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A generalization of the Lieb-Thirring inequalities in low dimensions

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ABSTRACT. We give an estimate for the moments of the negative eigenvalues of elliptic operators on $\mathbb{R}^n$ in low dimensions. The estimate is a generalization of the Lieb-Thirring inequalities in one or two dimensions. We use the φ-transform decomposition of Frazier and Jawerth.

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1 Introduction

For a real-valued measurable function V on $\mathbb{R}^n$ we set

$$V_+(x) = \max(V(x), 0) \quad \text{and} \quad V_-(x) = \max(-V(x), 0).$$

The Lieb-Thirring inequalities state

$$\sum_i |\lambda_i|^\gamma \le c_{n,\gamma} \int_{\mathbb{R}^n} V_-^{n/2+\gamma} dx \tag{1}$$

for suitable $\gamma \ge 0$, where $\lambda_1 \le \lambda_2 \le \cdots$ are the negative eigenvalues of the Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^n)$. The inequality (1) holds if and only if

$$\begin{aligned} \gamma &\ge \frac{1}{2} \qquad \text{for} \quad n = 1, \\ \gamma &> 0 \qquad \text{for} \quad n = 2, \\ \gamma &\ge 0 \qquad \text{for} \quad n \ge 3. \end{aligned}$$

The case $\gamma > 1/2, n = 1, \gamma > 0, n \geq 2$ was proved by Lieb and Thirring([8]). They applied the inequality (1) to the problem of the stability of matter. The case $\gamma = 1/2, n = 1$ was proved by Weidl([18]). The case $\gamma = 0, n \geq 3$ was established by Cwikel([1]), Lieb([7]) and Rozenbljum([12],[13]). Some generalizations and variations of the Lieb-Thirring inequalities are known([2],[6],[9],[14],[15]). In particular Egorov and Kondrat'ev([2]) studied the estimate for L_0+V where L_0 is an elliptic operator of order $2m$.

In the present paper we give a generalization of a result by Egorov and Kondrat'ev's for certain degenerate elliptic partial differential operator in low dimension, for which the rate of degeneracy is regulated by the weight $w \in A_2$. A generalization of the higer dimensional cases is given in [17]. In the proof of our main theorem we use the φ-transform of Frazier-Jawerth([3]).

First we recall the definition of A_p-weights. By a cube in $\mathbb{R}^n$ we mean a cube which sides are parallel to coordinate axes. A locally integrable and nonnegative function w on $\mathbb{R}^n$ is an A_p-weight for some $p \in (1, \infty)$ if there exists a positive constant C such that

$$\frac{1}{|Q|}\int_Q w(x)\,dx \left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^n$.

We say that w is an A_1-weight if there exists a positive constant C such that

$$\frac{1}{|Q|}\int_Q w(y)\,dy \leq Cw(x) \qquad a.e. x \in Q$$

for all cubes $Q \subset \mathbb{R}^n$. We write A_p for the class of A_p-weights. It turns out that $A_1 \subset A_p$ for $p > 1$.

Next we consider an elliptic partial differential operator of order $2m$. For $m \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{R}^n)$ let

$$L_0 f(x) = \sum_{|\alpha|=|\beta|=m} (-1)^m D^\alpha \left(a_{\alpha\beta}(x) D^\beta f(x)\right),$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \quad \text{for} \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}\cup\{0\})^n,$$

$$a_{\alpha\beta} \in H^m_{\mathrm{loc}}(\mathbb{R}^n), \quad \text{and} \quad a_{\alpha\beta} = \overline{a_{\beta\alpha}}.$$

In the above definition the space $H^m_{\mathrm{loc}}(\mathbb{R}^n)$ denotes the set of all $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ such that $D^\alpha f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ for all $|\alpha| \leq m$.

Let

$$a(f,g) = \int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\beta f(x) \overline{D^\alpha g(x)}\, dx$$

for $f, g \in C_0^\infty(\mathbb{R}^n)$ and $\|\cdot\|$ be the norm of $L^2(\mathbb{R}^n)$.

For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ the cube Q defined by

$$Q = \{(x_1, \dots, x_n) : k_i \leq 2^\nu x_i < k_i + 1,\ i = 1, \dots, n\}$$

is called a dyadic cube in $\mathbb{R}^n$. Let $\mathcal{Q}$ be the set of all dyadic cubes in $\mathbb{R}^n$. For any $Q \in \mathcal{Q}$ there exists a unique $Q' \in \mathcal{Q}$ such that $Q \subset Q'$ and the side-length of Q' is double of that of Q. We call Q' the parent of Q.
We have the following theorem.

Theorem 1.1. *Let $n \leq 2m, q \geq n/(2m), \gamma > 0$ and $q + \gamma > 1$. We assume that there exists a $w \in A_2$ such that*

$$(L_0 f, f) \geq \int_{\mathbb{R}^n} w(x) \sum_{|\alpha|=m} |D^\alpha f(x)|^2\, dx \tag{2}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ and

$$\int_{Q'} w\, dx \leq 2^{2m} \int_Q w\, dx \tag{3}$$

for all $Q \in \mathcal{Q}$ and its parent Q'.

For a $u \in A_{q+\gamma}$ we suppose that

$$|Q|^{2m/n+1} \leq c_1 \int_Q w\, dx \left(\int_Q u\, dx \right)^{1/q} \tag{4}$$

for all cubes $Q \subset \mathbb{R}^n$, where c_1 is a positive constant not depending on Q. For a real valued function V on $\mathbb{R}^n$ we assume that $V_+ \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} V_-^{q+\gamma} u\, dx < \infty. \tag{5}$$

Let $\mathcal{H}$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{\mathcal{H}} = \{a(f,f) + \int_{\mathbb{R}^n} V_+ |f|^2\, dx + \|f\|^2\}^{1/2}.$$

Then we have the following.
(i) There exists a unique self-adjoint operator L in $L^2(\mathbb{R}^n)$ with domain $\mathcal{D} \subset \mathcal{H}$ such that

$$(Lf, g) = a(f,g) + \int_{\mathbb{R}^n} V f \bar{g}\, dx$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{H}$.

(ii) The negative spectrum of L is discrete.

(iii) There exists a positive constant c such that

$$\sum_i |\lambda_i|^\gamma \leq c \int_{\mathbb{R}^n} V_-^{q+\gamma} u \, dx, \tag{6}$$

where $\{\lambda_i\}$ is the set of all negative eigenvalues of L counting multiplicity and c does not depend on V.

The inequality (6) is a generalization of the Lieb-Thirring inequality for the case $\gamma > 1/2, n = 1$ and $\gamma > 0, n = 2$. Our result does not include the case $\gamma = 1/2, n = 1$. The case $w \equiv 1$ and $u(x) = |x - x_0|^{2mq-n}$ is proved by Egorov and Kondrat'ev([2]). In [9] Netrusov and Weidl proved (6) for $w \equiv u \equiv 1, q = n/(2m) < 1, \gamma = 1 - n/(2m)$. Our result does not include their result.

We remark that the condition (4) is trivial by Hölder's inequality when $q = n/(2m)$ and $u = w^{-n/(2m)}$. We also remark that for a fixed n the condition (3) is satisfied for sufficiently large m because w satisfies the doubling condition, that is, (iv) of Proposition 2.1.

2 Preliminaries

First we recall some properties of A_p-weights which will be used in the following sections. Let M be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is taken over all cubes Q which contain x.

Proposition 2.1.

(i) Let $1 < p < \infty$ and w be a non-negative locally integrable function on $\mathbb{R}^n$. Then M is bounded on $L^p(w)$ if and only if $w \in A_p$.

(ii) Let $1 < p < \infty$ and $w \in A_p$. Then there exists a $q \in (1, p)$ such that $w \in A_q$.

(iii) Let $0 < \tau < 1$ and f be a locally integrable function on $\mathbb{R}^n$ such that $M(f)(x) < \infty$ a.e.. Then $(M(f))^\tau \in A_1$.

(iv) Let $1 \leq p < \infty$ and $w \in A_p$. Then there exists a positive constant c such that

$$\int_{2Q} w \, dx \leq c \int_Q w \, dx$$

for all cubes $Q \in \mathbb{R}^n$, where $2Q$ denotes the double of Q.

The proofs of these facts are in [4, Chapter IV] or [16, Chapter V]. Property (iv) is called the doubling property of A_p-weights.

Let φ be a function which satisfies the following conditions.

(A1) $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

(A2) $\operatorname{supp} \hat{\varphi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\}$

(A3) $|\hat{\varphi}(\xi)| \ge c > 0$ if $\frac{3}{5} \le |\xi| \le \frac{5}{3}$.

(A4) $\sum_{\nu \in \boldsymbol{Z}} |\hat{\varphi}(2^\nu \xi)|^2 = 1$ for all $\xi \neq 0$.

For a dyadic cube Q such that

$$Q = \{(x_1, \dots, x_n) : k_i \le 2^\nu x_i < k_i + 1,\ i = 1, \dots, n\}.$$

for $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we set

$$\varphi_Q(x) = 2^{\nu n/2} \varphi(2^\nu x - k).$$

3 Proof of Theorem 1.1

By (ii) of Proposition 2.1 there exists a constant s such that $1 < s < q + \gamma$ and $u \in A_{(q+\gamma)/s}$. It turns out that $V_- \in L^s_{\mathrm{loc}}(\mathbb{R}^n)$(c.f. [17, Section 3]).

Let $v(x) = (M(V_-^s)(x))^{1/s}$. We may assume that $v(x) > 0$ for all $x \in \mathbb{R}^n$. By the properties of the maximal operator we have $V_-(x) \le v(x)$ *a.e.*. By (i) of Proposition 2.1 we get

$$\int_{\mathbb{R}^n} v^{q+\gamma} u \, dx = \int_{\mathbb{R}^n} M(V_-^s)^{(q+\gamma)/s} u \, dx \le c_1 \int_{\mathbb{R}^n} V_-^{q+\gamma} u \, dx < \infty.$$

Furthermore v is an A_1-weight by (iii) of Proposition 2.1.

We have the following lemmas.

Lemma 3.1. *There exists a positive constant α such that*

$$\alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2m/n} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q w \, dx \le \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha| = m} |D^\alpha f|^2 \right\} w \, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

Lemma 3.2. *There exist positive constants β and β' such that*

$$\beta' \sum_{Q \in \mathcal{Q}} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx \le \int_{\mathbb{R}^n} |f|^2 v \, dx \le \beta \sum_{Q \in \mathcal{Q}} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

The proof of Lemma 3.1 is in [17, Proposition 2.2 and Lemma 3.2]. Lemma 3.2 is proved in [3].

Now we set

$$\mathcal{I} = \{Q \in \mathcal{Q} : \beta \int_Q v(x)\,dx > \alpha |Q|^{-2m/n} \int_Q w(x)\,dx\},$$

where α and β are constants in Lemmas 3.1 and 3.2. We remark that $\mathcal{I}$ is not empty. In fact, if $\mathcal{I}$ is empty, then we have

$$\beta \int_Q v(x)\,dx \leq \alpha |Q|^{-2m/n} \int_Q w(x)\,dx$$

for all $Q \in \mathcal{Q}$. Let $Q_0 \in \mathcal{Q}$ and $Q_0 \subset Q_1 \subset Q_2 \subset \cdots$ be the infinite sequence of dyadic cubes such that Q_{i+1} is the parent of Q_i for all $i = 1, 2, \ldots$. By (3) we have

$$|Q_{i+1}|^{-2m/n} \int_{Q_{i+1}} w(x)\,dx \leq |Q_i|^{-2m/n} \int_{Q_i} w(x)\,dx$$

for all i. Hence we have

$$\beta \int_{Q_i} v(x)\,dx \leq \alpha |Q_0|^{-2m/n} \int_{Q_0} w(x)\,dx$$

for all i. This is a contradiction because

$$\lim_{i \to \infty} \int_{Q_i} v(x)\,dx = \int_{\mathbb{R}^n} v(x)\,dx = \infty$$

by the doubling property of v (c.f.[16, p.39 or p.222]). Therefore $\mathcal{I}$ is not empty.

Let $Q \in \mathcal{I}$ and Q' be the parent of Q. Then we have

$$\alpha |Q'|^{-2m/n} \int_{Q'} w(x)\,dx \leq \alpha |Q|^{-2m/n} \int_Q w(x)\,dx < \beta \int_Q v(x)\,dx \leq \beta \int_{Q'} v(x)\,dx.$$

Hence we have $Q' \in \mathcal{I}$. This fact means that $\mathcal{I}$ is an infinite set.

Lemma 3.3. *There exists a $c > 0$ such that*

$$\sum_{Q \in \mathcal{I}} \left(\frac{1}{|Q|} \int_Q v\,dx \right)^{\gamma} \leq c \int_{\mathbb{R}^n} v^{q+\gamma} u\,dx$$

The proof of this lemma will be given later.

For $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int |f|^2 V_-\,dx \leq \int |f|^2 v\,dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q v\,dx,$$

where we used Lemma 3.2. The last quantity is bounded by

$$\beta \sum_{Q \in \mathcal{I}} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q v\,dx + \beta \sum_{Q \notin \mathcal{I}} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q v\,dx$$

$$\leq \beta K \sum_{Q \in \mathcal{I}} |(f, \varphi_Q)|^2 + \alpha \sum_{Q \notin \mathcal{I}} |(f, \varphi_Q)|^2 |Q|^{-2m/n} \frac{1}{|Q|} \int_Q w\,dx$$

$$\leq cK \|f\|_2^2 + \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha f|^2 \right\} w\,dx$$

where

$$K = \max_{Q \in \mathcal{I}} \frac{1}{|Q|} \int_Q v\,dx$$

and we used Lemma 3.1. We remark that K is finite by Lemma 3.3.

By the condition (2) we have

$$\int_{\mathbb{R}^n} |f|^2 V_-\,dx \leq \int_{\mathbb{R}^n} |f|^2 v\,dx \leq cK \|f\|_2^2 + (L_0 f, f). \tag{7}$$

Hence we have

$$a(f, f) + \int_{\mathbb{R}^n} V|f|^2\,dx \geq -cK \|f\|_2^2$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. Therefore

$$b(f, g) = a(f, g) + \int_{\mathbb{R}^n} V f \bar{g}\,dx$$

is a lower semi-bounded quadratic form on $\mathcal{H}$.

By the assumption of the coefficients of L_0 and $V_+ \in L^2_{loc}(\mathbb{R}^n)$ we can show that $b(f, g)$ is a closed form on $\mathcal{H}$(c.f. [17]). Since $b(f, g)$ is a closed and lower semi-bounded quadratic form on $\mathcal{H}$, there exists a unique self-adjoint operator L in $L^2(\mathbb{R}^n)$ with domain $\mathcal{D} \subset \mathcal{H}$ such that

$$(Lf, g) = a(f, g) + \int_{\mathbb{R}^n} V f \bar{g}\,dx$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{H}$([10, Theorem VIII.15]).

We set

$$\lambda_1 = \inf_{f \in \mathcal{D}, \|f\|=1} (Lf, f)$$

and

$$\lambda_k = \sup_{\phi_1, \ldots, \phi_{k-1} \in L^2} \inf_{\substack{f \in \mathcal{D}, \|f\|=1, \\ (f, \phi_j)=0, j=1, \ldots, k-1}} (Lf, f)$$

for $k \geq 2$.

For each fixed $k \in \mathbb{N}$ either:

(i) there are k eigenvalues counting multiplicity below the infimum of the essential spectrum of L, and λ_k is the kth eigenvalue of L;

or

(ii) λ_k is the infumum of the essential spectrum of L and $\lambda_k = \lambda_{k+1} = \lambda_{k+2} = \cdots$ and there are at most $k-1$ eigenvalues counting multiplicity below λ_k.

The proof of this fact is in [11, Theorem XIII.1].

We have the following lemma.

Lemma 3.4. *Let $A > 0$ and*

$$\mathcal{I}_A = \{Q \in \mathcal{I} : \alpha|Q|^{-1-2m/n} \int_Q w\,dx - \beta|Q|^{-1} \int_Q v\,dx \leq -A\}.$$

Then $\mathcal{I}_A$ is a finite set.

Proof. Let $Q \in \mathcal{I}_A$. Then we have

$$A \leq \frac{\beta}{|Q|} \int_Q v\,dx.$$

By Lemma 3.3 we conclude that $\mathcal{I}_A$ is a finite set. □

Let $\{\mu_k\}_{k=1}^{\infty}$ be the non-decreasing rearrangement of

$$\left\{\alpha|Q|^{-1-2m/n} \int_Q w\,dx - \beta|Q|^{-1} \int_Q v\,dx\right\}_{Q\in\mathcal{I}}.$$

Then

$$\mu_1 \leq \mu_2 \leq \cdots$$

and

$$\lim_{k\to\infty} \mu_k = 0$$

by Lemma 3.4.

When

$$\mu_k = \alpha|Q|^{-1-2m/n} \int_Q w\,dx - \beta|Q|^{-1} \int_Q v\,dx,$$

we define $\psi_k = \varphi_Q$.

By (7) and the density argument we have $\int_{\mathbb{R}^n} |f|^2 v\,dx < \infty$ for all $f \in \mathcal{D}$ and the inequalities in Lemmas 3.1 and 3.2 holds for $f \in \mathcal{D}$. Hence we have

$$
\begin{aligned}
(Lf, f) &= a(f,f) + \int_{\mathbb{R}^n} V|f|^2\,dx \\
&\geq \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha f|^2 \right\} w\,dx - \int_{\mathbb{R}^n} V_-|f|^2\,dx \\
&\geq \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha f|^2 \right\} w\,dx - \int_{\mathbb{R}^n} |f|^2 v\,dx \\
&\geq \sum_{Q\in\mathcal{Q}} |(f,\varphi_Q)|^2 \left\{ \alpha|Q|^{-2m/n-1} \int_Q w\,dx - \beta|Q|^{-1}\int_Q v\,dx \right\}
\end{aligned}
$$

for all $f \in \mathcal{D}$. Therefore we have

$$
\begin{aligned}
\lambda_k &\geq \inf_{\substack{f\in\mathcal{D},\|f\|=1,\\ (f,\psi_i)=0, i=1,\dots,k-1}} (Lf,f) \\
&\geq \inf_{\substack{f\in\mathcal{D},\|f\|=1,\\ (f,\psi_i)=0, i=1,\dots,k-1}} \sum_{j=1}^{\infty} |(f,\psi_j)|^2 \mu_j \\
&\geq \mu_k \sup_{\substack{f\in\mathcal{D},\|f\|=1,\\ (f,\psi_i)=0, i=1,\dots,k-1}} \sum_{j=k}^{\infty} |(f,\psi_j)|^2 \geq c\mu_k,
\end{aligned}
$$

where we used the fact $\mu_k < 0$ and $\sum_j |(f,\psi_j)|^2 \leq c\|f\|^2$.

Since $\lim_{k\to\infty} \mu_k = 0$, the negative spectrum of L is discrete. Furthermore we have

$$
\begin{aligned}
\sum_{k,\lambda_k<0} |\lambda_k|^\gamma &\leq c\sum_{k=1}^{\infty} |\mu_k|^\gamma \\
&= c\sum_{Q\in\mathcal{I}} \left(\beta|Q|^{-1}\int_Q v\,dx - \alpha|Q|^{-1-2m/n}\int_Q w\,dx \right)^\gamma \\
&\leq c\sum_{Q\in\mathcal{I}} \left(\beta|Q|^{-1}\int_Q v\,dx \right)^\gamma \leq c\int_{\mathbb{R}^n} v^{q+\gamma} u\,dx \leq c\int_{\mathbb{R}^n} V_-^{q+\gamma} u\,dx,
\end{aligned}
$$

where we used Lemma 3.3. This ends the proof of Theorem 1.1.

Proof of Lemma 3.3

For $Q \in \mathcal{I}$ we have

$$
\begin{aligned}
\alpha|Q|^{-2m/n} \int_Q w(x)\,dx &< \beta\int_Q v(x)\,dx \\
&\leq \beta\left(\int_Q v^{q+\gamma} u\,dx \right)^{1/(q+\gamma)} \left\{ \int_Q u^{-1/(q+\gamma-1)}\,dx \right\}^{(q+\gamma-1)/(q+\gamma)}.
\end{aligned}
$$

Since $u \in A_{q+\gamma}$ means

$$\frac{1}{|Q|}\int_Q u\,dx\left\{\frac{1}{|Q|}\int_Q u^{-1/(q+\gamma-1)}\,dx\right\}^{q+\gamma-1} \le c,$$

the last term is bounded by

$$\begin{aligned}
&c\left(\int_Q v^{q+\gamma}u\,dx\right)^{1/(q+\gamma)}|Q|\left(\int_Q u\,dx\right)^{-1/(q+\gamma)} \\
\le\,& c\left(\int_Q v^{q+\gamma}u\,dx\right)^{1/(q+\gamma)}|Q|\left(\int_Q u\,dx\right)^{\gamma/\{q(q+\gamma)\}-1/q} \\
\le\,& c\left(\int_Q v^{q+\gamma}u\,dx\right)^{1/(q+\gamma)}|Q|\left(\int_Q u\,dx\right)^{\gamma/\{q(q+\gamma)\}}|Q|^{-2m/n-1}\int_Q w\,dx,
\end{aligned} \tag{8}$$

where we used (4). Therefore we have

$$0 < c \le \int_Q v^{q+\gamma}u\,dx\left(\int_Q u\,dx\right)^{\gamma/q}.$$

By this inequality we conclude that if $Q_1 \supset Q_2 \supset \cdots$ are cubes in $\mathcal{I}$, then this sequence must have a minimal element with respect to the inclusion relation. Let $\mathcal{M}$ be the set of all such minimal cubes in $\mathcal{I}$.

Lemma 3.5. *Let $Q \in \mathcal{Q}$ and $Q_1, Q_2, \dots, Q_{2^n}$ be the half-size dyadic sub-cubes of Q. Then we have*

$$\left(\frac{1}{|Q|}\int_Q v\,dx\right)^{\gamma} \le 2^{-n\min\{1,\gamma\}}\sum_{i=1}^{2^n}\left(\frac{1}{|Q_i|}\int_{Q_i} v\,dx\right)^{\gamma}. \tag{9}$$

Proof. We have

$$\frac{1}{|Q|}\int_Q v\,dx = 2^{-n}\sum_{i=1}^{2^n}\frac{1}{|Q_i|}\int_{Q_i} v\,dx.$$

If $0 < \gamma < 1$, then we can get (9) easily. If $\gamma > 1$, then (9) is a consequence of the convexity of the function $y = x^{\gamma}, x > 0$. □

Let $\mathcal{N}$ be the set of all $Q \in \mathcal{Q}$ such that $Q \notin \mathcal{I}$ and its parent $Q' \in \mathcal{I} \setminus \mathcal{M}$. Then using Lemma 3.5 repeatedly we have

$$\begin{aligned}
&\sum_{Q\in\mathcal{I}}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^{\gamma} \\
\le &\sum_{Q\in\mathcal{M}}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^{\gamma}\left\{\sum_{k=0}^{\infty}2^{-kn\min\{1,\gamma\}}\right\} + \sum_{Q\in\mathcal{N}}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^{\gamma}\left\{\sum_{k=1}^{\infty}2^{-kn\min\{1,\gamma\}}\right\} \\
&\le c\sum_{Q\in\mathcal{M}}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^{\gamma} + c\sum_{Q\in\mathcal{N}}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^{\gamma}.
\end{aligned}$$

Let $Q \in \mathcal{I}$. Then by (4) and (8) we get

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^{\gamma} &\le c\left(\int_Q v^{q+\gamma}u\,dx\right)^{\gamma/(q+\gamma)}\left(\int_Q u\,dx\right)^{-\gamma/(q+\gamma)}\\ &\le c\left(\int_Q v^{q+\gamma}u\,dx\right)^{\gamma/(q+\gamma)}\left(|Q|^{-2m/n-1}\int_Q w\,dx\right)^{q\gamma/(q+\gamma)}\\ &\le c\left(\int_Q v^{q+\gamma}u\,dx\right)^{\gamma/(q+\gamma)}\left(|Q|^{-1}\int_Q v\,dx\right)^{q\gamma/(q+\gamma)}.\end{aligned}$$

Therefore we have

$$\left(\frac{1}{|Q|}\int_Q v\,dx\right)^{\gamma} \le c\int_Q v^{q+\gamma}u\,dx.$$

Similarly we have this inequality for $Q \in \mathcal{N}$ because the parent Q' of Q belongs to $\mathcal{I}$ and the inequality

$$|Q|^{-2m/n-1}\int_Q w\,dx \le c'|Q|^{-1}\int_Q v\,dx$$

holds by the doubling property of v.

Therefore we conclude

$$\begin{aligned}\sum_{Q\in\mathcal{I}}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^{\gamma} &\le c\sum_{Q\in\mathcal{M}}\int_Q v^{q+\gamma}u\,dx + c\sum_{Q\in\mathcal{N}}\int_Q v^{q+\gamma}u\,dx\\ &\le c\int_{\mathbb{R}^n} v^{q+\gamma}u\,dx\end{aligned}$$

where we used the fact that the cubes in $\mathcal{M}\cup\mathcal{N}$ are mutually disjoint. Hence Lemma 3.3 is proved.

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References

[1] Cwikel M., *Weak type estimates for singular values and the number of bound states of Schrödinger operators.* Ann. Math. **106** (1977), 93–100.

[2] Egorov Y.V. and Kondrat'ev V.A., *On moments of negative eigenvalues of an elliptic operator.* Math. Nachr. **174** (1995), 73–79.

[3] Frazier M. and Jawerth B., *A discrete transform and decompositions of distribution spaces.* J. Funct. Anal. **93** (1990), 34–170.

[4] García-Cuerva J. and Rubio de Francia J.L., *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies 116, North-Holland, 1985.

[5] Laptev A. and Netrusov Y., *On the negative eigenvalues of a class of Schrödinger operators.* Differential operators and spectral theory, Amer. Math. Soc. Transl. Ser.2 **189** 1999, pp. 173–186.

[6] Laptev A. and Weidl T., *Sharp Lieb-Thirring inequalities in high dimensions.* Acta Math. **184** (2000), 87–111.

[7] Lieb E., *Bounds on the eigenvalues of the Laplace and Schrödinger operators.* Bull. Amer. Math. Soc. **82** (1976), 751–753.

[8] Lieb E. and Thirring W., *Inequalities for the moments of the eigenvalues of the Schrödinger hamiltonian and their relation to Sobolev inequalities.* Studies in Mathematical Physics, Princeton University Press, 1976, pp. 269–303.

[9] Netrusov Y. and Weidl T., *On Lieb-Thirring inequalities for higher order operators with critical and subcritical powers.* Comm. Math. Phys. **182** (1996), 355–370.

[10] Reed M. and Simon B., *Methods of modern mathematical physics. I. Functional Analysis.* Academic Press, 1972.

[11] Reed M. and Simon B., *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press, 1978.

[12] Rozenbljum G.V., *Distribution of the discrete spectrum of singular differential operators.* Soviet Math. Dokl. **13** (1972), 245–249.

[13] Rozenbljum G.V., *Distribution of the discrete spectrum of singular differential operators.* Soviet Math. (Iz. VUZ) **20** (1976), 63–71.

[14] Solomyak M., *Piecewise-polynomial approximation of functions from* $H^l((0,1)^d)$, $2l = d$, *and applications to the spectral theory of the Schrödinger operator*. Israel J. Math. **86** (1994), 253–275.

[15] Solomyak M., *Spectral problems related to the critical exponent in the Sobolev embedding theorem.* Proc. London Math. Soc. **71** (1995), 53–75.

[16] Stein E.M., *Harmonic analysis : real-variable methods, orthogonality, and oscillatory integrals.* Princeton Mathematical Series 43, Princeton University Press, 1993.

[17] Tachizawa K., *On the moments of the negative eigenvalues of elliptic operators.* to appear in J. Fourier Analysis and Applications.

[18] Weidl T., *On the Lieb-Thirring constants* $L_{\gamma,1}$ *for* $\gamma \geq 1/2$. Comm. Math. Phys. **178** (1996), 135–146.