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Series #556. July 2002

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Singularities of line congruences

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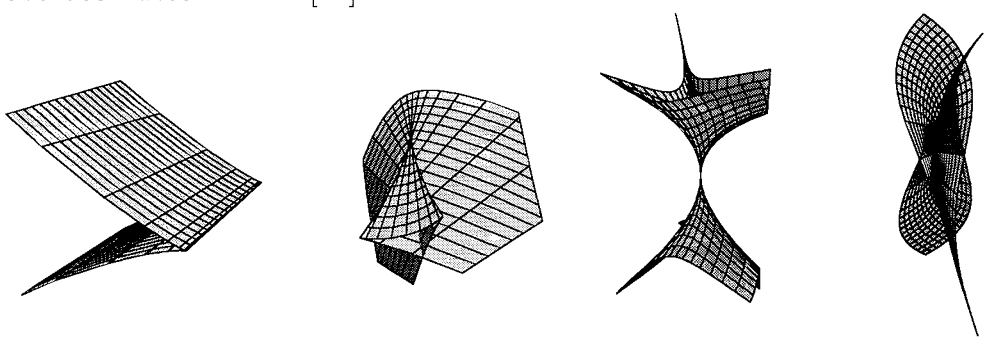
May 13, 2002

Abstract

A line congruence is a two parameter family of lines in $\mathbb{R}^3$. In this paper we study singularities of line congruences. We show that generic singularities of general line congruences are the same as those of stable mappings between three dimensional manifolds. Moreover, we also study singularities of normal congruences and equiaffine normal congruences from the view point of the theory of Lagrangian singularities.

1 Introduction

The study of line congruences in $\mathbb{R}^3$ is a classical area in line geometry. It has, however, much current interest (i.e., Projective differential geometry [17], Computing and Visualization [15], Geometry of Solitons [18, 19] etc.) One of the examples of line congruences is given by the normal lines of a regular surface. In this case the focal surface (i.e., the critical value set) of the line congruences are classically known as the evolute of the surface, also known as its caustic (i.e., the critical value set of a Lagrangian map). A line congruence is called *a normal congruence* if there exists a surface such that the line congruence is given as the normal lines of the surface. So the notion of normal congruences play an important role in the classical differential geometry of surfaces. The classification of singularities of the evolute of a generic surface have been described in [14].



cuspidal edge swallowtail pyramid purse

Fig.1

2000 Mathematics Subject classification. 58C27, 53A25, 53A05

Key Words and Phrases. line congruences, normal congruences, Lagrangian congruences, singularities

Briefly speaking, the cuspidal edge, the swallowtail, the elliptic umbilic (pyramid) and the hyperbolic umbilic (purse) are the generic singularities of evolutes of surfaces (cf., Fig. 1). These singularities are the same as the generic classification of general Lagrangian singularities in 3-space [1]. The evolute of a surface is the focal surface of a normal congruence which is a member of the special class of line congruences.

On the other hand, general line congruences arise in the classical method of transforming one surface to another by lines. Around 1875, Bäcklund and Bianchi ([2, 3]) studied such a transformation given by a line congruence which is now called the *Bäcklund transformation.* In this case the corresponding focal surfaces are pseudo spherical surfaces (i.e. surfaces of constant negative Gaussian curvature). In [18], Shephard draws several pictures of pseudo spherical surfaces by using another interpretation of the notion of line congruences. We can observe that only cuspidal edges and swallowtails appear on pseudo spherical surfaces in her pictures. There might be no pyramids and purses as focal surfaces of generic "general" line congruences. Therefore we have the following natural question:

Question. How are focal surfaces of normal congruences different from those of "general" line congruences?

In the first half of this paper we give a classification of singularities of general line congruences. Since we only consider the local classification of singularities of the focal surface of a line congruence, we adopt the following local analytic expression: *A line congruence* in $\mathbb{R}^3$ is (locally) the image of the map $F_{(x,e)} : U \times I \to \mathbb{R}^3$ defined by $F_{(x,e)}(u,v,t) = \boldsymbol{x}(u,v) + t\boldsymbol{e}(u,v)$, where $\boldsymbol{x} : U \to \mathbb{R}^3$, $\boldsymbol{e} : U \to \mathbb{R}^3 \setminus \{\boldsymbol{0}\}$ are smooth mappings, $U \subset \mathbb{R}^2$ is an open region and I is an open interval. We call $\boldsymbol{x}$ *a base surface* and $\boldsymbol{e}$ *a director surface.*

In order to describe the main result in the first half of this paper we need some preparatory material. Let $f_i : (N_i, x_i) \longrightarrow (P_i, y_i)$ $(i = 1, 2)$ be C^∞ map germs. We say that f, g are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi : (N_1, x_1) \longrightarrow (N_2, x_2)$ and $\psi : (P_1, y_1) \longrightarrow (P_2, y_2)$ such that $\psi \circ f_1 = f_2 \circ \phi$.

Let $C^\infty(U, \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\boldsymbol{0}\}))$ be the space of smooth mappings $(\boldsymbol{x}, \boldsymbol{e}) : U \longrightarrow \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\boldsymbol{0}\})$ equipped with Whitney C^∞-topology, where $U \subset \mathbb{R}^2$ is an open region. The following theorem gives a "generic" answer to the above question.

Theorem 1.1 *There exists an open dense subset $\mathcal{O} \subset C^\infty(U, \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\boldsymbol{0}\}))$ such that the germ of the line congruence $F_{(x,e)}$ at any point $(u_0, t_0) \in U \times I$ is an immersive germ, or $\mathcal{A}$-equivalent to the fold, the cuspidal edge or the swallowtail for any $(\boldsymbol{x}, \boldsymbol{e}) \in \mathcal{O}$.*

Here, the fold is the map germ defined by $(x,y,z) \mapsto (x,y,z^2)$, the cuspidal edge is the map germ defined by $(x,y,z) \mapsto (x,y,z^3+xz)$ and the swallowtail is defined by $(x,y,z) \mapsto (x,y,z^4+xz+yz^2)$.

It is well known that the critical value set for a generic smooth map germ between 3-manifolds is locally diffeomorphic to the fold, the cuspidal edge or the swallowtail (cf., [1, 5, 12]). The set of line congruences is a very small subset in the space of all C^∞-mappings between 3-spaces. The above theorem, however, asserts that the generic singularities of line congruences are the same as those of C^∞-mappings. We can summarize our key results as follows (cf., [1, 5, 7, 12, 14]) :

$$\{\text{Singularities of generic normal congruences}\} \neq \{\text{Singularities of generic line congruences}\},$$

$$\{\text{Singularities of generic line congruences}\} = \{\text{Singularities of generic } C^\infty\text{-mappings}\}.$$

In §2 we briefly review the classical theory of line congruences. The idea of the proof of Theorem 1.1 is that we may locally regard a line congruence as a one-dimensional unfolding of a map germ and apply the theory of unfoldings. In this case the parameter along the lines is considered to be the unfolding parameter. In §3 we explain the general theory of unfoldings. The proof of Theorem 1.1 is given in §4. The basic idea has been used by the authors in other papers on singularities of ruled surfaces and their generalizations [10, 16].

In the latter half of the paper we consider normal congruences and equiaffine normal congruences. In §5 we consider normal congruences from the view point of symplectic geometry and a classical characterization of normal congruences is given (cf., Proposition 5.1 and [8, 18]). The meaning of this characterization is interpreted in the framework of symplectic geometry. We show that the line congruence is a normal congruence if and only if it has a special Lagrangian lift to the cotangent bundle $T^*\mathbb{R}^3$ (cf., Proposition 5.5). Therefore we introduce the notion of Lagrangian congruences, which is equivalent to the notion of normal congruences. By using this fact we define the space of normal congruences and prove that the generic germs of normal congruences are the same as Lagrangian stable map germs. This result clarifies the fact that the generic singularities of evolutes are the same as the generic Lagrangian singularities. In §6 we consider another important class of line congruences. In the context of equiaffine differential geometry (cf., [4, 13]), the notion of equiaffine normal plays a principal role. Therefore we consider the notion of equiaffine normal congruences in the same way as an ordinary normal congruences. The definition of the equiaffine evolute is also given as the focal set of the equiaffine normal congruence. The assertion of Theorem 6.3 is that an equiaffine normal congruence is a Lagrangian congruence, so that it is a normal congruence if we consider the Euclidean scalar product. A generic classification of equiaffine normal congruences is, however, still an open problem. We give conjectures on the singularities for equiaffine normal congruences.

All manifolds and maps considered here are of class C^∞ unless otherwise stated.

2 Basic notions and a review of the classical theory

We now present basic concepts and properties of line congruences in $\mathbb{R}^3$. The classical theory has been given in [8]. However, line congruences are not so popular now, so that we review the classical framework.

For a line congruence $F_{(x,e)}$, if $\boldsymbol{e}$ has a constant direction, then all lines are parallel. Therefore, a line congruence $F_{(x,e)}$ is said to be *nonparallel* provided $(\boldsymbol{e}_1(u)\times\boldsymbol{e}(u), \boldsymbol{e}_2(u)\times\boldsymbol{e}(u))\neq\boldsymbol{0}$ for any $u\in U$, where $\boldsymbol{e}_i=\partial\boldsymbol{e}/\partial u_i$ $(i=1,2)$ and $\times$ is the vector product. Thus the lines are always changing directions on a nonparallel line congruence It is clear that the set consisting of nonparallel line congruences is an open dense subset in $C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\boldsymbol{0}\}))$.

In the theory of Bäcklund transformations, the notion of focal surfaces plays an principal role (cf., [18]). If we consider the line congruence which is given by the normal lines of a surface, the focal surface is the evolute of the surface. Therefore we concentrate on the study of focal surfaces for general line congruences. For a line congruence $F_{(x,e)}$, we consider a surface

$$\boldsymbol{y}(u^1,u^2)=\boldsymbol{x}(u^1,u^2)+t(u^1,u^2)\boldsymbol{e}(u^1,u^2).$$

We say that $\boldsymbol{y}$ is a *focal surface* of $F_{(x,e)}$ if $\langle\boldsymbol{e},\boldsymbol{y}_1\times\boldsymbol{y}_2\rangle=0$ and $\mathrm{Image}\,F_{(x,e)}=\mathrm{Image}\,F_{(y,e)}$, where $\boldsymbol{y}_i=\partial\boldsymbol{y}/\partial u^i$ $(i=1,2)$. Then we have the following lemma.

Lemma 2.1 *Let $F_{(x,e)}(t,u)$ be a line congruence. Suppose that there exists a smooth function $t(u^1,u^2)$ such that $\boldsymbol{y}(u^1,u^2)=\boldsymbol{x}(u^1,u^2)+t(u^1,u^2)\boldsymbol{e}(u^1,u^2)$ is a focal surface of the line* *congruence* $F_{(x,e)}$. *Then we have*

$$\langle \boldsymbol{e}, \boldsymbol{e}_1 \times \boldsymbol{e}_2 \rangle t^2 + \langle \boldsymbol{e}, \boldsymbol{x}_1 \times \boldsymbol{e}_2 + \boldsymbol{e}_2 \times \boldsymbol{x}_1 \rangle t + \langle \boldsymbol{e}, \boldsymbol{x}_1 \times \boldsymbol{x}_2 \rangle = 0.$$

Proof. We have $\boldsymbol{y}_i = \boldsymbol{x}_i + t_i \boldsymbol{e} + t \boldsymbol{e}_i$ $(i = 1, 2)$. Therefore we have

$$\begin{aligned} \boldsymbol{y}_1 \times \boldsymbol{y}_2 &= \boldsymbol{x}_1 \times \boldsymbol{x}_2 + t_2(\boldsymbol{x}_1 \times \boldsymbol{e}) + t_1(\boldsymbol{e} \times \boldsymbol{x}_2) + t(\boldsymbol{x}_1 \times \boldsymbol{e}_2 + \boldsymbol{e}_1 \times \boldsymbol{x}_2) \\ &+ tt_1(\boldsymbol{e} \times \boldsymbol{e}_2) + tt_2(\boldsymbol{e}_1 \times \boldsymbol{e}) + t^2(\boldsymbol{e}_1 \times \boldsymbol{e}_2). \end{aligned}$$

The assertion follows from this calculation directly. □

On the other hand, we can determine the singular set of $F_{(x,e)}$.

Lemma 2.2 *Let* $F_{(x,e)}$ *be a line congruence. A point* $p = (u^1, u^2, t)$ *is a singular point of* $F_{x,e)}$ *if and only if*

$$\langle \boldsymbol{e}, \boldsymbol{e}_1 \times \boldsymbol{e}_2 \rangle t^2 + \langle \boldsymbol{e}, \boldsymbol{x}_1 \times \boldsymbol{e}_2 + \boldsymbol{e}_2 \times \boldsymbol{x}_1 \rangle t + \langle \boldsymbol{e}, \boldsymbol{x}_1 \times \boldsymbol{x}_2 \rangle = 0.$$

Proof. We can calculate the partial derivatives of $F_{(x,e)}$ as follows:

$$\frac{\partial F_{(x,e)}}{\partial u^i}(u^1, u^2, t) = \boldsymbol{x}_i(u^1, u^2) + t\boldsymbol{e}_i(u^1, u^2) \ (i = 1, 2), \frac{\partial F_{(e,x)}}{\partial t}(u^1, u^2, t) = \boldsymbol{e}(u^1, u^2).$$

Therefore we have

$$\begin{aligned} &\det \left(\frac{\partial F_{(x,e)}}{\partial u^1}(u^1, u^2, t) \ \frac{\partial F_{(x,e)}}{\partial u^2}(u^1, u^2, t) \ \frac{\partial F_{(e,x)}}{\partial t}(u^1, u^2, t) \right) = \langle \boldsymbol{e}, (\boldsymbol{x}_1 + t\boldsymbol{e}_1) \times (\boldsymbol{x}_2 + t\boldsymbol{e}_2) \rangle \\ &= \langle \boldsymbol{e}, \boldsymbol{x}_1 \times \boldsymbol{x}_2 + (\boldsymbol{e}_1 \times \boldsymbol{x}_2 + \boldsymbol{x}_1 \times \boldsymbol{e}_2)t + (\boldsymbol{e}_1 \times \boldsymbol{e}_2)t^2 \rangle \\ &= \langle \boldsymbol{e}, \boldsymbol{e}_1 \times \boldsymbol{e}_2 \rangle t^2 + \langle \boldsymbol{e}, \boldsymbol{x}_1 \times \boldsymbol{e}_2 + \boldsymbol{e}_2 \times \boldsymbol{x}_1 \rangle t + \langle \boldsymbol{e}, \boldsymbol{x}_1 \times \boldsymbol{x}_2 \rangle \end{aligned}$$

Since $p = (u^1, u^2, t)$ is a singular point of $F_{(x,e)}$ if and only if

$$\det \left(\frac{\partial F_{(x,e)}}{\partial u^1}(u^1, u^2, t) \ \frac{\partial F_{(x,e)}}{\partial u^2}(u^1, u^2, t) \ \frac{\partial F_{(e,x)}}{\partial t}(u^1, u^2, t) \right) = 0,$$

the assertion holds. □

By Lemmas 2.1 and 2.2, the critical value set of $F_{(x,e)}$ is called the (generalized) *focal surface* of $F_{(x,e)}$.

3 Unfoldings

For the proof of Theorem 1.1, we need to review the theory of one-dimensional unfoldings of map germs. The definition of an *r-dimensional unfolding of* $f_0 : (\mathbb{R}^n, \boldsymbol{0}) \longrightarrow (\mathbb{R}^p, \boldsymbol{0})$ (originally due to Thom) is a germ $F : (\mathbb{R}^n \times \mathbb{R}^r, \boldsymbol{0}) \longrightarrow (\mathbb{R}^p \times \mathbb{R}^r, \boldsymbol{0})$ given by $F(x, u) = (f(x, u), u)$, where $f(x, u)$ is a germ of r dimensional parameterized families of germs with $f(x, \boldsymbol{0}) = f_0(x)$. This definition depends on the coordinates of both of spaces $(\mathbb{R}^n \times \mathbb{R}^r, \boldsymbol{0})$ and $(\mathbb{R}^p \times \mathbb{R}^r, \boldsymbol{0})$. For our purpose, we need the coordinate free definition of unfoldings given in [6]. Let $f : (N, x_0) \longrightarrow (P, y_0)$ be a map-germ between manifolds. An *unfolding* of f is a triple (F, i, j) of map germs, where $i : (N, x_0) \longrightarrow (N', x_0')$, $j : (P, y_0) \longrightarrow (P', y_0')$ are immersions and j is transverse to F, such that $F \circ i = j \circ f$ and $(i, f) : \{(x', y) \in N' \times P \mid F(x') = j(y)\} \longrightarrow N$ is a diffeomorphism germ. The *dimension* of (F, i, j) as an unfolding is $\dim N' - \dim N$. We can easily prove that the above two definitions are equivalent.

Lemma 3.1 *Let $F : (\mathbb{R}^{n-1} \times \mathbb{R}, (\mathbf{0}, 0)) \longrightarrow (\mathbb{R}^n, \mathbf{0})$ be a map germ with the components of the form*

$$F(\boldsymbol{x}, t) = (F_1(\boldsymbol{x}, t), \ldots, F_n(\boldsymbol{x}, t)).$$

Suppose that $\partial F_n/\partial t(\mathbf{0}, 0) \neq 0$. By the implicit function theorem, there exists a function germ $g : (\mathbb{R}^{n-1}, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ with

$$F_n^{-1}(0) = \{(\boldsymbol{x}, g(\boldsymbol{x})) \mid \boldsymbol{x} \in (\mathbb{R}^{n-1}, \mathbf{0})\}.$$

Let us consider immersion germs $i : (\mathbb{R}^{n-1}, \mathbf{0}) \longrightarrow (\mathbb{R}^n, (\mathbf{0}, 0))$ given by $i(\boldsymbol{x}) = (\boldsymbol{x}, g(\boldsymbol{x}))$, $j : (\mathbb{R}^{n-1}, \mathbf{0}) \longrightarrow (\mathbb{R}^n, (\mathbf{0}, 0))$ given by $j(\boldsymbol{y}) = (\boldsymbol{y}, 0)$ and a map germ $f : (\mathbb{R}^{n-1}, \mathbf{0}) \longrightarrow (\mathbb{R}^{n-1}, \mathbf{0})$ given by $f(\boldsymbol{x}) = (F_1(\boldsymbol{x}, g(\boldsymbol{x})), \ldots, F_{n-1}(\boldsymbol{x}, g(\boldsymbol{x})))$. Then the triple (F, i, j) is a one-dimensional unfolding of f.

Proof. It is clear that $F \circ i = j \circ f$. Since $\dfrac{\partial F_n}{\partial t}(\mathbf{0}, 0) \neq 0$, F is transverse to j. We can easily show that

$$\{(\boldsymbol{x}, u, \boldsymbol{y}) \mid F(\boldsymbol{x}, u) = j(\boldsymbol{y})\} = \{(\boldsymbol{x}, g(\boldsymbol{x}), f(\boldsymbol{x})) \mid \boldsymbol{x} \in (\mathbb{R}^{n-1}, \mathbf{0})\}.$$

Since $(i, f) : (\mathbb{R}^{n-1}, \mathbf{0}) \longrightarrow (\mathbb{R}^n \times \mathbb{R}^n, \mathbf{0})$ is given by $(i, f)(\boldsymbol{x}) = (\boldsymbol{x}, g(\boldsymbol{x}), f(\boldsymbol{x}))$, it maps diffeomorphically on to the above set. This completes the proof. $\square$

Since a cuspidal edge and a swallowtail are singularities of stable map germs $(\mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}^3, \mathbf{0})$, we now discuss the stability of unfoldings. Let $\mathcal{E}_n$ be the *local ring* of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal is denoted by $\mathcal{M}_n$. For a map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we say that f is *infinitesimally $\mathcal{A}$-stable* if the following equality holds:

$$\mathcal{E}(n, p) = \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle_{\mathcal{E}_n} + f^*\mathcal{E}(p, p),$$

where $\mathcal{E}(n, p)$ denotes the $\mathcal{E}_n$-module of map germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}^p$ and $f^* : \mathcal{E}(p, p) \longrightarrow \mathcal{E}(n, p)$ is the *pull back homomorphism* defined by $f^*(h) = h \circ f$. It is known that an infinitesimally $\mathcal{A}$-stable map germ $(\mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}^3, \mathbf{0})$ is an immersive germ, a cuspidal edge or a swallowtail [1, 5].

For map germs $f, g : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we say that they are *$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\phi : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^n, \mathbf{0})$ such that $f^*(\mathcal{M}_p)\mathcal{E}_n = \phi^* \circ g^*(\mathcal{M}_p)\mathcal{E}_n$. This $\mathcal{K}$-equivalence is an equivalence relation among map germs. Let $J^k(n, p)$ be the *k-jet space* of map germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$. For any $z = j^k f(\mathbf{0}) \in J^k(n, p)$, we set

$$\mathcal{K}^k(z) = \{j^k g(\mathbf{0}) \mid g \text{ is } \mathcal{K}\text{-equivalent to } f\}.$$

We call it *a $\mathcal{K}^k$-orbit* since it is the orbit of a certain Lie group action. For any map germ $f : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define a map germ $j_1^k f : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \longrightarrow J^k(n, p)$ by $j_1^k f(x_0, u_0) = j^k f_{u_0}(x_0)$, where $f_u(x) = f(x, u)$ and $j^k f_{u_0}(x_0) = j^k(f_{u_0}(x + x_0))(\mathbf{0})$. We have the following Lemma (cf., [12, 7, 6]).

Lemma 3.2 *Under the same notations as the above, $j_1^k f$ is transverse to $\mathcal{K}^k(j_1^k f_0(\mathbf{0}))$ for sufficiently large k if and only if*

$$\mathcal{E}(n, p) = \left\langle \frac{\partial f_0}{\partial x_1}, \ldots, \frac{\partial f_0}{\partial x_n} \right\rangle_{\mathcal{E}_n} + f_0^*(\mathcal{M}_p)\mathcal{E}(n, p) + \left\langle \frac{\partial f}{\partial u_1}(x, 0), \ldots, \frac{\partial f}{\partial u_r}(x, 0), \boldsymbol{e}_1, \ldots, \boldsymbol{e}_p \right\rangle_{\mathbb{R}},$$

where $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_p\}$ is the standard basis of $\mathbb{R}^p$.

The following lemma is implicitly well-known [5, 12]. The proof has been explicitly written in [10].

Lemma 3.3 *Let* $F : (\mathbb{R}^n \times \mathbb{R}^r, 0) \longrightarrow (\mathbb{R}^p \times \mathbb{R}^r, 0)$ *be an unfolding of* f_0 *of the form* $F(x,u) = (f(x,u), u)$. *If* $j_1^k f$ *is transverse to* $\mathcal{K}^k(j^k f_0(0))$ *for sufficiently large* k, *then* F *is infinitesimally* $\mathcal{A}$*-stable.*

4 Generic classifications

In this section we give the proof of Theorem 1.1. As we mentioned in the previous sections, an infinitesimally $\mathcal{A}$-stable map germ $(\mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}^3, \mathbf{0})$ is an immersion, a cuspidal edge or a swallowtail. Therefore, we prove that the germ of the line congruence $F_{(x,e)}$ at any point is infinitesimally $\mathcal{A}$-stable for a generic $(\boldsymbol{x}, \boldsymbol{e})$.

Since $\boldsymbol{e}(u) \neq \mathbf{0}$, the rank of the Jacobian matrix of $F_{(x,e)}$ is greater than or equal to one. We now regard the parameter u (i.e., the parameter along the line) of the line congruence as the parameter of a one-dimensional unfolding. For any nonparallel line congruence $(\boldsymbol{x}, \boldsymbol{e}) : U \longrightarrow \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\mathbf{0}\})$, we denote that $\boldsymbol{x}(u) = (x^1(u), x^2(u), x^3(u))$ and $\boldsymbol{e}(u) = (e^1(u), e^2(u), e^3(u))$ for $u = (u^1, u^2) \in U$, then we have the coordinate representation:

$$F_{(x,e)}(u,t) = (x^1(u) + te^1(u), x^2(u) + te^2(u), x^3(u) + te^3(u)).$$

For any fixed $(u_0, t_0) \in U \times I$ with $e^3(u_0) \neq 0$, we define a non empty open subset U_3 in U by $U_3 = \{u \in U \mid e^3(u) \neq 0\}$. We define a function $g^3(u)$ by

$$g^3(u) = -(x^3(u) - y_0)/e^3(u)$$

for any $u \in U_3$, where $y_0 = x^3(t_0) + t_0 e^3(u_0)$. Therefore, we have

$$F_{(x,e)}(u,t) = \boldsymbol{x}(u) + g^3(u)\boldsymbol{e}(u) + (t - g^3(u))\boldsymbol{e}(u) = \boldsymbol{x}(u) + g_3(u)\boldsymbol{e}(u) + T\boldsymbol{e}(u)$$

for $u = u = (u^1, u^2)$, $T = t - g^3(u)$. We denote the above map by $\widetilde{F}_{(x,e)}(u,T)$. We remark that the third component of $\widetilde{F}_{(x,e)}(u,0)$ is equal to $x^3(u) + g^3(u)e^3(u) = y_0$. It follows from Lemma 3.1 that the map germ $\widetilde{F}_{(x,e)}(u,T)$ at $(u_0, 0)$ is a one-dimensional unfolding of

$$\hat{\pi}_3 \circ \widetilde{F}_{(x,e)}(u,0) = (x^1(u) + g^3(u)e^1(u), x^2(u) + g^3(u)e^2(u)),$$

where $\hat{\pi}_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is the canonical projection given by $\hat{\pi}_3(y^1, y^2, y^3) = (y^1, y^2)$. The following lemma is the basis for the proof of Theorem 1.1.

Lemma 4.1 *Let* $W \subset J^k(2,2)$ *be a submanifold. For any fixed map germ* $\boldsymbol{e} : U \longrightarrow \mathbb{R}^3 \setminus \{\mathbf{0}\}$ *and any fixed point* $(u_0, t_0) \in U \times I$ *with* $e^3(u_0) \neq 0$, *the set*

$${}_3T^{e}_{W,(u_0,t_0)} = \{\boldsymbol{x} \mid j_1^k \hat{\pi}_3 \circ \tilde{F}_{(x,e)} \text{ is transverse to } W \text{ at } (u_0, t_0)\}$$

is a residual subset in $C^\infty(U, \mathbb{R}^3)$.

Here, we identify $C^\infty(U, \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\mathbf{0}\}))$ *with* $C^\infty(U, \mathbb{R}^3) \times C^\infty(U, \mathbb{R}^3 \setminus \{\mathbf{0}\})$ *and use the relative topology on* $C^\infty(U, \mathbb{R}^3) \times \{\boldsymbol{e}\} \cong C^\infty(U, \mathbb{R}^3)$.

For the proof of Lemma 4.1, we need, as usual, to apply the following fundamental transversality lemma of Thom (cf., [7], page 53, Lemma 4.6).

Lemma 4.2 *Let X, B and Y be C^∞-manifolds with W a submanifold of Y. Let $j : B \longrightarrow C^\infty(X,Y)$ be a mapping (not necessarily continuous) and define $\Phi : X \times B \longrightarrow Y$ by $\Phi(x,b) = j(b)(x)$.*

Assume that Φ is smooth and transverse to W. Then the set $\{b \in B \mid j(b) \text{ is transverse to } W\}$ is dense in B.

Proof of Lemma 4.1. Let $\{K_j\}_{j=1}^\infty$ be a countable open covering of W such that each closure $\bar{K}_j$ is compact. We define the following set

$$ {}_3T^e_{W,(u_0,t_0),K_j} = \Big\{ \boldsymbol{x} \mid j_1^k \hat{\pi}_3 \circ \tilde{F}_{(x,e)} \text{ is transverse to } W \text{ with } j_1^k\hat{\pi}_3 \circ \tilde{F}_{(x,e)}(u_0,t_0) \in \bar{K}_j \Big\}. $$

We now prove that ${}_3T^e_{W,(u_0,t_0),K_j}$ is an open subset. For the purpose, we consider the following mapping

$$ \hat{j}^k : C^\infty(U_3, \mathbb{R}^3) \longrightarrow C^\infty(U_3 \times I, J^k(2,2)) $$

defined by $\hat{j}^k(\boldsymbol{x}) = j^k\hat{\pi}_3 \circ \tilde{F}_{(x,e)}$. It is clear that the mapping $\hat{j}^k$ is continuous. We also define a subset

$$ O_{W,K_j} = \{g \in C^\infty(U_3 \times I, J^k(2,2)) \mid g \text{ is transverse to } W \text{at } (u_0,t_0) \text{ with } g(u_0,t_0) \in K_j\}, $$

which is open (cf., [7]). Since the restriction map $res_{U_3} : C^\infty(U, \mathbb{R}^3) \longrightarrow C^\infty(U_3, \mathbb{R}^3)$ is continuous, ${}_3T^e_{W,(u_0,t_0),K_j} = (res_{U_3})^{-1} \circ (\hat{j}^k)^{-1}(O_{W,K_j})$ is open. If we show that ${}_3T^e_{W,(u_0,t_0),K_j}$ is a dense subset in $C^\infty(U, \mathbb{R}^3)$, then ${}_3T^e_{W,(u_0,t_0)} = \bigcap_{j=1}^\infty {}_3T^e_{W,(u_0,t_0),K_j}$ is a residual subset.

Since res_{U_3} is surjective, it is enough to show that

$$ {}_3T^e_{W,(u_0,t_0),K_j,U_3} = \Big\{ \boldsymbol{x} \in C^\infty(U_3, \mathbb{R}^3) \mid j_1^k\hat{\pi}_3 \circ \tilde{F}_{(x,e)} \text{ is transverse to } W \text{at } (u_0,t_0) \text{ with } j_1^k\hat{\pi}_3 \circ \tilde{F}_{(x,e)}(u_0,t_0) \in \bar{K}_j \Big\}. $$

is a dense subset in $C^\infty(U_3, \mathbb{R}^3)$.

For any $\boldsymbol{x} \in C^\infty(U_3, \mathbb{R}^3)$ and $p = (p^1, p^2) \in P(2,2;k)$, we define a mapping $f_{(x,p)} : U_3 \times I \longrightarrow \mathbb{R}^2$ by

$$ f_{(x,p)}(u,t) = (x^1(u) + p^1(u) + g^3(u)e^1(u) + te^1(u), x^2(u) + p^2(u) + g^3(u)e^2(u) + te^2(u)), $$

where $P(2,2;k)$ denote the space of pairs of polynomials (p^1,p^2) with degrees at most k without constant terms. We also define a mapping

$$ \Phi : U_3 \times J \times P(2,2;k) \longrightarrow J^k(2,2) $$

by $\Phi(u,t,(p^1,p^2)) = j_1^k f_{(x,p)}(u,t) = j^k f_{(x,p),t}(u)$, where $f_{(x,p),t}(u) = f_{(x,p)}(u,t)$. We may regard $P(2,2;k)$ as a Euclidean space $\mathbb{R}^N$.

Since we can identify $P(2,2;k)$ with $J^k(2,2)$ and their tangent spaces, we can easily show that Φ is a submersion at any point, so that it is transverse to W. By Lemma 4.2,

$$ \{p = (p^1,p^2) \in P(2,2;k) \mid \Phi_{(p^1,p^2)} \text{ is transverse to } W \text{ at } (u_0,t_0) \text{ with } \Phi_{(p^1,p^2)}(u_0,t_0) \in \bar{K}_j\} $$

is dense in $P(2,2;k)$. Hence, we can find $(p^1,p^2)_1, (p^1,p^2)_2, (p^1,p^2)_3, \dots$ in $P(2,2;k)$ converging to $(0,0)$ so that $\Phi_{(p^1,p^2)_i}$ is transverse to W on K_j. Since $\lim_{i\to\infty}(\boldsymbol{x}+((p^1,p^2)_i,0)) = \boldsymbol{x}$ in $C^\infty(U_3,\mathbb{R}^3)$, $T_{W,(u_0,t_0),K_j,U_3}$ is dense in $C^\infty(U_3,\mathbb{R}^3)$. □

We remark that ${}_jT_{W,(u_0,t_0)}$ $(j=1,2)$ can also be defined for $(u_0,t_0)\in U\times I$ with $e^j(u_0)\neq 0$ and the assertion as the above holds for ${}_jT_{W,(u_0,t_0)}$. We set

$$\mathcal{O}_1 = \{\boldsymbol{e}\in C^\infty(U,\mathbb{R}^3\setminus\{\boldsymbol{0}\}) \mid (\boldsymbol{e}_1(u)\times\boldsymbol{e}(u), \boldsymbol{e}_2(u)\times\boldsymbol{e}(u))\neq \boldsymbol{0} \text{ for any } u\in U\ \},$$

then $\mathcal{O}_1$ is a residual subset of $C^\infty(U,\mathbb{R}^3\setminus\{\boldsymbol{0}\})$. By Lemma 4.1, the set

$${}_3\widetilde{T}_{W,(u_0,t_0)} = \{(\boldsymbol{x},\boldsymbol{e}) \mid j_1^k\hat{\pi}_3\circ\tilde{F}_{(x,e)} \text{ is transverse to } W \text{ at } (u_0,t_0) \text{ and } \boldsymbol{e}\in\mathcal{O}_1\}$$

is a residual subset in $C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\boldsymbol{0}\}))$.

Proof of Theorem 1.1. Let $\mathcal{K}_i$ be the $\mathcal{K}^k$-orbit (cf., §3) with codimension i in $J^k(2,2)$ for sufficiently large k. We also denote that $\Sigma(2,2)=\bigcap_{i\geq 4}\mathcal{K}_i\subset J^k(2,2)$. It has been known that $\Sigma(2,2)$ is a semi-algebraic subset in $J^k(2,2)$ with codimension greater than 3. Therefore we have the canonical stratification $\{\mathcal{S}_i\}_{i=1}^m$of $\Sigma(2,2)$ with $\operatorname{codim}\mathcal{S}_i>3$. For any (u_0,t_0) with $e^3(u_0)\neq 0$, we set ${}_3\widetilde{T}_{\Sigma(2,2),(u_0,t_0)}=\bigcap_{i=1}^m {}_3\widetilde{T}_{\mathcal{S}_i,(u_0,t_0)}$. Since ${}_3\widetilde{T}_{\mathcal{K}_i,(u_0,t_0)}$ and ${}_3\widetilde{T}_{\Sigma(2,2),(u_0,t_0)}$ are residual subsets in $C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\boldsymbol{0}\}))$,

$${}_3\mathcal{O}_{(u_0,t_0)} = \bigcap_{i=1}^3 {}_3\widetilde{T}_{\mathcal{K}_i,(u_0,t_0)} \cap_3 \widetilde{T}_{\Sigma(2,2),(u_0,t_0)}$$

is also a residual subset in $C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\boldsymbol{0}\}))$. By the remark after the proof of Lemma 4.1, ${}_j\mathcal{O}_{(u_0,t_0)}$ $(j=1,2)$ are also residual subsets in $C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\boldsymbol{0}\}))$ respectively. Therefore, for any fixed $(u_0,t_0)\in U\times I$, there exists a residual subset $\mathcal{O}_{(u_0,t_0)}\subset C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\boldsymbol{0}\}))$ such that the map germ $F_{(x,e)}$ at (u_0,t_0) is an infinitesimally $\mathcal{A}$-stable map germ for any $(\boldsymbol{x},\boldsymbol{e})\in\mathcal{O}_{(u_0,t_0)}$ by Lemma 3.3. Since an infinitesimally $\mathcal{A}$-stable map germ is an $\mathcal{A}$-stable map germ in the sense of Mather [12], there exists an open neighbourhood $U_{u_0}\times I_{t_0}$ of (u_0,t_0) in $U\times I$ such that $F_{(x,e)}|(U_{u_0}\times I_{t_0})$ is an $\mathcal{A}$-stable map. We can choose countably many points $(u_i,t_i)\in U\times I$ and their neigbourhoods $U_{u_i}\times I_{t_i}$ $(i=1,2,\dots)$ such that $F_{(x,e)}|(U_{u_i}\times I_{t_i})$ is an $\mathcal{A}$-stable map and $U\times I=\bigcup_{i=1}^\infty(U_{u_i}\times I_{t_i})$. Since each $\mathcal{O}_{(u_i,t_i)}$ is a residual subset of $C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\boldsymbol{0}\}))$, $\mathcal{O}_2=\bigcap_{i=1}^\infty\mathcal{O}_{(u_i,t_i)}$ is a residual subset of $C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\boldsymbol{0}\}))$. It is clear that the germ $F_{(x,e)}$ at any point $(u,t)\in U\times I$ is infinitesimally $\mathcal{A}$-stable for any $(\boldsymbol{x},\boldsymbol{e})\in\mathcal{O}_2$.

It is easy to show that the mapping $\mathcal{F}: C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\boldsymbol{0}\}))\longrightarrow C^\infty(U\times I,\mathbb{R}^3)$ defined by $\mathcal{F}(\boldsymbol{x},\boldsymbol{e})=F_{(x,e)}$ is continuous. Since the set

$$\mathcal{S}=\{f\in C^\infty(U\times I,\mathbb{R}^3) \mid f \text{ is infinitesimally } \mathcal{A}\text{-stable at any point } \in U\times I\}$$

is an open subset, $\mathcal{O}=\mathcal{F}^{-1}(\mathcal{S})$ is an open subset of $C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\boldsymbol{0}\}))$. By the previous arguments, we have $\mathcal{O}_2\subset\mathcal{O}$, so that $\mathcal{O}$ is open dense subset of $C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\boldsymbol{0}\}))$. This completes the proof of Theorem 1.1. □

5 Normal congruences and Lagrangian congruences

It has been known that the evolute of a surface is the caustic of a certain Lagrangian submanifold in the cotangent bundle $T^*\mathbb{R}^3$. We now try to understand this phenomenon in a general context for normal congruences. Let $F_{(x,e)} : U \times I \longrightarrow \mathbb{R}^3$ be a line congruence. We say that $F_{(x,e)}$ is *a normal congruence* if the following condition holds: for any point $u = (u^1, u^2) \in U$, there exist a neighbourhood $V \subset U$ of u and a regular surface $\boldsymbol{y} : V \longrightarrow \mathbb{R}^3$ such that the normal of $\boldsymbol{y}(v)$ is parallel to $\boldsymbol{e}(v)$ at any $v \in V$. We also say that $F_{(x,e)}$ is an *exact normal congruence* if $\boldsymbol{x}$ is a regular surface and $\boldsymbol{e}(u)$ is the normal vector $\boldsymbol{x}(u)$ at $u \in U$. Then we have the following classical characterization theorem for normal congruences (cf., [18]).

Proposition 5.1 *A line congruence $F_{(x,e)}$ is a normal congruence if and only if*

$$\left\langle \boldsymbol{x}_1, \left(\frac{\boldsymbol{e}}{\|\boldsymbol{e}\|}\right)_2 \right\rangle = \left\langle \boldsymbol{x}_2, \left(\frac{\boldsymbol{e}}{\|\boldsymbol{e}\|}\right)_1 \right\rangle.$$

Proof. Without the loss of generality, we consider the case $\|\boldsymbol{e}(u)\| = 1$. We consider a surface $\boldsymbol{y}(u) = \boldsymbol{x}(u) + t(u)\boldsymbol{e}(u)$. By a straightforward calculation, we have

$$\boldsymbol{y}_i = \boldsymbol{x}_i + t_i\boldsymbol{e} + t\boldsymbol{e}_i,$$

where $i = 1, 2$. Under the condition that $\boldsymbol{y}$ is regular at u, $\boldsymbol{e}(u)$ is a normal vector to $\boldsymbol{y}$ at $u \in U$ if and only if

$$0 = \langle \boldsymbol{e}, \boldsymbol{x}_i \rangle + t_i + \langle \boldsymbol{e}, \boldsymbol{e}_i \rangle.$$

Since $\boldsymbol{e}$ is a unit vector, the above condition is equivalent to the condition that

$$t_i + \langle \boldsymbol{e}, \boldsymbol{x}_i \rangle = 0.$$

We have the integrability condition $t_{12} = t_{21}$ of the above first order partial differential equations for the unknown function $t(u)$ which is equivalent to the condition

$$\langle \boldsymbol{x}_1, \boldsymbol{e}_2 \rangle = \langle \boldsymbol{x}_2, \boldsymbol{e}_1 \rangle.$$

The above function $t(u)$ exists at least locally if and only if the above condition holds. If the surface $\boldsymbol{y}$ is singular at u_0, there exists $\lambda > 0$ such that the surface $\boldsymbol{y}_\lambda(u) = \boldsymbol{x}(u) + (t(u) + \lambda)\boldsymbol{e}(u)$ is an immersion around u_0 by Lemmas 2.1, 2.2. The function $t(u) + \lambda$ is also a solution of the equation. □

Let $\mathrm{Emb}\,(U, \mathbb{R}^3) = \{\boldsymbol{x} \mid \boldsymbol{x} : U \longrightarrow \mathbb{R}^3 \text{ embedding }\}$ be the space of regular surfaces with the Whitney C^∞-topology. We consider the space of exact normal congruences defined by

$$\begin{aligned} \mathrm{EN}\,(U, \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\boldsymbol{0}\})) = \{(\boldsymbol{x}, \boldsymbol{e}) \mid \boldsymbol{x} \in \mathrm{Emb}\,(U, \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\boldsymbol{0}\})), \\ \boldsymbol{e}(u) \text{ is a normal vector of } \boldsymbol{x} \text{ at } \boldsymbol{x}(u)\}. \end{aligned}$$

We have the following well known theorem. Porteous might be the first person who presented the assertion in his celebrating paper [14]. However the proof of the theorem has been given by Looijenga [11].

Theorem 5.2 *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}\,(U, \mathbb{R}^3)$ such that the germ of the exact normal congruence $F_{(x,e)}$ at any point $(u_0, t_0) \in U \times I$ is a Lagrangian stable map germ for any $\boldsymbol{x} \in \mathcal{O}$.*

In other wards, $F_{(x,e)}$ is an immersive germ, $\mathcal{A}$-equivalent to the cuspidal edge, the swallowtail, the pyramid or the purse for any $\boldsymbol{x} \in \mathcal{O}$.

Here we refer to the article [1] for the definition and basic properties of Lagrangian stable map germs.

We now define a natural map $\pi : \mathrm{EN}\,(U, \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\mathbf{0}\})) \longrightarrow \mathrm{Emb}\,(U, \mathbb{R}^3)$ by $\pi(\boldsymbol{x}, \boldsymbol{e}) = \boldsymbol{x}$. It is a retraction, so that a continuous open map. Therefore we have the following corollary:

Corollary 5.3 *There exists an open dense subset $\mathcal{O} \subset \mathrm{EN}\,(U, \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\mathbf{0}\}))$ such that the germ of an exact normal congruence $F_{(x,e)}$ at any point $(u_0, t_0) \in U \times I$ is a Lagrangian stable map germ for any $(\boldsymbol{x}, \boldsymbol{e}) \in \mathcal{O}$.*

By the above theorems, we now pay attention to Lagrangian singularities. We consider the cotangent bundle $\pi : T^*\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ over $\mathbb{R}^3$ with the canonical symplectic structure $\omega = \sum_{i=1}^3 dx_i \wedge dp_i$, where $(x_1, x_2, x_3, p_1, p_2, p_3)$ is the canonical coordinate. For any line congruence $F_{(x,e)}$, we define a smooth mapping $L_{(x,e)} : U \times I \longrightarrow T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times (\mathbb{R}^*)^3$ by

$$L_{(x,e)}(u,t) = \left(\boldsymbol{x}(u) + t\frac{\boldsymbol{e}}{\|\boldsymbol{e}\|}(u), \frac{\boldsymbol{e}}{\|\boldsymbol{e}\|}(u) \right).$$

Lemma 5.4 *Suppose that $\boldsymbol{x} : U \longrightarrow \mathbb{R}^3$ is an immersion and $\boldsymbol{e}(u) \notin T_{x(u)}\boldsymbol{x}(U)$ for any $u \in U$, then $L_{(x,e)}$ is an immersion.*

Proof. We may also assume that $\|\boldsymbol{e}(u)\| = 1$. We have

$$\frac{\partial L_{(x,e)}}{\partial u_i}(u) = (\boldsymbol{x}_i(u) + t\boldsymbol{e}_i(u), \boldsymbol{e}_i(u))$$

and

$$\frac{\partial L_{(x,e)}}{\partial t}(u) = (\boldsymbol{e}(u), \mathbf{0}).$$

If $\boldsymbol{e}_1(u) \times \boldsymbol{e}_2(u) \neq \mathbf{0}$,

$$\frac{\partial F_{(x,e)}}{\partial u_i}(u,t) \ (i = 1, 2) \text{ and } \frac{\partial F_{(x,e)}}{\partial t}(u,t)$$

are linearly independent.

Suppose that $\boldsymbol{e}_1(u) \times \boldsymbol{e}_2(u) = \mathbf{0}$. If $\boldsymbol{e}_1(u) = \mathbf{0}$, then $\boldsymbol{x}_1(u)$ and $\boldsymbol{e}(u)$ are linearly independent. Therefore we consider the case when $\boldsymbol{e}_i(u) \neq \mathbf{0}$ $(i = 1, 2)$. In this case there exists a real number α that

$$\boldsymbol{e}_1(u) = \alpha \boldsymbol{e}_2(u).$$

Consider a linear relation

$$\lambda_1 (\boldsymbol{x}_1(u) + t\boldsymbol{e}_1(u), \boldsymbol{e}_1(u)) + \lambda_2 (\boldsymbol{x}_2(u) + t\boldsymbol{e}_2(u), \boldsymbol{e}_2(u)) + \mu(\boldsymbol{e}(u), \mathbf{0}) = \mathbf{0}.$$

Then we have $\alpha\lambda_1 + \lambda_2 = 0$, so that

$$\lambda_1 (\boldsymbol{x}_1(u) + t\boldsymbol{e}_1(u)) - \lambda_1\alpha (\boldsymbol{x}_2(u) + t\boldsymbol{e}_2(u)) + \mu\boldsymbol{e}(u) = \mathbf{0}.$$

This is equivalent to the condition that $\lambda_1(\boldsymbol{x}_1(u) - \alpha\boldsymbol{x}_2(u) + \mu\boldsymbol{e}(u) = \mathbf{0}$. Therefore we have $\lambda_1 = \lambda_2 = \mu = 0$. □

We say that $F_{(x,e)}$ is *a Lagrangian line congruence* if $L_{(x,e)}$ is a Lagrangian immersion (i.e, a immersion with $(L_{(x,e)})^*\omega = 0$).

Proposition 5.5 *Under the same situation as the above, $F_{(x,e)}$ is a normal congruence if and only if it is a Lagrangian line congruence.*

Proof. We consider the canonical 1-form $\theta = \sum_{i=1}^{3} p^i dx^i$, then we have $d\theta = \omega$. It follows that

$$\begin{aligned} L^*_{(x,e)}(\theta) &= \sum_{i=1}^{3} \frac{e^i}{\|\mathbf{e}\|}(u) d\left(x^i(u) + t\frac{e^i}{\|\mathbf{e}\|}(u)\right) \\ &= \sum_{i=1}^{3} \frac{e^i}{\|\mathbf{e}\|}(u)\left(dx^i(u) + \frac{e^i}{\|\mathbf{e}\|}(u)dt + td\frac{e^i}{\|\mathbf{e}\|}(u)\right) \\ &= \sum_{i=1}^{3}\left(\frac{e^i}{\|\mathbf{e}\|}(u)dx^i(u) + t\frac{e^i}{\|\mathbf{e}\|}(u)d\frac{e^i}{\|\mathbf{e}\|}(u)\right) + dt, \end{aligned}$$

where $\boldsymbol{x}(u) = (x^1(u), x^2(u), x^3(u))$ and $\mathbf{e}(u) = (e^1(u), e^2(u), e^3(u))$.

Therefore we have

$$\begin{aligned} L^*_{(x,e)}(w) &= dL^*_{(x,e)}(\theta) \\ &= \sum_{i=1}^{3}\left(d\frac{e^i}{\|\mathbf{e}\|}(u) \wedge dx^i(u) + \frac{e^i}{\|\mathbf{e}\|}(u)dt \wedge d\frac{e^i}{\|\mathbf{e}\|}(u)\right) \\ &= \left(\left\langle\left(\frac{\mathbf{e}}{\|\mathbf{e}\|}\right)_1, \boldsymbol{x}_2\right\rangle - \left\langle\left(\frac{\mathbf{e}}{\|\mathbf{e}\|}\right)_2, \boldsymbol{x}_1\right\rangle\right) du^1 \wedge du^2 \\ &+ \left\langle\frac{\mathbf{e}}{\|\mathbf{e}\|}, \left(\frac{\mathbf{e}}{\|\mathbf{e}\|}\right)_1\right\rangle du^1 \wedge dt \\ &+ \left\langle\frac{\mathbf{e}}{\|\mathbf{e}\|}, \left(\frac{\mathbf{e}}{\|\mathbf{e}\|}\right)_2\right\rangle du^2 \wedge dt \\ &= \left(\left\langle\left(\frac{\mathbf{e}}{\|\mathbf{e}\|}\right)_1, \boldsymbol{x}_2\right\rangle - \left\langle\left(\frac{\mathbf{e}}{\|\mathbf{e}\|}\right)_2, \boldsymbol{x}_1\right\rangle\right) du^1 \wedge du^2. \end{aligned}$$

It follows that $L^*_{(x,e)}(\omega) = 0$ if and only if

$$\left\langle\left(\frac{\mathbf{e}}{\|\mathbf{e}\|}\right)_1, \boldsymbol{x}_2\right\rangle = \left\langle\left(\frac{\mathbf{e}}{\|\mathbf{e}\|}\right)_2, \boldsymbol{x}_1\right\rangle.$$

□

By Proposition 3.3 and the proof of Proposition 3.1, $F_{(x,e)}$ is a Lagrangian line congruence if and only if there exists a smooth function $t : U \longrightarrow \mathbb{R}$ such that $\boldsymbol{x}(u)+t(u)\mathbf{e}(u)$ is an immersion and

$$(*) \quad \begin{cases} t_1(u) + \left\langle \frac{\mathbf{e}}{\|\mathbf{e}\|}(u), \boldsymbol{x}_1(u)\right\rangle = 0 \\ t_2(u) + \left\langle \frac{\mathbf{e}}{\|\mathbf{e}\|}(u), \boldsymbol{x}_2(u)\right\rangle = 0 \end{cases}$$

Therefore we define the space of Lagrangian (normal) line congruences as follows:

$$\mathrm{L}\left(U, \mathbb{R}^3 \times \mathbb{R} \times (\mathbb{R}^3 \setminus \{\mathbf{0}\})\right) = \{(\boldsymbol{x}, t, \mathbf{e}) \mid \boldsymbol{x}(u) + t(u)\mathbf{e}(u) \text{ is an immersion with the condition } (*) \text{ holds }\}.$$

We adopt the Whitney C^∞-topology on $\mathrm{L}\,(U,\mathbb{R}^3\times\mathbb{R}\times(\mathbb{R}^3\setminus\{\mathbf{0}\})$. We now define a map

$$Trp : C^\infty(U,\mathbb{R}^3\times\mathbb{R}\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))\longrightarrow C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))$$

by $Trp(\boldsymbol{x}(u),t(u),\boldsymbol{e}(u))=(\boldsymbol{x}(u)+t(u)\boldsymbol{e}(u),\boldsymbol{e}(u))$. We call Trp the *transitive projection.* Then we have the following proposition:

Proposition 5.6 *Under the same notations as the above paragraph, the mapping Trp is an open and continuous map under the Whitney C^∞-topology.*

Proof. For any natural number $k\in\mathbb{N}$, we define a map between k-jet spaces

$$Trp^k : J^k(U,\mathbb{R}^3\times\mathbb{R}\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))\longrightarrow J^k(U,\mathbb{R}\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))$$

by $Trp^k(j^k(\boldsymbol{x},t,\boldsymbol{e})(u_0))=j^k(\boldsymbol{x}+t\boldsymbol{e},\boldsymbol{e})(u_0)$.

By a straight forward calculation Trp^k is a submersion, so that it is an open map. It follows that Trp is an open continuous map. □

We set

$$\mathrm{N}\,(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))=Trp(\mathrm{L}\,(U,\mathbb{R}^3\times\mathbb{R}\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))).$$

By definition, we can regard $\mathrm{N}\,(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))$ as the space of normal congruences. Here, we consider the relative topology induced from the Whitney C^∞-topology of $C^\infty(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))$. Then we have the following theorem.

Theorem 5.7 *There exists an open dense subset $\mathcal{O}'\subset\mathrm{N}\,(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))$ such that the germ of the normal congruence $F_{(x,e/\|e\|)}$ at any point $(u_0,t_0)\in U\times I$ is a Lagrangian stable map germ for any $(\boldsymbol{x},\boldsymbol{e})\in\mathcal{O}'$.*

Therefore, $F_{(x,e)}$ is $\mathcal{A}$-equivalent to an immersion germ, the cuspidal edge, the swallowtail, the pyramid or the purse for any $(\boldsymbol{x},\boldsymbol{e})\in\mathcal{O}'$.

Proof. By Theorem 5.3, there exists an open dense subset $\mathcal{O}\subset\mathrm{EN}(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))$ such that the germ of the exact normal congruence $F_{(x,e)}$ at any point $(u_0,t_0)\in U\times I$ is a Lagrangian stable map germ for any $(\boldsymbol{x},\boldsymbol{e})\in\mathcal{O}$. Since the transitive projection Trp is an open map and $\mathcal{O}'=Trp(\mathcal{O})$ is an open dense subset in $\mathrm{N}\,(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))$. This completes the proof. □

The theorem asserts that generic singularities of normal congruences are the same as the generic classification of singularities of exact normal congruences.

6 Equiaffine normal congruences

We now consider another important class of line congruences. In [9] it has been studied the affine evolute of a nondegenerate plane curve. We observe that the affine evolute of a nondegenerate plane curve is the caustic of a certain Lagrangian submanifold in $T^*\mathbb{R}^2$. However, the similar calculations for nondegenerate surfaces as those for the curves in [9] cannot work because of the extremely complicated situation. Nevertheless, as an application of the results in §5, we will have an observation which authorizes that the similar phenomenon like as the curve case might occur for the surface case.

We consider $\mathbb{R}^3$ as a 3-dimensional affine space with a volume element ω given by

$$\omega(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = \det(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3),$$

where $\{ \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \}$ is the standard basis of $\mathbb{R}^3$. This volume element ω is parallel with respect to the standard flat affine connection D on $\mathbb{R}^3$. Let $\boldsymbol{x} : U \longrightarrow \mathbb{R}^3$ be a regular surface with $\boldsymbol{x}(U) = M$. If we consider a transversal vector field $\boldsymbol{\xi}$ along M, then for each $u \in U$, we decompose the tangent space $T_p\mathbb{R}^3$ as the direct sum of T_pM and $\langle \boldsymbol{\xi}(p) \rangle_{\mathbb{R}}$, where $p = \boldsymbol{x}(u)$. If $\boldsymbol{v}$ and $\boldsymbol{w}$ are vector fields on M, we decompose $D_v\boldsymbol{w}$ into tangential and transversal components as $D_v\boldsymbol{w} = \nabla_v\boldsymbol{w} + h(\boldsymbol{v}, \boldsymbol{w})\boldsymbol{\xi}$. Then ∇ is a torsion-free affine connection on M called the *induced affine connection,* and h is a tensor field which defines a symmetric bilinear form on each tangent space T_pM. We call h the *affine fundamental form* induced by $\boldsymbol{\xi}$. We can also decompose $D_v\boldsymbol{\xi}$ into tangential and transversal components as $D_v\boldsymbol{\xi} = -S\boldsymbol{v} + \tau(\boldsymbol{v})\boldsymbol{\xi}$. Here S is a tensor field of type $(1,1)$ called the *shape operator* determined by $\boldsymbol{\xi}$, and τ is a 1-form called the *transversal connection form.* Finally, we introduce a volume element θ on M setting $\theta(\boldsymbol{v}_1, \boldsymbol{v}_2) = \omega(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{\xi})$ for $\boldsymbol{v}_1, \boldsymbol{v}_2$ tangent to M. It has been known that $\nabla_v\theta = \tau(\boldsymbol{v})\theta$ for $\boldsymbol{v} \in T_pM$ (cf., [13]). The surface $\boldsymbol{x} : U \longrightarrow \mathbb{R}^3$ is said to be *nondegenerate* if h is nondegenerate at each point M. We remark that the notion of nondegeneracy is independent of the choice of transversal field $\boldsymbol{\xi}$ on M. If we choose the Euclidean normal as the transversal field, the surface is nondegenerate if and only if the Gaussian curvature never vanishes. Assume now that $\boldsymbol{x} : U \longrightarrow \mathbb{R}^3$ is a nondegenerate surface. A local transversal field $\boldsymbol{\xi}$ for which $\tau = 0$ (i.e. $D_v\boldsymbol{\xi}$ is tangent to M) is said to be *equiaffine.* Two important examples of equiaffine transversal fields are the Euclidean field of unit normals and the *Blaschke affine normal field* $\boldsymbol{\xi}$, which is determined up to a sign by the equiaffine condition and the requirement that θ equals the volume element for the bilinear form h associated to $\boldsymbol{\xi}$ (see, for example [13]). We now have important classes of line congruences as follows: Let $\boldsymbol{x} : U \longrightarrow \mathbb{R}^3$ be nondegenerate regular surface (i.e. the Gaussian curvature $K \neq 0$). Let $\boldsymbol{\xi}$ be an equiaffine normal field along $\boldsymbol{x}(U) = M$. Then the line congruence $F_{(x,\xi)}(u,t) = \boldsymbol{x}(u) + t\boldsymbol{\xi}(u)$ is called the *exact equiaffine normal congruence.* If $\boldsymbol{\xi}$ is the Blaschke normal field, we call $F_{(x,\xi)}$ the *exact Blaschke normal congruence.*

We now define the space of non-degenerate regular surfaces as follows:

$$\mathrm{Emb}_{ng}(U, \mathbb{R}^3) = \{ \boldsymbol{x} \in \mathrm{Emb}\,(U, \mathbb{R}^3) \mid \text{the Gaussian curvaure } K(u) \neq 0 \text{ for any } u \in U \ \}.$$

We also consider the space of exact affine normal congruences

$$\mathrm{EAN}\,(U, \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\boldsymbol{0}\})) = \{ (\boldsymbol{x}, \boldsymbol{\xi}) \mid \boldsymbol{x} \in \mathrm{Emb}_{ng}(U, \mathbb{R}^3),\ \boldsymbol{\xi}(u) \text{ is an affine normal of } \boldsymbol{x} \text{ at } \boldsymbol{x}(u) \ \}.$$

As in the case for exact normal congruences, the focal points of $F_{(x,\xi)}$ can be related to the critical point theory for a certain class of functions. For each $u = (u^1, u^2) \in U$, we decompose the vector $\boldsymbol{p} - \boldsymbol{x}(u)$ into tangential and transversal components as follows,

$$\boldsymbol{p} - \boldsymbol{x}(u) = \boldsymbol{v} + \rho_p(u)\boldsymbol{\xi}(u),$$

where $\boldsymbol{v} \in T_{x(u)}M$. The real valued function ρ_p is called an *affine support function* associated to the nondegenerate surface with equiaffine transversal field $\boldsymbol{\xi}$. Then we have the following proposition (cf., [13]).

Proposition 6.1 *Let $\boldsymbol{x} : U \longrightarrow \mathbb{R}^3$ be a nondegenerate surface with equiaffine transversal field $\boldsymbol{\xi}$. Then $\partial\rho_p/\partial u_i(u) = 0$ $(i = 1, 2)$ if and only if*

$$\boldsymbol{p} - \boldsymbol{x}(u) = \rho_p(u)\boldsymbol{\xi}(u).$$

Proof. We differentiate the equality $\boldsymbol{p} - \boldsymbol{x}(u) = \boldsymbol{v} + \rho_p(u)\boldsymbol{\xi}(u)$ and obtain

$$\begin{aligned} -\boldsymbol{x}_{u_i}(u) &= D_{\partial/\partial u_i}\boldsymbol{v} + \frac{\partial \rho_p}{\partial u_i}(u)\boldsymbol{\xi}(u) + \rho_p(u) D_{\partial/\partial u_i}\boldsymbol{\xi}(u) \\ &= \nabla_{\partial/\partial u_i}\boldsymbol{v} + h(\partial/\partial u_i, \boldsymbol{v})\boldsymbol{\xi}(u) + \frac{\partial \rho_p}{\partial u_i}(u)\boldsymbol{\xi}(u) - \rho_p(u) S(\partial/\partial u_i). \end{aligned}$$

It follows that

$$\begin{aligned} \nabla_{\partial/\partial u_i}\boldsymbol{v} &= -\boldsymbol{x}_{u_i}(u) + \rho_p(u) S(\partial/\partial u_i), \\ \frac{\partial \rho_p}{\partial u_i}(u) &= -h(\partial/\partial u_i, \boldsymbol{v}). \end{aligned}$$

Since h is nondegenerate, $\partial\rho_p/\partial u_i(u) = 0$ $(i = 1, 2)$ if and only if $\boldsymbol{v} = 0$. This means that $\boldsymbol{p} - \boldsymbol{x}(u) = \rho_p(u)\boldsymbol{\xi}(u)$. □

We now regard the affine support function as a generating family of a certain Lagrangian immersion. For the definition and basic properties of generating families of Lagrangian immersions, see [1]. By definition, the affine support function is given by

$$\rho_p(u) = \left\langle \boldsymbol{p} - \boldsymbol{x}(u), \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}(u) \right\rangle - \left\langle \boldsymbol{v}, \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}(u) \right\rangle,$$

where $\boldsymbol{v} \in T_{x(u)}M$ by using the canonical inner product. Therefore, we have

$$\frac{\partial \rho_p}{\partial p_i}(u) = \frac{\xi_i}{\|\boldsymbol{\xi}\|^2}(u),$$

where $\boldsymbol{p} = (p^1, p^2, p^3)$ and $\boldsymbol{\xi}(u) = (\xi^1(u), \xi^2(u), \xi^3(u))$. This means that the corresponding Lagrangian immersion is a map $L : U \times I \longrightarrow T^*\mathbb{R}^3$ defined by

$$L(u,t) = \left(\boldsymbol{x}(u) + t\boldsymbol{\xi}(u), \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}(u) \right).$$

We have the following characterization that $L(u,t)$ is a Lagrangian immersion.

Proposition 6.2 *Let $F_{(x,\xi)}$ be a line congruence. Then the following are equivalent:*

(1) $L(u,t) = (\boldsymbol{x}(u) + t\boldsymbol{\xi}(u), (\boldsymbol{\xi}/\|\boldsymbol{\xi}\|^2)(u))$ *is a Lagrangian immersion.*

(2) $\boldsymbol{x}$ *and* $\boldsymbol{\xi}$ *satisfy the condition that*

$$\langle \boldsymbol{\xi}_1, \boldsymbol{x}_2 \rangle = \langle \boldsymbol{\xi}_2, \boldsymbol{x}_1 \rangle \text{ and } \langle \boldsymbol{\xi}, \boldsymbol{\xi}_1 \rangle = \langle \boldsymbol{\xi}, \boldsymbol{\xi}_2 \rangle = 0.$$

(3) $\mathcal{L}(u,t) = (\boldsymbol{x}(u) + t\boldsymbol{\xi}(u), \boldsymbol{\xi}(u))$ *is a Lagrangian immersion.*

Proof. Let θ be the canonical 1-form, then we have

$$\begin{aligned} L^*\theta &= \left\langle \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}, (d\boldsymbol{x} + \boldsymbol{\xi} dt + t d\boldsymbol{\xi}) \right\rangle \\ &= \left\langle \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}, d\boldsymbol{x} \right\rangle + dt + \left\langle \frac{t\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}, d\boldsymbol{\xi} \right\rangle. \end{aligned}$$

It follows that

$$L^*\omega = d\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}\right)\wedge d\boldsymbol{x} + \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}dt\wedge d\boldsymbol{\xi} + td\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}\right)\wedge d\boldsymbol{\xi}.$$

Since $d(\boldsymbol{\xi}/\|\boldsymbol{\xi}\|^2)\wedge d\boldsymbol{\xi} = (\langle\boldsymbol{\xi},\boldsymbol{\xi}_1\rangle\langle\boldsymbol{\xi},\boldsymbol{\xi}_2\rangle - \langle\boldsymbol{\xi},\boldsymbol{\xi}_2\rangle\langle\boldsymbol{\xi},\boldsymbol{\xi}_1\rangle)du^1\wedge du^2 = 0$, $L^*\omega = 0$ if and only if

$$\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}\right)_1, \boldsymbol{x}_2\right\rangle = \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}\right)_2, \boldsymbol{x}_1\right\rangle$$

and

$$\left\langle\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}, \boldsymbol{\xi}_1\right\rangle = \left\langle\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}, \boldsymbol{\xi}_2\right\rangle = 0.$$

This condition is equivalent to the condition that

$$\langle\boldsymbol{\xi}_1, \boldsymbol{x}_2\rangle = \langle\boldsymbol{\xi}_2, \boldsymbol{x}_1\rangle \text{ and } \langle\boldsymbol{\xi}, \boldsymbol{\xi}_1\rangle = \langle\boldsymbol{\xi}, \boldsymbol{\xi}_2\rangle = 0.$$

On the other hand, we have

$$\begin{aligned}\mathcal{L}^*\theta &= \langle\boldsymbol{\xi}, d(\boldsymbol{x}+t\boldsymbol{\xi})\rangle \\ &= \langle\boldsymbol{\xi}, (d\boldsymbol{x}+\boldsymbol{\xi}dt+td\boldsymbol{\xi})\rangle \\ &= \langle\boldsymbol{\xi}, d\boldsymbol{x}\rangle + \|\boldsymbol{\xi}\|^2dt + t\langle\boldsymbol{\xi}, d\boldsymbol{\xi}\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{L}^*\omega &= d\boldsymbol{\xi}\wedge d\boldsymbol{x} + d\|\boldsymbol{\xi}\|^2\wedge dt + d(t\boldsymbol{\xi})\wedge d\boldsymbol{\xi} \\ &= d\boldsymbol{\xi}\wedge d\boldsymbol{x} + \langle\boldsymbol{\xi}, d\boldsymbol{\xi}\rangle\wedge dt \\ &= (\langle\boldsymbol{\xi}_1, \boldsymbol{x}_2\rangle - \langle\boldsymbol{\xi}_2, \boldsymbol{x}_1\rangle)du^1\wedge du^2 \\ &+ \langle\boldsymbol{\xi}, \boldsymbol{\xi}_1\rangle du^1\wedge dt + \langle\boldsymbol{\xi}, \boldsymbol{\xi}_2\rangle du^2\wedge dt = 0.\end{aligned}$$

This condition is equivalent to the condition that

$$\langle\boldsymbol{\xi}_1, \boldsymbol{x}_2\rangle = \langle\boldsymbol{\xi}_2, \boldsymbol{x}_1\rangle \text{ and } \langle\boldsymbol{\xi}, \boldsymbol{\xi}_1\rangle = \langle\boldsymbol{\xi}, \boldsymbol{\xi}_2\rangle = 0.$$

□

We have the following theorem as a corollary of the above propositions.

Theorem 6.3 *Let $\boldsymbol{x}: U \longrightarrow \mathbb{R}^3$ be a nondegenerate surface with equiaffine transversal field $\boldsymbol{\xi}$. Then the exact affine normal congruence $F_{(x,\xi)}(u,t) = \boldsymbol{x}(u) + t\boldsymbol{\xi}(u)$ is a normal congruence and hence it is a Lagrangian line congruence.*

Proof. By Propositions 6.1 and 6.2, we have

$$\langle\boldsymbol{\xi}_1, \boldsymbol{x}_2\rangle = \langle\boldsymbol{\xi}_2, \boldsymbol{x}_1\rangle \text{ and } \langle\boldsymbol{x}_i, \boldsymbol{\xi}\rangle = 0 \ (i = 1, 2).$$

It follows that

$$\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_i = \frac{\boldsymbol{\xi}_i}{\|\boldsymbol{\xi}\|} - \frac{\langle\boldsymbol{\xi}_i, \boldsymbol{\xi}\rangle}{\|\boldsymbol{\xi}\|^3}\boldsymbol{\xi} = \frac{\boldsymbol{\xi}_i}{\|\boldsymbol{\xi}\|}.$$

Therefore we have

$$\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_1, \boldsymbol{x}_2\right\rangle = \left\langle\frac{\boldsymbol{\xi}_1}{\|\boldsymbol{\xi}\|}, \boldsymbol{x}_2\right\rangle = \left\langle\frac{\boldsymbol{\xi}_2}{\|\boldsymbol{\xi}\|}, \boldsymbol{x}_1\right\rangle = \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_2, \boldsymbol{x}_1\right\rangle.$$

This completes the proof. $\square$

We now give a final remark on equiaffine normal congruences. For the purpose,we consider the following spaces:

$$\begin{aligned}\mathrm{AN}\,(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\})) = \{(\boldsymbol{x},\boldsymbol{e})\ | &\text{There exists } t(u) \text{ such that } \boldsymbol{x}(u)+t(u)\boldsymbol{e}(u) \\ &\text{is nondegenerate and there exists an equiaffine normal} \\ &\boldsymbol{\xi}(u) \text{ of } \boldsymbol{x}(u)+t(u)\boldsymbol{e}(u) \text{ such that } \boldsymbol{e}(u) \text{ and } \boldsymbol{\xi}(u) \text{ are parallel }\}.\end{aligned}$$

$$\mathrm{EN}_{nd}(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\})) = \{(\boldsymbol{x},\boldsymbol{e})\in \mathrm{EN}\,(U,\mathbb{R}^3\times(\mathbb{R}\setminus\{\mathbf{0}\})\ |\boldsymbol{x} \text{ is nondegenerate }\}.$$

$$\mathrm{N}_{nd}(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\})) = Trp(\mathrm{L}_{nd}(U,\mathbb{R}^3\times\mathbb{R}\times(\mathbb{R}\setminus\{\mathbf{0}\}))),$$

where

$$\begin{aligned}\mathrm{L}_{nd}(U,\mathbb{R}^3\times\mathbb{R}\times(\mathbb{R}^3\setminus\{\mathbf{0}\})) = \{(\boldsymbol{x},t,\boldsymbol{e})\in \mathrm{L}\,(U,\mathbb{R}^3\times\mathbb{R}\times(\mathbb{R}\setminus\{\mathbf{0}\}))\ | \\ \boldsymbol{x}(u)+t(u)\boldsymbol{e}(u) \text{ is nondegenerate }\}.\end{aligned}$$

By the previous arguments, we have the following relations:

$$\begin{aligned}\mathrm{EN}_{nd}(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\})) &\subset \mathrm{EAN}\,(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\})) \\ &\subset \mathrm{AN}\,(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))\subset \mathrm{N}_{nd}(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\})).\end{aligned}$$

We already have generic classifications of line congruences in both spaces $\mathrm{EN}_{nd}(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))$ and $\mathrm{N}_{nd}(U,\mathbb{R}^3\times(\mathbb{R}^3\setminus\{\mathbf{0}\}))$ in §5. As a consequence, germs at any point of generic congruences in both spaces are Lagrangian stable. We can expect that germs of generic equiaffine normal congruences are Lagrangian stable at any point. It is, however, still an open problem. Here, we can only assert that equiaffine normal congruences are Lagrangian congruences. The most important class of equiaffine normal congruences is the class of Blaschke affine normal congruences (cf., [4, 13].) The generic classification of such a class is also still unknown.

Conjectures (1) Germs of generic equiaffine normal congruences at each point are Lagrangian stable.

(2) Germs of generic Blaschke affine normal congruences at each point are Lagrangian stable.

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