STRONGLY SUPERCOMMUTING SELF-ADJOINT OPERATORS

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STRONGLY SUPERCOMMUTING SELF-ADJOINT OPERATORS

Tadahiro Miyao

We introduce the notion of strong supercommutativity of self-adjoint operators on a $\mathbb{Z}_2$-graded Hilbert space and give some basic properties. We clarify that strong supercommutativity is a unification of strong commutativity and strong anticommutativity. We also establish the theory of super quantization. Applications to supersymmetric quantum field theory and a fermion-boson interaction system are discussed.

1 Introduction

The notion of strong commutativity of self-adjoint operators is well-known in functional analysis. Theory of strongly anticommuting self-adjoint operators was established by Vasilescu [12] and some progress have been done by some authors [1, 2, 3, 4, 5, 9, 11]. (The authors of [9, 12] call the notion of strong anticommutativity simply anticommutativity, but to be definite, we call it strong anticommutativity. The same applies to commutativity of self-adjoint operators.)

The aim of this paper is to construct a theory which is able to deal with strong commutativity and strong anticommutativity simultaneously, and to give some applications of this theory. In costructing our theory, we will introduce the notion of strong supercommutativity which plays important roles in this paper. Roughly speaking, strong supercommutativity is a generalization of strong commutativity and strong anticommutativity of self-adjoint operators in a $\mathbb{Z}_2$-graded Hilbert space. By introducing this notion, we will see that strong supercommutativity unifies strong commutativity and strong anticommutativity. Since the theory of superalgebra can be interpreted as a unification of commutativity and anticommutativity, our theory can be regarded as the operator version theory of the superalgebra.

In quantum physics, bosons and fermions are fundamental objects. Bosons (resp. fermions) are deeply connected with commutativity (resp. anticommutativity). On the other hand, supersymmetric quantum physics is a theory which impartially deal with bosons and fermions. Therefore we can regard the supersymmetric quantum theory as the theory which unifies commutativity and anticommutativity. By this reason, we can expect that our theory is fitted for supersymmetric quantum theory. Indeed we will see this as an application of the theory of strong supercommutativity in Section 5. We note that in [3, 5] an application of the theory of strongly anticommuting self-adjoint operators is given.

The present paper is organized as follows. In Section 2 we will review some basic properties of the superalgebra and strong (anti)commuting self-adjoint operators. In Section 3 we introduce the notion of strong supercommutativity of self-adjoint operators in $\mathbb{Z}_2$-graded Hilbert space and prove some fundamental facts with respect to strongly supercommuting self-adjoint operators. Section 4 is concerned with the theory of super quantization which is associated with supersymmetric quantum mechanics. In Section 5 we will give some applications of our theory to quantum physics.

2 Preliminaries

2.1 $\mathbb{Z}_2$-graded structure

Let $\mathbb{Z}_2$ be the residue class ring mod 2, with the elements $\bar{0}$ and $\bar{1}$. When applied to elements of $\mathbb{Z}_2$, the symbol "+" always denotes addition modulo 2.

If V is a vector space over $\mathbb{C}$ then a *$\mathbb{Z}_2$-grading of V* is a decomposition of V into an direct sum $V_{\bar{0}} \oplus V_{\bar{1}}$ where $V_{\bar{0}}$ and $V_{\bar{1}}$ are subspaces of V. A vector space equipped with a $\mathbb{Z}_2$-grading is said to be a *$\mathbb{Z}_2$-graded vector space*. The elements of $V_{\bar{0}} \cup V_{\bar{1}}$ are said to be *homogeneous*. The elements of $V_{\bar{0}}$ are called *even*, those of $V_{\bar{1}}$ are *odd*. The zero element 0 of V is the unique element of $V_{\bar{0}} \cap V_{\bar{1}}$ and so is both odd and even. If $v \in V_{\bar{\alpha}}$ ($\alpha \in \{0,1\}$), then we can define a function $\rho : V_{\bar{0}} \cup V_{\bar{1}} \to \{0,1\}$ by

$$\rho(v) := \alpha$$

and we say that the value $\rho(v)$ is the *parity* of v. Throughout this paper, for any $\mathbb{Z}_2$-graded vector space or any structure with an underlying $\mathbb{Z}_2$-graded vector space (such as the $\mathbb{Z}_2$-graded algebra to be defined in this subsection) we denote the parity of homoeneous element v by $\rho(v)$.

Let A be an algebra over $\mathbb{C}$. We say that the algebra A is $\mathbb{Z}_2$-graded if the underlying vector space of A is $\mathbb{Z}_2$-graded, that is, there is a direct sum decomposition $A_{\bar{0}} \oplus A_{\bar{1}}$ as a vector space, and furtheremore, for α, β in $\mathbb{Z}_2$ we have

$$A_\alpha A_\beta \subset A_{\alpha+\beta}.$$

It is common for $\mathbb{Z}_2$-graded algebras to be called *superalgebras*.

A *Lie superalgebra* is a superalgebra A whose multiplication, which we denote "$\cdot$", satifies the following relations for arbitary homogeneous elements a, b, c in A:

$$a \cdot b = -(-1)^{\rho(a)\rho(b)} b \cdot a,$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c + (-1)^{\rho(a)\rho(b)} b \cdot (a \cdot c).$$

If $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is an associative superalgebra, then introducing a new multiplication $[\cdot, \cdot]_{\mathrm{S}}$ on A by the formula

$$[a, b]_{\mathrm{S}} := ab - (-1)^{\rho(a)\rho(b)} ba$$

for homogeneous elements a, b in A, we can easily check that the superalgebra A equipped with this multiplication is Lie superalgebra and denote this Lie superalgebra by A_{SLie}. The operation $[\cdot, \cdot]_{\mathrm{S}}$ is said to be *supercommutator*.

Suppose that $\mathcal{H}$ is a Hilbert space. If $\mathcal{H}$ is a direct sum of $\mathcal{H}_{\bar{0}}$ and $\mathcal{H}_{\bar{1}}$ (i.e., $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$), where $\mathcal{H}_{\bar{0}}$ and $\mathcal{H}_{\bar{1}}$ are closed subspaces of $\mathcal{H}$, then $\mathcal{H}$ is said to be $\mathbb{Z}_2$*-graded Hilbert space*. It is clear that $\mathbb{Z}_2$-graded Hilbert space is a $\mathbb{Z}_2$-graded vector space. Let $P_{\bar{0}}$ and $P_{\bar{1}}$ be orthogonal projections onto $\mathcal{H}_{\bar{0}}$ and $\mathcal{H}_{\bar{1}}$, respectively. We define an operator τ on $\mathcal{H}$ by

$$\tau := P_{\bar{0}} - P_{\bar{1}}.$$

It is not difficult to see that τ is self-adjoint and unitary. We refer to the operator τ as the *grading operator* for $\mathcal{H}$.

Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded operators on $\mathcal{H}$. If $B \in \mathcal{B}(\mathcal{H})$ satisfies

$$B = \tau B \tau,$$

then B is said to be *even operator*, on the other hand, if B satisfies

$$B = -\tau B \tau,$$

then B is said to be *odd operator*. It is easy to see that $B \in \mathcal{B}(\mathcal{H})$ is even and odd if and only if $B = 0$. A linear operator on $\mathcal{H}$ is said to be *homogeneous*, if it is even or odd. For a homogeneous operator B in $\mathcal{B}(\mathcal{H})$, we define

$$\rho(B) := \begin{cases} 0 & \text{if } B \text{ is even} \\ 1 & \text{if } B \text{ is odd} \end{cases}$$

and say that the value $\rho(B)$ is the *parity* of B.

Proposition 2.1 *Suppose that $B \in \mathcal{B}(\mathcal{H})$. Then B can be uniquely written as the sum of even operator $B_{\bar{0}}$ and odd oprator $B_{\bar{1}}$:*

$$B = B_{\bar{0}} + B_{\bar{1}}.$$

Proof. The even part $B_{\bar{0}}$ (resp. odd part $B_{\bar{1}}$) is given by

$$B_{\bar{0}} = P_{\bar{0}} B P_{\bar{0}} + P_{\bar{1}} B P_{\bar{1}} \quad (\text{resp. } B_{\bar{1}} = P_{\bar{0}} B P_{\bar{1}} + P_{\bar{1}} B P_{\bar{0}}).$$

To check the uniqueness is easy. □
For each α in $\mathbb{Z}_2$, we define

$$\mathcal{B}(\mathcal{H})_\alpha := \{B \in \mathcal{B}(\mathcal{H}) \mid B \text{ is homogeneous with } \rho(B) = \alpha\}.$$

Then it follows from Proposition 2.1 that $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H})_{\bar{0}} \oplus \mathcal{B}(\mathcal{H})_{\bar{1}}$. Hence $\mathcal{B}(\mathcal{H})$ is a $\mathbb{Z}_2$-graded associative algebra. We denote $\mathcal{B}(\mathcal{H})_{\mathrm{SLie}}$ the Lie superalgebra $\mathcal{B}(\mathcal{H})$ equipped with the supercommutator $[\cdot,\cdot]_{\mathrm{S}}$. We note that for each A, B in $\mathcal{B}(\mathcal{H})$,

$$[A, B]_{\mathrm{S}} = [A_{\bar{0}}, B_{\bar{0}}] + [A_{\bar{0}}, B_{\bar{1}}] + [A_{\bar{1}}, B_{\bar{0}}] + \{A_{\bar{1}}, B_{\bar{1}}\},$$

where $A_{\bar{0}}$ (resp. $B_{\bar{0}}$) is the even part of A (resp. B), $A_{\bar{1}}$ (resp. $B_{\bar{1}}$) is the odd part of A (resp. B) and $[a, b] := ab - ba$ (commutator) and $\{a, b\} := ab + ba$ (anticommutator).

Next we extend the notion of the parity to unbounded operators in $\mathcal{H}$. For this purpose, we define

$$\mathcal{L}(\mathcal{H}) := \{B : \text{linear operator on } \mathcal{H} \text{ s.t. } P_\alpha \mathrm{dom}(B) \subset \mathrm{dom}(B) \text{ for each } \alpha \in \mathbb{Z}_2\},$$

where we denote by $\mathrm{dom}(B)$ the domain of B. Note that for each B in $\mathcal{L}(\mathcal{H})$, $\tau \mathrm{dom}(B) = \mathrm{dom}(B)$. If $B \in \mathcal{L}(\mathcal{H})$ satisfies

$$B = \tau B \tau,$$

then B is said to be *even*. On the other hand, if

$$B = -\tau B \tau,$$

then B is said to be *odd*. The notions of homogeneous element and parity are defined by the same way in the case of bounded operators. For simplicity, we say that B is homogeneous (or even, odd) without mentioning $B \in \mathcal{L}(\mathcal{H})$.

It is well known that every linear operator on $\mathcal{H}$ is represented as a 2×2 matrix with entries being linear operator. For example, we have

$$P_{\bar{0}} = \begin{pmatrix} I_{\bar{0}} & 0 \\ 0 & 0 \end{pmatrix}, \ P_{\bar{1}} = \begin{pmatrix} 0 & 0 \\ 0 & I_{\bar{1}} \end{pmatrix}, \ \tau = \begin{pmatrix} I_{\bar{0}} & 0 \\ 0 & -I_{\bar{1}} \end{pmatrix},$$

where $I_\alpha (\alpha \in \mathbb{Z}_2)$ is the identity operator on $\mathcal{H}_\alpha$. We can easily see that an even (resp. odd) operator is represented as a diagonal (resp. off-diagonal) matrix.

For an element B in $\mathcal{L}(\mathcal{D})$, we introduce

$$B_{\bar{0}} := P_{\bar{0}} B P_{\bar{0}} + P_{\bar{1}} B P_{\bar{1}}, \ \ \mathrm{dom}(B_{\bar{0}}) = \mathrm{dom}(B),$$
$$B_{\bar{1}} := P_{\bar{0}} B P_{\bar{1}} + P_{\bar{1}} B P_{\bar{0}}, \ \ \mathrm{dom}(B_{\bar{1}}) = \mathrm{dom}(B).$$

Then it is clear that $B_{\bar{0}}$ (resp. $B_{\bar{1}}$) is even (resp. odd) and

$$B = B_{\bar{0}} + B_{\bar{1}}.$$

We say that $B_{\bar{0}}$ (resp. $B_{\bar{1}}$) is even (resp. odd) part of B. Note that if B is self-adjoint, then $B_{\bar{0}}$ and $B_{\bar{1}}$ are also self-adjoint. Since we have defined the notions of even part and odd part for any element in $\mathcal{L}(\mathcal{D})$, we can extend the supercommutator $[\cdot,\cdot]_S$ to each pair of elements in $\mathcal{L}(\mathcal{D})$.

Let $M(\mathbb{R})$ be the set of all complex valued Borel measurable functions on $\mathbb{R}$. If $f \in M(\mathbb{R})$ satisfies

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$, then f is said to be *even*. If f satisfies

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$, then f is said to be *odd*. We can also define the notions of homogenous element and parity of Borel measurable functions by the same way in the case of linear operators.

Proposition 2.2 *Suppose that A is a homogeneous self-adjoint operator on $\mathcal{H}$ and f is a homogeneous element in $M(\mathbb{R})$. Then $f(A)$ is also homogeneous with*

$$\rho(f(A)) = \rho(f)\rho(A).$$

Here $f(A)$ is given by the operational calculus.

Proof. Since τ is unitary and self-adjoint, we have

$$\tau f(A)\tau = f(\tau A\tau) = f((-1)^{\rho(A)}A) = (-1)^{\rho(A)\rho(f)}f(A)$$

by the operational calculus. □

Suppose that J is a subset of $\mathbb{R}$. If

$$J = -J,$$

J is said to be *even*, where $-J := \{-\lambda \mid \lambda \in J\}$. On the other hand, if

$$J \cap (-J) = \emptyset,$$

then J is said to be *odd*. The notions of homogeneous element and parity are clear.

Let $\mathbb{B}^1$ is the Borel field of $\mathbb{R}$. Following proposition is useful:

Proposition 2.3 *Suppose that $f \in M(\mathbb{R})$ is real valued and homogeneous and that J is in $\mathbb{B}^1$. If J is homogeneous, then $f^{-1}(J)$ is homogeneous with*

$$\rho(f^{-1}(J)) = \rho(f)\rho(J).$$

In particular, if $\rho(f) = 1$, then $-f^{-1}(J) = f^{-1}(-J)$ for each J in $\mathbb{B}^1$. Moreover, if $\rho(f) = 0$, then for each J in $\mathbb{B}^1$, we have $\rho(f^{-1}(J)) = 0$.

Proof. Suppose that $\rho(f) = 0$. Then for each J in $\mathbb{B}^1$,

$$\begin{aligned} -f^{-1}(J) &= \{-\lambda \in \mathbb{R} \mid f(\lambda) \in J\} \\ &= \{\lambda \in \mathbb{R} \mid f(-\lambda) \in J\} \\ &= \{\lambda \in \mathbb{R} \mid f(\lambda) \in J\} \\ &= f^{-1}(J). \end{aligned}$$

Thus $\rho(f^{-1}(J)) = 0$. Next suppose that $\rho(f) = 1$. Then by the similar argument in the above, we have

$$-f^{-1}(J) = f^{-1}(-J).$$

Hence we have

$$f^{-1}(J) \cap (-f^{-1}(J)) = f^{-1}(J \cap (-J)).$$

From this, we can conclude that $\rho(f^{-1}(J)) = \rho(J)$. □

Finally, we give a simple proposition:

Proposition 2.4 *Suppose that A is an odd self-adjoint operator. Then we have*

$$\rho(\sigma(A)) = 0,$$

where $\sigma(A)$ denote the spectrum of A.

Proof. $\sigma(A) = \sigma(\tau A \tau) = \sigma(-A) = -\sigma(A)$. □

2.2 Strong commutativity and strong anticommutativity

For the reader's convenience, we present the definitions of the strong commutativity and strong anticommutativity, and give some basic properties of these objects. More details on these objects can be found, e.g., in [9, 10, 11, 12].

Definition 2.5 Suppose that A and B are self-adjoint operators on a Hilbert space.

(i) A and B are said to be strongly commute if their spectral measure commute.

(ii) A and B are said to be strongly anticommute if $\mathrm{e}^{\mathrm{i}tB}A \subset A\mathrm{e}^{-\mathrm{i}tB}$ for all $t \in \mathbb{R}$.

The following proposition for strongly commuting self-adjoint operators is well-known:

Proposition 2.6 *Let A and B be self-adjoint operators on a Hilbert space. Then the following three statements are equivalent:*

(i) *A and B strongly commute.*

(ii) *For each $\lambda, \mu \in \mathbb{C}\backslash\mathbb{R}$, $R_\lambda(A)R_\mu(B) = R_\mu(B)R_\lambda(A)$, where $R_\nu(T) = (T - \nu)^{-1}$ for a self-adjoint operator T.*

(iii) *For all $s, t \in \mathbb{R}$, $\mathrm{e}^{\mathrm{i}tA}\mathrm{e}^{\mathrm{i}sB} = \mathrm{e}^{\mathrm{i}sB}\mathrm{e}^{\mathrm{i}tA}$.*

We can also charactrize strong commutativity as follows:

Proposition 2.7 *Let A and B be self-adjoint operators on a Hilbert space. Then the following three statements are equivalent:*

(i) *A and B strongly commute.*

(ii) $\mathrm{e}^{\mathrm{i}tA}B \subset B\mathrm{e}^{\mathrm{i}tA}$ *for all $t \in \mathbb{R}$.*

(iii) *For each $z \in \mathbb{C}\backslash\mathbb{R}$, $R_z(A)B \subset BR_z(A)$.*

Proposition 2.8 *Let A and B be self-adjoint operators on a Hilbert space. Then the following three statements are equivalent:*

(i) *A and B strongly anticommute.*

(ii) *$R_\lambda(A)B \subset BR_\lambda(-A)$ whenever λ is a non-real scalar.*

(iii) $\mathrm{e}^{\mathrm{i}tA}\mathrm{e}^{\mathrm{i}sB} = \cos tB\mathrm{e}^{\mathrm{i}sA} + \mathrm{i}\sin tB\mathrm{e}^{-\mathrm{i}sA}$ *for all $s, t \in \mathbb{R}$.*

3 Strongly Supercommuting Self-adjoint Operators

Throughout the remainder of this paper, we assume that $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ is a $\mathbb{Z}_2$-graded Hilbert space and τ is the grading operator for $\mathcal{H}$. Given a self-adjoint operator S on a Hilbert space, we denote its spectral measure by $E_S(J)$ for arbitary J in $\mathbb{B}^1$. We introduce a notion of strongly supercommutativity of self-adjoint operators. For this purpose, we give a simple lemma:

Lemma 3.1 *Let A be a homogeneous self-adjoint operator on $\mathcal{H}$. For each J in $\mathbb{B}^1$, we define $E_A^+(J)$ and $E_A^-(J)$ by*

$$E_A^+(J) = \frac{1}{2}\{E_A(J) + E_A(-J)\},\ E_A^-(J) = \frac{1}{2}\{E_A(J) - E_A(-J)\}.$$

Then we have

(i) *$E_A^\pm(J)$ is self-adjoint operators and $E_A(J) = E_A^+(J) + E_A^-(J)$,*

(ii) *$\rho(E_A^+(J)) = 0$ and $\rho(E_A^-(J)) = \rho(A)$,*

(iii) *$E_A^+(-J) = E_A^+(J)$ and $E_A^-(-J) = -E_A^-(J)$.*

Therefore, if A is an even operator, then $E_A(J)_{\bar{0}} = E_A(J)$ and $E_A(J)_{\bar{1}} = 0$. On the other hand, if A is an odd operator, we have $E_A(J)_{\bar{0}} = E_A^+(J)$ and $E_A(J)_{\bar{1}} = E_A^-(J)$.

Proof. (i) and (iii) are clear. For each $J \in \mathbb{B}^1$, we denote the characteristic function of J by χ_J. Let χ_J^+ and χ_J^- be the even part and odd part of χ_J, that is, $\chi_J^\pm := \frac{1}{2}\{\chi_J \pm \chi_{-J}\}$. Then we can easily check that

$$E_J^+(A) = \chi_J^+(A),\ E_J^-(A) = \chi_J^-(A).$$

By this and Proposition 2.2, we can conclude (ii) and (iii). □

With the help of the above lemma, we can now introduce the following notion.

Definition 3.2 Suppose that A and B are homogeneous self-adjoint operators on $\mathcal{H}$. We say that A and B are *strongly supercommuting*, when

$$[E_A(J_1)_\alpha, E_B(J_2)_\beta]_{\mathrm{S}} = 0 \quad (\alpha, \beta \in \mathbb{Z}_2)$$

for each J_1 and J_2 in $\mathbb{B}^1$, where $[\cdot,\cdot]_{\mathrm{S}}$ is the supercommutator introduced in the preceeding section.

Remark 3.3 By the above definition, we have

$$[E_A(J_1), E_B(J_2)]_{\mathrm{S}} = 0$$

for each J_1 and J_2 in $\mathbb{B}^1$. Therefore, if $\rho(A) = 0$ or $\rho(B) = 0$, then strong supercommutivity is equivalent to strong commutativity. Thus the notion of strong supercommutativity can be regarded as a generalization of that of strong commutativity. We will see that strong supercommutativity is also regarded as a generalization of strong anticomutativity.

Lemma 3.4 *Suppose that T is a self-adjoint operator and $J \in \mathbb{B}^1$ is homogeneous with $\rho(J) = 0$. Then we have*

$$E_T^-(J) = 0.$$

Proof. Easy. □

Proposition 3.5 *Suppose that A and B are homogeneous self-adjoint operators on $\mathcal{H}$. Suppose that $f, g \in M(\mathbb{R})$ are bounded. If A and B are strongly supercommuting, then*

$$[f(A), g(B)]_{\mathrm{S}} = 0.$$

Proof. Note first that it is sufficient to show the assertion when f and g are homogeneous and real valued.

Suppose that $h \in M(\mathbb{R})$ is real valued. For each $n \in \mathbb{N}$ and $k = 1, \cdots, 2^n n$, let

$$I_{n,k} := [\frac{k-1}{2^n}, \frac{k}{2^n}), \;\; F_n := [n, \infty).$$

Then we define

$$h_n := \sum_{k=1}^{2^n n} \frac{k-1}{2^n}\Big\{\chi_{h^{-1}(I_{n,k})} - \chi_{h^{-1}(-I_{n,k})}\Big\} + n\Big\{\chi_{h^{-1}(F_n)} - \chi_{h^{-1}(-F_n)}\Big\}.$$

It is easy to see that $h_n(\lambda) \to h(\lambda)$ $(n \to \infty)$. Let T be an arbitary homogeneous self-adjoint operator. If $\rho(h) = 0$, then by Proposition 2.3, $\rho(h^{-1}(\pm I_{n,k})) = \rho(h^{-1}(\pm F_n)) = 0$ for each n and k. Thus by the above lemma, we have

$$\begin{aligned} h_n(T) &= \sum_{k=1}^{2^n n} \frac{k-1}{2^n}\Big\{E_T^+(h^{-1}(I_{n,k})) - E_T^+(h^{-1}(-I_{n,k}))\Big\} \\ &+ n\Big\{E_T^+(h^{-1}(F_n)) - E_T^+(h^{-1}(-F_n))\Big\} \end{aligned}$$

and $\rho(h_n(T)) = 0$ for each n in $\mathbb{N}$.

If $\rho(h) = 1$, then by Proposition 2.3, we have $h^{-1}(-I_{n,k}) = -h^{-1}(I_{n,k})$. Thus we have

$$h_n(T) = \sum_{k=1}^{2^n n} \frac{k-1}{2^n} 2E_T^-(h^{-1}(I_{n,k})) + nE_T^-(h^{-1}(F_n))$$

and $\rho(h_n(T)) = \rho(T)$. Combing these fact, we can conclude that $\rho(h_n(T)) = \rho(T)\rho(h) = \rho(h(T))$ for each n in $\mathbb{N}$. Therefor if A and B are strongly supercommuting, and f, g are homogeneous real valued measurable function, it follows from the definition of strong supercommutativity that

$$[f_n(A), g_n(B)]_{\mathrm{S}} = 0$$

for each n in $\mathbb{N}$. By limiting argument, we have the desired result. □

Proposition 3.6 *Suppose that A and B are homogeneous self-adjoint operators on $\mathcal{H}$. Suppose that f and g are elements in $M(\mathbb{R})$ such that f is bounded and g is homogeneous. If A and B are strongly supercommuting and $E_B(\{\lambda \in \mathbb{R} |\ |g(\lambda)| = \infty\}) = 0$, then we have $f(A)\mathrm{dom}(g(B)) \subset \mathrm{dom}(g(B))$ and*

$$[f(A), g(B)]_{\mathrm{S}} = 0$$

on $\mathrm{dom}(g(B))$.

Proof. Note first that by assumption $E_B(\{\lambda \in \mathbb{R} |\ |g(\lambda)| = \infty\}) = 0$, $g(B)$ is closed. For each $a > 0$, let

$$g_a := \chi_{\{\lambda \in \mathbb{R} |\ |g(\lambda)| \leq a\}} \cdot g.$$

Then it is clear that

$$g_a(B) = \int_{|g| \leq a} g(\lambda) \mathrm{d}E_B(\lambda)$$

is bounded and $\rho(g_a(B)) = \rho(g)\rho(B)$ for each $a > 0$. Thus by Proposition 3.5, we get

$$[f(A), g_a(B)]_{\mathrm{S}} = 0 \tag{1}$$

for all $a > 0$. For each ψ in $\mathrm{dom}(g(B))$ and $a > 0$, we define a vector

$$\psi_a := E_B(\{|g| \leq a\})\psi = E_B(\{|g| \leq a\})_{\bar{0}}\psi,$$

where $\{|g| \leq a\} := \{\lambda \in \mathbb{R} |\ |g(\lambda)| \leq a\}$. Since $\rho(E_B(\{|g| \leq a\})) = 0$ and therefore $E_B(\{|g| \leq a\})(= \chi_{\{|g| \leq a\}}(B))$ commutes with $f(A)$ by Proposotion 3.5, we have

$$f(A)g_a(B)\psi = f(A)g(B)\psi_a, \quad g_a(B)f(A)\psi = g(B)f(A)\psi_a.$$

Combing this with (1), we obtain

$$f(A)g(B)\psi_a = (-1)^{\rho(f(A))\rho(g(B))} g(B)f(A)\psi_a.$$

Since $f(A)g(B)\psi_a \to f(A)g(B)\psi$ and $f(A)\psi_a \to f(A)\psi$ $(a \to \infty)$, it follows from the closedness of $g(B)$ that $f(A)\psi \in \mathrm{dom}(g(B))$ and

$$[f(A), g(B)]_{\mathrm{S}}\psi = 0. \quad \square$$

Proposition 3.7 *Suppose that A and B are homogeneous self-adjoint operators on $\mathcal{H}$. Suppose that f and g are homogeneous elements in $M(\mathbb{R})$. If A and B are strongly supercommuting and $E_A(\{\lambda \in \mathbb{R} | \ |f(\lambda)| = \infty\}) = 0$ and $E_B(\{\lambda \in \mathbb{R} | \ |g(\lambda)| = \infty\}) = 0$, then we have* $\mathrm{dom}(f(A)g(B)) \cap \mathrm{dom}(f(A)) \cap \mathrm{dom}(g(B)) = \mathrm{dom}(g(B)f(A)) \cap \mathrm{dom}(f(A)) \cap \mathrm{dom}(g(B))$ *and*

$$[f(A), g(B)]_{\mathrm{S}} = 0$$

on $\mathrm{dom}(f(A)g(B)) \cap \mathrm{dom}(f(A)) \cap \mathrm{dom}(g(B))$.

Proof. For each $a > 0$, let

$$f_a := \chi_{\{\lambda \in \mathbb{R} | \ |f(\lambda)| \leq a\}} \cdot f.$$

Then we have $\rho(f_a(A)) = \rho(f)\rho(A)$ for each $a > 0$.

Let

$$\mathcal{V} := \mathrm{dom}(f(A)g(B)) \cap \mathrm{dom}(f(A)) \cap \mathrm{dom}(g(B)).$$

Then, by Proposition 3.6, we get

$$[f_a(A), g(B)]_{\mathrm{S}}\phi = 0$$

for each ϕ in $\mathcal{V}$. Using the same argument as in the proof of Proposition 3.6, we obtain

$$[f(A), g(B)]_{\mathrm{S}}\phi_a = 0,$$

where $\phi_a := E_A(\{|f| \leq a\})\phi$. Since $f(A)g(B)\phi_a = E_A(\{|f| \leq a\})f(A)g(B)\phi \to f(A)g(B)\phi$ and $f(A)\phi_a \to f(A)\phi$ $(a \to \infty)$, we can conclude that $f(A)\phi \in \mathrm{dom}(g(B))$ and

$$[f(A), g(B)]_{\mathrm{S}}\phi = 0.$$

Hence we have $\mathcal{V} \subset \mathrm{dom}(g(B)f(A)) \cap \mathrm{dom}(f(A)) \cap \mathrm{dom}(g(B))$. By exchanging the role of A and B in the above argument, we have $\mathcal{V} = \mathrm{dom}(g(B)f(A)) \cap \mathrm{dom}(f(A)) \cap \mathrm{dom}(g(B))$. $\square$

Theorem 3.8 *Suppose that A and B are homogeneous self-adjoint operators on $\mathcal{H}$. The following conditions are equivalent to each other:*

(i) *A and B strongly supercommute.*
(ii) $[\mathrm{e}^{\mathrm{i}sA}, \mathrm{e}^{\mathrm{i}tB}]_{\mathrm{S}} = 0$ *for each s, t in $\mathbb{R}$.*
(iii) $[R_z(A), R_w(B)]_{\mathrm{S}} = 0$ *for each z, w in $\mathbb{C}\backslash\mathbb{R}$.*

Proof. (i) $\Rightarrow$ (ii) : This is a direct consequence of Proposition 3.5.

(ii)$\Rightarrow$(iii): For each $z \in \mathbb{C}$ such that $\mathrm{Im}z < 0$ and an arbitary self-adjoint operator T, it is well-known that

$$R_z(T) = -\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}sT}\mathrm{e}^{-\mathrm{i}sz}\mathrm{d}s$$

in operator norm. Hence if $\rho(T) = 1$, we have

$$R_z(T)_{\bar{0}} = -\mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}sz}\cos sT\mathrm{d}s, \quad R_z(T)_{\bar{1}} = \int_0^\infty \mathrm{e}^{-\mathrm{i}sz}\sin sT\mathrm{d}s. \tag{2}$$

On the other hand, if $\rho(T) = 0$, then $R_z(T)_{\bar{0}} = R_z(T)$, $R_z(T)_{\bar{1}} = 0$. Using these facts, we can prove (iii). A similar argument is hold for the case $\mathrm{Im}z \geq 0$.

(iii)$\Rightarrow$(i): For an arbitary self-adjoint operator T, let

$$F_T(a,b) := \frac{1}{2}[E_T([a,b]) + E_T((a,b))].$$

Then it is well-known that

$$F_T(a,b) = \text{s-}\lim_{\epsilon\downarrow 0}\frac{1}{2\pi\mathrm{i}}\int_a^b [R_{\lambda+\mathrm{i}\epsilon}(T) - R_{\lambda-\mathrm{i}\epsilon}(T)]\mathrm{d}\lambda.$$

On the other hand, we can easily check that

$$\text{s-}\lim_{\epsilon\downarrow 0}\frac{\epsilon}{2\mathrm{i}}\{R_{a+\mathrm{i}\epsilon}(T) - R_{a-\mathrm{i}\epsilon}(T)\} = E_T(\{a\}).$$

Hence we have

$$E_T((a,b)) = \text{s-}\lim_{\epsilon\downarrow 0}\left\{\frac{1}{2\pi\mathrm{i}}\int_a^b [R_{\lambda+\mathrm{i}\epsilon}(T) - R_{\lambda-\mathrm{i}\epsilon}(T)]\mathrm{d}\lambda - \frac{\epsilon}{4\mathrm{i}}[R_{a+\mathrm{i}\epsilon}(T) - R_{a-\mathrm{i}\epsilon}(T)]\right\}. \tag{3}$$

By this and (iii), we can conclude that

$$[E_A((a,b)), E_B((c,d))]_{\mathrm{S}} = 0$$

for each $a < b, c < d$. Hence (i) follows from the limiting argument. □

Remark 3.9 If $\rho(A) = \rho(B) = 1$, then (ii) can be rewritten as

$$\mathrm{e}^{\mathrm{i}sA}\mathrm{e}^{\mathrm{i}tB} = \cos tB\ \mathrm{e}^{\mathrm{i}sA} + \mathrm{i}\sin tB\ \mathrm{e}^{-\mathrm{i}sA}.$$

From Proposition 2.8, it follows that A and B are strong anticommuting in this case. Thus the strongly supercommutativity is a generalization of the strong anticommutativity. Combining this with Remark 3.3, the notion of strong supercommutativity is a unification of the strong commutativity and strong anticommutativity.

Theorem 3.10 *Suppose that A and B are homogeneous self-adjoint operators on $\mathcal{H}$. The following conditions are equivalent:*

(i) *A and B are strongly supercommuting.*

(ii) $[e^{isA}, B]_S = 0$ *on* $\mathrm{dom}(B)$ *for all s in $\mathbb{R}$.*

(iii) $[R_z(A), B]_S = 0$ *on* $\mathrm{dom}(B)$ *for all z in $\mathbb{C}\backslash\mathbb{R}$.*

(iv) $[E_A(J), B]_S = 0$ *on* $\mathrm{dom}(B)$ *for all J in $\mathbb{B}^1$.*

Remark 3.11 If the roles of A and B are exchanged in the above statement, the assertion is still hold.

Proof. (i)$\Rightarrow$(ii): This is a direct consequence of Proposition 3.6.

(ii)$\Rightarrow$(iii): By using (2) we have (iii).

(iii)$\Rightarrow$(i): By (iii), we have

$$R_z(A)_{\bar{1}}B \subset (-1)^{\rho(A)\rho(B)}BR_z(A)_{\bar{1}}, \quad R_z(A)_{\bar{0}}B \subset BR_z(A)_{\bar{0}}.$$

Thus we can conlude that

$$R_w((-1)^{\rho(A)\rho(B)}B)R_z(A)_{\bar{1}} = R_z(A)_{\bar{1}}R_w(B), \quad R_w(B)R_z(A)_{\bar{0}} = R_z(A)_{\bar{0}}R_w(B)$$

for each $w \in \mathbb{C}\backslash\mathbb{R}$. This implies (i) by Theorem 3.8.

(iii)$\Rightarrow$(iv): From (3), it follows that

$$[E_A((a,b)), B]_S = 0 \text{ on } \mathrm{dom}(B)$$

for $a \leq b$. This result can be extended to each J in $\mathbb{B}^1$.

(iv)$\Rightarrow$(iii): If $\rho(A) = 0$ or $\rho(B) = 0$, it is easy to show (iii) by (iv). Thus we discuss the case $\rho(A) = 1$ and $\rho(B) = 1$. Then (iv) implies $E_A(J)B = BE_A(-J)$. For each ψ and ϕ in $\mathrm{dom}(B)$, we have

$$\begin{aligned} & \langle R_z(A)B\phi, \psi\rangle \\ = & \int_{\mathbb{R}} (\bar{z} - \lambda)^{-1} \mathrm{d}\langle E_A(\lambda)B\phi, \psi\rangle \\ = & \int_{\mathbb{R}} (\bar{z} - \lambda)^{-1} \mathrm{d}\langle E_A(-\lambda)\phi, B\psi\rangle \\ = & \langle R_z(-A)\phi, B\psi\rangle. \end{aligned}$$

Hence we obtain

$$R_z(A)B \subset BR_z(-A).$$

Similarly we have

$$R_z(-A)B \subset BR_z(A).$$

Using these facts we can conclude

$$R_z(A)_{\bar{0}}B \subset BR_z(A)_{\bar{0}}, \quad R_z(A)_{\bar{1}}B \subset -BR_z(A)_{\bar{1}}.$$

□

Proposition 3.12 *Suppose that A and B are homogeneous self-adjoint operators on $\mathcal{H}$. If A and B are strongly supercommuting, then we have*

(i) $\mathrm{dom}(AB) \cap \mathrm{dom}(A) \cap \mathrm{dom}(B) = \mathrm{dom}(BA) \cap \mathrm{dom}(A) \cap \mathrm{dom}(B)$ *and*

$$[A, B]_{\mathrm{S}} = 0$$

on $\mathrm{dom}(AB) \cap \mathrm{dom}(A) \cap \mathrm{dom}(B)$*;*

(ii) *The operator $A + B$ is essentially self-adjoint. In particular, if $\rho(A) = \rho(B) = 1$, then $A + B$ is self-adjoint.*

Proof. (i) This follows from Proposition 3.7.
(ii) See [12] Theorem 2.1. □

Proposition 3.13 *Suppose that A and B are homogeneous, self-adjoint and bounded operators on $\mathcal{H}$. Then A and B are strongly supercommuting if and only if*

$$[A, B]_{\mathrm{S}} = 0. \tag{4}$$

Proof. If A and B strongly supercommute, then from Proposition 3.12, it follows that

$$[A, B]_{\mathrm{S}} = 0.$$

Conversely, if (4) is hold, then we can easily check that

$$[\mathrm{e}^{\mathrm{i}sA}, \mathrm{e}^{\mathrm{i}tB}]_{\mathrm{S}} = 0$$

for all s, t in $\mathbb{R}$. Hence, by Theorem 3.8, A and B strongly supercommute. □

For a self-adjoint operator S on $\mathcal{H}$, we denote by

$$S = U_S|S|$$

the polar decomposition of S with U_S a partial isometry. If S is homogeneous, we have

$$\rho(U_S) = \rho(S). \tag{5}$$

Indeed, it is well-known that

$$U_S = E_S((0, \infty)) - E_S((-\infty, 0)) = 2E_S^-((0, \infty)).$$

Thus we can conclude (5) by Lemma 3.1.

Proposition 3.14 *Suppose that A and B are homogeneous self-adjoint operators on $\mathcal{H}$. If A and B are strongly supercommuting, then the following* (i)-(v) *hold:*

(i) $[U_B, A]_{\mathrm{S}} = 0$ *on* $\mathrm{dom}(A)$ *and* $[U_A, B]_{\mathrm{S}} = 0$ *on* $\mathrm{dom}(B)$.
(ii) $[U_A, U_B]_{\mathrm{S}} = 0$.
(iii) $[U_B, |A|] = 0$ *on* $\mathrm{dom}(A)$ *and* $[U_A, |B|] = 0$ *on* $\mathrm{dom}(B)$.
(iv) $|A|$ *and* $|B|$ *strongly commute.*
(v) A *and* $|B|$ *strongly commute, and* B *and* $|A|$ *strongly commute.*

Proof. These assertions follow from a simple application of Proposition 3.7. □

4 Theory of Super Quantization

4.1 Quantization

For each n in $\mathbb{N}$, suppose that

$$\mathbb{Z}_2^n := \overbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}^{n}$$

and that $\otimes^n \mathcal{H}$ is the n-fold tensor product Hilbert space of $\mathcal{H}$. For arbitary $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}_2^n$, we introduce a closed subspace of $\otimes^n \mathcal{H}$ by

$$\mathcal{H}(\lambda) := \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n},$$

where we identify $\mathcal{H}_{\bar{0}}$ and $\mathcal{H}_{\bar{1}}$ with $\mathcal{H}_{\bar{0}} \oplus \{0\}$ and $\{0\} \oplus \mathcal{H}_{\bar{1}}$, respectively. For each α in $\mathbb{Z}_2$, let

$$\mathbb{Z}_{2,\alpha}^n := \left\{ \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}_2^n \;\middle|\; \sum_{j=1}^{n} \lambda_j = \alpha \right\}.$$

It is easily verified that $\mathbb{Z}_2^n = \mathbb{Z}_{2,\bar{0}}^n \cup \mathbb{Z}_{2,\bar{1}}^n$, $\mathbb{Z}_{2,\bar{0}}^n \cap \mathbb{Z}_{2,\bar{1}}^n = \emptyset$. Hence, introducing

$$(\otimes^n \mathcal{H})_\alpha := \bigoplus_{\lambda \in \mathbb{Z}_{2,\alpha}^n} \mathcal{H}(\lambda),$$

the Hilbert space $\otimes^n \mathcal{H}$ have the following $\mathbb{Z}_2$-grading structure:

$$\otimes^n \mathcal{H} = (\otimes^n \mathcal{H})_{\bar{0}} \oplus (\otimes^n \mathcal{H})_{\bar{1}}.$$

It is not difficult to show that the operator $\Gamma^{(n)} := \otimes^n \tau (= \overbrace{\tau \otimes \cdots \otimes \tau}^{n})$ is the grading operator for $\otimes^n \mathcal{H}$.

Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$. For convenience, we introduce

$$\mathcal{L}(\mathcal{D}) := \{ A \in \mathcal{L}(\mathcal{H}) \mid A\mathcal{D} \subset \mathcal{D},\ A^* \mathcal{D} \subset \mathcal{D} \}.$$

Note that each element in $\mathcal{L}(\mathcal{D})$ is closable.

Suppose that A is in $\mathcal{L}(\mathcal{D})$. If A is homogeneous, we define a linear operator $A^{[n;j]}$ on $\otimes^n \mathcal{H}$ by

$$A^{[n;j]} := \overbrace{\tau^{\rho(A)} \otimes \cdots \otimes \tau^{\rho(A)} \otimes \underbrace{A}_{j\text{ th}} \otimes I \otimes \cdots \otimes I}^{n}$$

for $1 \leq j \leq n$. For a non-homogeneous element A in $\mathcal{L}(\mathcal{D})$, we can also define the operator $A^{[n;j]}$ as follows. Let $A_{\bar{0}}$ and $A_{\bar{1}}$ are even and odd part of A, respectively. We introduce a subspace of $\otimes^n \mathcal{H}$ by

$$D^{[n;j]}(A) := \mathcal{H} \hat{\otimes} \cdots \hat{\otimes} \mathcal{H} \hat{\otimes} \overbrace{\operatorname{dom}(A)}^{j\text{ th}} \hat{\otimes} \mathcal{H} \hat{\otimes} \cdots \hat{\otimes} \mathcal{H}$$

(where the symbol $\hat{\otimes}$ means algebraic tensor product) and denote by $K_A^{[n;j]}$ the restriction of $A_{\bar{0}}^{[n;j]} + A_{\bar{1}}^{[n;j]}$ to $D^{[n;j]}(A)$. It is clear that $K_A^{[n;j]}$ is closable. Thus we can define the operator $A^{[n;j]}$ by the closure of $K_A^{[n;j]}$, that is,

$$A^{[n;j]} := \overline{K_A^{[n;j]}}.$$

Lemma 4.1 *Suppose that A is in $\mathcal{L}(\mathcal{D})$.*

(i) *If A is homogeneous, then $A^{[n;j]}$ is also homogeneous with $\rho(A^{[n;j]}) = \rho(A)$.*

(ii) *If A is self-adjoint, then $A^{[n;j]}$ is self-adjoint.*

(iii) *If A is non-negative, then $A^{[n;j]}$ is non-negative.*

(iv) *If A is homogeneous and self-adjoint, then $\{A^{[n;j]}\}_{j=1}^n$ is a family of strongly supercommuting self-adjoint operators on $\otimes^n \mathcal{H}$.*

Proof. (i)

$$\begin{aligned}\Gamma^{(n)} A^{[n;j]} \Gamma^{(n)} &= \tau^{\rho(A)} \otimes \cdots \otimes \tau^{\rho(A)} \otimes \tau A \tau \otimes I \otimes \cdots \otimes I \\ &= (-1)^{\rho(A)} A^{[n;j]}.\end{aligned}$$

Thus we can conclude that $\rho(A^{[n;j]}) = \rho(A)$.

(ii),(iii) When A is homogeneous, the assertions are easily verified. Hence we only discuss the case when A is non-homogeneous. For each $j = 1, \cdots, n$, we define

$$\mathcal{H}_{j,\bar{0}} := \bigoplus_{\lambda \in \mathbb{Z}_2^n,\ \sum_{i=1}^{j-1} \lambda_i = \bar{0}} \mathcal{H}(\lambda), \quad \mathcal{H}_{j,\bar{1}} := \bigoplus_{\lambda \in \mathbb{Z}_2^n,\ \sum_{i=1}^{j-1} \lambda_i = \bar{1}} \mathcal{H}(\lambda).$$

Then it is clear that $\otimes^n \mathcal{H} = \mathcal{H}_{j,\bar{0}} \oplus \mathcal{H}_{j,\bar{1}}$.

For each B in $\mathcal{L}(\mathcal{D})$, we define a linear operator on $\otimes^n \mathcal{H}$ by

$$B_j^{(n)} := \overbrace{I \otimes \cdots \otimes I \otimes \underbrace{B}_{j\text{ th}} \otimes I \otimes \cdots \otimes I}^{n}$$

Then it is easy to check that $B_j^{(n)}$ is reduced by $\mathcal{H}_{j,\alpha}$. We denote the reduced part of $B_j^{(n)}$ to $\mathcal{H}_{j,\alpha}$ by $B_{j,\alpha}^{(n)}$.

Relative to the direct sum decomposition $\otimes^n \mathcal{H} = \mathcal{H}_{j,\bar{0}} \oplus \mathcal{H}_{j,\bar{1}}$, we can show that

$$A^{[n;j]} = A_{j,\bar{0}}^{(n)} \oplus \tilde{A}_{j,\bar{1}}^{(n)}, \tag{6}$$

where $\tilde{A} = \tau A \tau$. Since A is self-adjoint (resp. non-negative), $A_{j,\bar{0}}^{(n)}$ and $A_{j,\bar{1}}^{(n)}$ are self-adjoint (resp. non-negative). Hence by (6), $A^{[n;j]}$ is self-adjoint (resp. non-negative).

(iv) Suppose that A is homogeneous. Then for each $t \in \mathbb{R}$, we have

$$\mathrm{e}^{\mathrm{i}tA^{[n;j]}} = I \otimes \cdots \otimes I \otimes \underbrace{\cos tA}_{j \text{ th}} \otimes I \otimes \cdots \otimes I + \mathrm{i}\tau^{\rho(A)} \otimes \cdots \otimes \tau^{\rho(A)} \otimes \underbrace{\sin tA}_{j \text{ th}} \otimes I \otimes \cdots \otimes I.$$

Therefore we obtain

$$[\mathrm{e}^{\mathrm{i}sA^{[n;j]}}, \mathrm{e}^{\mathrm{i}tA^{[n;k]}}]_{\mathrm{S}} = 0$$

for all $s, t \in \mathbb{R}$. By Theorem 3.8, we have desired result. □

Lemma 4.2 *Suppose that A and B are in $\mathcal{L}(\mathcal{D})$. Then we have*

$$[A^{[n;j]}, B^{[n;j]}]_{\mathrm{S}} = \delta_{jk}([A, B]_{\mathrm{S}})^{[n,j]} \tag{7}$$

on $\widehat{\otimes}^n \mathcal{D}$. Especially, for an odd self-adjoint operator $C = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$ in $\mathcal{L}(\mathcal{D})$, we have

$$\{C^{[n;j]}, C^{[n;k]}\} = 2\delta_{jk}(S^*S \oplus SS^*)^{[n;j]}. \tag{8}$$

Proof. We can easily show (7) when A and B are homogeneous. Since we have

$$A^{[n;j]} = A_{\bar{0}}^{[n;j]} + A_{\bar{1}}^{[n;j]}$$

on $\widehat{\otimes}^n \mathcal{D}$, we can conclude (7) for arbitary A and B in $\mathcal{L}(\mathcal{D})$.

Next we will show (8). For a linear operator T on a Hilbert space, we set

$$C^\infty(T) := \cap_{n=1}^\infty \mathrm{dom}(T^n).$$

It is well-known that if T is self-adjoint, then $C^\infty(T)$ is a core of T.

Let $\mathcal{D} = C^\infty(S^*S) \oplus C^\infty(SS^*)$. Then by (7), (8) is satisfied on $\widehat{\otimes}^n \mathcal{D}$. On the other hand, $\widehat{\otimes}^n \mathcal{D}$ is a core of $(S^*S \oplus SS^*)^{[n;j]}$. Hence we obtain (8). □

For each element A in $\mathcal{L}(\mathcal{D})$, the linear operator

$$(\sum_{j=1}^n A^{[n;j]}) \lceil \widehat{\otimes}^n \mathrm{dom}(A)$$

is closable. We denote its closure by $A^{[n]}$.

Proposition 4.3 *Suppose that $A \in \mathcal{L}(\mathcal{D})$ is homogeneous and self-adjoint. Then $A^{[n]}$ is also homogeneous and self-adjoint with $\rho(A^{[n]}) = \rho(A)$. Especially, if A is odd, then*

$$A^{[n]} = \sum_{j=1}^n A^{[n;j]}.$$

Proof. It is easy to check that

$$\Gamma^{(n)} A^{[n]} \Gamma^{(n)} = (-1)^{\rho(A)} A^{[n]}.$$

Hence $\rho(A^{[n]}) = \rho(A)$. Self-adjointness of $A^{[n]}$ follows from Proposition 3.12 (ii). □

Proposition 4.4 *Suppose that $A \in \mathcal{L}(\mathcal{D})$ is non-homogeneous. Then we have*

(i) *If A is self-adjoint and $A_{\bar{0}}$ or $A_{\bar{1}}$ is bounded, then $A^{[n]}$ is self-adjoint;*

(ii) *If A is non-negative, then $A^{[n]}$ is non-negative.*

Proof. These are direct consequences of Lemma 4.1. □

Proposition 4.5 *Suppose that $A, B \in \mathcal{L}(\mathcal{D})$ are homogeneous and self-adjoint. If A and B strongly supercommute, then $A^{[n]}$ and $B^{[n]}$ strongly supercommute for each $n \in \mathbb{N}$.*

Proof. First we will show the assertion when $\rho(A) = 0$ or $\rho(B) = 0$. Suppose that $\rho(A) = 0$. Then it is clear that

$$\mathrm{e}^{\mathrm{i}tA^{[n]}} = \overbrace{\mathrm{e}^{\mathrm{i}tA} \otimes \cdots \otimes \mathrm{e}^{\mathrm{i}tA}}^{n}$$

for each $t \in \mathbb{R}$. From Theorem 3.10 it follows that

$$[\mathrm{e}^{\mathrm{i}tA}, B]_{\mathrm{S}} = 0$$

on $\mathrm{dom}(B)$ for all t in $\mathbb{R}$. Thus we can easily see that

$$[\mathrm{e}^{\mathrm{i}tA^{[n]}}, B^{[n]}]_{\mathrm{S}} = [\mathrm{e}^{\mathrm{i}tA^{[n]}}, B^{[n]}] = 0$$

on $\mathrm{dom}(B^{[n]})$. From Theorem 3.10 it follows that $A^{[n]}$ and $B^{[n]}$ strongly supercommute.

Next we discuss the case $\rho(A) = \rho(B) = 1$. Since A and B strongly anticommute, $A \pm B$ are self-adjoint by Proposition 3.12. Hence $(A \pm B)^{[n]}$ are self-adjoint. Let

$$\mathcal{M}^{(n)} := \widehat{\otimes}^n (\mathrm{dom}(A) \cap \mathrm{dom}(B)).$$

Then by Proposition 4.3, $(A \pm B)^{[n]}$ is essentially self-adjoint on $\mathcal{M}^{(n)}$. On the other hand, since

$$(A \pm B)^{[n]} = A^{[n]} \pm B^{[n]}$$

on $\mathcal{M}^{(n)}$, $A^{[n]} \pm B^{[n]}$ are essentially self-adjoint on $\mathcal{M}_{\pm}^{(n)}$. Hence using [9] Theorem 4.3, we can conclude that $A^{[n]}$ and $B^{[n]}$ strongly anticommute. □

The full Fock space $\mathcal{F}(\mathcal{H})$ over $\mathcal{H}$ is defined by

$$\mathcal{F}(\mathcal{H}) := \oplus_{n=0}^{\infty}(\otimes^n \mathcal{H}),$$

where $\otimes^0 \mathcal{H} := \mathbb{C}$. The full Fock space $\mathcal{F}(\mathcal{H})$ has the following natural $\mathbb{Z}_2$-grading structure:

$$\mathcal{F}(\mathcal{H}) = \mathcal{F}(\mathcal{H})_{\bar{0}} \oplus \mathcal{F}(\mathcal{H})_{\bar{1}},$$

where $\mathcal{F}(\mathcal{H})_\alpha := \oplus_{n=0}^{\infty}(\otimes^n \mathcal{H})_\alpha$ for α in $\mathbb{Z}_2$. It is easily verified that the grading operator Γ for $\mathcal{F}(\mathcal{H})$ is given by $\Gamma := \oplus_{n=0}^{\infty}\Gamma^{(n)}$.

Definition 4.6 Let A be in $\mathcal{L}(\mathcal{D})$. A *quantization of* A is the operator $\mathrm{d}\Gamma(A)$ on $\mathcal{F}(\mathcal{H})$ defined by

$$\mathrm{d}\Gamma(A) := \oplus_{n=0}^{\infty} A^{[n]},$$

where $A^{[0]} := 0$.

Proposition 4.7 *Suppose that A and B are elements in $\mathcal{L}(\mathcal{D})$.*

(i) $\mathrm{d}\Gamma(A)$ *is a closed operator.*

(ii) *If A is homogeneous, then $\mathrm{d}\Gamma(A)$ is also homogeneous with $\rho(\mathrm{d}\Gamma(A)) = \rho(A)$.*

(iii) *Suppose that A is self-adjoint. If A is homogeneous, then $\mathrm{d}\Gamma(A)$ is self-adjoint. If A is non-homogenous and either even part $A_{\bar{0}}$ or odd part $A_{\bar{1}}$ is bounded, then $\mathrm{d}\Gamma(A)$ is self-adjoint.*

(iv) *If A is non-negative, then $\mathrm{d}\Gamma(A)$ is non-negative.*

(v) *If A and B are homogeneous and self-adjoint, then $\mathrm{d}\Gamma(A)$ and $\mathrm{d}\Gamma(B)$ strongly supercommute if and only if A and B strongly super-commute.*

(vi) $[\mathrm{d}\Gamma(A), \mathrm{d}\Gamma(B)]_{\mathrm{S}} = \mathrm{d}\Gamma([A, B]_{\mathrm{S}})$ *on $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$.*

(vii) *For an arbitary odd self-adjoint operator $A = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$ in $\mathcal{L}(\mathcal{D})$, we have*

$$\mathrm{d}\Gamma(A)^2 = \mathrm{d}\Gamma(S^*S \oplus SS^*).$$

Proof. These are direct consequences of Lemma 4.2 and Propositions 4.4, 4.5. □

4.2 Construction of the supersymmetrizer

4.2.1 The sign polynomial

Let S_n be the group of permutations of a set of cardinality n. We define the action of S_n on $\mathbb{Z}_2^n$ by

$$\sigma(\lambda) := (\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(n)})$$

for each $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}_2^n$ and $\sigma \in \mathsf{S}_n$. Clearly the mapping $\sigma(\cdot) : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ is bijective and for each $\sigma, \gamma \in \mathsf{S}_n$

$$\begin{aligned} \sigma(\gamma(\lambda)) &= \sigma(\lambda_{\gamma(1)}, \cdots, \lambda_{\gamma(n)}) \\ &= (\lambda_{\gamma(\sigma(1))}, \cdots, \lambda_{\gamma(\sigma(n))}) \\ &= (\gamma\sigma)(\lambda). \end{aligned} \tag{9}$$

For each $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}_2^n$ $(n \geq 2)$, we introduce a polynomial Δ_λ, which we call the *sign polynomial*, defined by

$$\Delta_\lambda(X_1, \cdots, X_n) := \Pi_{i<j}[X_i + (-1)^{\lambda_i \lambda_j} X_j].$$

(Throughout the rest of this paper, for notational convenience, we identify $(-1)^{\overline{\alpha}}$ $(\alpha = 0, 1)$ with $(-1)^{\alpha}$.) For each polynomial $f(X_1, \cdots, X_n)$ of order n, the action of S_n to f is defined by

$$(\sigma f)(X_1, \cdots, X_n) := f(X_{\sigma^{-1}(1)}, \cdots, X_{\sigma^{-1}(n)}), \ (\sigma \in \mathsf{S}_n).$$

Then we have

$$\sigma(\gamma f) = (\gamma\sigma) f \tag{10}$$

for any σ and γ in S_n. Indeed,

$$\begin{aligned} (\sigma(\gamma f))(X_1, \cdots, X_n) &= (\gamma f)(X_{\sigma^{-1}(1)}, \cdots, X_{\sigma^{-1}(n)}) \\ &= f(X_{\sigma^{-1}(\gamma^{-1}(1))}, \cdots, X_{\sigma^{-1}(\gamma^{-1}(n))}) \\ &= ((\gamma\sigma) f)(X_1, \cdots, X_n). \end{aligned}$$

Proposition 4.8 *For each σ in S_n and λ in $\mathbb{Z}_2^n$, we have*

$$\sigma \Delta_\lambda = \mathrm{sgn}(\sigma; \lambda) \Delta_{\sigma(\lambda)},$$

where

$$\mathrm{sgn}(\sigma; \lambda) := (-1)^{\sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} \lambda_{\sigma(i)} \lambda_{\sigma(j)}}.$$

Proof. For each σ in S_n, we have

$$(\sigma \Delta_\lambda)(X_1, \cdots, X_n) = \Pi_{i<j}[X_{\sigma^{-1}(i)} + (-1)^{\lambda_i \lambda_j} X_{\sigma^{-1}(j)}]. \tag{11}$$

On the right hand side of (11), we rewrite as

$$X_{\sigma^{-1}(i)} + (-1)^{\lambda_i \lambda_j} X_{\sigma^{-1}(j)} = \begin{cases} X_{\sigma^{-1}(i)} + (-1)^{\lambda_i \lambda_j} X_{\sigma^{-1}(j)} & \text{if } \sigma^{-1}(i) < \sigma^{-1}(j) \\ (-1)^{\lambda_i \lambda_j}[X_{\sigma^{-1}(j)} + (-1)^{\lambda_i \lambda_j} X_{\sigma^{-1}(i)}] & \text{if } \sigma^{-1}(i) > \sigma^{-1}(j) \end{cases}.$$

Then

$$\begin{aligned} \text{RHS of (11)} &= \Big(\Pi_{\substack{i<j \\ \sigma^{-1}(i)>\sigma^{-1}(j)}} (-1)^{\lambda_i \lambda_j}\Big) \Pi_{\sigma^{-1}(i)<\sigma^{-1}(j)}[X_{\sigma^{-1}(i)} + (-1)^{\lambda_i \lambda_j} X_{\sigma^{-1}(j)}] \\ &= \mathrm{sgn}(\sigma; \lambda) \Delta_{\sigma(\lambda)}(X_1, \cdots, X_n). \end{aligned}$$

Therefore we have desired result. □

Definition 4.9 For each σ in S_n and λ in $\mathbb{Z}_2^n$, the value $\mathrm{sgn}(\sigma;\lambda)$ described in the above proposition is said to be the *sign of σ relative to λ*. (If $n=1$, then we put $\mathrm{sgn}(\sigma;\lambda)=1$.)

Remark 4.10 Suppose that σ is an element in S_n.

(i) For $\tilde{\lambda}=(\bar{1},\cdots,\bar{1})\in\mathbb{Z}_2^n$, $\mathrm{sgn}(\sigma;\tilde{\lambda})$ equals to $\mathrm{sgn}(\sigma)$ the ordinary sign of σ.

(ii) For $\widehat{\lambda}=(\bar{0},\cdots,\bar{0})\in\mathbb{Z}_2^n$, $\mathrm{sgn}(\sigma;\widehat{\lambda})=1$.

Proposition 4.11 *Suppose that $\sigma,\tau\in\mathsf{S}_n$ and $\lambda\in\mathbb{Z}_2^n$.*

(i) $\mathrm{sgn}(\sigma\tau;\lambda)=\mathrm{sgn}(\sigma;\lambda)\mathrm{sgn}(\tau;\sigma(\lambda))$.

(ii) $\mathrm{sgn}(\sigma;\lambda)=\mathrm{sgn}(\sigma^{-1};\sigma(\lambda))$.

(iii) $\mathrm{sgn}(\sigma;\sigma^{-1}(\lambda))=\mathrm{sgn}(\sigma^{-1};\lambda)$.

Proof. (i) By (9),(10) and Proposition 4.8 we have

$$\begin{aligned}
\mathrm{sgn}(\sigma\tau;\lambda)\Delta_{(\sigma\tau)(\lambda)} &= (\sigma\tau)\Delta_\lambda \\
&= \tau(\sigma\Delta_\lambda) && \text{(by (10))} \\
&= \mathrm{sgn}(\sigma;\lambda)(\tau\Delta_{\sigma(\lambda)}) && \text{(by Proposition 4.8 and (9))} \\
&= \mathrm{sgn}(\sigma;\lambda)\mathrm{sgn}(\tau;\sigma(\lambda))\Delta_{(\sigma\tau)(\lambda)}.
\end{aligned}$$

Hence we have desired result. (ii) and (iii) follow from (i). □

4.2.2 The permutaion operator

For each $\sigma\in\mathsf{S}_n$ and $\lambda=(\lambda_1,\cdots,\lambda_n)\in\mathbb{Z}_2^n$, we define the linear operator $U_\sigma(\sigma(\lambda),\lambda)$ from $\mathcal{H}(\lambda)$ to $\mathcal{H}(\sigma(\lambda))$ by

$$U_\sigma(\sigma(\lambda),\lambda)(\phi_1^{\lambda_1}\otimes\cdots\otimes\phi_n^{\lambda_n}):=\phi_{\sigma(1)}^{\lambda_{\sigma(1)}}\otimes\cdots\otimes\phi_{\sigma(n)}^{\lambda_{\sigma(n)}},$$

where we take $\phi_j^{\lambda_j}\in\mathcal{H}_{\lambda_j}$ $(j=1,\cdots,n)$. This operator can be extended to the unitary operator from $\mathcal{H}(\lambda)$ to $\mathcal{H}(\sigma(\lambda))$. We denote it by the same symbol $U_\sigma(\sigma(\lambda),\lambda)$.

Lemma 4.12 *For each $\sigma,\tau\in\mathsf{S}_n$ and $\lambda\in\mathbb{Z}_2^n$, we have*

(i) $U_\sigma(\sigma(\lambda),\lambda)^*=U_{\sigma^{-1}}(\lambda,\sigma(\lambda))$;

(ii) $U_{\tau\sigma}((\tau\sigma)(\lambda),\lambda)=U_\sigma((\tau\sigma)(\lambda),\tau(\lambda))U_\tau(\tau(\lambda),\lambda)$.

Proof. Easy. □

Next we extend $U_\sigma(\sigma(\lambda),\lambda)$ to the linear operator $\tilde{U}_\sigma(\sigma(\lambda),\lambda)$ on $\otimes^n\mathcal{H}$ by

$$\tilde{U}_\sigma(\sigma(\lambda),\lambda):=\begin{cases} U_\sigma(\sigma(\lambda),\lambda) & \text{on } \mathcal{H}(\lambda) \\ 0 & \text{on } \mathcal{H}(\lambda)^\perp=\oplus_{\lambda'\neq\lambda}\mathcal{H}(\lambda') \end{cases}.$$

Now we can define the *permutation operator* U_σ on $\otimes^n \mathcal{H}$ by

$$U_\sigma := \sum_{\lambda \in \mathbb{Z}_2^n} \tilde{U}_\sigma(\sigma(\lambda), \lambda)$$

for σ in S_n.

Proposition 4.13 *For each* $\sigma, \tau \in \mathsf{S}_n$, *we have*

(i) U_σ *is a unitary operator on* $\otimes^n \mathcal{H}$;
(ii) $U_\sigma U_\tau = U_{\tau\sigma}$;
(iii) $U_{\sigma^{-1}} = U_\sigma^*$.

Proof. These follow from Lemma 4.12. □

4.2.3 The super-symmetrizer

Relative to the decomposition $\otimes^n \mathcal{H} = \oplus_{\lambda \in \mathbb{Z}_2^n} \mathcal{H}(\lambda)$, we define the linear operator $\mathrm{Sgn}(\sigma)$ by

$$\mathrm{Sgn}(\sigma) := \bigoplus_{\lambda \in \mathbb{Z}_2^n} \mathrm{sgn}(\sigma; \lambda) I_\lambda$$

for any σ in S_n, where I_λ is the identity on $\mathcal{H}(\lambda)$. We refer to $\mathrm{Sgn}(\sigma)$ as the *sign operator* relative to σ.

Lemma 4.14 *For each* σ *and* τ *in* S_n, *we have*

$$\mathrm{Sgn}(\tau) U_\sigma = U_\sigma \mathrm{Sgn}(\sigma\tau) \mathrm{Sgn}(\sigma).$$

Proof.

$$\begin{aligned}
\mathrm{Sgn}(\tau) U_\sigma &= \sum_{\lambda \in \mathbb{Z}_2} \mathrm{Sgn}(\tau) U_\sigma(\sigma(\lambda); \lambda) \\
&= \sum_{\lambda \in \mathbb{Z}_2} \mathrm{sgn}(\tau; \sigma(\lambda)) U_\sigma(\sigma(\lambda); \lambda) \\
&= \sum_{\lambda \in \mathbb{Z}_2} \mathrm{sgn}(\sigma\tau; \lambda) \mathrm{sgn}(\sigma; \lambda) U_\sigma(\sigma(\lambda); \lambda) \qquad \text{(by Proposition 4.11 (i))} \\
&= U_\sigma \mathrm{Sgn}(\sigma\tau) \mathrm{Sgn}(\sigma). \ \square
\end{aligned}$$

Definition 4.15 For each $n \geq 1$, the linear operator

$$W_n := \sum_{\sigma \in \mathsf{S}_n} \frac{1}{n!} U_\sigma \mathrm{Sgn}(\sigma)$$

is said to be the *supersymmetrizer* on $\otimes^n \mathcal{H}$.

Proposition 4.16 *Suppose that σ in S_n.*

(i) *W_n is an orthogonal projection operator on $\otimes^n \mathcal{H}$.*
(ii) *$W_n U_\sigma = W_n \mathrm{Sgn}(\sigma)$.*
(iii) *$U_\sigma W_n = \mathrm{Sgn}(\sigma^{-1}) W_n$.*

Proof. (i) On the one hand, we have

$$\begin{aligned} W_n^* &= \frac{1}{n!} \sum_{\sigma \in \mathsf{S}_n} \mathrm{Sgn}(\sigma) U_\sigma^* \\ &= \frac{1}{n!} \sum_{\sigma \in \mathsf{S}_n} \mathrm{Sgn}(\sigma) U_{\sigma^{-1}} && \text{(by Proposition 4.13)} \\ &= \frac{1}{n!} \sum_{\sigma \in \mathsf{S}_n} U_{\sigma^{-1}} \mathrm{Sgn}(\sigma^{-1}) && \text{(by Lemma 4.14)} \\ &= W_n. \end{aligned}$$

On the other hand,

$$\begin{aligned} W_n^2 &= \left(\frac{1}{n!}\right)^2 \sum_{\sigma,\tau \in \mathsf{S}_n} U_\sigma \mathrm{Sgn}(\sigma) U_\tau \mathrm{Sgn}(\tau) \\ &= \left(\frac{1}{n!}\right)^2 \sum_{\sigma,\tau \in \mathsf{S}_n} U_\sigma U_\tau \mathrm{Sgn}(\tau\sigma) \mathrm{Sgn}(\tau)^2 && \text{(by Lemma 4.14)} \\ &= \left(\frac{1}{n!}\right)^2 \sum_{\sigma,\tau \in \mathsf{S}_n} U_{\tau\sigma} \mathrm{Sgn}(\tau\sigma) && \text{(by Proposition 4.13 (ii))} \\ &= W_n. \end{aligned}$$

Thus W_n is an orthogonal projection.

(ii)

$$\begin{aligned} W_n U_\sigma &= \frac{1}{n!} \sum_{\tau \in \mathsf{S}_n} U_\tau \mathrm{Sgn}(\tau) U_\sigma \\ &= \frac{1}{n!} \sum_{\tau \in \mathsf{S}_n} U_\tau U_\sigma \mathrm{Sgn}(\sigma\tau) \mathrm{Sgn}(\sigma) && \text{(by Lemma 4.14)} \\ &= \Big(\frac{1}{n!} \sum_{\tau \in \mathsf{S}_n} U_{\sigma\tau} \mathrm{Sgn}(\sigma\tau)\Big) \mathrm{Sgn}(\sigma) && \text{(by Proposition 4.13 (ii))} \\ &= W_n \mathrm{Sgn}(\sigma). \end{aligned}$$

(iii) This follows from (ii). □

4.3 Super Quantization

4.3.1 The super-symmetric Fock space

Definition 4.17 For each $n \geq 1$, we call the closed subspace of $\otimes^n \mathcal{H}$ defined by

$$\otimes_{\mathrm{ss}}^n \mathcal{H} := W_n(\otimes^n \mathcal{H})$$

the *n-fold super-symmetric tensor product.*

The closed subspace of $\mathcal{F}(\mathcal{H})$ defined by

$$\mathcal{F}_{\mathrm{ss}}(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes_{\mathrm{ss}}^n \mathcal{H}$$

is said to be the *super-symmetric Fock space.*

For a subspace $\mathcal{D}$ of $\mathcal{H}$, we denote by $\mathcal{F}_{\mathrm{ss,fin}}(\mathcal{D})$ the subspace of $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$ defined by

$$\mathcal{F}_{\mathrm{ss,fin}}(\mathcal{D}) := \widehat{\oplus}_{n=0}^{\infty} W_n(\widehat{\otimes}^n \mathcal{D}),$$

where $\widehat{\oplus}_{n=0}^{\infty}$ means algebraic infinite direct sum. If $\mathcal{D}$ is dense in $\mathcal{H}$, then $\mathcal{F}_{\mathrm{ss,fin}}(\mathcal{D})$ is also dense in $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$.

Note that $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$ has the following $\mathbb{Z}_2$-grading structure:

$$\mathcal{F}_{\mathrm{ss}}(\mathcal{H}) = \mathcal{F}_{\mathrm{ss}}(\mathcal{H})_{\bar{0}} \oplus \mathcal{F}_{\mathrm{ss}}(\mathcal{H})_{\bar{1}},$$

where

$$\mathcal{F}_{\mathrm{ss}}(\mathcal{H})_\alpha := \oplus_{n=0}^{\infty} W_n(\otimes^n \mathcal{H})_\alpha, \quad (\alpha \in \mathbb{Z}_2).$$

The grading operator Γ for $\mathcal{F}(\mathcal{H})$ is reduced by $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$. Indeed, it is not difficult to check that the grading operator $\Gamma^{(n)}$ for $\otimes^n \mathcal{H}$ commutes with W_n. Hence, $\Gamma^{(n)}$ is reduced by $W_n(\otimes^n \mathcal{H})$ and the operator $\Gamma = \oplus_{n=0}^{\infty} \Gamma^{(n)}$ is reduced by $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$. From this fact, it follows that the grading operator for $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$ is the reduced part of Γ to $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$. We denote it by Γ_{ss}.

Proposition 4.18 *Suppose that $\{e_n^\alpha\}_{n=0}^{\infty}$ is a complete orthonormal system (CONS) of $\mathcal{H}_\alpha$ $(\alpha = \bar{0}, \bar{1})$. Then*

$$\mathcal{E}_n := \left\{ \sqrt{\frac{n!}{n_1! \cdots n_p!}} W_n\left((e_{i_1}^{\bar{0}})^{n_1} \otimes \cdots \otimes (e_{i_p}^{\bar{0}})^{n_p} \otimes e_{j_1}^{\bar{1}} \otimes \cdots \otimes e_{j_q}^{\bar{1}}\right) \,\middle|\, \sum_{i=1}^{p} n_i + q = n, \right.$$
$$\left. i_1 < \cdots < i_p,\ j_1 < \cdots < j_q \right\}$$

is a CONS of $\otimes_{\mathrm{ss}}^n \mathcal{H}$, where we use the following notation: $(e_i^{\bar{0}})^k := \overbrace{e_i^{\bar{0}} \otimes \cdots \otimes e_i^{\bar{0}}}^{k}$. Thus $\{\Omega_{\mathcal{H}}\} \cup (\cup_{n=1}^{\infty} \mathcal{E}_n)$ is a CONS of $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$.

Proof. It is not hard to see that $\mathcal{E}_n$ is an orthonormal sysem. Hence we only show the completness of $\mathcal{E}_n$.

Suppose that $\Psi \in \mathcal{L}(\mathcal{E}_n)^{\perp} \cap W_n(\otimes^n \mathcal{H})$, where $\mathcal{L}(\mathcal{E}_n)$ is a subspace of $\otimes^n \mathcal{H}$ generated by $\mathcal{E}_n$. Then we have

$$0 = \left\langle \Psi, W_n\left((e_{i_1}^{\bar{0}})^{n_1} \otimes \cdots \otimes (e_{i_p}^{\bar{0}})^{n_p} \otimes e_{j_1}^{\bar{1}} \otimes \cdots \otimes e_{j_q}^{\bar{1}}\right)\right\rangle.$$

for each $p, q \in \mathbb{N}$, $n_1, \cdots, n_p \in \mathbb{N}$ and $i_1 < \cdots < i_p$, $j_1 < \cdots < j_q$. Therefore

$$\left\langle \Psi, e_{k_1}^{\alpha_1} \otimes \cdots \otimes e_{k_n}^{\alpha_n} \right\rangle = 0$$

for each $k_i \in \mathbb{N}$ and $\alpha_i \in \mathbb{Z}_2$ $(i = 1, \cdots, n)$. Since $\{e_{k_1}^{\alpha_1} \otimes \cdots \otimes e_{k_n}^{\alpha_n} \mid k_i \in \mathbb{N},\ \alpha_i = \bar{0}, \bar{1},\ i = 1, \cdots, n\}$ is a CONS of $\otimes^n \mathcal{H}$, we can conclude that $\Psi = 0$. Therefore $\mathcal{E}_n$ is complete. □

Proposition 4.19 *Suppose that T is in $\mathcal{L}(\mathcal{D})$.*

(i) $T^{[n]}$ *is reduced by* $\otimes_{\mathrm{ss}}^n \mathcal{H}$ *for* $n \geq 2$.
(ii) $\mathrm{d}\Gamma(T)$ *is reduced by* $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$.

To prove this proposition, we need following lemma.

Lemma 4.20 *For each* $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}_2^n$, *let*

$$(\lambda)_j := (\lambda_1, \cdots, \lambda_{j-1}, \lambda_j + \bar{1}, \lambda_{j+1}, \cdots, \lambda_n).$$

Then we have

$$\mathrm{sgn}(\sigma; (\lambda)_j)(-1)^{\sum_{i<j} \lambda_i} = \mathrm{sgn}(\sigma; \lambda)(-1)^{\sum_{i<j} \lambda_{\sigma(i)}} \tag{12}$$

for each σ *in* S_n.

Proof. Note first that by a similar argument as in the proof of Proposition 4.8, we can show

$$\mathrm{sgn}(\sigma; \lambda) = (-1)^{\sum_{\substack{i>j \\ \sigma(i)<\sigma(j)}} \lambda_{\sigma(i)} + \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} \lambda_{\sigma(i)}} \mathrm{sgn}(\sigma; (\lambda)_j). \tag{13}$$

By (13), we have

$$\mathrm{sgn}(\sigma; (\lambda)_j)(-1)^{\sum_{i<j} \lambda_i} = (-1)^{\sum_{i<j} \lambda_j + \sum_{\substack{i>j \\ \sigma(i)<\sigma(j)}} \lambda_{\sigma(i)} + \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} \lambda_{\sigma(i)}} \mathrm{sgn}(\sigma; \lambda). \tag{14}$$

On the other hand, we have

$$\sum_{i<j} \lambda_j + \sum_{\substack{i>j \\ \sigma(i)<\sigma(j)}} \lambda_{\sigma(i)} = \sum_{i<j} \lambda_i + \sum_{\substack{i<j \\ \sigma^{-1}(i)>\sigma^{-1}(j)}} \lambda_i$$

$$
\begin{aligned}
&= \sum_{\substack{i<j \\ \sigma^{-1}(i)>\sigma^{-1}(j)}} \lambda_i + \sum_{\substack{i<j \\ \sigma^{-1}(i)<\sigma^{-1}(j)}} \lambda_i + \sum_{\substack{i<j \\ \sigma^{-1}(i)>\sigma^{-1}(j)}} \lambda_i \\
&= \sum_{\substack{i<j \\ \sigma^{-1}(i)<\sigma^{-1}(j)}} \lambda_i \\
&= \sum_{\substack{\sigma(i)<\sigma(j) \\ i<j}} \lambda_{\sigma(i)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\text{RHS of (14)} &= (-1)^{\sum_{\substack{\sigma(i)<\sigma(j) \\ i<j}} \lambda_{\sigma(i)} + \sum_{\substack{\sigma(i)>\sigma(j) \\ i<j}} \lambda_{\sigma(i)}} \operatorname{sgn}(\sigma;\lambda) \\
&= (-1)^{\sum_{i<j} \lambda_{\sigma(i)}} \operatorname{sgn}(\sigma;\lambda).
\end{aligned}
$$

Thus we have desired result. □

Proof of Proposition 4.19. (i) Since

$$
T^{[n]} = T^{[n]}_{\bar{0}} + T^{[n]}_{\bar{1}}
$$

on $\widehat{\otimes}^n \operatorname{dom}(T)$ and $\widehat{\otimes}^n \operatorname{dom}(T)$ is a core of $T^{[n]}$, it is suffices to show the assertion when T is homogeneous. Suppose that T is homogeneous. If $\rho(T) = 0$, then it is clear that $T^{[n]}$ is reduced by $\otimes^n_{\mathrm{ss}}\mathcal{H}$. Thus we will show the assertion when $\rho(T) = 1$.

Let $f_i^{\lambda_i} \in \operatorname{dom}(T)_{\lambda_i}$ ($\lambda_i = \bar{0}, \bar{1},\ i = 1, \cdots, n$). Then we have

$$
\begin{aligned}
&(T^{[n]} W_n)(f_1^{\lambda_1} \otimes \cdots \otimes f_n^{\lambda_n}) \\
&= \sum_{j=1}^{n} \sum_{\sigma \in \mathsf{S}_n} \frac{1}{n!} \operatorname{sgn}(\sigma;\lambda)(-1)^{\sum_{i<j} \lambda_{\sigma(i)}} f_{\sigma(1)}^{\lambda_{\sigma(1)}} \otimes \cdots \otimes T f_{\sigma(j)}^{\lambda_{\sigma(j)}} \otimes \cdots \otimes f_{\sigma(n)}^{\lambda_{\sigma(n)}},
\end{aligned}
$$

where $\lambda := (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}_2^n$. Hence, using the above lemma we have

$$
\begin{aligned}
&(W_n T^{[n]})(f_1^{\lambda_1} \otimes \cdots \otimes f_n^{\lambda_n}) \\
&= \sum_{j=1}^{n} \sum_{\sigma \in \mathsf{S}_n} \frac{1}{n!} \operatorname{sgn}(\sigma;(\lambda)_j)(-1)^{\sum_{i<j} \lambda_i} f_{\sigma(1)}^{\lambda_{\sigma(1)}} \otimes \cdots \otimes T f_{\sigma(j)}^{\lambda_{\sigma(j)}} \otimes \cdots \otimes f_{\sigma(n)}^{\lambda_{\sigma(n)}} \\
&= \sum_{j=1}^{n} \sum_{\sigma \in \mathsf{S}_n} \frac{1}{n!} \operatorname{sgn}(\sigma;\lambda)(-1)^{\sum_{i<j} \lambda_{\sigma(i)}} f_{\sigma(1)}^{\lambda_{\sigma(1)}} \otimes \cdots \otimes T f_{\sigma(j)}^{\lambda_{\sigma(j)}} \otimes \cdots \otimes f_{\sigma(n)}^{\lambda_{\sigma(n)}} \\
&= (T^{[n]} W_n)(f_1^{\lambda_1} \otimes \cdots \otimes f_n^{\lambda_n}).
\end{aligned}
$$

That is

$$
W_n T^{[n]} = T^{[n]} W_n
$$

on $\widehat{\otimes}^n \mathrm{dom}(T)$. Since $\widehat{\otimes}^n \mathrm{dom}(T)$ is a core of $T^{[n]}$, we can conlude that

$$W_n T^{[n]} \subset T^{[n]} W_n.$$

(ii) follows from (i). □

Definition 4.21 Let T be in $\mathcal{L}(\mathcal{D})$. We denote the reduced part of $\mathrm{d}\Gamma(T)$ to $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$ by $\mathrm{d}\Gamma_{\mathrm{ss}}(T)$. The operator $\mathrm{d}\Gamma_{\mathrm{ss}}(T)$ is called the *super quantization of* T.

Theorem 4.22 *Let A and B be elements in $\mathcal{L}(\mathcal{D})$.*

(i) $\mathrm{d}\Gamma_{\mathrm{ss}}(A)$ *is a closed operator.*

(ii) *If A is homogeneous, then $\mathrm{d}\Gamma_{\mathrm{ss}}(A)$ is also homogeneous with $\rho(\mathrm{d}\Gamma_{\mathrm{ss}}(A)) = \rho(A)$.*

(iii) *Suppose that A is self-adjoint. If A is homogeneous, then $\mathrm{d}\Gamma_{\mathrm{ss}}(A)$ is self-adjoint. If A is non-homogenous and either even part $A_{\bar{0}}$ or odd part $A_{\bar{1}}$ is bounded, then $\mathrm{d}\Gamma_{\mathrm{ss}}(A)$ is self-adjoint.*

(iv) *If A is non-negative, then $\mathrm{d}\Gamma_{\mathrm{ss}}(A)$ is non-negative.*

(v) *If A and B are homogeneous and self-adjoint, then $\mathrm{d}\Gamma_{\mathrm{ss}}(A)$ and $\mathrm{d}\Gamma_{\mathrm{ss}}(B)$ strongly supercommute if and only if A and B strongly supercommute.*

(vi) $[\mathrm{d}\Gamma_{\mathrm{ss}}(A), \mathrm{d}\Gamma_{\mathrm{ss}}(B)]_{\mathrm{S}} = \mathrm{d}\Gamma_{\mathrm{ss}}([A,B]_{\mathrm{S}})$ *on* $\mathcal{F}_{\mathrm{ss,fin}}(\mathcal{D})$.

(vii) *For an arbitary odd self-adjoint operator* $A = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$ *in $\mathcal{L}(\mathcal{D})$, we have*

$$\mathrm{d}\Gamma_{\mathrm{ss}}(A)^2 = \mathrm{d}\Gamma_{\mathrm{ss}}(S^*S \oplus SS^*).$$

Proof. These are simple applications of Propositions 4.7 and 4.19. □

4.3.2 Identification with the boson-fermion Fock space

Let $\mathcal{X}$ be a Hilbert space. The boson Fock space $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$ and the fermion Fock space $\mathcal{F}_{\mathrm{f}}(\mathcal{X})$ over $\mathcal{X}$ are respectively defined by

$$\mathcal{F}_{\mathrm{b}}(\mathcal{X}) := \oplus_{n=0}^{\infty} S_n(\otimes^n \mathcal{X}), \quad \mathcal{F}_{\mathrm{f}}(\mathcal{X}) := \oplus_{n=0}^{\infty} A_n(\otimes^n \mathcal{X}),$$

where we denote by S_n(resp.A_n) the symmetrizer (resp. the anti-symmetrizer) on $\otimes^n \mathcal{X}$. For a subspace $\mathcal{V}$ of $\mathcal{X}$, we define

$$\mathcal{F}_{\mathrm{b,fin}}(\mathcal{V}) := \widehat{\oplus}_{n=0}^{\infty} S_n(\widehat{\otimes}^n \mathcal{V}), \ \mathcal{F}_{\mathrm{f,fin}}(\mathcal{V}) := \widehat{\oplus}_{n=0}^{\infty} A_n(\widehat{\otimes}^n \mathcal{V}).$$

If $\mathcal{V}$ is dense, then $\mathcal{F}_{\#,\mathrm{fin}}(\mathcal{V})$ is also dense in $\mathcal{F}_{\#}(\mathcal{V})$ ($\# =$b,f).

Let $\Omega_{\mathrm{b}} := 1 \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{F}_{\mathrm{b}}(\mathcal{X})$ be the boson Fock vacuum in $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$. For each $f \in \mathcal{X}$, there exists a unique densely defined closed linear operator $a_{\mathrm{b}}(f)$ on $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$, called a boson annihilation operator (its adjoint $a_{\rm b}(f)^*$ is called a boson creation operator), such that (i) for all $f \in \mathcal{X}$, $a_{\rm b}(f)\Omega_{\rm B} = 0$, (ii) for all $n \in \mathbb{N}$, $f_j \in \mathcal{X}$, $j = 1, \cdots, n$,

$$a_{\rm b}(f)S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \langle f, f_j \rangle_{\mathcal{X}} S_{n-1}(f_1 \otimes \cdots \otimes \hat{f}_j \otimes \cdots f_n),$$

where $\hat{f}_j$ indicates the omission of f_j. (iii) $\mathcal{F}_{\rm b,fin}(\mathcal{H})$ is a core of $a_{\rm b}(f)$. We have for all $f_j \in \mathcal{X}$, $j = 1, \cdots, n$,

$$a_{\rm b}(f_1)^* \cdots a_{\rm b}(f_n)^* \Omega_{\rm b} = \sqrt{n!} S_n(f_1 \otimes \cdots \otimes f_n).$$

The set $\{a_{\rm b}(f) \mid f \in \mathcal{X}\}$ satisfies the canonical communtation relations (CCR)

$$[a_{\rm b}(f), a_{\rm b}(g)^*] = \langle f, g \rangle, \quad [a_{\rm b}(f), a_{\rm b}(g)] = 0, \quad [a_{\rm b}(f)^*, a_{\rm b}(g)^*] = 0$$

for all $f, g \in \mathcal{X}$ on $\mathcal{F}_{\rm b,fin}(\mathcal{X})$.

Boson Fock space objects have counter parts in the fermion Fock spcace. The fermion Fock vacuum $\Omega_{\rm f}$ in $\mathcal{F}_{\rm f}(\mathcal{X})$ is defined by $\Omega_{\rm f} := 1 \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{F}_{\rm f}(\mathcal{X})$. For each $g \in \mathcal{X}$, there exists a unique bounded linear operator $a_{\rm f}(g)$ on $\mathcal{F}_{\rm f}(\mathcal{X})$, called a fermion annihilation operator on $\mathcal{F}_{\rm f}(\mathcal{X})$ ($a_{\rm f}(g)^*$ is called a fermion creation operator), such that (i) for all $g \in \mathcal{X}$, $a_{\rm f}(g)\Omega_{\rm f} = 0$, (ii) for all $n \in \mathbb{N}$, $g_j \in \mathcal{X}$, $j = 1, \cdots, n$,

$$a_{\rm f}(g)A_n(g_1 \otimes \cdots g_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{j-1} \langle g, g_j \rangle_{\mathcal{X}} A_{n-1}(g_1 \otimes \cdots \otimes \hat{g}_j \otimes \cdots \otimes g_n).$$

We have for all $g_j \in \mathcal{X}$, $= 1, \cdots, n$,

$$a_{\rm f}(g_1)^* \cdots a_{\rm f}(g_n)^* \Omega_{\rm f} = \sqrt{n!} A_n(g_1 \otimes \cdots \otimes g_n).$$

The set $\{a_{\rm f}(g) \mid g \in \mathcal{X}\}$ satisfies the canonical anti-communtation relations (CAR)

$$\{a_{\rm f}(f), a_{\rm f}(g)^*\} = \langle f, g \rangle, \quad \{a_{\rm f}(f), a_{\rm f}(g)\} = 0, \quad \{a_{\rm f}(f)^*, a_{\rm f}(g)^*\} = 0$$

for all $f, g \in \mathcal{X}$.

Definition 4.23 *The boson-fermion Fock space* $\mathcal{F}_{\text{b-f}}(\mathcal{X}_1, \mathcal{X}_2)$ associated with the pair $(\mathcal{X}_1, \mathcal{X}_2)$ is defined by

$$\mathcal{F}_{\text{b-f}}(\mathcal{X}_1, \mathcal{X}_2) := \mathcal{F}_{\rm b}(\mathcal{X}_1) \otimes \mathcal{F}_{\rm f}(\mathcal{X}_2).$$

The linear operator V from $\mathcal{F}_{\rm ss}(\mathcal{H})$ to $\mathcal{F}_{\text{b-f}}(\mathcal{H}_{\bar{0}}, \mathcal{H}_{\bar{1}})$ defined by

$$V\Big(W_n\big((e^{\bar{0}}_{i_1})^{n_1} \otimes \cdots \otimes (e^{\bar{0}}_{i_p})^{n_p} \otimes e^{\bar{1}}_{j_1} \otimes \cdots \otimes e^{\bar{1}}_{j_q}\big)\Big)$$
$$:= \sqrt{\tfrac{p!}{q!n!}} S_{n_1+\cdots+n_p}\big((e^{\bar{0}}_{i_1})^{n_1} \otimes \cdots \otimes (e^{\bar{0}}_{i_p})^{n_p}\big) \otimes A_q\big(e^{\bar{1}}_{j_1} \otimes \cdots \otimes e^{\bar{1}}_{j_q}\big)$$

can be extended to the unitary operator,where $n := \sum_{i=1}^{p} n_i + q$. We denote it by the same symbol V. Relative to this identification, we have following identifications:

$$\otimes_{\rm ss}^n \mathcal{H} = \oplus_{p+q=n}[S_p(\otimes^p \mathcal{H}_{\bar 0}) \otimes A_q(\otimes^q \mathcal{H}_{\bar 1})],$$
$$\mathcal{F}_{\rm ss,fin}(\mathcal{D}_{\bar 0} \oplus \mathcal{D}_{\bar 1}) = \mathcal{F}_{\rm b,fin}(\mathcal{D}_{\bar 0}) \widehat{\otimes} \mathcal{F}_{\rm f,fin}(\mathcal{D}_{\bar 1}),$$

where $\mathcal{D}_{\bar 0} \oplus \mathcal{D}_{\bar 1}$ is a subspace of $\mathcal{H}$. The $\mathbb{Z}_2$-grading structure of $\mathcal{F}_{\rm ss}(\mathcal{H})$ induces the following $\mathbb{Z}_2$-grading of $\mathcal{F}_{\rm b\text{-}f}(\mathcal{H}_{\bar 0}, \mathcal{H}_{\bar 1})$:

$$\begin{aligned} \mathcal{F}_{\rm ss}(\mathcal{H})_{\bar 0} &= \mathcal{F}_{\rm b}(\mathcal{H}_{\bar 0}) \otimes \left[\oplus_{n=0}^{\infty} A_{2n}(\otimes^{2n} \mathcal{H}_{\bar 1}) \right], \\ \mathcal{F}_{\rm ss}(\mathcal{H})_{\bar 1} &= \mathcal{F}_{\rm b}(\mathcal{H}_{\bar 1}) \otimes \left[\oplus_{n=0}^{\infty} A_{2n+1}(\otimes^{2n+1} \mathcal{H}_{\bar 1}) \right]. \end{aligned}$$

Suppose that T is a closed operator densely defined on $\mathcal{X}$. Let

$$T^{(n)} := \sum_{j=1}^{n} \overbrace{I \otimes \cdots \otimes I \otimes \underbrace{T}_{j\text{th}} \otimes I \otimes \cdots \otimes I}^{n}.$$

Putting $T^{(0)} = 0$, one can define a closed operator

$$\mathrm{d}\Gamma(T) := \oplus_{n=0}^{\infty} \overline{T^{(n)}}$$

on the full Fock space $\mathcal{F}(\mathcal{X})$. (Though, in Section 4.1, we have used the symbol $\mathrm{d}\Gamma(\cdot)$ which is defferent from the one that we have just introduced, there will be no confusions.) It is easy to see that $\mathrm{d}\Gamma(T)$ is reduced by $\mathcal{F}_{\#}(\mathcal{X})$ ($\#$ =b,f). We denote the reduced part of $\mathrm{d}\Gamma(T)$ to $\mathcal{F}_{\#}(\mathcal{X})$ by $\mathrm{d}\Gamma_{\#}(T)$. We put

$$N_{\#} := \mathrm{d}\Gamma_{\#}(I),$$

called the number operator on $\mathcal{F}_{\#}(\mathcal{X})$.

Note that the grading operator $\Gamma_{\rm ss}$ can be identified with $(-1)^{I \otimes N_{\rm f}}$:

$$\Gamma_{\rm ss} = (-1)^{I \otimes N_{\rm f}}.$$

For each $f = f_{\bar 0} \oplus f_{\bar 1} \in \mathcal{H}$, we introduce an operator

$$a_{\rm s}(f) := a_{\rm b}(f_{\bar 0}) \otimes I + I \otimes a_{\rm f}(f_{\bar 1})$$

acting in $\mathcal{F}_{\rm b\text{-}f}(\mathcal{H}_{\bar 0}, \mathcal{H}_{\bar 1})$.

Proposition 4.24 *Let f, g be in $\mathcal{H}$.*

(i) *If f is a homogeous element in $\mathcal{H}$, then $a_{\rm s}^{\#}(f)$ is also homogeneous with $\rho(a_{\rm s}^{\#}(f)) = \rho(f)$.*

(ii) *The set* $\{a_s(f) \mid f \in \mathcal{H}\}$ *satisfies canonical super-commutation relations (CSR)*

$$[a_s(f), a_s(g)^*]_S = \langle f, g \rangle_{\mathcal{H}}, \quad [a_s(f), a_s(g)]_S = 0, \quad [a_s(f)^*, a_s(g)^*]_S = 0$$

on $\mathcal{F}_{b,fin}(\mathcal{H}_{\bar{0}})\hat{\otimes}\mathcal{F}_{f,fin}(\mathcal{H}_{\bar{1}})$.

(iii) *For a subspace* $\mathcal{V}$ *of* $\mathcal{H}$, *we have*

$$\begin{aligned} W_n(\hat{\otimes}^n \mathcal{V}) &= \mathcal{L}\{a_s(f_1)^* \cdots a_s(f_n)^* \Omega_b \otimes \Omega_f \mid f_1, \cdots, f_n \in \mathcal{V}\}, \\ \mathcal{F}_{ss,fin}(\mathcal{V}) &= \mathcal{L}\{a_s(f_1)^* \cdots a_s(f_n)^* \Omega_b \otimes \Omega_f, \Omega_b \otimes \Omega_f \mid f_1, \cdots, f_n \in \mathcal{V}, n \in \mathbb{N}\}. \end{aligned}$$

Proof. This is an easy exercise. □

Proposition 4.25 *Suppose that* $A \oplus B$ *is an even operator in* $\mathcal{L}(\mathcal{D})$. *Then relative to the identification* $\mathcal{F}_{ss}(\mathcal{H}) = \mathcal{F}_b(\mathcal{H}_{\bar{0}}) \otimes \mathcal{F}_f(\mathcal{H}_{\bar{1}})$, *we have*

$$d\Gamma_{ss}(A \oplus B) = \overline{d\Gamma_b(A) \otimes I + I \otimes d\Gamma_f(B)}.$$

Proof. Note first that from the construction of $d\Gamma_{ss}(A \oplus B)$ it follows that $d\Gamma_{ss}(A \oplus B)$ is essentially self-adjoint on $\mathcal{F}_{ss,fin}(\mathrm{dom}(A) \oplus \mathrm{dom}(B))(= \mathcal{F}_{b,fin}(\mathrm{dom}(A))\widehat{\otimes}\mathcal{F}_{f,fin}(\mathrm{dom}(B)))$. On the other hand, let

$$T(A, B) := [d\Gamma_b(A) \otimes I + I \otimes d\Gamma_f(B)]^-,$$

then $T(A, B)$ is essentially self-adjoint on $\mathcal{F}_{b,fin}(\mathrm{dom}(A))\widehat{\otimes}\mathcal{F}_{f,fin}(\mathrm{dom}(B))$.

It is not hard to check that

$$d\Gamma_{ss}(A \oplus B) = T(A, B)$$

on $\mathcal{F}_{ss,fin}(\mathrm{dom}(A) \oplus \mathrm{dom}(B)) = \mathcal{F}_{b,fin}(\mathrm{dom}(A))\hat{\otimes}\mathcal{F}_{f,fin}(\mathrm{dom}(B))$. Hence we have the desired result. □

By direct calculation, we can show the following two propositions:

Proposition 4.26 *Suppose that* A *is an element in* $\mathcal{L}(\mathcal{D})$ *and that* $\{e_n\}_{n=1}^{\infty} = \{e_n^{\bar{0}} \oplus 0,\ 0 \oplus e_n^{\bar{1}}\}$ *is a CONS of* $\mathcal{H}$ *such that* $\{e_n^{\alpha}\}_{n=1}^{\infty} \subset \mathcal{D}_{\alpha}$ *is a CONS of* $\mathcal{H}_{\alpha}$ $(\alpha = \bar{0}, \bar{1})$. *Then for each* Ψ *in* $\mathcal{F}_{ss,fin}(\mathcal{D})(= \mathcal{F}_{b,fin}(\mathcal{D}_{\bar{0}})\widehat{\otimes}\mathcal{F}_{f,fin}(\mathcal{D}_{\bar{1}}))$, *we have*

$$d\Gamma_{ss}(A)\Psi = \sum_{n=1}^{\infty} a_s(e_n)^* a_s(A^* e_n)\Psi.$$

Proposition 4.27 *Suppose that* A *is an element in* $\mathcal{L}(\mathcal{D})$. *Then for each* f *in* $\mathcal{D}$, *we have*

$$\begin{aligned} [d\Gamma_{ss}(A), a_s(f)^*]_S &= a_s(Af)^*, \\ [a_s(f), d\Gamma_{ss}(A)]_S &= a_s(A^* f) \end{aligned}$$

on $\mathcal{F}_{ss,fin}(\mathcal{D})$.

5 Applications

5.1 Supersymmetric quantum field theory

In [3, 5], A.Arai developed a mathematical theory about the supersymmetric quantum field theory. Here we will review this theory from a new point of view.

Suppose that S is a densely defined closed linear operator from $\mathcal{H}_{\bar{0}}$ to $\mathcal{H}_{\bar{1}}$. Then we can define a closed linear operator on $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$ by

$$d_S := \mathrm{d}\Gamma_{\mathrm{ss}}\left(\begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}\right).$$

Note that if we put $\mathcal{D} = \mathrm{dom}(S) \oplus \mathrm{dom}(S^*)$, then by Proposition 4.26 we have a following representation of d_S on $\mathcal{F}_{\mathrm{ss,fin}}(\mathcal{D})(= \mathcal{F}_{\mathrm{b,fin}}(\mathrm{dom}(S))\widehat{\otimes}\mathcal{F}_{\mathrm{f,fin}}(\mathrm{dom}(\mathcal{H}_{\bar{1}})))$

$$d_S = \sum_{n=1}^{\infty} a_{\mathrm{b}}(S^* e_n^{\bar{1}}) \otimes a_{\mathrm{f}}(e_n^{\bar{1}})^*,$$

where $\{e_n^{\bar{1}}\}_{n=1}^{\infty} \subset \mathrm{dom}(S^*)$ is a CONS of $\mathcal{H}_{\bar{1}}$.

Using the consequences of the preceeding section, we have the following proposition:

Proposition 5.1 *Let d_S be as above. Then we have following:*

(i) *d_s is an odd operator on $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$.*

(ii) $d_S^2 = 0, \quad (d_S^*)^2 = 0.$

Next we define a Dirac type operator on $\mathcal{F}_{\mathrm{ss}}(\mathcal{H})$ by

$$Q_S := d_S + d_S^*$$

with $\mathrm{dom}(Q_S) = \mathrm{dom}(d_S) \cap \mathrm{dom}(d_S^*)$.

Theorem 5.2 *Suppose that S, T are densely defined closed linear operator from $\mathcal{H}_{\bar{0}}$ to $\mathcal{H}_{\bar{1}}$ such that $\mathrm{dom}(S) \cap \mathrm{dom}(T) \oplus \mathrm{dom}(S^*) \cap \mathrm{dom}(T^*)$ is dense in $\mathcal{H}$. Then we have following:*

(i) $Q_S = \mathrm{d}\Gamma_{\mathrm{ss}}\left(\begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}\right)$. *Thus Q_S is self-adjoint.*

(ii) $d_S^* = \mathrm{d}\Gamma_{\mathrm{ss}}\left(\begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix}\right)$.

(iii)

$$\begin{aligned} d_S d_S^* + d_S^* d_S &= Q_S^2 \\ &= \mathrm{d}\Gamma_{\mathrm{ss}}(S^* S \oplus S S^*) \\ &= [\mathrm{d}\Gamma_{\mathrm{b}}(S^* S) \otimes I + I \otimes \mathrm{d}\Gamma_{\mathrm{f}}(S S^*)]^-. \end{aligned}$$

(iv) *Q_S and Q_T strongly anticommute if and only if two self-adjoint operators* $\begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ *strongly anticommute.*

Proof. (i) First we prove that Q_S is symmetric and closed. To show that Q_S is symmetric is very easy. So we only show the closedness of Q_S. Let $\{\Psi_n\}_{n=1}^{\infty}$ be a converging sequence in $\mathrm{dom}(Q_S)$ such that $\Psi_n \to \Psi \in \mathcal{F}_{\mathrm{ss}}(\mathcal{H})(n \to \infty)$ and $\{Q_S\Psi_n\}_{n=1}^{\infty}$ is a Caucy sequence. Then since

$$\|Q_S(\Psi_n - \Psi_m)\|^2 = \|d_S(\Psi_n - \Psi_m)\|^2 + \|d_S^*(\Psi_n - \Psi_m)\|^2,$$

we have

$$\|Q_S(\Psi_n - \Psi_m)\| \geq \|d_S(\Psi_n - \Psi_m)\|, \|d_S^*(\Psi_n - \Psi_m)\|.$$

Thus $\{d_S\Psi_n\}_{n=1}^{\infty}$ is a Caucy sequence. By closedness of d_S, we can conclude that $\Psi \in \mathrm{dom}(d_S)$ and $\lim_{n\to\infty} d_S\Psi_n = d_S\Psi$. Similarly we have $\Psi \in \mathrm{dom}(d_S^*)$ and $\lim_{n\to\infty} d_S^*\Psi_n = d_S^*\Psi$. Hence we have $\Psi \in \mathrm{dom}(Q_S)$ and $\lim_{n\to\infty} Q_S\Psi_n = Q_S\Psi$, which mean closedness of Q_S.

It is not hard to show that

$$Q_S = \mathrm{d}\Gamma_{\mathrm{ss}}(L_S)$$

on $\mathcal{F}_{\mathrm{ss,fin}}(\mathrm{dom}(S) \oplus \mathrm{dom}(S^*))$, where $L_A := \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$ for each densely defined closed linear operator A from $\mathcal{H}_{\bar{0}}$ to $\mathcal{H}_{\bar{1}}$. Since $\mathcal{F}_{\mathrm{ss,fin}}(\mathrm{dom}(S) \oplus \mathrm{dom}(S^*))$ is a core of $\mathrm{d}\Gamma_{\mathrm{ss}}(L_S)$ and Q_S is closed, we have $Q_S \supset \mathrm{d}\Gamma_{\mathrm{ss}}(L_S)$. By self-adjointness of $\mathrm{d}\Gamma_{\mathrm{ss}}(L_S)$, we have the desired result.

(ii) Let

$$\delta_S := \mathrm{d}\Gamma_{\mathrm{ss}}\left(\begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix}\right).$$

Then using the same argument in the proof of (i), we have

$$\mathrm{d}\Gamma_{\mathrm{ss}}(L_S) = d_S + \delta_S.$$

Combining this with (i), we have $\delta_S = d_S^*$.

(iii), (iv) These are simple applications of Theorem 4.22. □

5.2 A boson-fermion interaction model

Let ω_{b} (resp. ω_{f}) be a self-adjoint operator on $\mathcal{H}_{\bar{0}}$ (resp. $\mathcal{H}_{\bar{1}}$) and f_j (resp. g_j) ($j = 1, \cdots, n$) be vectors in $\mathcal{H}_{\bar{0}}$ (resp. $\mathcal{H}_{\bar{1}}$). We will discuss the following boson-fermion interaction Hamiltonian:

$$H := \mathrm{d}\Gamma_{\mathrm{b}}(\omega_{\mathrm{b}}) \otimes I + I \otimes \mathrm{d}\Gamma_{\mathrm{f}}(\omega_{\mathrm{f}}) + \alpha \sum_{j=1}^{n} \{a_{\mathrm{b}}(f_j)^* \otimes a_{\mathrm{f}}(g_j) + a_{\mathrm{b}}(f_j) \otimes a_{\mathrm{f}}(g_j)^*\},$$

where α in $\mathbb{R}$. To do this, we introduce a linear operator A defined by

$$A = A_{\bar{0}} + A_{\bar{1}},$$
$$A_{\bar{0}} = \omega_{\mathrm{b}} \oplus \omega_{\mathrm{f}}, \quad A_{\bar{1}} = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix},$$

where $S = \sum_{j=1}^{n} \alpha f_i \odot g_j \quad (\alpha \in \mathbb{R})$ and $f \odot g$ is a linear operator defined by $(f \odot g)\psi := \langle g, \psi \rangle f$. Then A is a self-adjoint operator with even (resp. odd) part $A_{\bar{0}}$ (resp. $A_{\bar{1}}$). From Propositions 4.25 and 4.26, it follows that

$$H := \mathrm{d}\Gamma_{\mathrm{ss}}(A)$$

on $\mathcal{F}_{\mathrm{ss,fin}}(\mathrm{dom}(A))$. Hence by Theorem 4.22, we have a following proposition:

Proposition 5.3 *Let A be as above. Then we have*

(i) *H is essentially self-adjoint on $\mathcal{F}_{\mathrm{ss,fin}}(\mathrm{dom}(A))$. Especially, $\overline{H} = \mathrm{d}\Gamma_{\mathrm{ss}}(A)$.;*

(ii) *If A has an eigenvalue $\mu(A)$ with eigenvector ψ, then $\mu(A)$ is also eigenvalue of $\overline{H}$ with eigenvector $0 \oplus \psi \oplus 0 \oplus \cdots \in \mathcal{F}_{\mathrm{ss}}(\mathcal{H})$;*

(iii) *If $A \geq 0$, then $\Omega_{\mathrm{b}} \otimes \Omega_{\mathrm{f}}$ is a ground state of $\overline{H}$.*

Example: The Wigner-Weisskopf model

Let $\mathcal{H}_{\bar{0}} = L^2(\mathbb{R}^d)$, $\mathcal{H}_{\bar{1}} = \mathbb{C}$ and $\omega : \mathbb{R}^d \to [0, \infty)$ be Borel measurable such that $0 < \omega(k) < \infty$ for almost everywhere $k \in \mathbb{R}^d$ with respect to the d-dimensional Lebesgue measure and $\hat{\omega}$ be the multiplication operator by the function ω, acting in $L^2(\mathbb{R}^d)$. Then the Wigner-Weisskopf Hamiltonian H_{WW} is defined by

$$H_{\mathrm{WW}} := \mathrm{d}\Gamma_{\mathrm{b}}(\hat{\omega}) \otimes I + I \otimes \mu_0 c^* c + \alpha\Big(a_{\mathrm{b}}(\lambda) \otimes c^* + a_{\mathrm{b}}(\lambda)^* \otimes c\Big),$$

where c is the fermion annihilation operator on $\mathcal{F}_{\mathrm{f}}(\mathbb{C}) = \mathbb{C}^2$. We will discuss some fundamental properties of this model here. More details on this model, see [6].

Let A be defined by

$$A = A_{\bar{0}} \oplus A_{\bar{1}},$$
$$A_{\bar{0}} := \hat{\omega} \oplus \mu_0, \quad A_{\bar{1}} := \alpha \begin{pmatrix} 0 & \lambda \odot 1 \\ 1 \odot \lambda & 0 \end{pmatrix},$$

where $\mu_0, \alpha \in \mathbb{R}$ is a constant and $\lambda \in L^2(\mathbb{R}^d)$. To describe spectral properties of A, we introduce a function

$$D(z) := -z + \mu_0 + \alpha^2 \int_{\mathbb{R}^d} \frac{|\lambda(k)|^2}{z - \omega(k)} \mathrm{d}k$$

defined for all $z \in \mathbb{C}$ such that $|\lambda(k)|^2/|z - \omega(k)|$ is Lebesgue integrable on $\mathbb{R}^d$. In particular, $D(z)$ is defined in the cut plane $\mathbb{C}\backslash[\mu, \infty)$ ($\mu := \mathrm{ess.\,inf}_{k \in \mathbb{R}^d}\, \omega(k)$) and analytic there. It is easy to see that $D(x)$ is monotone decreasing in $x < \mu$. Hence the limit

$$d_\mu := \lim_{x \uparrow \mu} D(x)$$

exsits, being allowed to be $-\infty$.

Lemma 5.4 *Assume that ω is continuous on $\mathbb{R}^d$ and $\omega(k) \to \infty$ as $|k| \to \infty$. Then we have*

(i) *Let $d_\mu \geq 0$. Then*
$$\sigma(A) = [\mu, \infty);$$

(ii) *Let $d_\mu < 0$. Then*
$$\sigma(A) = \{x_0\} \cup [\mu, \infty),$$
where x_0 is a simple eigenvalue of A.

Proof. See [6]. □

Proposition 5.5 *Assume that ω is continuous on $\mathbb{R}^d$ and $\omega(k) \to \infty$ as $|k| \to \infty$. Then we have*

(i) *H_{WW} is essentially self-adjoint on $\mathcal{F}_{\mathrm{b,fin}}(\mathrm{dom}(\omega))\hat{\otimes}\mathbb{C}^2$ and $\overline{H_{\mathrm{WW}}} = \mathrm{d}\Gamma_{\mathrm{ss}}(A)$;*

(ii) *Let $d_\mu \geq 0$. Then $\overline{H_{\mathrm{WW}}}$ is non-negative with ground state $\Omega_{\mathrm{b}} \otimes \Omega_{\mathrm{f}}$;*

(iii) *Let $d_\mu < 0$. If $x_0 \geq 0$, then $\overline{H_{\mathrm{WW}}}$ is non-negative with ground state $\Omega_{\mathrm{b}} \otimes \Omega_{\mathrm{f}}$. Moreover x_0 is eigenvalue of $\overline{H_{\mathrm{WW}}}$ with eigenvector $0 \oplus \phi_0 \oplus 0 \oplus \cdots \in \mathcal{F}_{\mathrm{ss}}(\mathcal{H})$, where ϕ_0 is the eigenvector with respect to x_0.*

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