The horospherical geometry of surfaces
in Hyperbolic 4-space

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The horospherical geometry of surfaces in Hyperbolic 4-space

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Abstract

We study some geometrical properties associated to the contacts of surfaces with hyperhorospheres in $H_+^4(-1)$. We introduce the concepts of osculating hyperhorospheres, horobinormals, horoasymptotic directions and horospherical points and provide conditions ensuring their existence. We show that totally semiumbilical surfaces have orthogonal horoasymptotic directions.

1   Introduction

A hypersurface given by the intersection of the hyperbolic n-space $H_+^n(-1)$ with a spacelike, timelike or lightlike hyperplane of $I\!\!R_1^{n+1}$ is respectively called *hypersphere, equidistant hyperplane* or *hyperhorosphere*. The last ones have the curious property of inheriting a euclidean geometry as submanifolds of the hyperbolic space. A great deal of the properties of submanifolds of euclidean spaces can be studied at the light of their contacts with invariant subsets, such as hyperplanes or hyperspheres, of the ambient space (see for instance, [7], [8] or [9]). In a similar way, given any submanifold $M \subset H_+^n(-1)$, the study of its contacts with hyperhorospheres leads to certain properties that give rise to the *horospherical geometry* of M. An introduction to this for hypersurfaces in $H_+^n(-1)$ has been given in [3]. On the other hand, the geometry associated to the contacts of surfaces with lightlike hyperplanes in $I\!\!R_1^4$ has been described in [4] through the analysis of the singularities of the *lightcone height functions family* $H : M \times S_+^2 \to I\!\!R$. These are tightly related to those of the lightcone Gauss map. The contacts that concern us here, namely, those of a surface

$M = g(U) \subset H_+^4(-1)$ with hyperhorospheres can be similarly described by means of the *lightcone height functions family* $H : M \times S_+^3 \to \mathbb{R}$. This setting allows us to see that M may have at some points a stronger contact with certain hyperhorospheres, that we call *osculating hyperhorospheres*. They play an equivalent role to that of osculating hyperplanes of surfaces immersed in euclidean 4-space introduced in [7]. We notice that there is an essential difference: whereas in the euclidean case the maximum number of osculating hyperplanes at a point of a generic surface is 2, we show in our case that there may be up to four osculating hyperhorospheres at each point of a generic surface immersed in $H_+^4(-1)$ (Proposition 4.4). We define the *horospherical points* as those at which some osculating hyperhorosphere has a contact of corank 2 with the surface. They are the analogous of the inflection points in the euclidean case. We show in Proposition 4.6 that the curvature ellipse at these points is degenerate (i. e., they are semiumbilic, see [5]). We characterize them as critical points of certain direction fields on M that we introduce and call *horoasymptotic* here. We obtain some conditions guaranteeing the global existence of such fields on the surface (Theorem 5.2) and from these that of horospherical points (Corollary 5.3). This leads us to put forward the following horospherical version of Caratheodory's conjecture:

Any 2-sphere immersed as an everywhere horohyperbolic surface in hyperbolic 4-space has at least 2 horospherical points.

Finally, we show that, provided the horoasymptotic fields are globally defined on M, total semiumbilicity implies orthogonality of their integral lines (theorem 5.4).

2   Basic concepts and notations

We consider the $(n + 1)$-dimensional Minkowski space $(\mathbb{R}^{n+1}, \langle , \rangle)$, with the pseudo scalar product given by $\langle (x_1, x_2, ..., x_{n+1}), (y_1, y_2, ..., y_{n+1}) \rangle = -x_1 y_1 + x_2 y_2 + ... + x_{n+1} y_{n+1}$. We shall denote this space by $\mathbb{R}_1^{n+1}$.

We say that a vector $\mathbf{x} = (x_1, ..., x_{n+1}) \in \mathbb{R}_1^{n+1} \backslash \{\mathbf{0}\}$ is *spacelike, timelike* or *lightlike* provided $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $= 0$ or < 0 respectively. The norm or *length* of a vector $\mathbf{x} \in \mathbb{R}_1^4$ is defined by $||\mathbf{x}|| = (|\langle \mathbf{x}, \mathbf{x} \rangle|)^{\frac{1}{2}}$.

Given a vector $\mathbf{v} \in \mathbb{R}_1^{n+1}$ and a real number c, we define the *hyperplane with pseudonormal* $\mathbf{v}$ as

$$HP(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} | \langle \mathbf{x}, \mathbf{v} \rangle = c\}.$$

This hyperplane is said to be *spacelike, timelike* or *lightlike* according to $\mathbf{v}$ is timelike, spacelike or lightlike.

We define the *Hyperbolic n-space* by

$$H_+^n(-1) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} | \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_1 \geq 1\}$$

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The hypersurfaces of $H^n_+(-1)$ determined by intersecting it with a space-like, a timelike or a lightlike hyperplane are respectively called *hyperspheres*, *equidistant hyperplanes* and *hyperhorospheres*.

We also consider the set

$$LC_{\mathbf{a}} = \{\mathbf{x} \in I\!R^{n+1}_1 | \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0\}$$

known as the *n-dimensional lightcone with vertex* $\mathbf{a} \in I\!R^{n+1}_1$. The subset,

$$S^{n-1}_+(\mathbf{a}) = \{\mathbf{x} = (x_1, x_2, ..., x_{n+1}) | \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0, x_1 = a_1 + 1\}$$

is the *lightcone* $(n-1)$-*sphere* centered at $\mathbf{a} = (a_1, a_2, \ldots, a_{n+1})$. It shall be denoted as S^{n-1}_+ when centered at the origin.

Suppose that M is a surface immersed in $I\!R^{n+1}_1$. We say that M is a *spacelike surface* if the tangent plane $T_{\mathbf{x}}M$ is spacelike (i.e., consists of spacelike vectors) and thus a euclidean plane $(T_{\mathbf{x}}M, \langle , \rangle)$ for every point $\mathbf{x} \in M$. In this case, the normal space $N_{\mathbf{x}}M$ is a Lorentz $(n-1)$-space $((N_{\mathbf{x}}M, \langle , \rangle)$.

3 Second fundamental form and curvature ellipses

Given a smooth oriented surface M immersed in $I\!R^5_1$, we denote by $\mathcal{X}(M)$ and $\mathcal{N}(M)$ the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Consider the second fundamental map,

$$\alpha : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{N}(M), \ \alpha(X, Y) = \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y,$$

where $\bar{\nabla}$ denotes the pseudoriemannian connection of $I\!R^5_1$ and $\bar{X}$ and $\bar{Y}$ are local extensions to $I\!R^5_1$ of tangent vector fields X and Y on M. This map is well defined, symmetric and bilinear. Given any normal field $\nu \in \mathcal{N}(M)\backslash\{0\}$, we have for each $\mathbf{x} \in M$ a function

$$H_\nu : T_{\mathbf{x}}M \times T_{\mathbf{x}}M \to I\!R, \ H_\nu(\mathbf{v}, \mathbf{w}) = \langle \alpha(\mathbf{v}, \mathbf{w}), \nu(\mathbf{x}) \rangle.$$

which is also symmetric and bilinear. The *second fundamental form* of M at $\mathbf{x}$ is the associated quadratic form,

$$II_\nu : T_{\mathbf{x}}M \to I\!R, \ II_\nu() = H_\nu \mathbf{v}, \mathbf{v}).$$

Suppose that M is locally defined at $\mathbf{x}$ by $g : I\!R^2 \to I\!R^5_1$ such that $g(0,0) = \mathbf{x}$ and choose local isothermic coordinates $\{x, y\}$ on M and a pseudo-orthonormal frame, $\{e_1, e_2, e_3, e_4, e_5\}$ in a neighbourhood of $\mathbf{x} = g(0,0) \in M$, such that $\{e_1, e_2, e_3\}$ is a normal frame and $\{e_4, e_5\}$ a tangent frame, with $\langle e_1, e_1 \rangle = -1$

and $\langle e_i, e_i \rangle = 1$, $i = 2, ..., 5$. Then the matrix of the bilinear form H_{e_i} is given by

$$H_{e_i}(\mathbf{x}) = \begin{bmatrix} a_i & b_i \\ b_i & c_i \end{bmatrix},$$

where if $ds^2 = E(dx^2 + dy^2)$ is the first fundamental form, we have

$$a_i = \frac{1}{E} \langle \frac{\partial^2 g}{\partial x^2}(0,0), e_i \rangle, \quad b_i = \frac{1}{E} \langle \frac{\partial^2 g}{\partial x \partial y}(0,0), e_i \rangle, \quad c_i = \frac{1}{E} \langle \frac{\partial^2 g}{\partial y^2}(0,0), e_i \rangle,$$

for $i = 1, 2, 3$.

Given $\mathbf{x} \in M$, consider the linear map induced by the second fundamental form on M, $A_{\mathbf{x}} : N_{\mathbf{x}}M \to Q^2$, where Q^2 is the space of quadratic forms in two variables. That is, $A_{\mathbf{x}}(\mathbf{v}) = II_{\mathbf{v}}, \forall \mathbf{v} \in N_{\mathbf{x}}M$.

In the same coordinates as above, we have that if $\mathbf{v} = v_1 e_1 + v_2 e_2 + v_3 e_3$, then

$$A_{\mathbf{x}}(\mathbf{v}) = \frac{1}{E} \left(v_1 \langle d^2 g(0,0), e_1 \rangle + v_2 \langle d^2 g(0,0), e_2 \rangle + v_3 \langle d^2 g(0,0), e_3 \rangle \right)$$

and thus the matrix of $A_{\mathbf{x}}$ is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

We say that a point $\mathbf{x} \in M$ into the type $M_i, i = 3, 2, 1, 0$ provided rank$A_{\mathbf{x}} = i$. It was shown in ([8], Propositions 2 and 3) that for generic surfaces in euclidean 5-space the M_3-points fill an open and dense submanifold, whereas the M_2 points form closed regular curves and the M_1 and M_0 points can be avoided. We remark that the arguments there used can be easily adapted to the case that concerns us here, so that we can conclude that the same assertions hold for generic spacelike surfaces immersed in 5-dimensional Minkowski space.

Given $\mathbf{x} \in M$, consider the unit circle in $T_{\mathbf{x}}M$ parametrized by the angle $\theta \in [0, 2\pi]$. Denote by γ_θ the spacelike curve obtained by intersecting M with the timelike hyperplane defined by the direct sum of the normal subspace $N_{\mathbf{x}}M$ and the straight line in the tangent direction represented by θ. The curvature vector $\eta(\theta)$ of γ_θ in $\mathbf{x}$ lies in the timelike hyperplane $N_{\mathbf{x}}M$. Varying θ from 0 to 2π, the vector $\eta(\theta)$ describes an ellipse in $N_{\mathbf{x}}M$, called the *curvature ellipse* of M at $\mathbf{x}$. This ellipse is the image of the affine map (see [5] for the case $n = 3$, the case $n \geq 4$ is a straightforward generalization)

$$\eta : S^1 \subset T_{\mathbf{x}}M \longrightarrow N_{\mathbf{x}}M$$

given by

$$\theta \longmapsto \eta(\theta) = \sum_{i=1}^{3} \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \cdot \begin{bmatrix} a_i & b_i \\ b_i & c_i \end{bmatrix} \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \cdot e_i,$$

that is,

$$\eta(\theta) = H_{\mathbf{x}} + B_{\mathbf{x}} \cos 2\theta + C_{\mathbf{x}} \sin 2\theta,$$

with

$$H_{\mathbf{x}} = \frac{1}{2}(a_1 + c_1)e_1 - \frac{1}{2} \sum_{i=2}^{3} (a_i + c_i) \cdot e_i,$$

$$B_{\mathbf{x}} = \frac{1}{2}(a_1 - c_1)e_1 - \frac{1}{2} \sum_{i=2}^{3} (a_i - c_i) \cdot e_i,$$

$$C_{\mathbf{x}} = b_1 e_1 - \sum_{i=2}^{3} b_i \cdot e_i.$$

Lemma 3.1 *Given a spacelike surface $M \subset I\!\!R_1^5$, the subspace $\mathrm{Ker} A_{\mathbf{x}}$ determined by the kernel of $A_{\mathbf{x}}$ in $N_{\mathbf{x}}M$ is pseudo-orthogonal to the vectors $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$ that define the curvature ellipse.*

Proof: Suppose that $\mathbf{x} \in M_i, i < 3$, for otherwise $\mathrm{Ker} A_{\mathbf{x}} = \{0\}$ and the result is trivial. Given $\mathbf{v} = v_1 e_1 + v_2 e_2 + v_3 e_3 \in \mathrm{Ker} A_{\mathbf{x}}$, we have that $a_1 v_1 + a_2 v_2 + a_3 v_3 = b_1 v_1 + b_2 v_2 + b_3 v_3 = c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. Then $2\langle \mathbf{v}, B_{\mathbf{x}} \rangle = -v_1(a_1 - c_1) - v_2(a_2 - c_2) - v_3(a_3 - c_3) = 0$ and $\langle \mathbf{v}, C_{\mathbf{x}} \rangle = -v_1 b_1 - v_2 b_2 - v_3 b_3 = 0$ $\square$

The curvature ellipse at $\mathbf{x}$ is contained in the Lorentz 3-space $N_{\mathbf{x}}M$ and it may be degenerate (a segment or even a point) at certain points of $\mathbf{x} \in M$ that are called *semiumbilics*. A semiumbilic point $\mathbf{x}$ is said to be *spacelike, timelike* or *lightlike* provided the curvature segment defines respectively a spacelike, timelike or lightlike direction in $N_{\mathbf{x}}M$. The points at which the curvature ellipse becomes a point are degenerate semiumbilics known as *umbilics*. It is a straightforward exercise to verify that any semiumbilic point is of type $M_i, i < 3$. We notice that although M_1 points are either semiumbilic or umbilic, not every point of type M_2 needs to be a semiumbilic. Moreover, it was shown in [10] that the semiumbilics of generically immersed surfaces in euclidean 5-space are isolated points (lying on curves of M_2- points) and it is not difficult to see that similar arguments apply to the case of surfaces generically immersed in Minkowski 5-space. A surface all whose points are semiumbilic is said to be *totally semiumbilical*. Some of the properties of totally semiumbilical surfaces in $I\!\!R_1^{n+1}$ were studied in [5]. In particular, for surfaces contained in hyperbolic 4-space we have,

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Proposition 3.2 *([5]) Given a surface $M \subset H_+^4(-1)$, the curvature ellipse of M at a point $\mathbf{x} \in M$ is contained in an affine plane of $N_{\mathbf{x}}M$ parallel to $T_{\mathbf{x}}H_+^4(-1) \cap N_{\mathbf{x}}M$.*

We shall see as a consequence of this, in the next section, that the generic behaviour of semiumbilic points of surfaces contained in $H_+^4(-1)$ differs from the case of surfaces in $I\!R_1^5$.

The *shape operator* associated to a normal field ν is defined as

$$S_\nu : TM \to TM, \ S_\nu(X) = -(\bar{\nabla}_{\bar{X}}\bar{\nu})^\top,$$

where $\bar{\nu}$ is a local extension to $I\!R_1^5$ of the normal vector field ν at $\mathbf{x}$ and $()^\top$ means the tangent component. This operator is bilinear, self-adjoint and satisfies the following equation: $\langle S_\nu(X), Y \rangle = H_\nu(X, Y), \forall X, Y \in \mathcal{X}(M)$. So, we have that $II_\nu(X) = \langle S_\nu(X), X \rangle$.

We can find for each $\mathbf{x} \in M$, an orthonormal basis of $T_{\mathbf{x}}M$ consisting of eigenvectors of S_ν, for which the restriction of the second fundamental form to the unitary vectors, $II_\nu|_{S^1}$, takes its maximal and minimal values. The corresponding eigenvalues k_1, k_2 are the ν-*principal curvatures*, respectively. A point $\mathbf{x}$ is said to be ν-*umbilic* if both ν-principal curvatures coincide at $\mathbf{x}$. Let $\mathcal{U}_\nu$ be the set of ν-umbilics in M. For any $\mathbf{x} \in M \backslash \mathcal{U}_\nu$ there are two ν-principal directions defined by the eigenvectors of S_ν, these fields of directions are smooth and integrable, then they define two families of orthogonal curves, its integrals, which are called the ν-*principal lines of curvature*. The two orthogonal foliations with the ν-umbilics as its singularities form the ν-*principal configuration* of M. We say that the surface M is ν-*umbilical* if each point of M is ν-umbilic. Some umbilicity properties of surfaces immersed in Minkowski spaces have been studied in [5]. It was there proved (Proposition 5.1) that a point $\mathbf{x}$ of a surface $M \subset I\!R_1^5$ is ν-umbilic for some normal field ν if and only if $\nu(\mathbf{x})$ is pseudo-orthogonal to the vectors $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$ that define the curvature ellipse at $\mathbf{x}$. In particular, in the case of a surface $M \subset H_+^4(-1)$ we have that in virtue of Proposition 3.2 as follows:

If $\rho(\mathbf{x}) = \mathbf{x}$ is the position (normal) field on M, then each point of M is ρ-umbilical.

Moreover,

Proposition 3.3 *([5]) A surface $M \subset H_+^4(-1)$ is totally semiumbilical if and only if M is umbilical with respect to some lightlike normal field.*

We also have

Corollary 3.4 *([5]) A surface $M \subset H_+^n(-1)$ lies in a hyperhorosphere if and only if it is umbilical with respect to some lightlike normal field ν with constant zero curvature.*

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4 Contacts with lightlike hyperplanes and hyperhorospheres

Given submanifolds M and N of $I\!\!R^{n+1}$, locally defined by $M = g(I\!\!R^m)$ and $N = f^{-1}(0)$, where $g : I\!\!R^m \to I\!\!R^{n+1}$ is an embedding, and $f : I\!\!R^{n+1} \to I\!\!R^q$ is a submersion, we can "measure their contact at a common point $p \in M \cap N$ by analyzing the singularities of the composed map $f \circ g : I\!\!R^m \to I\!\!R^q$ ("contact map"). In fact, we say that two submanifolds M and N of a manifold Z have the same contact at a point p as two submanifolds M' and N' of a manifold Z' at a point p' if and only if there is a diffeomorphism germ $\phi : (Z, p) \to (Z', p')$ taking M to M' and N to N'. In this case we write, $K(M, N) = K(M', N')$. J.A. Montaldi proved [9] that this holds if and only if their respective contact map-germs are $\mathcal{K}$-equivalent, where we say that two map-germs $f_i : (I\!\!R^m, x_i) \to (I\!\!R^p, y_i)$, $i = 1, 2$ are contact-equivalent or $\mathcal{K}$-equivalent (denoted $\mathcal{K}(f_1) = \mathcal{K}(f_2)$) if there is a diffeomorphism-germ (contact-equivalence), $H : (I\!\!R^m \times I\!\!R^p, (x_1, y_1)) \to (I\!\!R^m \times I\!\!R^p, (x_2, y_2))$ of the form $H(x, t) = (h(x), \theta(x, t))$, such that $H(x, y_1) = (h(x), y_2)$ and $H(x, f_1(x)) = (h(x), f_2(h(x)))$. We refer to [2] or [6] for the definition and details on $\mathcal{K}$-equivalence.

Therefore, to study the contact of a spacelike surface locally given as $M = g(I\!\!R^2) \subset I\!\!R_1^5$ with a hyperplane $H(\mathbf{v}, \mathbf{x}) = H(\mathbf{v}, \langle \mathbf{x}, \mathbf{v} \rangle)$ at a point $\mathbf{x} = g(u) \in M \cap H$, the map f has to be chosen in such a way that $H(\mathbf{v}, \mathbf{x}) = f^{-1}(0)$, that is,

$$f(x_1, ..., x_5) = -x_1 \cdot v_1 + ... + x_5 \cdot v_5 - \langle \mathbf{x}, \mathbf{v} \rangle,$$

where $\mathbf{v} = (v_1, ..., v_5)$ is a pseudonormal vector to $H(\mathbf{v}, \mathbf{x})$.

And hence, to analyze all the possible contacts of the submanifold $M = g(I\!\!R^2)$ with the lightlike hyperplanes of $I\!\!R^5$, we must describe the singularities of the *lightcone height functions family*

$$H : I\!\!R^2 \times S_+^3 \longrightarrow I\!\!R$$
$$(u, \mathbf{v}) \longmapsto < g(u), \mathbf{v} >$$

We shall denote by $h_\mathbf{v}$ the function obtained when fixing the parameter $\mathbf{v}$. Clearly, u is a singular point of $h_\mathbf{v}$ if and only if $\mathbf{v} \in N_{\mathbf{x}(u)}M$.

Suppose now that M lies in $H_+^4(-1)$. Given $\mathbf{v} \in S_+^3$ and $\mathbf{x} \in M$, let us denote by $\Omega(\mathbf{v}, \mathbf{x})$ the hyperhorosphere determined on $H_+^4(-1)$ by the lightlike hyperplane with pseudo-normal $\mathbf{v}$ passing trough the point $\mathbf{x} = g(u)$, $H(\mathbf{v}, \mathbf{x})$. We have that $\Omega(\mathbf{v}, \mathbf{x})$ is tangent to M at $\mathbf{x}$ if and only if u is a singular point of $h_\mathbf{v}$. Furthermore,

Lemma 4.1 *Given a surface $M = g(I\!\!R^2) \subset H^4_+(-1)$ and $\mathbf{x} \in M$, the contact map-germs of the pairs $(M, \Omega(\mathbf{v}, \mathbf{x}))$ and $(M, H(\mathbf{v}, \mathbf{x}))$ at $\mathbf{x}$ coincide.*

Proof: We have that $H(\mathbf{v}, \mathbf{x}) = h^{-1}_{(\mathbf{v}, \mathbf{x})}(0)$, where $h_{(\mathbf{v}, \mathbf{x})} : I\!\!R^5_1 \to I\!\!R$ is given by $h_{(\mathbf{v}, \mathbf{x})}(\mathbf{p}) = \langle \mathbf{v}, \mathbf{p} \rangle - \langle \mathbf{v}, \mathbf{x} \rangle$. So, if we represent by $i : H^4_+(-1) \to I\!\!R^5_1$ the canonical inclusion, we have that the contact map for $H(\mathbf{v}, \mathbf{x})$ and M is given by $h_{(\mathbf{v}, \mathbf{x})} \circ i \circ g$. On the other hand, if we denote $\bar{h}_{(\mathbf{v}, \mathbf{x})} = h_{(\mathbf{v}, \mathbf{x})}|_{H^4_+(-1)}$, we have that $\bar{h}^{-1}_{(\mathbf{v}, \mathbf{x})}(0) = H(\mathbf{v}, \mathbf{x}) \cap H^4_+(-1) = \Omega(\mathbf{v}, \mathbf{x})$, and hence $\bar{h}_{(\mathbf{v}, \mathbf{x})} \circ g$ is the contact map for M and $\Omega(\mathbf{v}, \mathbf{x})$. But, clearly, $\bar{h}_{(\mathbf{v}, \mathbf{x})} \circ g = h_{(\mathbf{v}, \mathbf{x})} \circ i \circ g$. □

Given a singular point u of the function $h_\mathbf{v}$, if the hessian matrix $Hessh_\mathbf{v}(u)$ defines a degenerate quadratic form we say that $\mathbf{v}$ is a *horobinormal* direction for M at $\mathbf{x} = g(u)$. In this case we have that $\Omega(\mathbf{v}, \mathbf{x})$ has higher order contact with M at $\mathbf{x}$ and we call it *osculating hyperhorosphere*. A normal field ν defined on some open subset V of M and such that $\nu(\mathbf{x})$ is a horobinormal direction at $\mathbf{x}, \forall \mathbf{x} \in V$ is called a *horobinormal field on V*.

Given $\mathbf{x} \in M$, consider the linear map $A_\mathbf{x} : N_\mathbf{x} M \to Q^2$ and denote by C the cone of degenerate quadratic forms in Q^2. We observe that $\mathbf{v} \in N_\mathbf{x} M$ determines a horobinormal if and only if $\mathbf{v} \in A^{-1}_\mathbf{x}(C) \cap LC_\mathbf{x}$.

A particular feature of the spacelike surfaces contained in hyperbolic 4-space is the following.

Lemma 4.2 *The points of type M_2 of a surface $M \subset H^4_+(-1)$ are all semium-bilic.*

Proof: Take a point $\mathbf{x} \in M$ of type M_2 and suppose that M is locally defined at $\mathbf{x}$ by an embedding $g : I\!\!R^2 \to H^4_+(-1)$, such that $\mathbf{x} = g(0, 0)$. Then the height function $h_\mathbf{x}(u) = \langle g(u), \mathbf{x} \rangle - \langle g(0, 0), \mathbf{x} \rangle = \langle g(u), \mathbf{x} \rangle + 1$ describes the contact of M at $\mathbf{x}$ with the hyperplane $H_\mathbf{x}$ pseudo-orthogonal to the vector $\mathbf{x}$. Observe that if we take g in the Monge form $g(u) = (u, g_1(u), g_2(u), g_3(u))$, it is not difficult to verify that the hessian matrix of $h_\mathbf{x}$ at $(0, 0)$ coincides with that of $A_\mathbf{x}(\mathbf{x})$. If we assume now that $\mathbf{x}$ is not semiumbilic, it follows from Lemma 3.1 together with Proposition 3.2 that $\text{Ker} A_\mathbf{x}$ is spanned by the position vector $\mathbf{x}$ and hence $Hessh_\mathbf{x}(0, 0)$ is the null matrix. This means that $(0, 0)$ is a non stable singularity of $h_\mathbf{x}$, which implies that the extension of this function to $H^4_+(-1)$ also has a non stable singularity at $\mathbf{x} \in H^4_+(-1)$. But it can be seen that the contacts of $H^4_+(-1)$ with all its tangent hyperplanes are non degenerate in the sense that they have the minimal possible order, or in other words, they lead to a stable height function. So we arrive to a contradiction. □

This leads to the following result concerning the distribution of semiumbilic points over surfaces generically immersed in $H^4_+(-1)$, which supposes an interesting difference with respect to the generic behaviour of surfaces immersed in Minkowski 5-space.

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Proposition 4.3 *Given a surface M generically immersed in $H_+^4(-1)$, the points of type M_3 fill an open and dense submanifold and the semiumbilic points are all of type M_2 and define a closed curves embedded in M. Umbilic points do not appear on these surfaces.*

Proof: Let $\Delta(\mathbf{x}) = det A_{\mathbf{x}}$. It is clear that $\Delta^{-1}(0) = M - M_3$. Since Δ is a continuous function on M, we have that M_3 must be an open region in M. The condition that $\mathbf{x} \in M_2$ implies by Lemma 4.2 that $\mathbf{x}$ is semiumbilic. But this is equivalent to requiring that the normal vectors $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$ be linearly dependent. Since by Proposition 3.2 we know that $B_{\mathbf{x}}, C_{\mathbf{x}} \in T_{\mathbf{x}} H_+^4(-1)$, it follows that the position vector $\mathbf{x}$ must be pseudo-orthogonal to both $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$. We can give M by a local embedding $g : I\!\!R^2 \to H_+^4(-1) \subset I\!\!R_1^5$ in the Monge form at $\mathbf{x}$ and take a pseudo-orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$ for M in a neighbourhood of the point $\mathbf{x}$ in such a way that e_1 is the position vectorfield, e_2 and e_3 are normal vectorfields and e_4, e_5 generate the tangent planes. In these coordinates, we can write $B_{\mathbf{x}} = -\frac{1}{2}(a_2 - c_2) \cdot e_2 - \frac{1}{2}(a_3 - c_3) \cdot e_3$ and $C_{\mathbf{x}} = -b_2 \cdot e_2 - b_3 \cdot e_3$. Then the linearly dependence of these two vectors is given by the requirement

$$\frac{a_2 - c_2}{b_2} = \frac{a_3 - c_3}{b_3},$$

which defines a 1-codimensional algebraic variety of the jet space $J^2(I\!\!R^2, I\!\!R^5)$. It follows now from the Thom Transversality Theorem ([2]) that the 2-jet extension, $j^2 g : I\!\!R^2 \to J^2(I\!\!R^2, I\!\!R^5)$, meets this submanifold transversally and therefore, the considered points determine an algebraic subset of codimension 1 in M.

On the other hand, the condition that $\mathbf{x} \in M_1 \cup M_0$ is equivalent to asking that $rank A_{\mathbf{x}} \le 1$. This means that the vector $H_{\mathbf{x}}$ must also be parallel to both $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$. This provides at least three independent quadratic equations in $J^2(I\!\!R^2, I\!\!R^5)$ and by using again Thom Transversality Theorem we can conclude that, for a generic g, $j^2 g$ does not meet the corresponding algebraic variety, and thus $M_1 \cup M_0 = \emptyset$.

So $\Delta^{-1}(0) = M_2$ is completely made of semiumbilic points. We see now that they form embedded curves. In fact, let

$$
\begin{aligned}
g \quad : \quad & I\!\!R^2, 0 \quad \longrightarrow \quad I\!\!R_1^5 \\
& (u, v) \quad \longmapsto \quad (u, v, g_1(u, v), g_2(u, v), g_3(u, v))
\end{aligned}
$$

be the local representation of M in the Monge form at $\mathbf{x} \in M_2$. In these coordinates $\Delta(\mathbf{x}) = g_{1uu} g_{2uv} g_{3vv} - g_{1uv} g_{2uu} g_{3vv} - g_{1uu} g_{2vv} g_{3uv} + g_{1vv} g_{2uu} g_{3uv} + g_{1uv} f_{2vv} f_{3uu} - f_{1vv} f_{2uv} f_{3uu}$. It follows from this expression that, under appropriate transversality conditions on the 3-jet of g, the set $\Delta = 0$ represents a curve possibly with isolated singular points determined by the vanishing of the derivatives of the function Δ. We observe that the pseudo-orthogonality property of the frame $\{e_1, e_2, e_3, e_4, e_5\}$ is irrelevant for our study. For a change of

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basis in $N_{\mathbf{x}}M$ preserves the relative position of $Im(A_{\mathbf{x}})$ with the cone C in Q^2, and thus the sets M_3 and M_2. So we can take $\{e_1, e_2, e_3\}$ such that e_1 generates $Ker(A_{\mathbf{x}})$.

If $p \in M_2$, we have three possibilities:

i) $Im(A_{\mathbf{x}}) \cap C$ is a couple of lines,

ii) $Im(A_{\mathbf{x}}) \cap C$ is a line, and

iii) $Im(A_{\mathbf{x}}) \cap C$ is just the origin.

In case i) we can choose $\{e_2, e_3\}$ as the two (degenerate) directions lying in $A^{-1}(C) \subset N_p M$. Furthermore, we can also make a change of coordinates in the source, such that the two degenerate directions correspond to the quadratic forms u^2 and v^2 in C. Thus g can be locally written as

$$g(u,v) = (u, v, u^2 + R_1(u,v), v^2 + R_2(u,v), R_3(u,v)),$$

where $R_i \in m^3$, i.e., all the derivatives of the R_i vanish up to order 3, $i = 1, 2, 3$.

In case ii) $Im(A_{\mathbf{x}})$ is tangent to C and we take e_3 as the generator of $A_{\mathbf{x}}^{-1}(Im(A_{\mathbf{x}}) \cap C)$. With additional change of coordinates in the source, g can be written as

$$g(u,v) = (u, v, u^2 - v^2 + R_1(u,v), uv + R_2(u,v), R_3(u,v)),$$

Finally, in case iii), all the quadratic forms $A_{\mathbf{x}}(\mathbf{v}$ are hyperbolic, and g can be written as

$$g(u,v) = (u, v, u^2 + R_1(u,v), uv + R_2(u,v), R_3(u,v)),$$

In each of the above cases it is a simple (but tedious) calculation to verify that under generic conditions on the 3-jet of g at $(0,0)$, the derivatives of the function Δ do not vanish at $\mathbf{x}$ and thus it is a regular point of $\Delta^{-1}(0)$. $\square$

We analyse next the possibilities that we may have for the sets $A_{\mathbf{x}}^{-1}(C) \cap LC_{\mathbf{x}}$ at different points $\mathbf{x} \in M_i, i = 3, 2, 1, 0$:

a) If $\mathbf{x} \in M_3$, then $A_{\mathbf{x}}^{-1}(C)$ is a nondegenerate cone. The intersection $A_{\mathbf{x}}^{-1}(C) \cap LC_{\mathbf{x}}$ depends on the relative position of both cones and may thus consist in either four, three, two, one or no lines in $N_{\mathbf{x}}M$. We remark that it is also possible that both cones $A_{\mathbf{x}}^{-1}(C)$ and $LC_{\mathbf{x}}$ coincide, but this is a extremely degenerate phenomena that can be generically avoided, so we shall not consider it here.

b) If $\mathbf{x} \in M_2$, as we have seen before, the plane $ImA_{\mathbf{x}}$ intersects the cone C in either i) two lines, ii) one line or iii) just the origin. In this case $A_{\mathbf{x}}^{-1}(C)$ is respectively made of i) two planes with the common line $KerA_{\mathbf{x}}$, ii) a plane containing the line $KerA_{\mathbf{x}}$, or iii) just the line $KerA_{\mathbf{x}}$. Again, the intersection $A_{\mathbf{x}}^{-1}(C) \cap LC_{\mathbf{x}}$ depends on the relative position of both subsets and will thus consist in at most four lines in $N_{\mathbf{x}}M$.

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c) When $\mathbf{x} \in M_1$, $Im A_\mathbf{x}$ is a line that may either lie on C or intersect it just at the origin. Correspondingly, $A_\mathbf{x}^{-1}(C)$ is the whole normal space $N_\mathbf{x} M$, or a plane. It follows that $A_\mathbf{x}^{-1}(C) \cap LC_\mathbf{x}$ may consist of the whole $LC_\mathbf{x}$ in the first case, or at most two lines in the second.

d) Finally, in case $\mathbf{x} \in M_0$, $A_\mathbf{x}^{-1}(C)$ is the whole normal space and $A_\mathbf{x}^{-1}(C) \cap LC_\mathbf{x} = LC_\mathbf{x}$.

Therefore, by taking into account that generic surfaces in $H_+^4(-1)$ are exclusively made of points of types M_3 and semiumbilics (M_2), we can state the following

Proposition 4.4 *The number of osculating hyperhorospheres at any point of a surface M generically immersed in $H_+^4(-1)$ is at most four.*

We say that a point $\mathbf{x} \in M$ of type M_3 or M_2 is *horoelliptic* provided the subsets $A_\mathbf{x}^{-1}(C)$ and $LC_\mathbf{x}$ intersect only at the origin. On the other hand, it is said to be *horohyperbolic or horoparabolic* according to they have transversal or nontransversal intersections off the origin. It is not difficult to verify that, generically, horoelliptic and horohyperbolic points determine open submanifolds of M separated by horoparabolic curves. For the particular case of a non semiumbilic M_2-point x we observe that x is necessarily horohyperbolic, having four horobinormals in case b,i), horoparabolic with two horobinormals in case b,ii) and horoelliptic with no horobinormals in case b,iii). On the other hand, if $x \in M_2$ is semiumbilic then both cases, b,i) and b,ii) may correspond to either horohyperbolic, horoparabolic or horoelliptic points. Here we observe that horohyperbolic points may have either four or two horobinormal directions, due to the fact that the line $Ker A_x$ does not need to lie inside the lightcone. Taking into account these considerations, we can conclude

Proposition 4.5 *If $M \subset H_+^4(-1)$ is exclusively made of horohyperbolic points then it has either four or two globally defined horobinormal fields.*

It can be shown that, as a consequence of the methods developed by Montaldi ([9]) and analogously to what happens in the case of surfaces generically immersed in Euclidean space, the rank of $Hessh_\mathbf{v}(u)$, for any horobinormal $\mathbf{v}$ at most points $\mathbf{x} \in M$ is 1. The points at which this rank is 0 are those at which the surface is better approached by the same hyperhorosphere in all the tangent directions. These can be seen as the analogous, in horospherical geometry terms, of the inflection points of surfaces in Euclidean 4-space studied in [1] and shall be called *horospherical points*.

Proposition 4.6 *The horospherical points of a surface M immersed in $H_+^4(-1)$ are either of semiumbilic or umbilic type. Moreover, every point of type M_1 or M_0 is a horospherical point.*

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Proof: Suppose that M is given in the Monge form at a horospherical point $\mathbf{x} = g(0,0)$ and take a pseudo-orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$ in a neighbourhood of $\mathbf{x}$ such that $\{e_4, e_5\}$ is a tangent frame and $\{e_1, e_2, e_3\}$ is a normal frame in this neighbourhood with $\langle e_1, e_1 \rangle = -1$, as above. Then, for any normal vector $\mathbf{v} \in N_{\mathbf{x}}M$, we can write $\mathbf{v} = v_1 e_1 + v_2 e_2 + v_3 e_3$. We observe that in this situation the matrices $A_{\mathbf{x}}(\mathbf{v})$ and $Hessh_{\mathbf{v}}(0,0)$ coincide. Now, under the assumption that $\mathbf{x}$ is horospherical, we can choose a horobinormal vector $\mathbf{v} \in N_{\mathbf{x}}M \cap S^3_+(\mathbf{x})$, such that all the entries of the matrix $Hessh_{\mathbf{v}}(0,0)$ vanish, which implies that $II_{\mathbf{v}}$ also vanishes at $\mathbf{x}$. So we have that $-v_1 a_1 + v_2 a_2 + v_3 a_3 = -v_1 b_1 + v_2 b_2 + v_3 b_3 = -v_1 c_1 + v_2 c_2 + v_3 c_3 = 0$, where the $a_i, b_i, c_i, i = 1, 2, 3$ are as in the previous section. But this means that the vectors $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$, that determine the curvature ellipse at $\mathbf{x}$ are both pseudo-orthogonal to the lightlike direction $\mathbf{v}$. On the other hand, it follows from proposition 3.2 that they are also pseudo-orthogonal to the timelike normal direction, $\mathbf{x}$, to $H^4_+(-1)$ at $\mathbf{x}$. Therefore $\mathrm{rank}(B_{\mathbf{x}}, C_{\mathbf{x}}) \leq 1$, which implies that the curvature ellipse is degenerated at $\mathbf{x}$, which shows the first assertion. As for the second, we observe that provided $\mathbf{x} \in M_1$, we know from Lemma 3.1 that the plane $Ker A_{\mathbf{x}}$ is pseudo-orthogonal to the direction determined by the (linearly dependent) vectors $B_{\mathbf{x}}$ and $C_{\mathbf{x}}$. But Proposition 3.2 implies that this direction is contained in the plane $T_{\mathbf{x}}H^4_+(-1)$. Therefore we get that $Ker A_{\mathbf{x}}$ must contain the line $\langle \mathbf{x} \rangle$ and thus cut the lightcone at $\mathbf{x}$. This determines two horobinormals for which the Hessian of the corresponding lightcone height function has rank 1, and hence $\mathbf{x}$ is a horospherical point. In case $\mathbf{x} \in M_0$ we have that all the horobinormals give rise to lightcone height functions whose Hessian has rank 1 at $\mathbf{x}$ and the result also follows. $\square$

In particular, we remark that the surfaces contained in a hyperhorosphere of $H^4_+(-1)$ are a special case of totally semiumbilical surface as can be concluded from Proposition 3.4 and Corollary 3.5.

5 Horoasymptotic directions

Given a surface M immersed in $H^4_+(-1)$, if $\mathbf{v} \in S^3_+(\mathbf{x})$ is a horobinormal of M at $\mathbf{x} = \mathbf{X}(u)$ we have that u is a degenerate singularity for the height function $h_{\mathbf{v}}$. Here, we consider a local parametrization $\mathbf{X} : U \to H^4_+(-1)$ of M at $\mathbf{x}$. Therefore $Ker Hess(h_{\mathbf{v}})(u) \neq \{0\}$. The non zero directions lying in $Ker Hess(h_{\mathbf{v}})(u)$ are called *horoasymptotic* directions at $\mathbf{x}$. We observe that these are the tangent directions at $\mathbf{x}$ along which the higher order contact of M and the hyperhorosphere $\Omega(\mathbf{v}, \mathbf{x})$ occurs. Clearly, any horobinormal field determines a (tangent) horoasymptotic field on the region of M over which it is defined. It follows from the definition of both, horoasymptotic directions and horospherical points, that the last are the critical points of the horoasymptotic fields. We investigate next the possibilities of having some globally defined horoasymptotic field on M.

Lemma 5.1 *i) Two horobinormals* $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *at a point* $\mathbf{x} \in M$ *of type* M_3 *cannot share horoasymptotic directions.*

ii) Provided $\mathbf{x} \in M$ *is of type* $M_i, i < 3$, *two horobinormals,* $\mathbf{b}_1$ *and* $\mathbf{b}_2$, *share a horoasymptotic direction if and only if they belong to some linear subspace contained in* $A_{\mathbf{x}}^{-1}(C)$.

Proof: i) Suppose that θ is a common asymptotic direction for $\mathbf{b}_1$ and $\mathbf{b}_2$ at $\mathbf{x}$. In such case we can choose coordinates on M at $\mathbf{x}$ in such a way that

$$Hess(h_{\mathbf{b}_1})(\mathbf{x}) = \left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & 0 \end{array} \right] \quad \text{and} \quad Hess(h_{\mathbf{b}_2})(\mathbf{x}) = \left[\begin{array}{cc} \lambda_2 & 0 \\ 0 & 0 \end{array} \right].$$

But then the normal direction $\mathbf{b} = \lambda_2 \mathbf{b}_1 - \lambda_1 \mathbf{b}_2$ has vanishing Hessian matrix at $\mathbf{x}$, which means that $\mathbf{x}$ is a horospherical point. By Proposition 4.3 this implies that $\mathbf{x}$ is a semiumbilic point and these are of type $M_i, i < 3$ we have arrived to a contradiction.

ii) A similar argument to the used in i) tells us that any common horoasymptotic direction for $\mathbf{b}_1$ and $\mathbf{b}_2$ at $\mathbf{x}$ lies in the kernel of any linear combination of them. Therefore $\mathbf{b}_1$ and $\mathbf{b}_2$ define a plane made of degenerate directions of $N_{\mathbf{x}}M$ in the sense that all of them are mapped by $A_{\mathbf{x}}$ into the cone C. Conversely, take any $\mathbf{b} \in A_{\mathbf{x}}^{-1}(C)$ that does not belong to $KerA_{\mathbf{x}}$ (if all the degenerate directions lie in $KerA_{\mathbf{x}}$ we have that all of them share all the tangent directions in $T_{\mathbf{x}}M$ as horoasymptotic directions and the result is trivially true). Then given any other $\mathbf{b}'$ lying in the same subspace than $\mathbf{b}$ in $A_{\mathbf{x}}^{-1}(C)$, we can always write $\mathbf{b}' = \lambda_1 \mathbf{b} + \lambda_2 \mathbf{b}''$, for some $\mathbf{b}'' \in KerA_{\mathbf{x}}$ and real numbers $\lambda_i, i = 1, 2$. It is not difficult to see that then $\mathbf{b}$ and $\mathbf{b}'$ share horoasymptotic directions. $\square$

Theorem 5.2 *i) A generic surface* $M \subset H_+^4(-1)$ *all whose points are horohyperbolic has either two or four globally defined horoasymptotic fields that may eventually coincide pairwise over a closed subset of semiumbilic points.*

ii) If M *is totally semiumbilical with isolated horospherical points and such that all its points are horohyperbolic, then* M *has either one or two horoasymptotic fields globally defined that may eventually coincide over some closed subset.*

Proof: i) In this case the horohyperbolicity ensures the existence of either two or four globally defined horobinormal fields which cannot be coincident at any point, for in this case it would be horoparabolic. Then Lemma 5.1 a) guarantees the existence of either two or four horoasymptotic fields respectively over the open submanifold M_3. Part ii) of Lemma 5.1 tells us that the horoasymptotic directions coincide by pairs over the semiumbilic points. ii) In this case the

horohyperbolicity condition ensures the existence of either two or four horobinormals at each point. Due to the absence of horoparabolic points, we can assert that these determine globally defined horobinormal fields on M, which in turn determine either one or two horoasymptotic fields. Again, we have that the

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horobinormal pairs may be coincident at points for which the image of $A_\mathbf{x}$ is tangent to the cone C and this determines that the two horoasymptotic directions will also be coincident at such points. $\square$

Corollary 5.3 *A generic surface in $H_+^4(-1)$ exclusively made of horohyperbolic points which is compact without boundary and has nonvanishing Euler number has horospherical points.*

Proof: This follows immediately by applying the Poincaré-Bendixon index formula to any of the horoasymptotic direction fields guaranteed by part i) of Proposition 5.2. $\square$

We observe that the same assertion is valid for surfaces satisfying either of the conditions ii) or iii) in Proposition 5.2.

It is worth to recall at this point the *Carathéodory's conjecture*. This asserts that a 2-sphere immersed in $I\!\!R^3$ has at least two umbilic points. We observe that the inverse of stereographic projection takes umbilics of surfaces in 3-space to inflection points of their images in the S^3 considered as surfaces in euclidean 4-space. This naturally leads to the following *generalized Carathéodory's conjecture: 2-spheres convexly embedded in $I\!\!R^4$ have at least two inflection points*, where we observe that the convexity property is equivalent to the global existence of asymptotic directions (see [1] for a proof in the case of generic surfaces in euclidean 4-space). Now, by considering the analogy of this situation with the case treated here, we establish the following

Conjecture: *A 2-sphere immersed as an everywhere horohyperbolic surface in hyperbolic 4-space has at least two horospherical points.*

We finally find a necessary condition for the semiumbilicity of a spacelike surface in terms of the horoasymptotic directions.

Theorem 5.4 *Suppose that M is a surface in $H_+^4(-1)$ with two globally defined horoasymptotic fields and isolated horospherical points. If M is totally semiumbilical then the horoasymptotic directions are mutually orthogonal everywhere except at the horospherical points.*

Proof: First of all we observe that M is totally semiumbilical if and only there exist two normal fields ν_1, ν_2, locally defined and linearly independent at every non umbilical point of M, such that M is ν_i-umbilical ([5], Theorem 5.6). On the other hand, we have that M admits two linearly independent normal fields locally defined for each $\mathbf{x} \in M$ for which M is umbilical if and only if M admits a unique principal configuration. In fact, given $\mathbf{x} \in M$, consider isothermal coordinates $\{u, v\}$ in a neighbourhood $U_\mathbf{x}$ of $\mathbf{x}$ and suppose that ν^1 and ν^2 are normal fields defined on $U_\mathbf{x}$ such that M is ν^j-umbilical. Without loss of generality we can take ν^1 as the position vectorfield on M and ν^2 as a vectorfield

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ν tangent to $H^4_+(-1)$. We can take now another normal field ξ such that $\{\rho, \xi, \nu\}$ defines a pseudo-orthonormal frame for the normal bundle $NU_{\mathbf{x}}$. Then given any normal field η, we can write $\eta = k_1\rho + k_2\xi + k_3\nu$, for appropriate smooth functions $k_1, k_2, k_3 : U \to I\!\!R$. The coefficients of the second fundamental form in the direction of η are given by

$$e_\eta = \left\langle \frac{\partial^2 \mathbf{X}}{\partial x^2}, k_1\rho + k_2\xi + k_2\nu \right\rangle = k_1 e_\rho + k_2 e_\xi + k_3 e_\nu,$$
$$f_\eta = \left\langle \frac{\partial^2 \mathbf{X}}{\partial x \partial y}, k_1\rho + k_2\xi + k_3\nu \right\rangle = k_1 f_\rho + k_2 f_\xi + k_3 f_\nu,$$
$$g_\eta = \left\langle \frac{\partial^2 \mathbf{X}}{\partial y^2}, k_1\rho + k_2\xi + k_3\nu \right\rangle = k_1 g_\rho + k_2 g_\xi + k_3 g_\nu.$$

And the equation of the curvature lines in this coordinates becomes (see [5], [11])

$$h\left(f_\xi du^2 + (g_\xi - e_\xi)dudv - f_\xi dv^2\right) + k_1\left(f_{\nu^1}du^2 + (g_{\nu^1} - e_{\nu^1})dudv - f_{\nu^1}dv^2\right) + k_2\left(f_{\nu^2}du^2 + (g_{\nu^2} - e_{\nu^2})dudv - f_{\nu^2}dv^2\right) = 0.$$

Since M is ρ- and ν-umbilical, we have that $e_\rho(\mathbf{x}) = g_\rho(\mathbf{x})$, $f_\rho(\mathbf{x}) = 0$ and $e_\nu(\mathbf{x}) = g_\nu(\mathbf{x})$, $f_\nu(\mathbf{x}) = 0$, for all $\mathbf{x} \in M$ and thus $f_\rho du^2 + (g_\rho - e_\rho)dudv - f_\rho dv^2 = 0$ and $f_\nu du^2 + (g_\nu - e_\nu)dudv - f_\nu dv^2 = 0$ Therefore, the principal configuration associated to η is given by $h\left(f_\xi du^2 + (g_\xi - e_\xi)dudv - f_\xi dv^2\right) = 0$. So both fields η and ξ have the same principal configurations.

Conversely, given $M \subset H^4_+(-1)$, we know that it is ρ-umbilical, where ρ is the position field. Take $\mathbf{x} \in M$ and let η_1 and η_2 be normal fields on a neighbourhood $U_{\mathbf{x}}$ of $\mathbf{x}$ in M, lying in $T_{\mathbf{x}}H^4_+(-1)$ (i.e., pseudo-orthogonal to ρ) that are linearly independent on $U_{\mathbf{x}}$. Their respective principal configurations are given by the equations $f_{\eta_i}du^2 + (g_{\eta_i} - e_{\eta_i})dudv - f_{\eta_i}dv^2 = 0$, for $i = 1, 2$. Since M admits a unique principal configuration, we must have that $f_{\eta_1} = \lambda f_{\eta_2}$ and $g_{\eta_1} - e_{\eta_1} = \lambda\left(g_{\eta_2} - e_{\eta_2}\right)$, for some function λ on $U_{\mathbf{x}}$. Taking $\bar{\nu} = \eta_1 - \lambda\eta_2$ we have that $f_{\bar{\nu}} = f_{\eta_1} - \lambda f_{\eta_2} = 0$ and $g_{\bar{\nu}} - e_{\bar{\nu}} = g_{\eta_1} - e_{\eta_1} - \lambda\left(g_{\eta_2} - e_{\eta_2}\right) = 0$. Therefore M is $\bar{\nu}$-umbilical. We observe that $\bar{\nu}$ and ρ are linearly independent on $U_{\mathbf{x}}$, for they are pseudo-orthogonal.

Finally, suppose that θ_1 and θ_2 are the two distinct horoasymptotic fields globally defined on M and let $b_i, i = 1, 2$ be the corresponding horobinormal fields, which must be distinct too. By taking appropriate coordinates on M we can see that the direction θ_i is a principal direction for the shape operator S_{b_i} and that its corresponding principal curvature vanishes everywhere. Since M has a unique principal configuration, we have that the principal configurations of b_1 and b_2 coincide and thus θ_2 (resp. θ_1) must be the principal direction of b_1 (resp. b_2) corresponding to the nonvanishing principal curvature. But this means that θ_1 and θ_2 must be everywhere orthogonal, except at the critical points of the principal configurations. $\square$

Remark: In the case of a surface M immersed in Euclidean 4-space, we have that the above is also a sufficient condition for total semiumbilicity ([12]). This is due to the fact that it can be shown that if $b_i, i = 1, 2$ are the horobinormal fields on M and $k_i, i = 1, 2$ are the corresponding non vanishing curvatures, then the normal field $\nu = k_2 b_1 - k_1 b_2$ is umbilical over M, and in this case this is a

sufficient condition for semiumbilicity of M. In the horospherical case, we need to require the existence of some umbilical field over M that is everywhere tangent to $H^4_+(-1)$, and to be able to ensure this we should have that $\nu = k_2 b_1 - k_1 b_2$ is not a multiple of the position vectorfield ρ over M. It is not clear at all that this needs to be true on any surface having everywhere orthogonal horoasymptotic fields. In any case we can assert that under the orthogonality assumption on the horoasymptotic fields M is totally semiumbilical provided the normal field $\nu = k_2 b_1 - k_1 b_2$ is not a multiple of ρ.

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