quad s \geq 0. \tag{5.10}$$

Remark 5.1. The quantity $\mathcal{E}_\psi[w](0) = \mathcal{E}_\psi[w_0]$ is estimated as

$$\begin{aligned} \mathcal{E}_\psi[w_0] &\leq M_0^2 \left(E[w_0] + \frac{1}{p+1} \|w_0; L_\rho^{p+1}(\mathbf{R}^n)\|^{p+1} \right) \\ &\leq M_0^2 (E[w_0] + C_1 \|w_0\|_\infty^{p+1}), \end{aligned}$$

where $C_1 = 1/(p+1)\int_{\mathbf{R}^n} \rho dy$. Thus

$$\mathcal{E}_\psi[w](s) \leq E_1 \quad \text{for} \quad s \geq 0$$

with E_1 depending only on n, p, M_0, $\|\nabla\psi\|_\infty$, and a bound for $E[w_0]$ and for $\|w_0\|_\infty$.

5.3 Lower bound for local energy

In this section we prove a lower bound for the local energy $\mathcal{E}_\psi[w]$. The identity (5.6) implies that E_ψ cannot be very negative but we have to prove a lower bound for $\mathcal{E}_\psi$ not E_ψ. We first compare E_ψ and $\mathcal{E}_\psi$. Then using a quasi monotone property of $\mathcal{E}_\psi$ in s (Proposition 5.4), we get a lower bound for $\mathcal{E}_\psi$.

Proposition 5.6. *Let w be a solution of (3.4) with (3.3). For $\psi \in BC^2(\mathbf{R}^n)$ with* supp $\psi \subset B_R$ *there exists a positive constant J_1 depending only on n, p, M_j $(j = 0, 1, 2)$ and a bound for $E[w](0)$ such that*

$$\frac{1}{2}\frac{d}{ds}\int_{\mathbf{R}^n} |\psi w|^2 \rho dy \geq -2\mathcal{E}_\psi - J_1 + \frac{p-1}{p+1}\int_{\mathbf{R}^n} \psi^2 |w|^{p+1}\rho dy. \qquad (5.11)$$

Here $M_0 = \|\psi\|_\infty$, $M_1 = \|\nabla\psi\|_\infty$ and $M_2 = \|\Delta\psi\|_\infty$.

Proof. We shall estimate the difference $|E_\psi - \mathcal{E}_\psi|$. We integrate the first term of E_ψ by parts to get

$$E_\psi - \mathcal{E}_\psi = \int_{\mathbf{R}^n} \psi w(\nabla\psi \cdot \nabla w)\rho dy.$$

We integrate the right side by parts again:

$$\begin{aligned}\int_{\mathbf{R}^n} \psi w(\nabla\psi\cdot\nabla w)\rho dy &= -\int_{\mathbf{R}^n} w\nabla\cdot(\psi w\rho\nabla\psi)dy \\ &= -\int_{\mathbf{R}^n} |w|^2|\nabla\psi|^2\rho dy - \int_{\mathbf{R}^n}\psi w(\nabla\psi\cdot\nabla w)\rho dy \\ &\quad -\int_{\mathbf{R}^n}\psi|w|^2\Delta\psi\rho dy + \int_{\mathbf{R}^n}\psi|w|^2\frac{y}{2}\cdot\nabla\psi\rho dy.\end{aligned}$$

Therefore, we are able to get

$$\begin{aligned}\int_{\mathbf{R}^n} \psi w(\nabla\psi\cdot\nabla w)\rho dy &= -\frac{1}{2}\int_{\mathbf{R}^n} |w|^2|\nabla\psi|^2\rho dy \\ &\quad -\frac{1}{2}\int_{\mathbf{R}^n}\psi|w|^2\Delta\psi\rho dy + \frac{1}{4}\int_{\mathbf{R}^n}\psi|w|^2 y\cdot\nabla\psi\rho dy.\end{aligned}$$

By this identity and (3.7) we obtain

$$|E_\psi - \mathcal{E}_\psi| \leq \frac{1}{4}\int_{\mathbf{R}^n}\psi|w|^2 y\cdot\nabla\psi\rho dy + \frac{1}{2}\int_{\mathbf{R}^n}\psi|w|^2|\Delta\psi|\rho dy \leq J_1,$$

where

$$J_1 = K^2 M_0 M_1 R + K^2 M_0 M_2.$$

This inequality together with (5.6) yields (5.11). □

Lemma 5.2. *Assume that $h \in L^1(0,\infty)$ is nonnegative. Let L and C be positive constants and $p > 1$. There exists a positive constant M depending on p, C, L and $m = \int_0^\infty h d\tau$ such that for any $g \in C^1(0,\infty) \cap C[0,\infty)$*

$$g \geq -M \qquad \text{on} \quad [0,\infty)$$

provided that g satisfies

$$(1) \quad g' \leq L + h \qquad \text{on} \quad (0,\infty)$$

and

$$(2) \quad f' \geq -g + Cf^p \qquad \text{on} \quad (0,\infty)$$

for some nonnegative $f \in C^1(0,\infty) \cap C[0,\infty)$.

Proof. For $z > 0$, let F be the solution of

$$\begin{cases} F' &= z + cF^p \\ F(0) &= 0 \end{cases}.$$

Then F blows up at the time

$$T_z := \int_0^\infty \frac{dr}{z + cr^p}.$$

We shall show that

$$g \geq -M$$

for $M = z + T_z L + m$. If not, there exists a point $t_0 \in (0,\infty)$ such that $g(t_0) < -M$. By (1) we see that

$$g(t + t_0) < -M + Lt + m = -z$$

for $t \in [0, T_z]$. By (2) we observe that

$$\begin{cases} f' &\geq z + cf^p \quad \text{on} \quad [t_0, t_0 + T_z] \\ f(t_0) &\geq 0 \end{cases}.$$

We compare $F(t + t_0)$ by f and conclude that

$$f(t) \geq F(t - t_0) \qquad \text{on} \quad [t_0, t_0 + T_z).$$

However, this yields a contradiction since F blows up at T_z.□

We apply Lemma 5.2 with $g(s) = \mathcal{E}_\psi[w](s)$ and $f(s) = \int_{\mathbf{R}^n} \psi^2 |w(y,s)|^2 \rho(y) dy$. By (5.8) with (3.6) and (5.11) we obtain a lower bound for $\mathcal{E}_\psi$.

Theorem 5.2. *Let w be a solution of (3.4) with (3.3). For $\psi \in BC^1(\mathbf{R}^n) \cap C^2(\mathbf{R}^n)$ there exists a positive constant L_2 depending only on n, p, $M_j (j = 0,1,2)$ and a bound for $E[w](0)$ such that*

$$\mathcal{E}_\psi[w](s) \geq -L_2 \quad \text{for all} \quad s \geq 0. \tag{5.12}$$

6 Integral estimate

In this section we shall prove Lemma 4.1 using Theorem 5.1 and Theorem 5.2. For $R > 0$ we shall fix φ so that it satisfies

$$\varphi(y) \in C_0^\infty(\mathbf{R}^n),\ 0 \le \varphi(y) \le 1, \quad \text{and} \quad \varphi(y) = \begin{cases} 1 & \text{on} \quad B_R \\ 0 & \text{on} \quad \mathbf{R}^n \setminus B_{2R} \end{cases}.$$

Lemma 6.1. *Let w be a solution of (3.4) with (3.3). Then*

$$\|w; L_\rho^{p+1}(B_R)\|^{p+1} \le L_3(1 + \|w; W_\rho^{1,2}(B_{2R})\|^2) \qquad \textit{for all} \quad s \ge 0 \tag{6.1}$$

with $L_3 = (p+1)\max\{1/2, L_2\}$.

Proof. The estimate (6.1) follows from the lower bound of local energy $\mathcal{E}_\varphi[w]$ and properties of the cutoff function φ. Indeed,

$$\begin{aligned} \|w; L_\rho^{p+1}(B_R)\|^{p+1} &= \frac{p+1}{2}\int_{\mathbf{R}^n} (|\nabla w|^2 + \beta|w|^2)\varphi^2 \rho dy - (p+1)\mathcal{E}_\varphi[w] \\ &\le L_3(1 + \|w; W_\rho^{1,2}(B_{2R})\|^2). \square \end{aligned}$$

By Lemma 6.1 we observe that to prove Lemma 4.1 it is sufficient to prove an integral estimate for local $W^{1,2}$-norm of w described below.

Lemma 6.2. *Let w be a solution of (3.4) with (3.3). Then for each $q \ge 2$ there exists a positive constant C_q depending only on p, q, R, n and a bound for $E[w](0)$ and for $\|w_0; BC^2(\mathbf{R}^n)\|$ such that*

$$(\mathrm{B}_q) \qquad \int_s^{s+1} \|w; W_\rho^{1,2}(B_R)\|^{2q} d\tau \le C_q \qquad \textit{for all} \quad s \ge 0. \tag{6.2}$$

In this section we prove (6.2) using a bootstrap argument. First, we have to show the first step $q = 2$ of the bootstrap argument. Fortunately, by [9] this step can be easily proved.

Proposition 6.1. *Let w be a solution of (3.4) with (3.3). Then there exists a positive constant K_3 depending only on n, p and a bound for $E[w](0)$ such that*

$$\int_s^{s+1} \|w; W_\rho^{1,2}(B_R)\|^4 d\tau \le K_3 \qquad \textit{for all} \quad s \ge 0.$$

Proof. We square both sides of (5.7), integrate on $[s, s+1]$ and use (3.6) to get

$$\begin{aligned}\int_s^{s+1} \|w; W_\rho^{1,2}(B_R)\|^4 d\tau &\leq \int_s^{s+1} K_1^2(1+\|w_s; L_\rho^2(\mathbf{R}^n)\|)^2 d\tau \\ &\leq 2K_1^2(1+E[w](0)) =: K_3.\end{aligned}$$

The proof is now complete.□

We introduce several constants $\lambda,\ \lambda_2,\ \theta, p_1,\ \tilde{q}$ and α satisfying

$$\lambda_2(q) = \max\left\{2, \frac{2\{(p+1)q-2\}(p+1)(q-1)}{2q(p^2-1)+\{(p+1)q+2\}(2q-p-1)}\right\}. \tag{6.3}$$

$$\lambda \in (\lambda_2(q), \lambda_1(q)). \tag{6.4}$$

$$\theta = (p+1)(\lambda-2)/(p-1)\lambda. \tag{6.5}$$

$$p_1 = (p+1)/p. \tag{6.6}$$

$$q < \tilde{q} < q+2/(p+1). \tag{6.7}$$

$$\alpha = 2/(1-\theta)\tilde{q}. \tag{6.8}$$

We show relations of these constants to prove (B_q) for all $q \geq 2$ by using the bootstrap argument. We also need to recall $\lambda_1(q) = p+1-(p-1)/(q+1)$.

Proposition 6.2. *For $q \geq 2$ we have*

$$\lambda_2(q) < \lambda_1(q),\ \theta \in (0,1),\ q + \frac{2}{p+1} < \frac{2q}{\theta+(1-\theta)q} < \frac{2}{1-\theta} \quad \textit{and} \quad 1 < \theta\tilde{q}\alpha' < 2q.$$

Here α' is the conjugate exponent of α, i.e. $1/\alpha + 1/\alpha' = 1$.

Proof. The proof is just an algebra but we present it for completeness and convenience. Frist we show $\lambda_2(q) < \lambda_1(q)$. Since $p > 1$ we observe that

$$\lambda_1(q) = \frac{(p+1)q+2}{q+1} > \frac{2q+2}{q+1} = 2.$$

So, we may assume that $\lambda_2(q) = 2\{(p+1)q-2\}(p+1)(q-1)/l(q)$ when we set $l(q) = 2q(p^2-1)+\{(p+1)q+2\}(2q-p-1)$. We claim that $l(q) > 0$ since $p > 1$ and $q \geq 2$. Indeed,

$$\begin{aligned} l(q) &= 2q(p^2-1)+\{(p+1)q+2\}(2q-p-1) \\ &= (p+1)\{(p+q-3)q+q^2-2\}+4q > 0.\end{aligned}$$

Thus $\lambda_1(q) - \lambda_2(q)$ equals

$$\frac{(p+1)q+2}{q+1} - \frac{2\{(p+1)q+2\}(p+1)(q-1)}{l(q)} = \frac{q\{(p+1)q+2\}(p-1)^2}{(q+1)l(q)}$$

which is evidently positive.

Next we show $\theta \in (0,1)$. It is clear $\theta > 0$ because $p > 1$ and $\lambda > 2$. Since $p > 1$ and $\lambda < \lambda_1(q) = p + 1 - (p-1)/(q+1)$ we get

$$\begin{aligned} 1-\theta &= 1 - \frac{(p+1)(\lambda-2)}{(p-1)\lambda} \\ &= \frac{2(p+1-\lambda)}{(p-1)\lambda} > 0. \end{aligned}$$

We next show that

$$q + \frac{2}{p+1} < \frac{2q}{\theta + (1-\theta)q} < \frac{2}{1-\theta}.$$

By $\theta > 0$ we easily get

$$\frac{2q}{\theta + (1-\theta)q} < \frac{2q}{(1-\theta)q} = \frac{2}{1-\theta}.$$

By the relation of λ, p and q we observe that

$$\begin{aligned} \frac{2q}{\theta + (1-\theta)q} - \left(q + \frac{2}{p+1}\right) &= \frac{\lambda l(q) - 2\{(p+1)q - 2\}(p+1)(q-1)}{\{(p+1)(\lambda-2) + 2q(p+1-\lambda)\}(p+1)} \\ &> 0 \end{aligned}$$

so that $q + 2/(p+1) < 2q/\{\theta + (1-\theta)q\}$.

We finally show $1 < \theta\tilde{q}\alpha' < 2q$. By the definition of α we see that

$$\theta\tilde{q}\alpha' = \frac{2\theta\tilde{q}}{2 - (1-\theta)\tilde{q}}.$$

Since $q+2/(p+1) < 2/(1-\theta)$ so that $\tilde{q} < 2/(1-\theta)$, the quantity $\theta\tilde{q}\alpha'$ is well-defined as a positive number. Since $\theta \in (0,1)$ and $2 \le q < \tilde{q} < q+2/(p+1) < 2q/\{\theta+(1-\theta)q\}$ we observe that

$$2\theta\tilde{q} - \{2 - (1-\theta)\tilde{q}\} = (1+\theta)\tilde{q} - 2 > 0,$$
$$2q\{2 - (1-\theta)\tilde{q}\} - 2\theta\tilde{q} = 2[2q - \{\theta + (1-\theta)q\}\tilde{q}] > 0.$$

so that $1 < \theta\tilde{q}\alpha' < 2q$.

The proof is now complete.□

Our next goal is to prove (6.2) for all $q \ge 2$. We adjust a bootstrap argument which is used in [18] to prove a priori bound for a global solution u of (1.1) in a bounded domain with zero Dirichlet condition. We recall a well-known $L^p - L^q$ estimate for the heat equation.

Lemma 6.3. ($L^p - L^q$ estimate for heat equation) *Let $v(x,t)$ be a solution of*

$$\begin{cases} v_t &= \Delta v + f & in \quad \Omega \times (0,s), \\ v(x,0) &= v_0(x) \in L^\infty(\Omega) & in \quad \Omega, \\ v(x,t) &= 0 & on \quad \partial\Omega \times (0,s), \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^n$. Assume that v, Δv, and $f \in L^p((0,s);L^q(\Omega))$ for some $1 < p,\ q < \infty$. Then

$$\left(\int_0^s (\|v_t(s);L^q(\Omega)\|^p + \|\Delta v(s);L^q(\Omega)\|^p)ds\right)^{1/p}$$
$$\leq C\left(\left(\int_0^s (\|f;L^q(\Omega)\|^p ds\right)^{1/p} + \|v_0;C^2(\overline{\Omega})\|\right)$$

with a positive constant C depending only on p, q, n and the domain Ω.

For the proof see e.g.[1].

We next observe that φw solves

$$(\varphi w)_s - \Delta(\varphi w) = f \quad \text{with}$$
$$f = -2\nabla\varphi \cdot \nabla w - w\Delta\varphi - \frac{\varphi}{2} y \cdot \nabla w - \beta\varphi w + \varphi|w|^{p-1}w.$$

We apply Lemma 6.3 with $\Omega = B_{2R}$ and obtain Lemma 6.4.

Lemma 6.4. *Let w be a solution of (3.4) with (3.3). Assume that $q \geq 2$. θ, $\tilde{q}$ and α are assumed to satisfy the relation (6.5), (6.7) and (6.8). If* (B_q) *holds then there exists a positive constant J_2 depending only on n, p, q, $\tilde{q}$, R, M_1, M_2 and a bound for $E[w](0)$ and for $\|w_0;C^2(\overline{B_{2R}})\|$ and for C_q such that*

$$\int_s^{s+1} \|\varphi w_s(\tau);L^{p_1}_\rho(B_{2R})\|^{\theta\tilde{q}\alpha'} d\tau \leq J_2 \left(1 + \int_s^{s+1} \|\varphi|w|^{p-1}w;L^{p_1}_\rho(B_{2R})\|^{\theta\tilde{q}\alpha'} d\tau\right) \tag{6.9}$$

for all $s \geq 0$ provided that $\lambda \in (\lambda_2(q), \lambda_1(q))$.

Proof. We apply Lemma 6.3 to get

$$\int_s^{s+1} \|(\varphi w)_s;L^{p_1}_\rho(B_{2R})\|^{\theta\tilde{q}\alpha'} d\tau \leq C\left(\int_s^{s+1} \|f;L^{p_1}_\rho(B_{2R})\|^{\theta\tilde{q}\alpha'} d\tau + \|w_0;C^2(\overline{B_{2R}})\|^{\theta\tilde{q}\alpha'}\right) \tag{6.10}$$

with a positive constant C depending only on p, q, n, $\tilde{q}$ and R. We shall estimate the term involving f.

We apply Hölder's inequality and (3.7) to get

$$\begin{aligned}\|w\Delta\varphi; L^{p_1}_\rho(B_{2R})\| &\le \left(\int_{B_{2R}} |w|^2 \rho dy\right)^{1/2} \left(\int_{B_{2R}} |\Delta\varphi|^{\frac{p^2-1}{2p^2}} \rho dy\right)^{2p^2/(p^2-1)} \\ &\le KM_2\sigma,\end{aligned}$$

where $\sigma = (4\pi)^{np^2/(p^2-1)}$. Similarly we get

$$\begin{aligned}\|\beta\varphi w; L^{p_1}_\rho(B_{2R})\| &\le \frac{1}{p-1}K\sigma, \\ \|\nabla\varphi\cdot\nabla w; L^{p_1}_\rho(B_{2R})\| &\le \sigma M_1\|\nabla w; L^2_\rho(B_{2R})\|, \\ \|(\varphi/2)\ y\cdot\nabla w; L^{p_1}_\rho(B_{2R})\| &\le \sigma R\|\nabla w; L^2_\rho(B_{2R})\|.\end{aligned}$$

Since $\theta\tilde{q}\alpha' < 2q$ by Proposition 6.2, we apply Hölder's inequality with (B_q) to get

$$\begin{aligned}\int_s^{s+1} \|\nabla\varphi\cdot\nabla w; L^{p_1}_\rho(B_{2R})\|^{\theta\tilde{q}\alpha'} d\tau &\le (\sigma M_1)^{\theta\tilde{q}\alpha'} \left(\int_s^{s+1} \|w; L^2_\rho(B_{2R})\|^{2q} d\tau\right)^{\theta\tilde{q}\alpha'/2q} \\ &\le (\sigma M_1 C_q^{1/2q})^{\theta\tilde{q}\alpha'}. \\ \int_s^{s+1} \|\frac{\varphi}{2}y\cdot\nabla w; L^{p_1}_\rho(B_{2R})\|^{\theta\tilde{q}\alpha'} d\tau &\le (\sigma R C_q^{1/2q})^{\theta\tilde{q}\alpha'}.\end{aligned}$$

Combining these estimates, we see that

$$\begin{aligned}&\left(\int_s^{s+1} \|f; L^{p_1}_\rho(B_{2R})\|^{\theta\tilde{q}\alpha} d\tau\right)^{1/\theta\tilde{q}\alpha} \\ &\le \sigma\left(KM_2 + \frac{K}{p-1} + C_q^{1/2q}(M_1+R)\right) + \left(\int_s^{s+1} \|\varphi|w|^{p-1}w; L^{p_1}_\rho(B_{2R})\|^{\theta\tilde{q}\alpha} d\tau\right)^{1/\theta\tilde{q}\alpha}.\end{aligned}$$

Thus by (6.10), the proof of (6.9) is now complete.□

We next apply an upper bound for $\mathcal{E}_\varphi$ to control $W^{1,2}_\rho(B_R)$-norm by $\|\varphi w\varphi w_s; L^1_\rho(B_{2R})\|$.

Proposition 6.3. *Let w be a solution of (3.4) with (3.3). There exists a positive constant J_3 depending only on n, p, R, M_1, M_2 and a bound for $E[w](0)$ such that*

$$\|w(s); W^{1,2}_\rho(B_R)\|^2 \le J_3(1 + \|(\varphi w\varphi w_s)(s); L^1_\rho(B_{2R})\|) \quad \text{for all} \quad s \ge 0. \qquad (6.11)$$

Proof. We use the local energy identity (5.4) :

$$\frac{p-1}{2}\int_{\mathbf{R}^n} \varphi^2(|\nabla w|^2 + \beta|w|^2)\rho dy = \int_{\mathbf{R}^n} \varphi^2 w w_s \rho dy + (p+1)\mathcal{E}_\varphi[w] + \int_{\mathbf{R}^n} 2\varphi w\nabla\varphi\cdot\nabla w\rho dy.$$

The last term equals $2(E_\varphi - \mathcal{E}_\varphi)$ as observed in the proof of Proposition 5.6, which claimes that

$$|E_\varphi - \mathcal{E}_\varphi| \le J_1.$$

We use an upper bound for the local energy $\mathcal{E}_\varphi$ (Theorem 5.1 and Remark 5.1) to get

$$\|w; W^{1,2}_\rho(B_R)\|^2 \le \frac{2}{p-1}(\|\varphi w \varphi w_s; L^1_\rho(B_{2R})\| + (p+1)E_1 + 2J_1).$$

We set $J_3 = 2/(p-1)\max\{1, (p+1)E_1 + 2J_1\}$ and obtain (6.11).□

To proceed our bootstrap argument we observe that if (B_q) is fulfilled for a fixed $q \ge 2$, then by Leemma 6.1 (A_q) holds. This $(\mathrm{A_q})$ with (3.6) implies

$$\|w; L^\lambda_\rho(B_R)\| \le C'_q \tag{6.12}$$

for all $\lambda \in (\lambda_2(q), \lambda_1(q))$ by the interpolation Lemma 4.2.

Lemma 6.5. *Let w be a solution of (3.4) with (3.3). Assume that $q \ge 2$. If (B_q) holds with B_R replaced by B_{4R}, then there exists a positive constant J_4 depending only on n, p, q, $\tilde q$, R, M_1, M_2 and a bound for $E[w](0)$, $\|w_0\|_\infty$ and for C_q such that*

$$\int_s^{s+1} \|w; W^{1,2}_\rho(B_R)\|^{2\tilde q} d\tau \le J_4\left(1 + \left(\int_s^{s+1} \|w; W^{1,2}_\rho(B_{4R})\|^{2\theta\tilde q\alpha' p/(p+1)} d\tau\right)^{1/\alpha'}\right) \tag{6.13}$$

for all $s \ge 0$. Here θ, $\tilde q$ and α are assumed to satisfying (6.5), (6.7) and (6.8).

Proof. We estimate (6.11) by using Hölder's inequality twice with (6.12) since $\theta \in (0,1)$

$$\begin{aligned} \|w; W^{1,2}_\rho(B_R)\|^2 &\le J_3(1 + \|\varphi w; L^\lambda_\rho(B_{2R})\| \|\varphi w_s; L^{\lambda'}_\rho(B_{2R})\|) \\ &\le J_3(1 + C'_q \|\varphi w_s; L^{p_1}_\rho(B_{2R})\|^\theta \|\varphi w_s; L^2_\rho(B_{2R})\|^{1-\theta}), \end{aligned}$$

where λ' is the conjugate exponent of λ. We set $J_4 = J_3 \max\{1, C'_q\}$ to get

$$\begin{aligned} &\int_s^{s+1} \|w; W^{1,2}_\rho(B_R)\|^{2\tilde q} d\tau \\ \le\ & \frac{1}{2}(2J_4)^{\tilde q}\left(1 + \int_s^{s+1} \|\varphi w_s; L^{p_1}_\rho(B_{2R})\|^{\theta\tilde q} \|\varphi w_s; L^2_\rho(B_{2R})\|^{(1-\theta)\tilde q} d\tau\right). \end{aligned}$$

We shall apply Hölder's inequality to the second term of the rightest side. By definition of α we see that $(1-\theta)\tilde{q}\alpha = 2$. So by (3.6) we observe that

$$\begin{aligned}
&\int_s^{s+1} \|\varphi w_s; L^{p_1}_\rho(B_{2R})\|^{\theta\tilde{q}} \|\varphi w_s; L^2_\rho(B_{2R})\|^{(1-\theta)\tilde{q}} d\tau \\
\leq\ & \left(\int_s^{s+1} \|\varphi w_s; L^{p_1}_\rho(B_{2R})\|^{\theta\tilde{q}\alpha'} d\tau\right)^{1/\alpha'} \left(\int_s^{s+1} \|\varphi w_s; L^2_\rho(B_{2R})\|^{(1-\theta)\tilde{q}\alpha} d\tau\right)^{1/\alpha} \\
\leq\ & (E[w](0))^{1/\alpha} \left(\int_s^{s+1} \|\varphi w_s; L^{p_1}_\rho(B_{2R})\|^{\theta\tilde{q}\alpha'} d\tau\right)^{1/\alpha'}.
\end{aligned}$$

We use (6.9) and (6.1) to obtain

$$\int_s^{s+1} \|w; W^{1,2}_\rho(B_R)\|^{2\tilde{q}} d\tau \leq J_5\left(1 + \left(\int_s^{s+1} \|w; W^{1,2}_\rho(B_{4R})\|^{2\theta\tilde{q}\alpha' p/(p+1)} d\tau\right)^{1/\alpha'}\right)$$

with

$$J_5 = \frac{1}{2}(2J_4)^{\tilde{q}}(1 + E[w](0)^{1/\alpha} J_2(1 + L_3^{1/(p+1)})).$$

Thus we obtain (6.13).□

Proof of Lemma 6.2. We show that (6.2):

$$(\mathrm{B}_q) \qquad \int_s^{s+1} \|w; W^{1,2}_\rho(B_R)\|^{2q} d\tau \leq C_{q,R} \qquad \text{for all} \quad q \geq 2 \quad R > 0.$$

By Proposition 6.1 we see that (B_2) holds. Assume that (B_q) holds for a fixed $q \geq 2$ and for $R > 0$. Then (6.13) holds for all $\tilde{q} \in (q, q + 2/(p+1))$. Since $\tilde{q}$ and λ satisfy (6.7) and (6.4), we observe that

$$\frac{2\theta\tilde{q}\alpha' p}{p+1} < 2q. \tag{6.14}$$

We now apply Hölder's inequality to obtain

$$\begin{aligned}
\int_s^{s+1} \|w; W^{1,2}_\rho(B_R)\|^{2\tilde{q}} d\tau &\leq J_4\left(1 + (\int_s^{s+1} \|w; W^{1,2}_\rho(B_{4R})\|^{2q} d\tau)^{1/2q\alpha'}\right) \\
&\leq J_5(1 + C_q^{1/2q\alpha'})
\end{aligned}$$

since (6.14) holds. Thus $(\mathrm{B}_{\tilde{q}})$ holds for all $\tilde{q} \in (q, q + 2/(p+1))$ and all R.

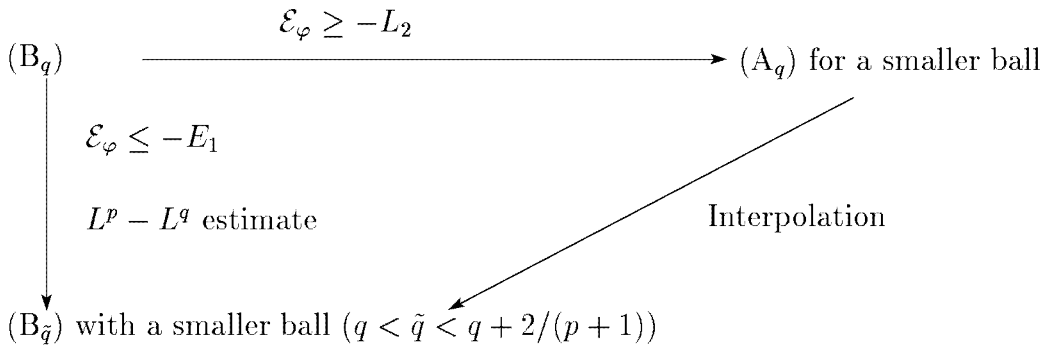
Let $\tilde{q}$ be $\tilde{q} > 2$. Let R_1 be a positive constant. We start (B_2) with $R = 4^m R_1$ and $m = [(q_1 - 2)/\{1/(p+1)\}] + 1$, where $[\ \cdot\ ]$ denotes the integer part. Since (B_q) with $R = R_0$ implies $(\mathrm{B}_{\overline{q}})$ with $\overline{q} = q + 1/(p+1)$ and $R = R_0/4$, repeating this argument m times to get $(\mathrm{B}_{\overline{q}})$ with $R = R_1$. So that (B_q) holds for all $q \geq 2$ and $R > 0$.□

Proof of Lemma 4.1. Since (B_q) holds for all $q \geq 2$, Lemma 6.1 implies (A_q).□

Let us summarize the way of the proof in a figure.

When $q = 2$, (B_q) holds.

If (B_q) holds for a given $q \geq 2$, then the way to prove $(\mathrm{B}_{\tilde{q}})$ is as follows.



References

[1] H. Amann, *Linear and Quasilinear Parabolic Problems,* Volume I: Abstract Linear Theory, Birkhäuser, Basel, 1995.

[2] T. Cazenave and P. -L. Lions, *Solution globales d'équations de la chaleur semi linéaies*, Comm. Partial Diff. Equ. **9** (1984), 955-978.

[3] S. Filippas, M. A. Herrero and J .J L. Velázquez, *Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity*, Proc. R. Soc. Lond. A **456** (2000), 2957-2982.

[4] A. Friedman and B. McLeod, *Blowup of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. **34** (1985), 425-447.

[5] M. Fila and P. Souplet, *The blow-up rate for semilinear parabolic problems on general domains*, Nonlinear Differ. Equ. Appl. **8** (2001), 473-480.

[6] M. Fila, P. Souplet and F. Weissler, *Linear and nonlinear heat equation in L^p_δ spaces and universal bounds for global solutions,* Math. Ann. **320** (2001), 87-113.

[7] Y. Giga, *Solutions for semilinear parabolic equations L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations **62** (1986), 541-554.

[8] Y. Giga and R. V. Kohn, *Asymptotically self-similar blowup of semilinear heat equation*, Comm. Pure Appl. Math. **38** (1985), 297-319.

[9] Y. Giga and R. V. Kohn, *Characterizing blowup using similarity variables*, Indiana Univ. Math J. **36** (1987), 1-40.

[10] Y. Giga and R. V. Kohn, *Nondegeneracy of blowup for semilinear heat equations,* Comm. Pure Appl. Math. **42** (1989), 845-884.

[11] V. A. Galaktionov and J. L. Vázquez, *The problem of blow-up in nonlinear parabolic equations,* Discrete Contin. Dyn. Syst. **8** (2002) 399-433.

[12] M. A. Herrero and J. J. L. Velázquez, *Explosion de solutions d'équations paraboliques semilinéaires supercritiques*, C. R. Acad. Sci. Paris Sér. I Math., **319** (1994), 141-145.

[13] K. Ishige and N. Mizoguchi, *Blow-up behavior for semilinear heat equation with Neumann boundary conditions*, preprint.

[14] C. F. Kamerer, F. Merle and H. Zaag, *Stability of the blow-up profile of nonlinear heat equations from the dynamical system point of view,* Math Ann. **317** (2000) 347-387.

[15] O. A. Ladyzenskaya, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Amer. Math. Soc., Providence, 1968.

[16] J. Matos and P. Souplet, *Universal blow-up rates for a semilinear heat equation and application*, Adv. Diff. Eq., to appear.

[17] F. Merle and H. Zaag, *O.D.E type behavior of blow-up solution of nonlinear heat equations*, Discrete Contin. Dyn. Syst. **8** (2002) 435-450.

[18] P. Quittner, *A priori bounds for global solutions of a semilinear parabolic problem*, Acta Math. Univ. Comenianae Vol. LXVIII, **2** (1999), 195-203.

[19] P. Quittner, *Universal bound for global positive solutions of a superlinear parabolic problem,* Math. Ann. **320** (2001), 299-305.

[20] J. J. Velázquez, *Estimate on (N-1)-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation,* Indiana Univ. Math. Journal **42** (1993), 445-476.

[21] J. J. Velázquez, *Classification of singularities for blowing up solutions in higher dimensions,* Trans. Amer. Math. Soc. **338** (1993), 441-464.