Berg's Effect

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Series #553. July 2002

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Berg's Effect

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June 14, 2002

Abstract. A Neumann problem for the Laplace equation is considered outside a three dimensional straight cylinder. The value of a solution σ at space infinity is prescribed. The Neumann data $\partial\sigma/\partial\mathbf{n}$ ($\mathbf{n}$ is the outer normal of the cylinder) is assumed to be independent of the spatial variables on the top and the bottom and also on the lateral part of the boundary of the cylinder. The behavior of the value of σ on the boundary is studied. In particular, it is shown that σ is an increasing function of the distance from the center of the top (respectively, the bottom) if $\partial\sigma/\partial\mathbf{n} > 0$ on the lateral part and $\partial\sigma/\partial\mathbf{n}$ is the same constant on the top and (respectively, the bottom). An analogous statement is shown for σ on the lateral part.

In the theory of crystal growth σ is interpreted as a supersaturation and cylinder is a crystal. The value $\partial\sigma/\partial\mathbf{n}$ is the growth speed. The main contribution of this paper is considered as the first rigorous proof of Berg's effect when the crystal shape is a cylinder.

1. Introduction and presentation of the problem

In the theory of crystal growth the behavior of the supersaturation (which we will denote by σ) on the crystal surface plays an important role in determining the shape of a growing crystals. It has been observed that in certain experimental setting the supersaturation exhibits a kind of regular behavior, known as Berg's effect, [B]. A simple way of explaining this phenomenon follows. Let us suppose that we consider an evolving crystal of the shape of a straight cylinder. Its facets (i.e. the top, the bottom and the lateral surface) move with velocity which is constant on each facet. The signs of these constants are chosen so that the crystal is growing. Then the supersaturation on the top (respectively, the bottom) facet is an increasing function of the distance from the center of the top (respectively, the bottom). An analogous statement is valid for the lateral part of the boundary of the evolving crystal.

Our goal is to present a rigorous proof for this phenomenon. We do this in section 3. As we shall explain, our proof is independent of the evolution law. Strictly speaking, at a fixed instance of time evolution, the supersaturation σ is a solution to the Neumann problem in the complement of the cylinder with a prescribed value at the space infinity. We study the behavior of σ when the boundary data are piecewise constant. The Neumann data are considered as the growth velocity of the facets. In our proof we are forced to assume that the supersaturation is symmetric with respect to the symmetry plane perpendicular to the rotation axis of the cylinder. We find this restriction peculiar. Without this assumption our proof breaks down, but we do not know if this is a genuine effect or a technical deficiency.

Until now, the only known rigorous justification of Berg's effect have been obtained for evolving crystal whose evolution laws permitted explicit solutions, for example when the crystal shape is a regular polygon so that supersaturation is given by hypergeometric functions (see [Se]). However, here is a weaker version saying that the supersaturation at center of a facet is the smallest and that its value at the edge of the facet is the largest. This version is well-known in the physics literature (see e.g. [Ne, §3 eq.(6)]).

We mention that the Berg's effect plays a key role in explaining the morphological instability in the theory of crystal growth (e.g. [YK]).

We now present the mathematical background of our problem. Namely, we consider the following evolution law for a crystal $\Omega(t)$ and the supersaturation σ with a specific value at infinity,

$$\Delta\sigma = 0 \quad \text{in } \mathbb{R}^3 \setminus \Omega(t), \qquad \lim_{|x|\to\infty} \sigma(x) = \sigma^\infty, \tag{1.1}$$

$$\frac{\partial \sigma}{\partial \mathbf{n}} = V \quad \text{on } \partial\Omega(t), \tag{1.2}$$

$$-\sigma = -\gamma \operatorname{div} \xi - \beta V \quad \text{on } \partial\Omega(t), \tag{1.3}$$

where $\mathbf{n}$ is the outer unit normal to $\Omega(t)$. In our notation V is the normal velocity of the surface in the direction of $\mathbf{n}$; ξ is a Cahn-Hoffman vector, $\beta \geq 0$ is the kinetic coefficient and $\gamma \geq 0$ is a constant. The theoretical background necessary to derive the above system is provided by Seeger, see [Se]. A more detailed explanation is provided in [GR]. Before proceeding let us comment on the Cahn-Hoffman vector ξ. For smooth surfaces S and a smooth energy density function $\gamma_0 : \mathbb{R}^3 \to \mathbb{R}$, which is 1-homogeneous, we have

$$\xi(x) = \nabla\gamma_0(\mathbf{n}(x)),$$

which is a well-defined quantity. However, for energy density functions γ_0 which are only Lipschitz continuous and surfaces S with corners, some care is necessary while defining ξ, (see [GPR] and [GR] for related studies). Nonetheless, we always assume that γ_0 is convex so that the equation (1.3) is at least degenerate parabolic for the evolution of $\partial\Omega(t)$.

For a convex function γ_0 its subdifferential $\partial\gamma_0$ is a well-defined nonempty convex set. We require that

$$\xi(x) \in \partial\gamma_0(\mathbf{n}(x)).$$

It seems that there is a lot of freedom to choose ξ. However, one can see in related works, [GG], [GGK], it is expected that the value div ξ is unique and V in (1.3) is constant on each facet, provided that σ is nearly constant. In this situation $\Omega(t)$ stays as a cylinder, i.e.

$$\Omega(t) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \; x_1^2 + x_2^2 \leq R^2(t), \quad |x_3| \leq L(t)\}, \tag{1.4}$$

if $\Omega(0)$ is a cylinder and t is small. If σ is very large at the edge of the facet compared with the value at the center, then one may expect that V is no longer constant on facets and the crystal ceases to be a cylinder. We shall discuss this point in a forthcoming paper.

A cylindrical shape is important in studying the ice crystal growth (see, e.g. [YSF]). It is also considered as a simplification of a hexagonal prism. This sort of assumptions is usually made by physicists in order to facilitate their calculations, see e.g. [Ne, §3].

Here we do not pay any particular attention to the specific form of the law (1.3), i.e. we just assume γ, $\beta \ge 0$. We only consider $\Omega(t)$ being a cylinder of the form (1.4). Nonetheless we need to assume that $\sigma(t)$ is well-defined for each $t \ge 0$ as a solution to (1.1-2) (1.3 is dropped) for V being constant on each S_i, $i = L, T, B$, i.e.

$$V|_{S_i} \equiv V_i = const.$$

Here,

$$S_L = \{(x_1, x_2, x_3) \in \Omega(t) : x_1^2 + x_2^2 = R^2(t)\},$$

$$S_T = \{(x_1, x_2, x_3) \in \Omega(t) : x_3 = L(t)\}, \quad S_B = \{(x_1, x_2, x_3) \in \Omega(t) : x_3 = -L(t)\}.$$

The facets S_T, S_B and S_L are called respectively: the top, the bottom and the lateral part. In this paper we do not study the existence results. However, sometimes existence theorems for (1.1)–(1.3) are available, e.g. for $\gamma = 1$, $\beta > 0$, and (1.3) replaced by its average, i.e.

$$-\int_{S_i} \sigma \, dS = \Gamma_i - \beta_i V_i |S_i|, \quad i = T, B, L,$$

see [GR]. We expect that this solution is actually a solution of (1.1)–(1.3) at least for a short time if the size of crystal is small for some nonsmooth choice of γ_0. We shall discuss this point in our forthcoming paper. We note that for γ_0 which is not C^1 the meaning of (1.3) is not at all clear, see [GG], [GGK].

In our present analysis the law (1.3) shall play no role. We fix t and study the Neumann problem (1.1)–(1.2) in the set $\mathbb{R}^3 \setminus \Omega(t)$ (which we sometimes denote by Ω^c, especially in Section 3). Our goal is to prove $\partial\sigma/\partial r > 0$ on the top S_T if $V_T = V_B$ and $V_L > 0$ and $\partial\sigma/\partial x_3 > 0$ on $S_L \cap \{x_3 > 0\}$ if $V_T = V_B > 0$. The main idea is to apply the maximum principle for $\partial\sigma/\partial r$ and $\partial\sigma/\partial x_3$. For example $\partial\sigma/\partial x_3$ is harmonic in Ω^c and it is zero on $\{x_3 = 0\}$. To carry out this idea we invoke the condition $V_T = V_B$, so that we control the behavior of $\nabla\sigma$ at space infinity. Also we use continuity of $\nabla\sigma$ near the edge of the cylinder. This regularity is established by adjusting the result of Grisvard [Gv1, 2] and estimating difference quotients of σ. The regularity is discussed in Section 2. The main results (Berg's effect) as well as its proof based on the regularity is given in Section 3.

2. Auxiliary regularity results

We want to establish higher regularity up to the boundary of solutions to

$$\begin{cases} \Delta u = g & \text{in } \mathbb{R}^3 \setminus \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega^c \end{cases} \tag{2.1}$$

where g is smooth, ν is the outer normal to $\mathbb{R}^3 \setminus \Omega$, i.e. inner normal to Ω. The set Ω was defined previously in (1.4) with t dependence dropped.

The higher regularity up to the boundary is in fact true away from edges of Ω. Thus multiplication of (2.1) by a suitable test function leads us to a problem

$$\begin{cases} \Delta u = g & \text{in } D \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D \end{cases} \tag{2.2}$$

where D is a bounded axisymmetric domain and its rough shape is $(B(0, R_{\text{out}})\backslash B(0, R_{\text{in}}))\times [-\Lambda, \Lambda]\backslash\Omega$. We assume that $0 < R_{\text{in}} < R < R_{\text{out}}$, i.e. D does not contain the axis of revolution $\{r = 0\}$, moreover $\Lambda > 0$ is large, i.e $\Lambda > L$. A part of the boundary of D, namely $\partial D \setminus \partial\Omega$ is assumed to be smooth and the function g in (2.1) is suitably redefined.

Due to the assumed axial symmetry of D and g we may reduce the number of variables to two, i.e. $r = \sqrt{x_1^2 + x_2^2}$ and x_3. We shall define the reduced sets in the following way,

$$\hat{\Omega} = \{(r, x_3) \in \mathbb{R}^2 : \ (x_1, x_2, x_3) \in \Omega \text{ and } r = \sqrt{x_1^2 + x_2^2}\},$$

$$\hat{D} = \{(r, x_3) \in \mathbb{R}^2 : \ (x_1, x_2, x_3) \in D \text{ and } r = \sqrt{x_1^2 + x_2^2}\},$$

(see fig. 1 for the picture of $\hat{D}$ and $\hat{\Omega}$). The above choice of D and especially the right angles at the corners will help performing the analysis of difference quotients below.

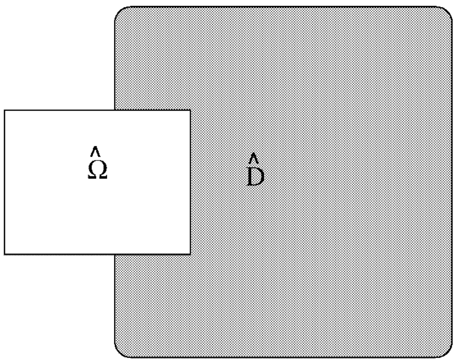

Fig. 1. The reduced domains.

Lemma 1. *Let us suppose that D is as above. We assume that $g \in C_0^\infty(D)$ and g is axially symmetric i.e. $g(x_1, x_2, x_3) = \bar{g}(\sqrt{x_1^2 + x_2^2}, x_3)$ and u is a unique variational solution to (2.2), i.e. $u \in H^1(D)$, and $u(x_1, x_2, x_3) = \bar{u}(\sqrt{x_1^2 + x_2^2}, x_3)$. Then*

$$\frac{\partial}{\partial x_3}u, \ \frac{\partial u}{\partial r} \in W^{1,p} \text{ for all } p < 6$$

i.e. they are continuous.

Proof. The first step in our analysis is writing (2.2) in the cylindrical coordinates (r, x_3), thus reducing the dimension of the problem to two:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + u_{x_3x_3} = g & \text{in } \hat{D} \\ \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\hat{D}. \end{cases} \tag{2.3}$$

This is permitted due to the assumed symmetry of g. A weak form of (2.3) which takes into account that $u_r(0, x_3) = 0$ is

$$\int_{\hat{D}} (-\nabla_2 u \nabla_2 \varphi + \frac{1}{r} u_r \varphi - g\varphi) dr dx_3 = 0 \quad \text{for all } \varphi \in H^1(\hat{D}), \tag{2.4}$$

where ∇_2 denotes the gradient in two variables (r, x_3).

Before we recall the summation by parts formula we make some additional remarks. We define an auxiliary family of sets Q_h, $h \in \mathbb{R}$, by

$$Q_h = \begin{cases} [R, R + |h|] \times [-L, L] & h > 0 \\ \\ [R - |h|, R] \times [-L, L] & h < 0, \end{cases}$$

(see fig. 2 below).

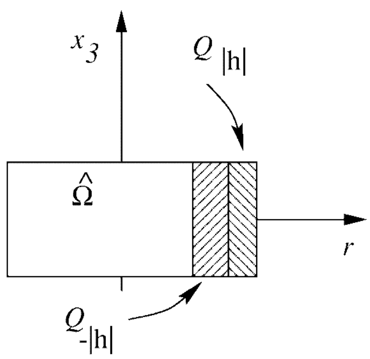

Fig. 2. Sets Q_h.

Since $g \in C_0^\infty(\hat{D})$ we extend g to $\hat{D} \cup Q_{-R/2}$ by zero. We assume in the formula below that $w \in H^{1+s}(\hat{\Omega})$ for some $s > 0$ and w is defined on $\hat{D} \cup Q_{-R/2}$. We set $\Delta^h w(x) = \frac{w(h+he_r)-w(x)}{h}$, when $e_r = (1, 0)$. We recall

$$\begin{aligned} \int_{\hat{D}} w\Delta^{-h} v \, dx &= \int_{\hat{D}} w(x) \frac{v(x - he_r) - v(x)}{-h} dx \\ &= \int_{\hat{D}} \frac{v(x)w(x)}{h} dx_3 dr - \int_{\hat{D}-he_r} \frac{1}{h} w(x + he_r)v(x) dr dx_3 \\ &= -\int_{\hat{D}} \Delta^h w(x)v(x) dr dx_3 + \epsilon \int_{Q_h} \frac{1}{h} w(x + e_r h)v(x) dx_3 dr \end{aligned} \tag{2.5}$$

where $\epsilon = -1$ or $\epsilon = +1$ depending upon the sign of h, i.e. $\epsilon = -\operatorname{sign} h$. Before we make an application of (2.5) to (2.4) we assume that the test function appearing in (2.4) belongs not only to $H^1(\hat{D})$ but also to

$$H^{1+s}(\hat{D}) \cap \{\varphi \in H^1(\hat{D}) : \ \varphi\,|_{S_L} = 0\},$$

for some $s > \frac{1}{2}$. We will extend the elements of the above space by 0 to $\hat{D} \cup Q_{-R/2}$. Of course such a v will not be in H^1 due to a possible jump at $\partial\hat{D} \cap \partial Q_{-R/2}$. That is why we say that the domain $\hat{D} \cup Q_{-R/2}$ has a cut. We extend also the weak solution to the cut domain. But first we establish that $u(r, x_3)$ solving (2.4) is in $H^{1+s}(\hat{D})$. We note that $\frac{1}{r}$ is a bounded function in a neighborhood of the point $p = (R, L)$, say in a ball $B(p, \delta)$. Moreover, $u_r \in L^2(B(p, \delta))$ as well as $\frac{1}{r}u_r \in L^2(B(p, \delta))$. Thus by [Gv2, Corollary 2.4.4] it follows that $u \in H^{1+s}(\hat{D}), s < \frac{2}{3}$ because u is a weak solution to a Neumann problem

$$\Delta h = g + \frac{1}{r}u_r,$$

where the RHS is in $L^2(B(p, \delta))$. The exponent s is related to the measure $\omega = 3\pi/2$ of the angle at the corner of $\hat{D}$, namely $s < \pi/\omega$, (see [Gv2, §2.3]).

Now, let us set $u_1 = u\,|_{[R,\infty)\times[-L,L]}$. We extend u_1 in $H^1([R,\infty) \times [-L, L]$ in a standard way to a function $E(u_1)$ in $H^{1+s}(Q_{-R/2} \cup [R, \infty) \times [-L, L])$. Finally, we set

$$\tilde{u}(x) = \begin{cases} u(x), & \text{if } x \in \hat{D} \\ E(u_1)(x), & \text{if } x \in Q_{-R/2}. \end{cases}$$

Of course there is no ground to claim that $\tilde{u} \in H^1(\hat{D} \cup Q_{-R/2})$. Now, we are ready to apply (2.5) to (2.4). It is apparent from (2.5) that we have to use an extention of u to Q_h, if $h < 0$. We will use $\tilde{u}$ for this purpose. In the foregoing calculations v is a test function from the function space which we have described above. We insert $\varphi = \Delta^{-h}v$ into (2.4) and we apply (2.5) to the result

$$0 = \int_{\hat{D}} -\nabla_2\Delta^h u \nabla_2 v + \frac{\epsilon}{h}\int_{Q_h} \nabla_2\tilde{u}(x + he_r)\nabla_2 v(x) +$$

$$+ \int_{\hat{D}} \frac{1}{r}u_r\Delta^{-h}v + \int_{\hat{D}} \Delta^h gv - \frac{\epsilon}{h}\int_{Q_h} g(x + he_r)v(x)$$

$$= I_1 + I_2 + I_3 + I_4 + I_5. \tag{2.6}$$

Since $\operatorname{supp} g \cap \hat{D} = \emptyset$ we immediately see that I_5 vanishes for sufficiently small $|h|$. We notice that for $h < 0$, $v\,|_{Q_h} = 0$, then $I_2 \equiv 0$ for all $h < 0$ which are sufficiently small. For this reason we shall restrict our attention only to $h < 0$. To stress this we shall write

$$\Delta^h_- u$$

when h is allowed only to be negative.

We add $\int_{\hat{D}} \Delta_-^h uv$ to both sides of (2.6). We note that

$$\int_{\hat{D}} \nabla v \cdot \nabla w + \int_{\hat{D}} v \cdot w =: \langle v, w\rangle$$

is the inner product in $H^1(\hat{D})$. Thus (2.6) yields

$$\langle v, \Delta_-^h u\rangle = \int_{\hat{D}} \frac{1}{r} u_r \Delta_-^{-h} v + \int_{\hat{D}} \Delta_-^h g v$$

It is quite obvious that the RHS converges, as $h \to 0$, to

$$\int_{\hat{D}} [\frac{1}{r} u_r \frac{\partial}{\partial r} v + \frac{\partial}{\partial r} g v].$$

Thus we conclude that $\Delta_-^h u$ has a weak limit in H^1, which is the one-sided derivative $\frac{\partial^- u}{\partial r}$. In particular,

$$\frac{\partial^- u}{\partial r} \in H^1(\hat{D}).$$

Since we do not allow $h > 0$, we are not able at this stage to conclude that $\frac{\partial u}{\partial r} \in H^1(\hat{D})$. At this point, we refer to the general structure of the solutions to

$$\begin{cases} \Delta\psi & = F \text{ in } \hat{D} \\ \frac{\partial \psi}{\partial \nu} & = 0 \text{ on } \partial\hat{D}. \end{cases}$$

We wish to recall two facts:

(i) if $F \in L^2$, then $\psi \in H^{1+s}(\hat{D})$, for any $s < 2/3$, because the inner angle of $\hat{D}$ at the corner is $\frac{3\pi}{2}$. This is the conclusion of [Gv2, Corollary 2.4.4]

(ii) if $F \in L^p$, where $p \geq 2$, then

$$\psi = \psi_0 + \sum_{i \in I} S_i,$$

where $\psi_0 \in W^{2,p}$ is the regular part of the solution and S_i's are singular (i.e. $S_i \in H^1(\hat{D}) \setminus W^{2,p}(\hat{D})$, see [Gv1, Lemma 4.4.3.5]), I is the set of corners of $\partial\hat{D}$ (see [Gv1, Theorem 4.4.4.13]). In our case above we have $\psi = u$, $F = \frac{1}{r} u_r + g$ and u_0 is the regular part of u. By (i) $u_r \in H^s(\hat{D})$, for any $s < 2/3$. It follows from the embedding theorem (see [Gv2, Theorem 1.2.10 and Theorem 1.2.14]) that $u_r \in L^p$ for any $p < 6$ i.e. $u_0 \in W^{2,p}$, for $p < 6$.

We showed that

$$\frac{\partial^- u}{\partial r} = \frac{\partial^-}{\partial r} u_0 + \frac{\partial^-}{\partial r} \sum_{i \in I} S_i \in H^1(\hat{D}).$$

This means that $\frac{\partial^-}{\partial r} S_i \in H^1$ and due to the structure of S_i explained above this implies that $S_i \equiv 0$, $i \in I$. It follows that $u \in W^{2,p}(\hat{D})$, $p < 6$ and $\nabla u \in C^{0,\alpha}$ for some $\alpha > 0$.

Let us finally note that $u \in W^{2,p}(\hat{D})$ implies that $u(r(x_1, x_2), x_3)$ is also in $W^{2,p}(D)$. This finishes the proof of Lemma 1. □

3. Berg's Effect

Once we establish higher regularity of our solutions (i.e. $\sigma \in W^{2,p}_{\rm loc}(\mathbb{R}^3 \setminus \hat{\Omega})$), $p < 6$, we are in a position to show the so-called Berg's effect which is well-known in the physics community (see [B], [Se]) but lacked rigorous proof in a general setting, see however [Ne, §3 eq. (6)] for related results.

It states that the supersaturation attains its minimum in an interior point on a facet and it monotonically increases towards corners. Surprisingly we can show this only under our symmetry assumptions. The proof breaks down without them.

Theorem (Berg's effect) *Suppose that σ is a unique solution to*

$$\begin{cases} \Delta\sigma = 0 & \text{in } \Omega^c, \quad \sigma(\infty) = \sigma^\infty \\ \frac{\partial\sigma}{\partial\mathbf{n}} = V_i & \text{on } S_i, \quad i = L, T, B, \end{cases}$$

where $\sigma = \sigma(r, x_3)$, $\sigma(r, -x_3) = \sigma(r, x_3)$, $\mathbf{n}$ is the outer normal to Ω and V_i, $i = L, T, B$ are constants, moreover $V_T = V_B$.

(a) If $V_T > 0$, then $\frac{\partial\sigma}{\partial x_3} > 0$ for $x_3 > 0$ and $\frac{\partial\sigma}{\partial x_3} < 0$ for $x_3 < 0$ on S_L.

(b) If $V_L > 0$, then $\frac{\partial\sigma}{\partial r} > 0$ on $S_T \cup S_B$.

Remarks.
(1) A similar statement holds if we reverse the inequality signs.
(2) Once we established regularity of $\nabla\sigma$, it is continuous up to the boundary of Ω^c.

The proof of this theorem requires establishing behavior of $\nabla\sigma$ at infinity.

Lemma 2. *Let us suppose that σ is regular at infinity (see [F, Chapter 2, §H]) and $\sigma = \sigma(r, x_3)$, $\sigma(r, -x_3) = \sigma(r, x_3)$, then*

$$\nabla\sigma = O(\rho^{-1}) \textit{ when } \rho^2 = r^2 + x_3^2 \to \infty.$$

Proof. Due to our assumptions σ is regular at ∞ meaning that the Kelvin transform of σ (see [F, Chapter 2, §H]), which is defined by

$$\tilde{\sigma}(y) = \frac{1}{|y|}\sigma(\frac{y}{|y|^2})$$

is smooth at $y = 0$ (see [F, Theorem 2.69]). Moreover, by (see [F, Theorem 2.69])

$$\sigma(x) - \sigma^\infty = O(\rho^{-1}).$$

However, to make our notation simpler, we will assume $\sigma^\infty = 0$. It is also well-known that the radial derivative

$$\frac{\partial}{\partial\rho}\sigma = \frac{\partial}{\partial e_\rho}e_\rho, \quad e_\rho = \frac{x}{|x|} = \frac{x}{\rho}$$

goes to zero when $\rho \to \infty$, namely $\frac{\partial}{\partial \rho}\sigma = O(\rho^{-2})$, see [F,Theorem 2.73].

We note that the symmetry relation for σ imply the following relations for $\tilde{\sigma}$

$$\begin{aligned} \tilde{\sigma}(-y_1, 0, \delta) &= \tilde{\sigma}(y_1, 0, 0) \\ \tilde{\sigma}(0, y_2, 0) &= \tilde{\sigma}(0, y_2, 0) \\ \tilde{\sigma}(0, 0, y_3) &= \tilde{\sigma}(0, 0, y_3). \end{aligned}$$

It follows that

$$\nabla_y \tilde{\sigma}(0,0,0) = 0$$

and $\nabla_y \tilde{\sigma}(y) \to 0$ or $y \to 0$. We calculate $\nabla_y \tilde{\sigma}$ in terms of σ

$$\nabla_y \tilde{\sigma}(y) = -\frac{y}{|y|^3}\sigma(\frac{y}{|y|^2}) - \frac{2}{|y|^3}\frac{\partial \sigma}{\partial R}\cdot\frac{y}{|y|} + \frac{1}{|y|^3}\nabla_x \sigma(\frac{y}{|y|^2})$$

we conclude that

$$\nabla_x \sigma = O(\rho^{-1}).$$

□

Remark. The symmetry of σ implying $\nabla_y \tilde{\sigma}(0) = 0$ was crucial for concluding $|\nabla_x \sigma| \to 0$, when $\rho \to \infty$. It is not clear what happens if the symmetry does not hold.

Proof of Theorem 1. Let us introduce three cut-off functions $\eta_i \in C_0^\infty(\mathbb{R}^3)$, $1 \geq \eta_i \geq 0$, $i \in I$. We require that $\eta_L|_{\{r<\varepsilon\}} = 0 = \eta_L|_{\{x_3^2+r^2>\rho^2\}}$ for some $\varepsilon > 0$ and sufficiently large ρ, and η_L restricted to the set $\{r \in [R-\varepsilon, R+\varepsilon]\} \cap \{|x_3| < \frac{\rho}{2}\}$ is 1. We need that

$$\eta_T(r, x_3) = \begin{cases} 0 & \text{if } x_3 \leq 0 \text{ or } r^2 + x_3^2 \geq \rho^2 \\ 1 & \text{if } x_3 \in [L-\varepsilon, L+\varepsilon] \text{ and } r < \frac{\rho}{2}, \end{cases}$$

finally $\eta_B(r, x_3) = \eta_T(r, -x_3)$. We also define

$$\begin{aligned} \overline{f_L}(r, x_3) &= -\eta_L(r, x_3) R \ln r \\ \overline{f_T}(r, x_3) &= -\eta_T x_3, \quad \overline{f_B}(r, x_3) = -\eta_B x_3. \end{aligned}$$

Obviously $\overline{f_i} \in C^\infty$ and if ν is the inner normal to Ω, then

$$\frac{\partial \overline{f_i}}{\partial \nu} = \begin{cases} 1 & \text{on } S_i, \\ 0 & \text{on } S_j, \quad j \neq i. \end{cases}$$

Subsequently, we define

$$u = \sigma - \sum_{i \in \{L,B,T\}} V_i \overline{f_i}.$$

We can now see that u satisfies the assumptions of Lemma 1. Thus $u \in W^{2,p}(\hat{D})$, $p < 6$, this implies that $u \in W^{2,p}_{loc}(\Omega^c)$, $p < 6$ as well as $\nabla u \in C^{0,\alpha}$ up to the boundary.

(a) Let set $w = \frac{\partial\sigma}{\partial x_3}$, then $w \in C^0(\bar{\Omega}^c)$ due to the symmetry of σ

$$\frac{\partial\sigma}{\partial x_3}(R,0) = 0 = w(R,0)$$

by assumption $w > 0$ on S_T and $w < 0$ on S_B. We claim $w(R,x_3) > 0$ for all $x_3 > 0$ (resp. $w(R,x_3) < 0$ for $x_3 < 0$). Let us suppose otherwise, then due to continuity of w

$$\min_{L>\zeta>0} w(R,\zeta) = w(R,z_0) < 0.$$

Since $w(R,0) = 0,\ w(R,L) > 0$ we conclude that $z_0 \in (0,L)$. Let us denote by Ω^c_ϵ the set

$$\{p \in \Omega^c : \mathrm{dist}\,(p,((S_T \cup S_B) \cap S_L) \cup \{(x_1,x_2,x_3) \in \Omega^c : x_1 = x_2 = 0\}) > \epsilon\},$$

thus by standard regularity theory we conclude that

$$\sigma \in C^\infty(\bar{\Omega}^c_\epsilon) \text{ and } w \in C^\infty(\bar{\Omega}^c_\epsilon).$$

At this point we recall that due to Lemma 1 w is in fact a harmonic function continuous up to the boundary of Ω^c. Thus, if we set

$$\Gamma_\rho = [(\partial B(0,\rho) \cup \partial\Omega^c_\epsilon) \cap \{x_3 > 0\}] \cup [\{x_3 = 0\} \cap B(0,\rho) \cap \Omega^c_\epsilon]$$

for some large ρ, then we see

$$\inf_{B(0,\rho)\cap\Omega^c_\epsilon\cap\{x_3>0\}} w = \min_{\Gamma_\rho} w = \min\{\min_{\partial\Omega^c_\epsilon} w,\ \min_{\partial B(0,\rho)},\ \min_{\{x_3=0\}} w\}.$$

But $w(r(x_1,x_2),0) = 0$, and we showed the $\nabla\sigma \to 0$ as $\rho \to \infty$.

We conclude that for sufficiently large ρ

$$\inf_{B(0,\rho)\cap\Omega^c_\epsilon\cap\{x_3>0\}} w = \min_{\partial\Omega^c_\epsilon\cap\{x_3>0\}} w.$$

We may pass to the limit with $\rho \to \infty$, then with $\epsilon \to 0$,

$$\inf_{\Omega^c\cap\{x_3>0\}} w = \min_{\partial\Omega^c\cap\{x_3>0\}} w = w(R,z_0)$$

Now, Hopf's maximum principle (see e.g. [PW, Chapter 2, §3, Theorem 7]) implies that

$$0 > \frac{\partial w}{\partial\nu}(R,x_3) = -\frac{\partial w}{\partial\mathbf{n}}(R,x_3) = -\frac{\partial\sigma}{\partial r\partial x_3}(R,x_3)$$

but due to boundary condition on σ, $\frac{\partial\sigma}{\partial r} = \text{const.}$ we have $\frac{\partial\sigma}{\partial r\partial x_3} \equiv 0$ on S_L, thus we have reached a contradiction and our claim follows.

(b) We set $w = \frac{\partial \sigma}{\partial r}$, we recall that for axis symmetric σ the Laplace equation for σ takes the form

$$\frac{\partial^2}{\partial r^2}\sigma + \frac{\partial^2 \sigma}{\partial x_3^2} + \frac{1}{r}\frac{\partial}{\partial r}\sigma = 0 \qquad \text{in} \quad \Omega^c,$$

hence after differentiation of this equation with respect to r, we see that $\frac{\partial \sigma}{\partial r}$ satisfies

$$\frac{\partial^2}{\partial r^2} w + \frac{\partial^2}{\partial x_3^2} w - \frac{1}{r^2} w + \frac{1}{r}\frac{\partial w}{\partial r} = 0$$

and the symmetry implies $w(0, x_3) = 0$. We claim that $w > 0$ on $S_T \cup S_B$. Otherwise there would exist a point (r_0, L) such that

$$w(r_0, L) < 0 \quad \text{and} \quad \min_{\rho \in (0,R)} w(\rho, L) < 0.$$

By maximum principle (see e.g. [PW, Chapter 2, §3, Theorem 6]) we have

$$\inf_{\Omega^c_\epsilon \cap B(0,\rho)} w = \min_{\partial \Omega^c_\epsilon \cup \partial B(0,\rho)} w < 0$$

with the same definition of Ω^c_ϵ, as before. Due to Lemma 2, $w \to 0$ as $\rho \to \infty$, then we see

$$\inf_{\Omega^c_\epsilon} w = \min_{\partial \Omega^c_\epsilon} w < 0$$

after passing to the limit with $\epsilon \to 0$. We obtain

$$\inf_{\Omega^c} w = \min_{\partial \Omega^c} w$$

since $w \mid_{S_L} > 0$, the minimum of w is attained on $S_T \cup S_B$ at (ρ_0, L).

By Hopf's maximum principle, (see [PW, Chapter 2, §3, Theorem 7]),

$$0 > \frac{\partial}{\partial \nu} w = -\frac{\partial w}{\partial x_3}$$

but

$$\frac{\partial w}{\partial x_3} = \frac{\partial^2 \sigma}{\partial x_3 \partial r} = \frac{\partial}{\partial r}\,\text{const.} = 0$$

which yields a contradiction and our claim follows. Our theorem is proven. □

Acknowledgment. A part of this research was done during the visits of P. R. to Hokkaido University and University of Minnesota. Hospitality of both institutions is gratefully acknowledged. The work of the first author was partly supported by a Grant-in-Aid for Scientific Research, No. 14204011, 12874024 the Japan Society for the Promotion of Science. The second author was in part supported by KBN grant 2 P03A 035 17.

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