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Gauss-Manin System and the Virtual Structure Constants

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Abstract

In this paper, we discuss some applications of Givental's differential equations to enumerative problems on rational curves in projective hypersurfaces. Using this method, we prove some of the conjectures on the structure constants of quantum cohomology of projective hypersurfaces, proposed in our previous article. Moreover, we clarify the correspondence between the virtual structure constants and Givental's differential equations when the projective hypersurface is Calabi-Yau or general type.

1 Introduction

The main ingredient of this paper is the well-known ordinary differential equation:

$$\Big((\partial_x)^{N-1} - k\cdot e^x\cdot(k\partial_x+k-1)(k\partial_x+k-2)\cdots(k\partial_x+1)\Big)w(x)=0, \tag{1.1}$$

with arbitrary N and k. First, we derive the solutions of (1.1) that can be expressed as asymptotic expansion around $x=-\infty$. To this aim, it is convenient to introduce the following rational function in z:

$$\phi_d^{N,k}(z) := \frac{(kd)!}{(d!)^N}\prod_{j=1}^{kd}(1+\frac{k}{j}z)\prod_{j=1}^{d}(1+\frac{1}{j}z)^{-N}. \tag{1.2}$$

Then we introduce the generating function of $\phi_d^{N,k}(z)$:

$$\phi^{N,k}(x,z) := \sum_{d=0}^{\infty}\exp(dx)\frac{(kd)!}{(d!)^N}\prod_{j=1}^{kd}(1+\frac{k}{j}z)\prod_{j=1}^{d}(1+\frac{1}{j}z)^{-N}. \tag{1.3}$$

Next, we introduce the power series in $\exp(x)$,

$$w_j^{N,k}(x) := (\partial_z)^j\phi^{N,k}(x,z)|_{x=0}. \tag{1.4}$$

If we multiply $\phi^{N,k}(x,z)$ by $\exp(zx)$,

$$\Phi^{N,k}(x,z) := \exp(zx)\phi^{N,k}(x,z), \tag{1.5}$$

$\Phi^{N,k}(x,z)$ satisfies,

$$\Big((\partial_x)^{N-1} - k\cdot e^x \cdot (k\partial_x + k - 1)(k\partial_x + k - 2)\cdots(k\partial_x + 1)\Big)\Phi^{N,k}(x,z) = z^{N-1}\cdot \exp(zx). \quad (1.6)$$

Thus, we obtain the following $N-1$ solution of (1.1):

$$u_j^{N,k}(x) := \frac{1}{j!}(\partial_z)^j(\Phi^{N,k}(x,z))|_{z=0}, \quad (j = 0, 1, \cdots, N-2). \quad (1.7)$$

When $N-k \geq 2$, these solutions are the generating functions of a certain type of two-point correlation functions of topological sigma model on M_N^k : the degree k hypersurface in $\mathbf{P}^{N-1}$. In [4], Bertram and Kley showed that all the rational correlation functions (inserted operators are restricted to Kähler sub-ring) are reconstructed from these solutions in the $N-k \geq 2$ case. Up to now, we also know how to modify $u_j^{N,k}(x)$ to construct the corresponding generating function when $N-k=1$.

In this paper, we take a different path to reconstruct small quantum cohomology rings (Kähler sub-rings) from (1.1). Our idea is very simple. We just look at the differential equation instead of looking at the solution. We will first show that the Gauss-Manin system associated with the quantum Kähler sub-ring of M_N^k have the same informations as the ones of (1.1) if $N-k \geq 2$. Precisely speaking, if we assume the topological selection rule, we can determine all the structure constants of the quantum Kähler sub-ring of M_N^k from (1.1) via the Gauss-Manin system. Conversely, we can derive (1.1) by usual reduction of the Gauss-Manin system associated with the quantum Kähler sub-ring [6]. We can extend our discussion to the $N-k \leq 0$ case. In this case, direct relation between (1.1) and the Gauss-Manin system associated with quantum Kähler sub-ring of M_N^k is lost. But we can still construct a kind of Gauss-Manin system which is directly connected to (1.1). In the $N=k$ case, this Gauss-Manin system is nothing but the B-model used in the mirror computation.

On the other hand, we conjectured the recursive formulas that evaluate the structure constants of the quantum Kähler sub-ring of M_N^k in terms of the ones of M_{N+1}^k when $N-k \geq 2$ [5], [11]. These recursive formula is strong enough to determine all the structure constants of M_N^k in this region. Then we find that the above reconstruction process via the Gauss-Manin system is useful enough to give a proof of the recursive formulas. The proof of them is one of the main results of this paper. In [5] and [9], we also conjectured that the virtual structure constants, that are obtained from iterated use of these recursive formulas into the $N-k \leq 0$ region, can be regarded as analogue of the B-model in the mirror computation. We then constructed the generalized mirror transformation, that evaluate the structure constants of the quantum Kähler sub-ring of M_N^k $(N-k \leq 0)$ from the virtual structure constants, up to some lower degrees of rational curves [9], [8].

Another result of this paper is the assertion that the virtual structure constants are nothing but the structure constants of the Gauss-Manin system associated with (1.1). Thus, we find a stronger evidence of the existence of the analogue of the mirror theorem for the general type projective hypersurface.

This paper is organized as follows. In Section 2, we first introduce our notation for the quantum Kähler sub-ring of projective hypersurfaces. Next, we overview the conjectures proposed in our previous article, some of which are proved in this paper. In Section 3, we first introduce the Gauss-Manin system associated with the quantum Kähler sub-ring of M_N^k $(N-k \geq 2)$ and explain how all the structure constants of the quantum Kähler sub-ring are reconstructed from the Gauss-Manin system and (1.1). Next, we prove the recursive formulas for the structure constants of the quantum Kähler sub-ring of M_N^k $(N-k \geq 2)$, introduced in the previous section. Lastly, we extend our discussion in the $(N-k \geq 2)$ case to the cases $N-k=1$, $N-k=0$ and $N-k<0$ and show that the virtual structure constants introduced in Section 2 is nothing but the structure constants of the Gauss-Manin system associated with (1.1).

2 Quantum Kähler Sub-ring of Projective Hypersurfaces

2.1 Notation

In this section, we introduce the quantum Kähler sub-ring of the quantum cohomology ring of a degree k hypersurface in $\mathbf{P}^{N-1}$. Let M_N^k be a hypersurface of degree k in $\mathbf{P}^{N-1}$. We denote by $QH_e^*(M_N^k)$ the sub-ring of the quantum cohomology ring $QH^*(M_N^k)$ generated by $\mathcal{O}_e$ induced from the Kähler form e (or, equivalently the intersection $H\cap M_N^k$ between a hyperplane class H of $\mathbf{P}^{N-1}$ and M_N^k). Additive basis of $QH_e^*(M_N^k)$ is given by $\mathcal{O}_{e^j}$ $(j=0,1,\cdots,N-2)$, which is induced from $e^j\in H^{j,j}(M_N^k)$. The multiplication rule of $QH_e^*(M_N^k)$ is determined by the Gromov-Witten invariant of genus 0 $\langle\mathcal{O}_e\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)d}}\rangle_{d,M_N^k}$ and it is given as follows:

$$
\begin{aligned}
L_m^{N,k,d} &:= \frac{1}{k}\langle\mathcal{O}_e\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)d}}\rangle_d,\\
\mathcal{O}_e\cdot 1 &= \mathcal{O}_e,\\
\mathcal{O}_e\cdot\mathcal{O}_{e^{N-2-m}} &= \mathcal{O}_{e^{N-1-m}}+\sum_{d=1}^{\infty}L_m^{N,k,d}q^d\mathcal{O}_{e^{N-1-m+(k-N)d}},\\
q &:= \exp(t),
\end{aligned}
\tag{2.8}
$$

where the subscript d counts the degree of the rational curves measured by e. Therefore, $q=\exp(t)$ is the degree counting parameter.

Definition 1 *We call $L_n^{N,k,d}$ the structure constant of weighted degree d.*

Since M_N^k is a complex $(N-2)$ dimensional manifold, we see that a structure constant $L_m^{N,k,d}$ is non-zero only if the following condition is satisfied:

$$
\begin{aligned}
&1\leq N-2-m\leq N-2, 1\leq m-1+(N-k)d\leq N-2,\\
\Longleftrightarrow\ &\text{max.}\{0,2-(N-k)d\}\leq m\leq\text{min.}\{N-3,N-1-(N-k)d\}.
\end{aligned}
\tag{2.9}
$$

We can rewrite (2.9) into the form:

$$
\begin{aligned}
(N-k\geq 2) &\Longrightarrow 0\leq m\leq(N-1)-(N-k)d\\
(N-k=1,d=1) &\Longrightarrow 1\leq m\leq N-3\\
(N-k=1,d\geq 2) &\Longrightarrow 0\leq m\leq N-1-(N-k)d\\
(N-k\leq 0) &\Longrightarrow 2+(k-N)d\leq m\leq N-3.
\end{aligned}
\tag{2.10}
$$

From (2.10), we easily see that the number of the non-zero structure constants $L_m^{N,k,d}$ is finite except for the case of $N=k$. Moreover, if $N\geq 2k$, the non-zero structure constants come only from the $d=1$ part and the non-vanishing $L_m^{N,k,1}$ is determined by k and independent of N. The $N\geq 2k$ region is studied by Beauville [2], and his result plays the role of an initial condition of our discussion later. Explicitly, they are given by the formula :

$$
\sum_{n=0}^{k-1}L_n^{N,k,1}w^n=k\prod_{j=1}^{k-1}(jw+(k-j)),
\tag{2.11}
$$

and the other $L_n^{N,k,d}$'s all vanishes. In the $N=k$ case, the multiplication rule of $QH_e^*(M_k^k)$ is given as follows:

$$
\begin{aligned}
\mathcal{O}_e\cdot 1 &= \mathcal{O}_e,\\
\mathcal{O}_e\cdot\mathcal{O}_{e^{k-2-m}} &= (1+\sum_{d=1}^{\infty}q^dL_m^{k,k,d})\mathcal{O}_{e^{k-1-m}}\ \ (m=2,3,\cdots,k-3),\\
\mathcal{O}_e\cdot\mathcal{O}_{e^{k-3}} &= \mathcal{O}_{e^{k-2}}.
\end{aligned}
\tag{2.12}
$$

Hence it is useful to introduce the generating function of the structure constants of the Calabi-Yau hypersurface M_k^k:

$$L_m^{k,k}(e^t) := 1 + \sum_{d=1}^{\infty} L_m^{k,k,d} e^{dt} \ \ (m = 2, \cdots, k-3). \tag{2.13}$$

2.2 Overview of the Results for Fano and Calabi-Yau Hypersurfaces and Introduction of the Virtual Structure Constants

Let us summarize the conjectures proposed in [5], [11], some of which will be proved in this paper. In [5], we conjectured that the structure constants $L_m^{N,k,d}$ of $QH_e^*(M_N^k)$ for $(N-k \geq 2)$ can be obtained by applying the recursive formulas which describe $L_m^{N,k,d}$ in terms of $L_{m'}^{N+1,k,d'}$ $(d' \leq d)$, with the initial conditions of $L_m^{N,k,1}$ given by (2.11) and $L_m^{N,k,d} = 0$ $(d \geq 2)$ in the $N \geq 2k$ region. Let us introduce the construction of the recursive formulas given in [11]. First, we introduce the polynomial $Poly_d$ in $x, y, z_1, z_2, \cdots, z_{d-1}$ defined by the formula:

$$\begin{aligned}
&Poly_d(x, y, z_1, z_2, \cdots, z_{d-1}) \\
&= \frac{1}{(2\pi\sqrt{-1})^{d-1}} \int_{C_1} \frac{dt_1}{t_1} \cdots \int_{C_{d-1}} \frac{dt_{d-1}}{t_{d-1}} \prod_{j=1}^{d-1} \Big(\frac{(d-j)x + jy}{d} + \sum_{i=1}^{j} \frac{d-j}{d-i} t_i + \sum_{i=j+1}^{d-1} \frac{j}{i} t_i + \\
&z_j \big(\frac{(d-j)x + jy}{d} + \sum_{i=1}^{j} \frac{d-j}{d-i} t_i + \sum_{i=j+1}^{d-1} \frac{j}{i} t_i \big) / \big(\frac{(d-j)x + jy}{d} + \sum_{i=1}^{j} \frac{d-j}{d-i} t_i + \sum_{i=j+1}^{d-1} \frac{j}{i} t_i - z_j \big) \Big) \\
&= \frac{d}{(2\pi\sqrt{-1})^{d-1}} \int_{D_1} du_1 \cdots \int_{D_{d-1}} du_{d-1} \prod_{j=1}^{d-1} \Big(\big(\frac{1}{2u_j - u_{j-1} - u_{j+1}} \big) \cdot \big(u_j + z_j \frac{u_j}{u_j - z_j} \big) \Big) \\
&= \frac{d}{(2\pi\sqrt{-1})^{d-1}} \int_{D_1} du_1 \cdots \int_{D_{d-1}} du_{d-1} \prod_{j=1}^{d-1} \Big(\frac{(u_j)^2}{(2u_j - u_{j-1} - u_{j+1})(u_j - z_j)} \Big),
\end{aligned} \tag{2.14}$$

where we denote x (resp. y) by u_0 (resp. u_d) in the last two lines. In (2.14), the path D_i goes around both poles $u_i = \frac{u_{i-1}+u_{i+1}}{2}, u_i = z_i$. Next, let us consider the monomial $x^{d_{i_0}} z_{i_1}^{d_{i_1}} \cdots z_{i_{l-1}}^{d_{i_{l-1}}} y^{d_{i_l}}$ $(\sum_{j=0}^{l} d_{i_j} = d-1)$, that appear in $Poly_d$, associated with the following ordered partition of a positive integer d [3]:

$$0 = i_0 < i_1 < i_2 < \cdots < i_{l-1} < i_l = d. \tag{2.15}$$

Then we prepare some elements in (a free Abelian group) $\mathbf{Z}^l$, which are determined for each monomial $x^{d_{i_0}} z_{i_1}^{d_{i_1}} \cdots z_{i_{l-1}}^{d_{i_{l-1}}} y^{d_{i_l}}$, as follows:

$$\begin{aligned}
\alpha &:= (l-d, l-d, \cdots, l-d), \\
\beta &:= (0, i_1 - 1, i_2 - 2, \cdots, i_{l-1} - l + 1), \\
\gamma &:= (0, i_1(N-k), i_2(N-k), \cdots, i_{l-1}(N-k)), \\
\epsilon_1 &:= (1, 0, 0, 0, \cdots, 0), \\
\epsilon_2 &:= (1, 1, 0, 0, \cdots, 0), \\
\epsilon_3 &:= (1, 1, 1, 0, \cdots, 0), \\
&\cdots \\
\epsilon_l &:= (1, 1, 1, 1, \cdots, 1).
\end{aligned} \tag{2.16}$$

Now we define $\delta = (\delta_1, \cdots, \delta_l) \in \mathbf{Z}^l$ by the formula:

$$\delta := \alpha + \beta + \gamma + \sum_{j=1}^{l-1} (d_{i_j} - 1)\epsilon_j + d_{i_l}\epsilon_l. \tag{2.17}$$

With these set-up, we state the following theorem:

Theorem 1 *The recursive formulas are given as follows:*

$$L_n^{N,k,d} = \phi(Poly_d), \tag{2.18}$$

where ϕ is a $\mathbf{Q}$-linear map from the $\mathbf{Q}$-vector space of the homogeneous polynomials of degree $d-1$ in $x, y, z_1, \cdots, z_{d-1}$ to the $\mathbf{Q}$-vector space of the weighted homogeneous polynomials of degree d in $L_m^{N+1,k,d'}$. And it is defined on the basis by:

$$\phi(x^{d_0} y^{d_d} z_{i_1}^{d_{i_1}} \cdots z_{i_{l-1}}^{d_{i_{l-1}}}) = \prod_{j=1}^{l} L_{n+\delta_j}^{N+1,k,i_j - i_{j-1}}. \tag{2.19}$$

The proof will be given in the next section.

The structure constant $L_m^{k,k,d}$ for a Calabi-Yau hypersurface does not obey the recursive formulas (2.18). Instead, we introduce here the virtual structure constants $\tilde{L}_m^{N,k,d}$ as follows.

Definition 2 *Let $\tilde{L}_m^{N,k,d}$ be the rational number obtained by applying the recursion formulas (2.18) for arbitrary N and k with the initial condition $L_n^{N,k,1}$ $(N \geq 2k)$ and $L_n^{N,k,d} = 0$ $(d \geq 2,\ N \geq 2k)$.*

Remark 1 *In the $N - k \geq 2$ region, $\tilde{L}_m^{N,k,d} = L_m^{N,k,d}$.*

Remark 2 *$\tilde{L}_m^{N,k,d}$ is non-zero if $0 \leq m \leq N - 1 - (N-k)d$, and we have infinite number of $\tilde{L}_m^{N,k,d}$'s when $N - k \leq 0$.*

Definition 3 *We call $\tilde{L}_n^{N,k,d}$ the virtual structure constant of weighted degree d.*

We define the generating function of the virtual structure constants of the Calabi-Yau hypersurface M_k^k as follows:

$$\begin{aligned} \tilde{L}_n^{k,k}(e^x) &:= 1 + \sum_{d=1}^{\infty} \tilde{L}_n^{k,k,d} e^{dx}, \\ & (n = 0, 1, \cdots, k-1). \end{aligned} \tag{2.20}$$

In [5], we observed that $\tilde{L}_n^{k,k}(e^x)$ gives us the information of the B-model of the mirror manifold of M_k^k. In this paper, we prove the theorem:

Theorem 2

$$\begin{aligned} &\tilde{L}_0^{k,k}(e^x) = w_0^{k,k}(x) = \sum_{d=0}^{\infty} \frac{(kd)!}{(d!)^k} e^{dx}, \\ &\tilde{L}_1^{k,k}(e^x) = \partial_x\left(x + \frac{w_1^{k,k}(x)}{w_0^{k,k}(x)}\right) = \partial_x\left(x + \left(\sum_{d=1}^{\infty} \frac{(kd)!}{(d!)^k} \cdot \left(\sum_{i=1}^{d}\sum_{m=1}^{k-1} \frac{m}{i(ki-m)}\right) e^{dx}\right) / \left(\sum_{d=0}^{\infty} \frac{(kd)!}{(d!)^k} e^{dx}\right)\right), \end{aligned} \tag{2.21}$$

where $w_j^{k,k}(x)$ is the function introduced in (1.4).

Of course, we can extend the theorem (2.21) to the general $\tilde{L}_n^{k,k}(e^x)$ if we compare the $\tilde{L}_n^{k,k}(e^x)$ with the B-model three point functions in [7]. In particular, this theorem asserts that we can obtain the mirror map $t = t(x)$ used in the mirror computation without assuming the mirror conjecture.

$$t(x) = x + \int_{-\infty}^{x} dx' (\tilde{L}_1^{k,k}(e^{x'}) - 1) = x + \sum_{d=1}^{\infty} \frac{\tilde{L}_1^{k,k,d}}{d} e^{dx}. \tag{2.22}$$

With the above theorem, we can construct the mirror transformation that transforms the virtual structure constants of the Calabi-Yau hypersurface into the real ones as follows:

$$L_{m}^{k,k}(e^{t}) = \frac{\tilde{L}_{m}^{k,k}(e^{x(t)})}{\tilde{L}_{1}^{k,k}(e^{x(t)})}. \quad (m = 2, \cdots, k-3) \tag{2.23}$$

After some combinatorial computation, we can rewrite (2.23) into the following form:

$$L_{n}^{k,k,d} = \sum_{m=0}^{d-1} \text{Res}_{z=0}(z^{-m-1}\exp(-d\sum_{j=1}^{\infty}\frac{\tilde{L}_{1}^{k,k,j}}{j}z^{j}))\cdot(\tilde{L}_{n}^{k,k,d-m} - \tilde{L}_{1}^{k,k,d-m}). \tag{2.24}$$

In [9], we argued that this formula must have deep connection with toric compactification of the moduli space of rational curves in $\mathbf{P}^{N-1}$. With this idea, we speculated that we can generalize the formula (2.24) to the $N-k<0$ case. In [9] and [8], we gave some numerical evidence of this generalization up to some lower degree of rational curves.

3 Gauss-Manin System

Let us first introduce the Gauss-Manin system associated with the quantum Kähler sub-ring of M_{N}^{k}:

Definition 4 *We call the following rank 1 ODE for vector valued function $\psi_{m}(t)$, $(m = 0, 1, \cdots, N-2)$:*

$$\begin{aligned} \partial_{t}\psi_{N-2-m}(t) &= \psi_{N-1-m}(t) + \sum_{d=1}^{\infty}\exp(dt)\cdot L_{m}^{N,k,d}\cdot\psi_{N-1-m-(N-k)d}(t), \\ \partial_{t}\psi_{N-2}(t) &= \sum_{d=1}^{\infty}\exp(dt)\cdot L_{0}^{N,k,d}\cdot\psi_{N-1-(N-k)d}(t), \end{aligned} \tag{3.25}$$

the Gauss-Manin system associated with the quantum Kähler sub-ring of M_{N}^{k}.

This definition can be applied to any M_{N}^{k}.

3.1 Fano case ($N-k\geq 2$)

In this case, we already have the celebrated theorem of Givental [6]:

Theorem 3 (Givental)
If $N-k\geq 2$, (3.25) can be reduced to the rank $N-1$ ODE for $\psi_{0}(t)$:

$$\Big((\partial_{t})^{N-1} - k\cdot e^{t}\cdot(k\partial_{t}+k-1)\cdots(k\partial_{t}+2)\cdot(k\partial_{t}+1)\Big)\psi_{0}(t) = 0. \tag{3.26}$$

Conversely, we can determine all the structure constants of the quantum Kähler sub-ring explicitly using (3.25) and (3.26) as the starting point.

Corollary 1 *The structure constants $L_{n}^{N,k,d}$ are fully reconstructed from (3.26). In particular, we have,*

$$\sum_{n=0}^{k-1}\tilde{L}_{n}^{N,k,1}w^{n} = k\cdot\prod_{j=1}^{k-1}(jw+(k-j)). \tag{3.27}$$

proof)
Using some algebra, we can represent $\psi_{N-1-m}(t)$ in terms of $\psi_0(t)$ as the form:

$$\psi_{N-1-m}(t) = (\partial_t)^{N-1-m}\psi_0(t) - \sum_{d=1}^{\infty}\exp(dt)\sum_{j=0}^{N-1-m-(N-k)d}\gamma_{m,j}^{N,k,d}(\partial_t)^{N-1-m-(N-k)d-j}\psi_0(t), \tag{3.28}$$

when $1 \leq m \leq N-1$. Moreover, we can obtain the ODE for $\psi_0(t)$ by introducing $\psi_{N-1}(t)$ formally, which satisfies

$$\partial_t\psi_{N-2}(t) = \psi_{N-1}(t) + \sum_{d=1}^{\infty}\exp(dt)\cdot L_0^{N,k,d}\cdot\psi_{N-1-(N-k)d}(t). \tag{3.29}$$

If we represent $\psi_{N-1}(t)$ as the form of (3.28), the ODE is just given by the equation:

$$\psi_{N-1}(t) = 0. \tag{3.30}$$

Substitution of (3.28) into (3.25) leads us to the recursive formula for $\gamma_{m,j}^{N,k,d}$,

$$\gamma_{m,0}^{N,k,d} - \gamma_{m+1,0}^{N,k,d} = L_m^{N,k,d} - \sum_{f+g=d} L_m^{N,k,f}\gamma_{m+(N-k)f,0}^{N,k,g}, \tag{3.31}$$

$$\gamma_{m,j}^{N,k,d} - \gamma_{m+1,j}^{N,k,d} = d\cdot\gamma_{m+1,j-1}^{N,k,d} - \sum_{f+g=d} L_m^{N,k,f}\gamma_{m+(N-k)f,j}^{N,k,g}. \tag{3.32}$$

Here, we introduce the generating function,

$$\gamma_m^{N,k,d}(w) := \sum_{j=0}^{N-1-(N-k)d}\gamma_{m,j}^{N,k,d}w^j. \tag{3.33}$$

Then, the recursive formulas (3.31) and (3.32) are reduced to one recursive formula for $\gamma_m^{N,k,d}(w)$,

$$\gamma_m^{N,k,d}(w) = (1+dw)\gamma_{m+1}^{N,k,d}(w) + L_m^{N,k,d} - \sum_{f+g=d} L_m^{N,k,f}\gamma_{m+(N-k)f}^{N,k,g}(w). \tag{3.34}$$

Multiplying (3.34) by $(1+dw)^m$ makes the recursive formula more tractable.

$$(1+dw)^m\gamma_m^{N,k,d}(w) - (1+dw)^{m+1}\gamma_{m+1}^{N,k,d}(w) = (1+dw)^m L_m^{N,k,d} - \sum_{f+g=d}(1+dw)^m L_m^{N,k,f}\gamma_{m+(N-k)f}^{N,k,g}(w). \tag{3.35}$$

We can easily solve (3.35) inductively. The answer is given by the formula:

$$\gamma_m^{N,k,d}(w) =$$
$$\sum_{l=1}^{d}(-1)^{l-1}\sum_{(d_1,\cdots,d_l)\in OP_d}\sum_{j_l=m}^{N-1-(N-k)d}\cdots\sum_{j_2=m}^{j_3}\sum_{j_1=m}^{j_2}\prod_{i=1}^{l}\Bigl((1+(\sum_{n=i}^{l}d_n)w)^{j_i-j_{i-1}}\cdot L_{j_i+(N-k)(\sum_{n=1}^{i-1}d_n)}^{N,k,d_i}\Bigr), \tag{3.36}$$

where we formally identify j_0 with m and denote by OP_d the set of the ordered partitions of d; $\{(d_1,d_2,\cdots,d_l)|\ d_j \geq 1,\ \sum_{j=1}^{l}d_j = d\}$. At this point, we look back at Theorem 3. It merely says that

$$\begin{aligned}\gamma_0^{N,k,1}(w) &= k\cdot\prod_{j=1}^{k-1}(k+jw),\\ \gamma_0^{N,k,d}(w) &= 0, \quad (d\geq 2).\end{aligned} \tag{3.37}$$

Hence we obtain from (3.36),

$$\sum_{m=0}^{k-1} L_m^{N,k,1}(1+w)^m = k\cdot\prod_{j=1}^{k-1}(k+jw), \tag{3.38}$$

and,

$$\sum_{m=0}^{N-1-(N-k)d} L_m^{N,k,d}(1+dw)^m =$$

$$\sum_{l=2}^{d}(-1)^l \sum_{(d_1,\cdots,d_l)\in OP_d} \sum_{j_l=0}^{N-1-(N-k)d}\cdots\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2}\prod_{i=1}^{l}\Bigl((1+(\sum_{n=i}^{l}d_n)w)^{j_i-j_{i-1}}\cdot L^{N,k,d_i}_{j_i+(N-k)(\sum_{n=1}^{i-1}d_n)}\Bigr). \tag{3.39}$$

Substitution of $\frac{z-1}{d}$ into w leads us to the formulas:

$$\sum_{m=0}^{k-1} L_m^{N,k,1}z^m = k\cdot\prod_{j=1}^{k-1}((k-j)+jz), \tag{3.40}$$

$$\sum_{m=0}^{N-1-(N-k)d} L_m^{N,k,d}z^m =$$

$$\sum_{l=2}^{d}(-1)^l \sum_{(d_1,\cdots,d_l)\in OP_d} \sum_{j_l=0}^{N-1-(N-k)d}\cdots\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2}\prod_{i=1}^{l}\Bigl((1+(\sum_{n=i}^{l}d_n)(\frac{z-1}{d}))^{j_i-j_{i-1}}\cdot L^{N,k,d_i}_{j_i+(N-k)(\sum_{n=1}^{i-1}d_n)}\Bigr). \tag{3.41}$$

It is obvious that we can completely determine all the $L_n^{N,k,d}$'s by induction of d, because the r.h.s. includes only the $L_m^{N,k,d'}$'s with $d'<d$. Q.E.D.

Example

M_7^5 model

The corresponding Gauss-Manin system is given by,

$$\begin{aligned}
\partial_t\psi_0(t) &= \psi_1(t),\\
\partial_t\psi_1(t) &= \psi_2(t)+a\cdot e^t\cdot\psi_0(t),\\
\partial_t\psi_2(t) &= \psi_3(t)+b\cdot e^t\cdot\psi_1(t),\\
\partial_t\psi_3(t) &= \psi_4(t)+c\cdot e^t\cdot\psi_2(t)+d\cdot e^{2t}\cdot\psi_0(t),\\
\partial_t\psi_4(t) &= \psi_5(t)+b\cdot e^t\cdot\psi_3(t)+g\cdot e^{2t}\cdot\psi_1(t),\\
\partial_t\psi_5(t) &= a\cdot e^t\cdot\psi_4(t)+d\cdot e^{2t}\cdot\psi_2(t)+f\cdot e^{3t}\cdot\psi_0(t),
\end{aligned} \tag{3.42}$$

where we used a trivial equality $L_m^{N,k,d}=L^{N,k,d}_{N-1-(N-k)d-m}$. We reduce (3.42) into an ordinary equation for $\psi_0(t)$ and obtain,

$$\begin{aligned}
&\Bigl((\partial_t)^6 - e^t\cdot((2a+2b+c)(\partial_t)^4+(4a+4b+2c)(\partial_t)^3+(6a+3b+c)(\partial_t)^2+(4a+b)(\partial_t)+a)\\
&+e^{2t}(a^2+b^2+2ab+2ac-2d-g)(\partial_t)^2+e^{2t}(2a^2+2b^2+4ab+4ac-4d-2g)(\partial_t)\\
&+e^{2t}(a^2+2ab+4ac-4d)+e^{3t}(2ad-a^2c-f)\Bigr)\psi_0(t)=0.
\end{aligned} \tag{3.43}$$

Then Theorem 1 asserts that the equation (3.43) equals the equation:

$$\Big((\partial_t)^6 - e^t\cdot(3125(\partial_t)^4 + 6250(\partial_t)^3 + 4375(\partial_t)^2 + 1250(\partial_t) + 120)\Big)\psi_0(t) = 0. \qquad (3.44)$$

By comparing (3.43) with (3.44), we obtain,

$$a = 120,\ b = 770,\ c = 1345,\ d = 211200,\ g = 692500,\ f = 31320000, \qquad (3.45)$$

which agree with our previous results in [10].

This corollary enables us to prove Theorem 1. As the first step, we prove the following theorem:

Theorem 4 *Let φ be the recursive formula in (2.18) considered as a homomorphism from the polynomial ring of $L_n^{N,k,d}$ to the one of $L_n^{N+1,k,d}$. Then we have,*

$$\varphi(\gamma_0^{N,k,d}(w)) = \Big(\prod_{j=1}^{d-1}(1+jw)\Big)\cdot\gamma_0^{N+1,k,d}(w). \qquad (3.46)$$

proof) From now on, we use another notation $0 = i_0 < i_1 < i_2 < \cdots < i_{l-1} < i_l = d$ of ordered partition. Correspondence to the previous notation $(d_1,\cdots,d_l)\in OP_d$ is given by $d_j = i_j - i_{j-1}$. We denote by $f^{d_{i_1}d_{i_2}\cdots d_{i_{l-1}}}_{(i_1-i_0,\cdots,i_l-i_{l-1})}(x,y)$ the coefficient polynomial of $z_{i_1}^{d_{i_1}}z_{i_2}^{d_{i_2}}\cdots z_{i_{l-1}}^{d_{i_{l-1}}}$ in the generating polynomial $Poly_d$. Using (2.14), it is explicitly given as follows:

$$\begin{aligned}
f^{d_{i_1}d_{i_2}\cdots d_{i_{l-1}}}_{(i_1-i_0,\cdots,i_l-i_{l-1})}(x,y) &= \sum_{j=0}^{d-1-\sum_{m=1}^{l-1}d_{i_m}} a_{d_{i_1}d_{i_2}\cdots d_{i_{l-1}}j}(i_1-i_0,\cdots,i_l-i_{l-1})x^j y^{d-1-\sum_{m=1}^{l-1}d_{i_m}-j}\\
&:= \frac{d}{(2\pi\sqrt{-1})^{d-1}}\int_{D_1}du_1\cdots\int_{D_{d-1}}du_{d-1}\prod_{m=1}^{d-1}\frac{u_m}{(2u_m-u_{m-1}-u_{m+1})}\cdot\prod_{n=1}^{l-1}\frac{1}{u_{i_n}^{d_{i_n}}}\\
&= \frac{d}{(2\pi\sqrt{-1})^{l-1}}\int_{D_{i_1}}du_{i_1}\cdots\int_{D_{i_{l-1}}}du_{i_{l-1}}\prod_{j=1}^{l}f_{(i_j-i_{j-1})}(u_{i_{j-1}},u_{i_j})\times\\
&\times\prod_{j=1}^{l-1}\frac{1}{((i_{j+1}-i_{j-1})u_{i_j}-(i_j-i_{j-1})u_{i_{j+1}}-(i_{j+1}-i_j)u_{i_{j-1}})\cdot u_{i_j}^{d_{i_j}-1}}\prod_{j=1}^{l-2}(i_{j+1}-i_j).
\end{aligned} \qquad (3.47)$$

where we have introduced the following polynomial:

$$\sum_{j=0}^{d-1}a_j(d)x^jy^{d-1-j} := \prod_{j=1}^{d-1}(\frac{jx+(d-j)y}{d}) = f_{(d)}(x,y). \qquad (3.48)$$

The path D_j in the second line of (3.47) goes around $u_j = \frac{u_{j-1}+u_{j+1}}{2}, u_j = 0$ if $j\in\{i_1,\cdots,i_{l-1}\}$ and $u_j = \frac{u_{j-1}+u_{j+1}}{2}$ otherwise. The last equality in (3.47) is obtained from integrating out the variable u_j $(j\in\{1,2,\cdots,d-1\}\setminus\{i_1,\cdots,i_{l-1}\})$.

With this definition, we can write down the form of the recursive formula as follows,

$$\begin{aligned}
L_n^{N,k,d} &= \sum_{l=1}^{d}\sum_{0=i_0<\cdots<i_l=d}\sum_{(d_{i_1},\cdots,d_{i_{l-1}})}\sum_{j=0}^{d-1-\sum_{m=1}^{l-1}d_{i_m}} a_{d_{i_1}d_{i_2}\cdots d_{i_{l-1}}j}(i_1-i_0,\cdots,i_l-i_{l-1})\times\\
&\times\prod_{h=1}^{l}L^{N+1,k,i_h-i_{h-1}}_{n-\sum_{m=1}^{h-1}d_{i_m}-j+i_{h-1}(N-k+1)}.
\end{aligned} \qquad (3.49)$$

On the other hand, we can rewrite $\gamma_0^{N,k,d}(w)$ using the notation $0=i_0<\cdots<i_l=d$ into,

$$\sum_{l=1}^{d}(-1)^{l-1}\sum_{0=i_0<\cdots<i_l=d}\sum_{j_l=0}^{N-1-(N-k)d}\sum_{j_{l-1}=0}^{j_l}\cdots\sum_{j_1=0}^{j_2}\prod_{n=1}^{l}\Bigl((1+(d-i_{n-1})w)^{j_n-j_{n-1}}L^{N,k,i_n-i_{n-1}}_{j_n+(N-k)i_{n-1}}\Bigr). \tag{3.50}$$

Substituting (3.49) into (3.50), we obtain,

$$\begin{aligned}
&\varphi(\gamma_0^{N,k,d}(w))=\\
&\sum_{l=1}^{d}(-1)^{l-1}\sum_{l=1}^{d}\sum_{0=i_0<\cdots<i_l=d}\sum_{j_l=0}^{N-1-(N-k)d}\sum_{j_{l-1}=0}^{j_l}\cdots\sum_{j_1=0}^{j_2}\\
&\prod_{n=1}^{l}\Bigl((1+(d-i_{n-1})w)^{j_n-j_{n-1}}\sum_{l_n=1}^{i_n-i_{n-1}}\sum_{i_{n-1}=i_0^n<\cdots<i_{l_n}^n=i_n}\sum_{(d_{i_1^n},\cdots,d_{i_{l_n-1}^n})}\sum_{c_n=0}^{i_n-i_{n-1}-1-\sum_{m_n=1}^{l_n-1}d_{i_{m_n}^n}}\times\\
&\times a_{d_{i_1^n}\cdots d_{i_{l_n-1}^n}c_n}(i_1^n-i_0^n,\cdots,i_{l_n}^n-i_{l_n-1}^n)\prod_{h_n=1}^{l_n}L^{N+1,k,i_{h_n}^n-i_{h_n-1}^n}_{j_n-c_n-i_{n-1}-\sum_{m_n=1}^{h_n-1}d_{i_{m_n}^n}+i_{h_n-1}^n(N-k+1)}\Bigr).
\end{aligned} \tag{3.51}$$

In the above formula, we can see appearance of iterated ordered partition:

$$0=i_0=i_0^1<i_1^1<\cdots<i_{l_1}^1=i_1=i_0^2<\cdots<i_{l_2}^2=i_2<\cdots<i_{l_l}^l=i_l=d. \tag{3.52}$$

Then we pick up the terms whose iterated ordered partition is equal to the ordered partition $0=i_0<i_1<\cdots<i_l=d$. The result is conveniently written in terms of ordered partition $0=h_0<h_1<\cdots<h_s=l$, and the statement of the theorem is reduced to the following equality:

$$\begin{aligned}
&\sum_{s=1}^{l}(-1)^{s-1}\sum_{0=h_0<\cdots<h_s=l}\sum_{j_s=0}^{N-1-(N-k)d}\sum_{j_{s-1}=0}^{j_s}\cdots\sum_{j_1=0}^{j_2}\sum_{\substack{(d_{i_1},\cdots,d_{i_{l-1}})\\(d_{i_{h_j}}=0)}}\prod_{n=1}^{s}\Bigl((1+(d-i_{h_{n-1}})w)^{j_n-j_{n-1}}\times\\
&\times\sum_{c_n=0}^{i_{h_n}-i_{h_{n-1}}-1-\sum_{j=h_{n-1}}^{h_n}d_{i_j}}a_{d_{i_{h_{n-1}+1}}\cdots d_{i_{h_n-1}}c_n}(i_{h_{n-1}+1}-i_{h_{n-1}},\cdots,i_{h_n}-i_{h_n-1})\times\\
&\times\prod_{a=h_{n-1}+1}^{h_n}L^{N+1,k,i_a-i_{a-1}}_{j_n-c_n-i_{h_{n-1}}-\sum_{j=h_{n-1}+1}^{a-1}d_{i_j}+i_{a-1}(N-k+1)}\Bigr)\\
&=\Bigl(\prod_{j=1}^{d-1}(1+jw)\Bigr)\cdot\Bigl((-1)^{l-1}\sum_{j_l=0}^{N-(N-k+1)d}\sum_{j_{l-1}=0}^{j_l}\cdots\sum_{j_1=0}^{j_2}\prod_{n=1}^{l}(1+(d-i_{n-1})w)^{j_n-j_{n-1}}L^{N,k,i_n-i_{n-1}}_{j_n+(N-k+1)i_{n-1}}\Bigr).
\end{aligned} \tag{3.53}$$

Next, we carefully look at the summand in the l. h. s. of (3.53) coming from the ordered partition $0=h_0<h_1<\cdots<h_s=l$:

$$\begin{aligned}
&(-1)^{s-1}\sum_{(d_{i_1},\cdots,d_{i_{l-1}}),\,(d_{i_{h_j}}=0)}\sum_{c_1=0}^{i_{h_1}-i_{h_0}-1-\sum_{j=h_0}^{h_1}d_{i_j}}\cdots\sum_{c_s=0}^{i_{h_s}-i_{h_{s-1}}-1-\sum_{j=h_{s-1}}^{h_s}d_{i_j}}\\
&\prod_{n=1}^{s}a_{d_{i_{h_{n-1}+1}}\cdots d_{i_{h_n-1}}c_n}(i_{h_{n-1}+1}-i_{h_{n-1}},\cdots,i_{h_n}-i_{h_n-1})\times\sum_{j_s=0}^{N-1-(N-k)d}\sum_{j_{s-1}}^{j_s}\cdots\sum_{j_1=0}^{j_2}
\end{aligned}$$

$$\prod_{n=1}^{s}\left((1+(d-i_{h_{n-1}})w)^{j_n-j_{n-1}}\prod_{a=h_{n-1}+1}^{h_n}L^{N+1,k,i_a-i_{a-1}}_{j_n-c_n-i_{h_{n-1}}-\sum_{j=h_{n-1}+1}^{a-1}d_{i_j}+i_{a-1}(N-k+1)}\right). \tag{3.54}$$

Changing j_n into $j'_n=j_n-c_n-i_{h_{n-1}}+\sum_{j=1}^{h_{n-1}}d_{i_j}$, we can separate (3.54) into a bulk part:

$$(-1)^{s-1}\sum_{(d_{i_1},\cdots,d_{i_{l-1}}),\,(d_{i_{h_j}}=0)}\ \sum_{c_1=0}^{i_{h_1}-i_{h_0}-1-\sum_{j=h_0}^{h_1}d_{i_j}}\cdots\sum_{c_s=0}^{i_{h_s}-i_{h_{s-1}}-1-\sum_{j=h_{s-1}}^{h_s}d_{i_j}}$$
$$\prod_{n=1}^{s}a_{d_{i_{h_{n-1}+1}}\cdots d_{i_{h_n-1}}c_n}(i_{h_{n-1}+1}-i_{h_{n-1}},\cdots,i_{h_n}-i_{h_n-1})\times\sum_{j_s=0}^{N-(N-k+1)d}\ \sum_{j_{s-1}=0}^{j_s}\cdots\sum_{j_1=0}^{j_2}$$
$$\prod_{n=1}^{s}\left((1+(d-i_{h_{n-1}})w)^{j_n-j_{n-1}+c_n-c_{n-1}+i_{h_{n-1}}-i_{h_{n-2}}-\sum_{j=h_{n-2}}^{h_{n-1}}d_{i_j}}\prod_{a=h_{n-1}+1}^{h_n}L^{N+1,k,i_a-i_{a-1}}_{j_n-\sum_{j=1}^{a-1}d_{i_j}+i_{a-1}(N-k+1)}\right), \tag{3.55}$$

and boundary parts:

$$(-1)^{s-1}\sum_{m=1}^{s-1}\ \sum_{(d_{i_1},\cdots,d_{i_{l-1}}),\,(d_{i_{h_j}}=0)}\ \sum_{c_1=0}^{i_{h_1}-i_{h_0}-1-\sum_{j=h_0}^{h_1}d_{i_j}}\cdots\sum_{c_s=0}^{i_{h_s}-i_{h_{s-1}}-1-\sum_{j=h_{s-1}}^{h_s}d_{i_j}}$$
$$\prod_{n=1}^{s}a_{d_{i_{h_{n-1}+1}}\cdots d_{i_{h_n-1}}c_n}(i_{h_{n-1}+1}-i_{h_{n-1}},\cdots,i_{h_n}-i_{h_n-1})\sum_{l_m=1}^{c_{m+1}-c_m+i_{h_m}-i_{h_{m-1}}-\sum_{j=h_{m-1}}^{h_m}d_{i_j}}$$
$$\sum_{j_s=-c_s-i_{h_{s-1}}+l_m+\sum_{j=1}^{h_{s-1}}d_{i_j}}^{N-1-(N-k)d-c_s-i_{h_{s-1}}+l_m+\sum_{j=1}^{h_{s-1}}d_{i_j}}\cdots\sum_{j_{m+2}=-c_{m+2}-i_{h_{m+1}}+l_m+\sum_{j=1}^{h_{m+1}}d_{i_j}}^{j_{m+3}+c_{m+3}-c_{m+2}+i_{h_{m+2}}-i_{h_{m+1}}-\sum_{j=h_{m+1}}^{h_{m+2}}d_{i_j}}$$
$$\sum_{j_m=0}^{j_{m+2}+c_{m+2}-c_{m+1}+i_{h_{m+1}}-i_{h_m}-\sum_{j=h_m}^{h_{m+1}}d_{i_j}}\ \sum_{j_{m-1}=0}^{j_m}\ \sum_{j_{m-2}=0}^{j_{m-1}}\cdots\sum_{j_1=0}^{j_2}$$
$$\left(\prod_{n=1}^{m}(1+(d-i_{h_{n-1}})w)^{j_n-j_{n-1}}\right)\cdot(1+(d-i_{h_{m+1}})w)^{j_{m+2}-j_m}\cdot\left(\prod_{n=m+3}^{s}(1+(d-i_{h_{n-1}})w)^{j_n-j_{n-1}}\right)$$
$$\prod_{n=1}^{m}\left(((1+(d-i_{h_{n-1}})w)^{c_n-c_{n-1}+i_{h_{n-1}}-i_{h_{n-2}}-\sum_{j=h_{n-2}}^{h_{n-1}}d_{i_j}}\prod_{a=h_{n-1}+1}^{h_n}L^{N+1,k,i_a-i_{a-1}}_{j_n-\sum_{j=1}^{a-1}d_{i_j}+i_{a-1}(N-k+1)}\right)$$
$$\left(((1+(d-i_{h_m})w)^{-l_m+c_{m+1}-c_m+i_{h_m}-i_{h_{m-1}}-\sum_{j=h_{m-1}}^{h_m}d_{i_j}}\prod_{a=h_m+1}^{h_{m+1}}L^{N+1,k,i_a-i_{a-1}}_{j_m-l_m-\sum_{j=1}^{a-1}d_{i_j}+i_{a-1}(N-k+1)}\right)$$
$$\prod_{n=m+2}^{s}\left(((1+(d-i_{h_{n-1}})w)^{c_n-c_{n-1}+i_{h_{n-1}}-i_{h_{n-2}}-\sum_{j=h_{n-2}}^{h_{n-1}}d_{i_j}}\prod_{a=h_{n-1}+1}^{h_n}L^{N+1,k,i_a-i_{a-1}}_{j_n-l_m-\sum_{j=1}^{a-1}d_{i_j}+i_{a-1}(N-k+1)}\right). \tag{3.56}$$

To derive (3.56), we further replace j'_n $(n > m)$ by $j''_n = j'_n + l_m$ and omit the dashes in j'_n, j''_n in the final form. Since we can easily see that the identity:

$$\sum_{c_1=0}^{i_1-i_0-1}\cdots\sum_{c_l=0}^{i_l-i_{l-1}-1} a_{c_1}(i_1-i_0)\cdots a_{c_l}(i_l-i_{l-1})\prod_{j=1}^{l}\left((1+(d-i_{j-1})w)^{c_j-c_{j-1}+i_{j-1}-i_{j-2}}\right)=\prod_{j=1}^{d-1}(1+jw), \tag{3.57}$$

holds true, the bulk part coming from the ordered partition $h_j = j$ $(j = 1, 2, \cdots, l-1)$ is nothing but the r.h.s. of (3.53). Therefore, what remains to show is cancellation of the remaining terms. At this stage, we add some comments on boundary parts. Looking at the first boundary part in (3.56) which corresponds to the operation to remove m $(m = 1, 2, \cdots, s-1)$ from the set $\{1, 2, \cdots, s-1\}$, we can further pick up the second boundary part which corresponds to remove n $(n = m+1, m+2, \cdots, s-1)$ from $\{m+1, m+2, \cdots, s-1\}$. Explicitly, the first boundary part separated from the second boundary parts is given by the formula:

$$\begin{aligned}
&(-1)^{s-1}\sum_{(d_{i_1},\cdots,d_{i_{l-1}}),\,(d_{i_{h_j}}=0)}\ \sum_{c_1=0}^{i_{h_1}-i_{h_0}-1-\sum_{j=h_0}^{h_1}d_{i_j}}\cdots\sum_{c_s=0}^{i_{h_s}-i_{h_{s-1}}-1-\sum_{j=h_{s-1}}^{h_s}d_{i_j}}\\
&\prod_{n=1}^{s}a_{d_{i_{h_{n-1}+1}}\cdots d_{i_{h_n-1}}c_n}(i_{h_{n-1}+1}-i_{h_{n-1}},\cdots,i_{h_n}-i_{h_n-1})\sum_{l_m=1}^{c_{m+1}-c_m+i_{h_m}-i_{h_{m-1}}-\sum_{j=h_{m-1}}^{h_m}d_{i_j}}\\
&\sum_{j_{s-1}=0}^{N-(N-k+1)d}\ \sum_{j_{s-2}=0}^{j_{s-1}}\cdots\sum_{j_1=0}^{j_2}\left(\prod_{n=1}^{m}(1+(d-i_{h_{n-1}})w)^{j_n-j_{n-1}}\right)\cdot\left(\prod_{n=m+1}^{s-1}(1+(d-i_{h_n})w)^{j_n-j_{n-1}}\right)\\
&\prod_{n=1}^{m}\left((1+(d-i_{h_{n-1}})w)^{c_n-c_{n-1}+i_{h_{n-1}}-i_{h_{n-2}}-\sum_{j=h_{n-2}}^{h_{n-1}}d_{i_j}}\prod_{a=h_{n-1}+1}^{h_n}L^{N+1,k,i_a-i_{a-1}}_{j_n-\sum_{j=1}^{a-1}d_{i_j}+i_{a-1}(N-k+1)}\right)\\
&\left((1+(d-i_{h_m})w)^{-l_m+c_{m+1}-c_m+i_{h_m}-i_{h_{m-1}}-\sum_{j=h_{m-1}}^{h_m}d_{i_j}}\prod_{a=h_m+1}^{h_{m+1}}L^{N+1,k,i_a-i_{a-1}}_{j_m-l_m-\sum_{j=1}^{a-1}d_{i_j}+i_{a-1}(N-k+1)}\right)\\
&\prod_{n=m+2}^{s}\left((1+(d-i_{h_{n-1}})w)^{c_n-c_{n-1}+i_{h_{n-1}}-i_{h_{n-2}}-\sum_{j=h_{n-2}}^{h_{n-1}}d_{i_j}}\prod_{a=h_{n-1}+1}^{h_n}L^{N+1,k,i_a-i_{a-1}}_{j_n-l_m-\sum_{j=1}^{a-1}d_{i_j}+i_{a-1}(N-k+1)}\right).
\end{aligned}\tag{3.58}$$

Continuing the same operation, we can observe that the summand of (3.54) produce $\binom{s-1}{t}$ t-th boundary parts and that they have the same structure of summation on $j'_n s$ as the bulk part coming from the ordered partition:

$$\begin{aligned}
&0 = h_{p_0} < h_{p_1} < h_{p_2} < \cdots < h_{p_{s-t}} = l\\
&(0 = p_0 < p_1 < \cdots < p_{s-t} = s,\ \ 1 \leq t \leq s-1).
\end{aligned}\tag{3.59}$$

We then separate the set $\{1, 2, \cdots, s-1\}$ into disjoint union of two sets associated with (3.59).

$$\begin{aligned}
&\{1,2,\cdots,s-1\} = \{p_1,p_2,\cdots,p_{s-t-1}\}\coprod\{r_1,r_2,\cdots,r_t\},\\
&(r_1 < r_2 < \cdots < r_t).
\end{aligned}\tag{3.60}$$

With these set-up, we can write down the t-th boundary part as the generalization of (3.58),

$$(-1)^{s-1}\sum_{(d_{i_1},\cdots,d_{i_{l-1}}),\,(d_{i_{h_j}}=0)}\ \sum_{c_1=0}^{i_{h_1}-i_{h_0}-1-\sum_{j=h_0}^{h_1}d_{i_j}}\cdots\sum_{c_s=0}^{i_{h_s}-i_{h_{s-1}}-1-\sum_{j=h_{s-1}}^{h_s}d_{i_j}}$$

$$\prod_{n=1}^{s} a_{d_{i_{h_{n-1}+1}}\cdots d_{i_{h_{n}-1}}c_n}(i_{h_{n-1}+1}-i_{h_{n-1}},\cdots,i_{h_n}-i_{h_n-1})$$

$$\sum_{l_{r_1}=1}^{c_{r_1+1}-c_{r_1}+i_{h_{r_1}}-i_{h_{r_1-1}}-\sum_{j=h_{r_1-1}}^{h_{r_1}}d_{i_j}}\cdots\sum_{l_{r_t}=1}^{c_{r_t+1}-c_{r_t}+i_{h_{r_t}}-i_{h_{r_t-1}}-\sum_{j=h_{r_t-1}}^{h_{r_t}}d_{i_j}}$$

$$\sum_{j_{s-t}=0}^{N-(N-k+1)d}\sum_{j_{s-t-1}=0}^{j_{s-t}}\cdots\sum_{j_1=0}^{j_2}\Bigl(\prod_{n=1}^{s-t}(1+(d-i_{h_{p_{n-1}}})w)^{j_n-j_{n-1}}\Bigr)\cdot\Bigl(\prod_{n=1}^{t}(1+(d-i_{h_{r_n}})w)^{-l_{r_n}}\Bigr)$$

$$\prod_{n=1}^{s-t}\prod_{m=p_{n-1}+1}^{p_n}\Bigl((1+(d-i_{h_{m-1}})w)^{c_m-c_{m-1}+i_{h_{m-1}}-i_{h_{m-2}}-\sum_{j=h_{m-2}}^{h_{m-1}}d_{i_j}}\times$$

$$\times\prod_{a=h_{m-1}+1}^{h_m}L^{N+1,k,i_a-i_{a-1}}_{j_n-\sum_{j=1}^{a-1}d_{i_j}-\sum_{r_j<m}l_{r_j}+i_{a-1}(N-k+1)}\Bigr).\tag{3.61}$$

Now, what we have to show is that the bulk part coming from the ordered partition $0=h_0<h_1<\cdots<h_{s-1}<h_s=l$ cancels with the boundary parts coming from the ordered partition $0=q_0<q_1<\cdots<q_{t-1}<q_t=l$, $(\{h_0,h_1,\cdots,h_{s-1},h_s\}\subset\{q_0,q_1,\cdots,q_{t-1},q_t\})$. Before general discussion on cancellation of these terms, we carry out computations for some lower l's as warming-up's.

$0=i_0<i_1=d$ sector:
In this case, there are no boundary contributions and the bulk part is given by,

$$\sum_{c_1=0}^{d-1}\sum_{j=-c_1}^{N-1-(N-k)d-c_1}a_{c_1}(d)(1+dw)^{j+c_1}L_j^{N+1,k,d}$$
$$=\sum_{c_1=0}^{d-1}\sum_{j=0}^{N-(N-k+1)d}a_{c_1}(d)(1+dw)^{j+c_1}L_j^{N+1,k,d}=\prod_{j=1}^{d-1}(1+jw)\cdot\sum_{j=0}^{N-(N-k+1)d}(1+dw)^jL_j^{N+1,k,d}\tag{3.62}$$

where we used (3.57) and the fact that $L_j^{N+1,k,d}=0$ unless $0\leq j\leq N-(N+1-k)d$.

$0=i_0<i_1<i_2=d$ sector:
In this sector, the summand coming from $0=h_0<h_1=1<h_2=2$ is separated into one bulk contribution and one boundary contribution corresponding to $0=h_0<h_2=2$:

$$-\prod_{j=1}^{d-1}(1+jw)\cdot\sum_{j_2=0}^{N-(N-k+1)d}\sum_{j_1=0}^{j_2}(1+dw)^{j_1}(1+(d-i_1)w)^{j_2-j_1}L^{N+1,k,i_1}_{j_1+(N-k+1)i_1}L^{N+1,k,d-i_1}_{j_2+(N-k+1)i_2}$$
$$-\sum_{c_1=0}^{i_1-1}\sum_{c_2=0}^{d-i_1-1}\sum_{j_1=0}^{N-(N-k+1)d}(1+dw)^{j_1}\sum_{l_1=1}^{c_2-c_1+i_1}a_{c_1}(i_1)a_{c_2}(d-i_1)(1+(d-i_1)w)^{-l_1+c_2-c_1+i_1}(1+dw)^{c_1}\times$$
$$\times L^{N+1,k,i_1}_{j_1}L^{N+1,k,d-i_1}_{j_1-l_1+(N-k+1)d_1}.\tag{3.63}$$

On the other hand, we have to prove that the second summand of the r.h.s of (3.63) cancels with the summand (3.54) coming from $0=h_0<h_1=2$,

$$\sum_{j_1=0}^{N-d(N-k+1)}\sum_{d_{i_1}=1}^{d-1}\sum_{c_1=0}^{d-1-d_{i_1}}(1+dw)^{j_1}a_{d_{i_1}c_1}(i_1,d-i_1)(1+dw)^{c_1}L^{N+1,k,i_1}_{j_1}L^{N+1,k,d-i_1}_{j_1-d_{i_1}+(N-k+1)d_1},$$

(3.64)

where

$$\sum_{j=0}^{d-1-d_{i_1}} a_{d_{i_1}j}(i_1,d-i_1)x^jy^{d-1-d_{i_1}-j}$$
$$=\frac{1}{2\pi\sqrt{-1}}\int_{D(0,\frac{i_1y+(d-i_1)x}{d})}du_{i_1}\frac{1}{(u_{i_1}-\frac{i_1y+(d-i_1)x}{d})}\frac{f_{(i_1)}(x,u_{i_1})\cdot f_{(d-i_1)}(u_{i_1},y)}{u_{i_1}^{d_{i_1}-1}}$$
$$=f^{d_{i_1}}_{(i_1,d-i_1)}(x,y). \tag{3.65}$$

Since $-l_1+c_2-c_1+i_1\geq 0$ in (3.63), the assertion of the theorem in this sector reduces to the following polynomial identity in this sector:

$$\frac{f_{(i_1)}(x,u)f_{(d-i_1)}(u,y)}{u^{d_{i_1}-1}}|_{\deg(u)\geq 0,\ u=\frac{(d-i_1)x+i_1y}{d}}=f^{d_{i_1}}_{(i_1,d-i_1)}(x,y), \tag{3.66}$$

where $\frac{f_{(i_1)}(x,u)f_{(d-i_1)}(u,y)}{u^{d_{i_1}-1}}|_{\deg(u)\geq 0}$ means the operation of picking up monomials, whose degree in u is non-negative, from $\frac{f_{(i_1)}(x,u)f_{(d-i_1)}(u,y)}{u^{d_{i_1}-1}}$. Now, we prove the above equality. Using the residue integral in u-plane, we have,

$$\frac{f_{(i_1)}(x,u)f_{(d-i_1)}(u,y)}{u^{d_{i_1}-1}}|_{\deg(u)\geq 0}=\frac{1}{2\pi\sqrt{-1}}\int_{D_v(0)}du\frac{f_{(i_1)}(x,v)f_{(d-i_1)}(v,y)}{v^{d_{i_1}-1}}\frac{1}{v}\sum_{n=0}^{\infty}(\frac{u}{v})^n$$
$$=\frac{1}{2\pi\sqrt{-1}}\int_{D_v(0,u)}du\frac{f_{(i_1)}(x,v)f_{(d-i_1)}(v,y)}{v^{d_{i_1}-1}}\frac{1}{v-u}. \tag{3.67}$$

Therefore, we can rewrite the l. h. s. of (3.66) as follows:

$$\frac{f_{(i_1)}(x,u)f_{(d-i_1)}(u,y)}{u^{d_{i_1}-1}}|_{deg(u)\geq 0,\ u=\frac{(d-i_1)x+i_1y}{d}}$$
$$=(\frac{1}{2\pi\sqrt{-1}})\int_{D_u(0,\frac{(d-i_1)x+i_1y}{d})}du\frac{1}{(u-\frac{(d-i_1)x+i_1y}{d})}\frac{f_{(i_1)}(x,u)f_{(d-i_1)}(u,y)}{u^{d_{i_1}-1}}. \tag{3.68}$$

But the last line is nothing but the definition of $f^{d_{i_1}}_{(i_1,d-i_1)}(x,y)$.

$0=i_0<i_1<i_2<i_3=d$ sector:
In this case, we have four choices of partitions:

$$\begin{aligned}&0=h_0<1=h_1<2=h_2<h_3=3,\\&0=h_0<h_1=1<h_2=3,\\&0=h_0<h_1=2<h_2=3,\\&0=h_0<h_1=3.\end{aligned} \tag{3.69}$$

The summand in (3.54) coming from the ordered partition $0=h_0<1=h_1<2=h_2<h_3=3$ is decomposed as follows:

$$\prod_{j=1}^{d-1}(1+jw)\cdot\sum_{j_3=0}^{N-(N-k+1)d}\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2}(1+dw)^{j_1}(1+(d-i_1)w)^{j_2-j_1}(1+(d-i_2)w)^{j_3-j_2}\times$$
$$\times L^{N+1,k,i_1}_{j_1}L^{N+1,k,i_2-i_1}_{j_2+i_1(N-k+1)}L^{N+1,k,d-i_2}_{j_3+i_2(N-k+1)}$$

$$+\sum_{c_1=0}^{i_1-1}\sum_{c_2=0}^{i_2-i_1-1}\sum_{c_3=0}^{i_3-i_2-1} a_{c_1}(i_1)a_{c_2}(i_2-i_1)a_{c_3}(d-i_2)\times$$
$$\times\sum_{l_1=1}^{c_2-c_1+i_1}\sum_{j_3=0}^{N-(N-k+1)d}\sum_{j_1=0}^{j_3}(1+dw)^{j_1+c_1}(1+(d-i_1)w)^{-l_1+c_2-c_1+i_1}\times$$
$$\times(1+(d-i_2)w)^{j_3-j_1+c_3-c_2+i_2-i_1}L_{j_1}^{N+1,k,i_1}L_{j_1-l_1+i_1(N-k+1)}^{N+1,k,i_2-i_1}L_{j_3-l_1+i_2(N-k+1)}^{N+1,k,d-i_2}$$
$$+\sum_{c_1=0}^{i_1-1}\sum_{c_2=0}^{i_2-i_1-1}\sum_{c_3=0}^{i_3-i_2-1} a_{c_1}(i_1)a_{c_2}(i_2-i_1)a_{c_3}(d-i_2)\times$$
$$\times\sum_{l_2=1}^{c_3-c_2+i_2-i_1}\sum_{j_2=0}^{N-(N-k+1)d}\sum_{j_1=0}^{j_2}(1+dw)^{j_1+c_1}(1+(d-i_1)w)^{j_2-j_1+c_2-c_1+i_1}\times$$
$$\times(1+(d-i_2)w)^{-l_2+c_3-c_2+i_2-i_1}L_{j_1}^{N+1,k,i_1}L_{j_2+i_1(N-k+1)}^{N+1,k,i_2-i_1}L_{j_2-l_2+i_2(N-k+1)}^{N+1,k,d-i_2}$$
$$+\sum_{c_1=0}^{i_1-1}\sum_{c_2=0}^{i_2-i_1-1}\sum_{c_3=0}^{i_3-i_2-1} a_{c_1}(i_1)a_{c_2}(i_2-i_1)a_{c_3}(d-i_2)\times$$
$$\times\sum_{l_1=1}^{c_2-c_1+i_1}\sum_{l_2=1}^{c_3-c_2+i_2-i_1}\sum_{j_1=0}^{N-(N-k+1)d}(1+dw)^{j_1+c_1}(1+(d-i_1)w)^{-l_1+c_2-c_1+i_1}\times$$
$$\times(1+(d-i_2)w)^{-l_2+c_3-c_2+i_2-i_1}L_{j_1}^{N+1,k,i_1}L_{j_1-l_1+i_1(N-k+1)}^{N+1,k,i_2-i_1}L_{j_1-l_1-l_2+i_2(N-k+1)}^{N+1,k,d-i_2}. \qquad (3.70)$$

Note that the first summand, the second and the third ones, and the last one correspond to the bulk part, the first boundary parts, and the second boundary part respectively. Next, we decompose the summand coming from $0=h_0<h_1=1<h_2=3$,

$$-\sum_{d_2=1}^{i_2-i_1-1}\sum_{c_1=0}^{i_1-1}\sum_{c_2=0}^{i_3-i_1-1-d_2} a_{c_1}(i_1)a_{d_2c_2}(i_2-i_1,d-i_2)\times$$
$$\times\sum_{j_2=0}^{N-(N-k+1)d}\sum_{j_1=0}^{j_2}(1+dw)^{j_1+c_1}(1+(d-i_1)w)^{j_2-j_1+c_2-c_1+i_1}\times$$
$$\times L_{j_1}^{N+1,k,i_1}L_{j_2+i_1(N-k+1)}^{N+1,k,i_2-i_1}L_{j_2-d_2+i_2(N-k+1)}^{N+1,k,d-i_2}$$
$$-\sum_{d_2=1}^{i_2-i_1-1}\sum_{c_1=0}^{i_1-1}\sum_{c_2=0}^{i_3-i_1-1-d_2} a_{c_1}(i_1)a_{d_2c_2}(i_2-i_1,d-i_2)\times$$
$$\times\sum_{l_1=1}^{c_2-c_1+i_1}\sum_{j_1=0}^{N-(N-k+1)d}(1+dw)^{j_1+c_1}(1+(d-i_1)w)^{-l_1+c_2-c_1+i_1}\times$$
$$\times L_{j_1}^{N+1,k,i_1}L_{j_1-l_1+i_1(N-k+1)}^{N+1,k,i_2-i_1}L_{j_1-l_1-d_2+i_2(N-k+1)}^{N+1,k,d-i_2}, \qquad (3.71)$$

and the one from $0=h_0<h_1=2<h_2=3$,

$$-\sum_{d_1=1}^{i_2-1}\sum_{c_1=0}^{i_2-1-d_1}\sum_{c_2=0}^{i_3-i_2-1} a_{d_1c_1}(i_1,i_2-i_1)a_{c_2}(d-i_2)\times$$
$$\times\sum_{j_2=0}^{N-(N-k+1)d}\sum_{j_1=0}^{j_2}(1+dw)^{j_1+c_1}(1+(d-i_2)w)^{j_2-j_1+c_2-c_1+i_2-d_1}\times$$

$$\times L^{N+1,k,i_1}_{j_1} L^{N+1,k,i_2-i_1}_{j_1-d_1+i_1(N-k+1)} L^{N+1,k,d-i_2}_{j_2-d_1+i_2(N-k+1)}$$

$$-\sum_{d_1=1}^{i_2-1}\sum_{c_1=0}^{i_2-1-d_1}\sum_{c_2=0}^{i_3-i_2-1} a_{d_1c_1}(i_1,i_2-i_1)a_{c_2}(d-i_2)\times$$

$$\times \sum_{l_1=1}^{c_2-c_1+i_2-d_1}\sum_{j_1=0}^{N-(N-k+1)d}(1+dw)^{j_1+c_1}(1+(d-i_2)w)^{-l_1+c_2-c_1+i_2-d_1}\times$$

$$\times L^{N+1,k,i_1}_{j_1} L^{N+1,k,i_2-i_1}_{j_1-d_1+i_1(N-k+1)} L^{N+1,k,d-i_2}_{j_1-d_1-l_1+i_2(N-k+1)}. \tag{3.72}$$

The summand coming from $0=h_0<h_3=3$ is given by,

$$\sum_{j_1=0}^{N-d(N-k+1)}\sum_{d_1=1}^{d-1}\sum_{d_2=1}^{d-1-d_1}\sum_{c_1=0}^{d-1-d_1-d_2} a_{d_1d_2c_1}(i_1,i_2-i_1,d-i_2)(1+dw)^{j_1+c_1}L^{N+1,k,i_1}_{j_1}\times$$

$$\times L^{N+1,k,i_2-i_1}_{j_1-d_1+i_1(N-k+1)} L^{N+1,k,d-i_2}_{j_1-d_1-d_2+i_2(N-k+1)}. \tag{3.73}$$

With these results, we can easily see that the second (resp. third) summand in (3.70) cancels with the first summand in (3.72) (resp. (3.71)) due to the identity proved in the $l=2$ case. Therefore, the new identity we have to prove comes from the cancellation of the fourth summand in (3.70) , the second summand in (3.71) and in (3.72), and (3.73). This can be translated into the polynomial equality:

$$\sum_{c_1=0}^{i_1-1}\sum_{c_2=0}^{i_2-i_1-1}\sum_{c_3=0}^{i_3-i_2-1} a_{c_1}(i_1)a_{c_2}(i_2-i_1)a_{c_3}(d-i_2)\times$$

$$\times\sum_{l_1=1}^{c_2-c_1+i_1}\sum_{l_2=1}^{c_3-c_2+i_2-i_1}(1+dw)^{c_1}(1+(d-i_1)w)^{-l_1+c_2-c_1+i_1}(1+(d-i_2)w)^{-l_2+c_3-c_2+i_2-i_1}$$

$$-\sum_{d_2=0}^{i_2-i_1-1}\sum_{c_1=0}^{i_1-1}\sum_{c_2=0}^{i_3-i_1-1-d_2} a_{c_1}(i_1)a_{d_2c_2}(i_2-i_1,d-i_2)\sum_{l_1=1}^{c_2-c_1+i_1}(1+dw)^{c_1}(1+(d-i_1)w)^{-l_1+c_2-c_1+i_1}$$

$$-\sum_{d_1=0}^{i_2-1}\sum_{c_1=0}^{i_1-1-d_1}\sum_{c_2=0}^{i_3-i_2-1} a_{d_1c_1}(i_1,i_2-i_1)a_{c_2}(d-i_2)\sum_{l_1=1}^{c_2-c_1+i_2-d_1}(1+dw)^{c_1}(1+(d-i_2)w)^{-l_1+c_2-c_1+i_2-d_1}$$

$$+\sum_{d_1=1}^{d-1}\sum_{d_2=1}^{d-1-d_1}\sum_{c_1=0}^{d-1-d_1-d_2} a_{d_1d_2c_1}(i_1,i_2-i_1,d-i_2)(1+dw)^{c_1}=0. \tag{3.74}$$

We can rewrite the above condition in a more compact form,

$$\Bigg(\frac{f_{(i_1)}(x,u)f^{n}_{(i_2-i_1,d-i_2)}(u,y)}{u^{m-1}}|_{\deg(u)\geq 0}+\frac{f^{m}_{(i_1,i_2-i_1)}(x,v)f_{(d-i_2)}(v,y)}{v^{n-1}}|_{\deg(v)\geq 0}$$

$$-\frac{f_{(i_1)}(x,u)f_{(i_2-i_1)}(u,v)f_{(d-i_2)}(v,y)}{u^{m-1}v^{n-1}}|_{\deg(u)\geq 0,\ \deg(v)\geq 0}\Bigg)|_{u=\frac{i_1y+(d-i_1)x}{d},\,v=\frac{(i_2)y+(d-i_2)x}{d}}$$

$$=f^{mn}_{(i_1,i_2-i_1,d-i_2)}(x,y). \tag{3.75}$$

On the other hand, the definition of $f^{mn}_{(i_1,i_2-i_1,d-i_2)}(x,y)$ tells us,

$$f^{mn}_{(i_1,i_2-i_1,d-i_2)}(x,y):=$$

$$(\frac{1}{2\pi\sqrt{-1}})^2\int_{D_u}\int_{D_v}\frac{d\cdot(i_2-i_1)\cdot(dudv)}{(i_2u-i_1v-(i_2-i_1)x)((d-i_1)v-(d-i_2)u-(i_2-i_1)y)}\times$$

$$\times\frac{f_{(i_1)}(x,u)f_{(i_2-i_1)}(u,v)f_{(d-i_2)}(v,y)}{u^{m-1}v^{n-1}}. \tag{3.76}$$

Hence what remains to show in the $l=3$ case is the following equality:

$$\Bigg(\frac{f_{(i_1)}(x,u)f^n_{(i_2-i_1,d-i_2)}(u,y)}{u^{m-1}}|_{\deg(u)\geq 0}+\frac{f^m_{(i_1,i_2-i_1)}(x,v)f_{(d-i_2)}(v,y)}{v^{n-1}}|_{\deg(v)\geq 0}$$
$$-\frac{f_{(i_1)}(x,u)f_{(i_2-i_1)}(u,v)f_{(d-i_2)}(v,y)}{u^{m-1}v^{n-1}}|_{\deg(u)\geq 0,\ \deg(v)\geq 0}\Bigg)|_{u=\frac{i_1y+(d-i_1)x}{d},v=\frac{(i_2)y+(d-i_2)x}{d}}$$
$$=(\frac{1}{2\pi\sqrt{-1}})^2\int_{D_u}\int_{D_v}\frac{d\cdot(i_2-i_1)\cdot(dudv)}{(i_2u-i_1v-(i_2-i_1)x)((d-i_1)v-(d-i_2)u-(i_2-i_1)y)}\times$$
$$\times\frac{f_{(i_1)}(x,u)f_{(i_2-i_1)}(u,v)f_{(d-i_2)}(v,y)}{u^{m-1}v^{n-1}}. \tag{3.77}$$

First, we consider the following part:

$$(\frac{1}{2\pi\sqrt{-1}})^2\int_{D_u}\int_{D_v}\Bigg(\frac{f_{(i_1)}(x,u)f_{(i_2-i_1)}(u,v)f_{(d-i_2)}(v,y)}{u^{m-1}v^{n-1}}|_{\deg(u)\leq -1,\deg(v)\leq -1}\Bigg)\times$$
$$\times\frac{d\cdot(i_2-i_1)\cdot(dudv)}{((i_2)u-i_1v-(i_2-i_1)x)((d-i_1)v-(d-i_2)u-(i_2-i_1)y)}. \tag{3.78}$$

But we can easily see with some computation,

$$(\frac{1}{2\pi\sqrt{-1}})^2\int_{D_u}\int_{D_v}\frac{1}{u^kv^l}\cdot\frac{d\cdot d_2\cdot(dudv)}{((i_2)u-i_1v-(i_2-i_1)x)((d-i_1)v-(d-i_2)u-(i_2-i_1)y)}$$
$$=\frac{1}{2\pi\sqrt{-1}}\int_{D_u}\frac{du}{u^k}\cdot\frac{(i_1)^l}{((i_2)u-(i_2-i_1)x)^l}\cdot\frac{1}{(u-\frac{i_1y+(d-i_1)x}{d})}$$
$$=0, \tag{3.79}$$

where $k,l\geq 1$. The last equality follows from the fact that D_u goes around all the poles of the integrand. Hence we have

$$(\frac{1}{2\pi\sqrt{-1}})^2\int_{D_u}\int_{D_v}\Bigg(\frac{f_{(i_1)}(x,u)f_{(i_2-i_1)}(u,v)f_{(d-i_2)}(v,y)}{u^{m-1}v^{n-1}}|_{\deg(u)\leq -1,\deg(v)\leq -1}\Bigg)\times$$
$$\times\frac{d\cdot(i_2-i_1)\cdot(dudv)}{((i_2)u-i_1v-(i_2-i_1)x)((d-i_1)v-(d-i_2)u-(i_2-i_1)y)}$$
$$=0. \tag{3.80}$$

Using (3.80), we can rewrite the r.h.s. of (3.77) as follows,

$$(\frac{1}{2\pi\sqrt{-1}})^2\int_{D_u}\int_{D_v}\frac{d\cdot(i_2-i_1)\cdot(dudv)}{(i_2u-i_1v-(i_2-i_1)x)((d-i_1)v-(d-i_2)u-(i_2-i_1)y)}\times$$
$$\times\frac{f_{(i_1)}(x,u)f_{(i_2-i_1)}(u,v)f_{(d-i_2)}(v,y)}{u^{m-1}v^{n-1}}$$
$$=(\frac{1}{2\pi\sqrt{-1}})^2\int_{D_u}\int_{D_v}\frac{d\cdot(i_2-i_1)\cdot(dudv)}{(i_2u-i_1v-(i_2-i_1)x)((d-i_1)v-(d-i_2)u-(i_2-i_1)y)}\times$$
$$\times\frac{f_{(i_1)}(x,u)f_{(i_2-i_1)}(u,v)f_{(d-i_2)}(v,y)}{u^{m-1}v^{n-1}}$$
$$-(\frac{1}{2\pi\sqrt{-1}})^2\int_{D_u}\int_{D_v}\frac{d\cdot(i_2-i_1)\cdot(dudv)}{(i_2u-i_1v-(i_2-i_1)x)((d-i_1)v-(d-i_2)u-(i_2-i_1)y)}\times$$
$$\times\Bigg(\frac{f_{(i_1)}(x,u)f_{(i_2-i_1)}(u,v)f_{(d-i_2)}(v,y)}{u^{m-1}v^{n-1}}|_{\deg(u)\leq -1,\ g(v)\leq -1}\Bigg)$$
$$=(\frac{1}{2\pi\sqrt{-1}})^2\int_{D_u}\int_{D_v}\frac{d\cdot(i_2-i_1)\cdot(dudv)}{(i_2u-i_1v-(i_2-i_1)x)((d-i_1)v-(d-i_2)u-(i_2-i_1)y)}\times$$

$$\times\left(\frac{f_{(i_1)}(x,u)f_{(i_2-i_1)}(u,v)f_{(d-i_2)}(v,y)}{u^{m-1}v^{n-1}}|_{\deg(u)\geq 0}\right)$$
$$+(\frac{1}{2\pi\sqrt{-1}})^2\int_{D_u}\int_{D_v}\frac{d\cdot(i_2-i_1)\cdot(dudv)}{(i_2u-i_1v-(i_2-i_1)x)((d-i_1)v-(d-i_2)u-(i_2-i_1)y)}\times$$
$$\times\left(\frac{f_{(i_1)}(x,u)f_{(i_2-i_1)}(u,v)f_{(d-i_2)}(v,y)}{u^{m-1}v^{n-1}}|_{\deg(v)\geq 0}\right)$$
$$-(\frac{1}{2\pi\sqrt{-1}})^2\int_{D_u}\int_{D_v}\frac{d\cdot(i_2-i_1)\cdot(dudv)}{(i_2u-i_1v-(i_2-i_1)x)((d-i_1)v-(d-i_2)u-(i_2-i_1)y)}\times$$
$$\times\left(\frac{f_{(i_1)}(x,u)f_{(i_2-i_1)}(u,v)f_{(d-i_2)}(v,y)}{u^{m-1}v^{n-1}}|_{\deg(v)\geq 0,\ \deg(u)\geq 0}\right). \tag{3.81}$$

At this stage, we look at the first integral of the last line of (3.81). Due to the condition $\deg(u)\geq 0$, u variable has only one pole at $u=\frac{i_1v+(i_2-i_1)x}{i_1}$. And if we integrate out the u variable first, the integral turns into,

$$(\frac{1}{2\pi\sqrt{-1}})\int_{D_v}\frac{dv}{(v-\frac{i_2y+(d-i_2)x}{d})}\left(\frac{f^m_{(i_1,i_2-i_1)}(x,v)f_{(d-i_2)}(v,y)}{v^{n-1}}\right). \tag{3.82}$$

This is nothing but the second term in the l.h.s. of (3.77). Using the same operation, we can show that the second and the third integrals in the last line of (3.81) equal the first and the third terms in the l.h.s. of (3.77). Thus, the proof of $l=3$ case is completed.

With these preparation, we turn into the general proof of the theorem. In this case, we have to consider the integral,

$$f^{d_{i_1}d_{i_2}\cdots d_{i_{l-1}}}_{(i_1-i_0,\cdots,i_l-i_{l-1})}(x,y)=$$
$$\frac{d}{(2\pi\sqrt{-1})^{l-1}}\int_{D_{i_1}}\cdots\int_{D_{i_{l-1}}}\left(\prod_{j=1}^{l}f_{(i_j-i_{j-1})}(u_{i_{j-1}},u_{i_j})\prod_{j=1}^{l-1}\frac{1}{u_{i_j}^{d_{i_j}-1}}\right)\times$$
$$\times\prod_{j=1}^{l-1}\frac{du_{i_j}}{((i_{j+1}-i_{j-1})u_{i_j}-(i_j-i_{j-1})u_{i_{j+1}}-(i_{j+1}-i_j)u_{i_{j-1}})}\prod_{j=1}^{l-2}(i_{j+1}-i_j). \tag{3.83}$$

For convenience of space, we introduce the definition:

Definition 5 *Let $\alpha_j(x,y)$ $(j=1,2,\cdots,l)$ be a homogeneous polynomial in x and y. We define two types of l-product, which are both non-commutative and non-associative as follows,*

$$(\alpha_1\circ\alpha_2\circ\cdots\circ\alpha_l)(u_{i_0},u_{i_l}):=$$
$$\frac{(i_l-i_0)}{(2\pi\sqrt{-1})^{l-1}}\int_{D_{i_1}}\cdots\int_{D_{i_{l-1}}}\left(\prod_{j=1}^{l}\alpha_j(u_{i_{j-1}},u_{i_j})\prod_{j=1}^{l-1}\frac{1}{u_{i_j}^{d_{i_j}-1}}\right)\times$$
$$\times\prod_{j=1}^{l-1}\frac{du_{i_j}}{((i_{j+1}-i_{j-1})u_{i_j}-(i_j-i_{j-1})u_{i_{j+1}}-(i_{j+1}-i_j)u_{i_{j-1}})}\prod_{j=1}^{l-2}(i_{j+1}-i_j), \tag{3.84}$$
$$(\alpha_1*\alpha_2*\cdots*\alpha_l)(u_{i_0},u_{i_l}):=$$
$$\left(\prod_{j=1}^{l}\alpha_j(u_{i_{j-1}},u_{i_j})\prod_{j=1}^{l-1}\frac{1}{u_{i_j}^{d_{i_j}-1}}\right)|_{\deg(u_{i_1})\geq 0,\ \cdots,\ \deg(u_{i_{l-1}})\geq 0,\ u_{i_j}=\frac{(i_j-i_0)u_{i_l}+(i_l-i_j)u_{i_0}}{(i_l-i_0)}}. \tag{3.85}$$

In the same way as the $l=3$ cases, we can show the following two lemmas.

Lemma 1

$$\frac{(i_l-i_0)}{(2\pi\sqrt{-1})^{l-1}}\int_{D_{i_1}}\cdots\int_{D_{i_{l-1}}}\left(\prod_{j=1}^{l}\alpha_j(u_{i_{j-1}},u_{i_j})\prod_{j=1}^{l-1}\frac{1}{u_{i_j}^{d_{i_j}-1}}|_{\deg(u_{i_1})\le -1,\cdots,\deg(u_{i_{l-1}})\le -1}\right)\times$$
$$\times\prod_{j=1}^{l-1}\frac{du_{i_j}}{((i_{j+1}-i_{j-1})u_{i_j}-(i_j-i_{j-1})u_{i_{j+1}}-(i_{j+1}-i_j)u_{i_{j-1}})}\prod_{j=1}^{l-2}(i_{j+1}-i_j)=0. \tag{3.86}$$

proof) By expanding $\frac{1}{((i_{j+1}-i_{j-1})u_{i_j}-(i_j-i_{j-1})u_{i_{j+1}}-(i_{j+1}-i_j)u_{i_{j-1}})}$ into

$$\frac{1}{(i_{j+1}-i_{j-1})u_{i_j}}\cdot\left(1-\frac{(i_j-i_{j-1})u_{i_{j+1}}+(i_{j+1}-i_j)u_{i_{j-1}}}{(i_{j+1}-i_{j-1})u_{i_j}}\right)^{-1}$$
$$=\frac{1}{(i_{j+1}-i_{j-1})u_{i_j}}\cdot\sum_{n=0}^{\infty}\left(\frac{(i_j-i_{j-1})u_{i_{j+1}}+(i_{j+1}-i_j)u_{i_{j-1}}}{(i_{j+1}-i_{j-1})u_{i_j}}\right)^n, \tag{3.87}$$

we can reduce the integral in the l.h.s. of (3.86) to (infinite) linear combination of the integral:

$$\frac{1}{(2\pi\sqrt{-1})^{l-1}}\int_{C_{i_1}}du_{i_1}\cdots\int_{C_{i_{l-1}}}du_{i_{l-1}}\prod_{j=0}^{l}(u_{i_j})^{m_{i_j}}, \tag{3.88}$$

where the path C_{i_j} goes around $u_{i_j}=0$. But we can easily see $m_{i_0},m_{i_l}\ge 0$ and $\sum_{j=0}^{l}m_{i_j}\le -l$, due to the condition $\deg(u_{i_1})\le -1,\cdots,\deg(u_{i_{l-1}})\le -1$. Therefore, $\sum_{j=1}^{l-1}m_{i_j}\le -l$. It follows that there exists $j\in\{1,2,\cdots,l-1\}$ such that $\deg(u_{i_j})$ is less than -1. Hence the integral (3.88) vanishes and the lemma is proved. Q.E.D.

Lemma 2

$$\alpha_1\circ\cdots\circ\alpha_l$$
$$=\sum_{s=1}^{l-1}(-1)^{l-1-s}\sum_{0=h_0<\cdots<h_s=l}(\alpha_1*\cdots*\alpha_{h_1})\circ(\alpha_{h_1+1}*\cdots*\alpha_{h_2})\circ\cdots\circ(\alpha_{h_{s-1}+1}*\cdots*\alpha_{h_s}). \tag{3.89}$$

proof) We denote by $A_j(F)$ the operation picking up the monomials with $\deg(u_{i_j})\ge 0$ from $F=\prod_{j=1}^{l}\alpha_j(u_{i_{j-1}},u_{i_j})\prod_{j=1}^{l-1}\frac{1}{u_{i_j}^{d_{i_j}-1}}$. Using Lemma 1 and the inclusion-exclusion principle, we obtain,

$$(\alpha_1\circ\cdots\circ\alpha_l)(u_{i_0},u_{i_l})$$
$$=\frac{(i_l-i_0)}{(2\pi\sqrt{-1})^{l-1}}\int_{D_{i_1}}\cdots\int_{D_{i_{l-1}}}\left(\prod_{j=1}^{l}\alpha_j(u_{i_{j-1}},u_{i_j})\prod_{j=1}^{l-1}\frac{1}{u_{i_j}^{d_{i_j}-1}}\right)\times$$
$$\times\prod_{j=1}^{l-1}\frac{du_{i_j}}{((i_{j+1}-i_{j-1})u_{i_j}-(i_j-i_{j-1})u_{i_{j-1}}-(i_{j+1}-i_j)u_{i_{j+1}})}\prod_{j=1}^{l-2}(i_{j+1}-i_j)$$
$$=\frac{(i_l-i_0)}{(2\pi\sqrt{-1})^{l-1}}\int_{D_{i_1}}\cdots\int_{D_{i_{l-1}}}\left((\cup_{j=1}^{l-1}A_j)(F)\right)\times$$
$$\times\prod_{j=1}^{l-1}\frac{du_{i_j}}{((i_{j+1}-i_{j-1})u_{i_j}-(i_j-i_{j-1})u_{i_{j-1}}-(i_{j+1}-i_j)u_{i_{j+1}})}\prod_{j=1}^{l-2}(i_{j+1}-i_j)$$

$$= \sum_{s=1}^{l-1}(-1)^s \sum_{1\leq h_1<\cdots<h_s\leq l-1} \frac{(i_l-i_0)}{(2\pi\sqrt{-1})^{l-1}} \int_{D_{i_1}} \cdots \int_{D_{i_{l-1}}} \Bigl((\cap_{j=1}^{s} A_{h_j})(F)\Bigr) \times$$
$$\times \prod_{j=1}^{l-1} \frac{du_{i_j}}{((i_{j+1}-i_{j-1})u_{i_j} - (i_j - i_{j-1})u_{i_{j-1}} - (i_{j+1}-i_j)u_{i_{j+1}})} \prod_{j=1}^{l-2}(i_{j+1}-i_j). \qquad (3.90)$$

Integrating out $u_{i_{h_j}}$'s in the last line of (3.90) leads us to the assertion of the lemma. Q.E.D.

At this stage, we look back at the computation for the $l \leq 3$ cases and the structure of the boundary parts given in (3.61). Then we can see that the bulk part (3.55) coming from the ordered partition $0 = h_0 < h_1 < h_2 < \cdots < h_{s-1} < h_s = l$ cancels with boundary parts obtained from the ordered partition $0 = q_0 < q_1 < q_2 < \cdots < q_{t-1} < q_t = l$, $(\{h_0, h_1, \cdots, h_s\} \subset \{q_0, q_1, \cdots, q_t\})$ if the following equality holds true.

Proposition 1

$$\sum_{s=1}^{l}(-1)^{s-1} \sum_{0=h_0<\cdots<h_s=l} \Bigl(\prod_{k=1}^{s} f^{d_{i_{h_{k-1}+1}}\cdots d_{i_{h_k-1}}}_{(i_{h_{k-1}+1}-i_{h_{k-1}},\cdots,i_{h_k}-i_{h_k-1})}(u_{i_{h_{k-1}}}, u_{i_{h_k}}) \times$$
$$\times \prod_{j=1}^{s-1} \frac{1}{(u_{i_{h_j}})^{d_{i_{h_j}}-1}}\Bigr)\Big|_{\deg(u_{i_{h_1}})\geq 0, \deg(u_{i_{h_2}})\geq 0, \cdots, \deg(u_{i_{h_s}})\geq 0,\ u_{i_{h_j}} = \frac{i_{h_j} y + (d - i_{h_j})x}{d}}$$
$$= 0. \qquad (3.91)$$

proof) For the proof of the assertion of the proposition, it is sufficient to show the following relation,

$$\alpha_1 \circ \cdots \circ \alpha_l$$
$$= \sum_{s=2}^{l}(-1)^s \sum_{0=h_0<\cdots<h_s=l} (\alpha_1 \circ \cdots \circ \alpha_{h_1}) * (\alpha_{h_1+1} \circ \cdots \circ \alpha_{h_2}) * \cdots * (\alpha_{h_{s-1}+1} \circ \cdots \circ \alpha_{h_s}). \qquad (3.92)$$

We have to notice here that we can represent $\alpha_1 \circ \cdots \circ \alpha_l$ in terms $*$-product by using iteratively (3.92) only, or by iterative use of (3.89). Therefore, to show the equivalence between (3.92) and (3.89), it is enough for us to prove that both $*$-product representations of $\alpha_1 \circ \cdots \circ \alpha_l$ obtained from (3.92) and (3.89) coincide for all l. Since the $*$-product $\alpha_1 * \cdots * \alpha_l$ has different meaning for each l, we have to take care of the way of insertion of parenthesis ( ) into $\alpha_1 * \cdots * \alpha_l$. For example, we have to distinguish $((\alpha_1 * \alpha_2) * \alpha_3) * \alpha_4$ from $(\alpha_1 * \alpha_2 * \alpha_3) * \alpha_4$. Using this fact, we give here some symbolic discussion. First, we denote by Q_l the set of all the non-trivial ways of inserting parentheses into $\alpha_1 * \cdots * \alpha_l$. Next, for $\pi_l \in Q_l$, we use the notation $\pi_l(\alpha_1 * \cdots * \alpha_l)$ for the result of insertion of parentheses. For example,

$$\pi_4(\alpha_1 * \alpha_2 * \alpha_3 * \alpha_4) = (\alpha_1 * \alpha_2) * (\alpha_3 * \alpha_4) \qquad (3.93)$$

We also denote by $|\pi_l|$ the number of parentheses inserted by π_l.

With these preparation, we can easily obtain from (3.92) the formula:

$$\alpha_1 \circ \cdots \circ \alpha_l = \sum_{\pi_l \in Q_l} (-1)^{(l-|\pi_l|)} \pi_l(\alpha_1 * \cdots * \alpha_l), \qquad (3.94)$$

by induction of l. Therefore, what remains to show is that we can derive the formula (3.94) only by using (3.89). We show this by induction of l. In the $l = 2$ case, (3.89) reduces to $\alpha_1 \circ \alpha_2 = \alpha_1 * \alpha_2$, and (3.94) trivially holds. Then we assume that (3.94) holds for $l = 1, 2, \cdots, m-1$ cases. By the assumption of induction, it is clear that all the $\pi_m(\alpha_1 * \cdots * \alpha_m)$ $(\pi_m \in Q_m)$ appear in the process of rewriting $\alpha_1 \circ \cdots \circ \alpha_m$ using (3.89). Therefore, we only have to show that the coefficient of $\pi_m(\alpha_1 * \cdots \alpha_m)$ becomes $(-1)^{(m-|\pi_m|)}$ after adding up all the contributions.

Now, we fix one $\pi_m(\alpha_1 * \cdots * \alpha_m)$. By assumption, the terms coming from one term in (3.89):

$$(-1)^{l-1-s}(\alpha_1 * \cdots * \alpha_{h_1}) \circ (\alpha_{h_1+1} * \cdots * \alpha_{h_2}) \circ \cdots \circ (\alpha_{h_{s-1}+1} * \cdots * \alpha_{h_s}), \qquad (3.95)$$

are all different from each other, and we first determine the term (3.95) that produces $\pi_m(\alpha_1 * \cdots * \alpha_m)$. Here, we have to notice that the terms coming from (3.95) have no insertion of parentheses inside $(\alpha_{h_{n-1}+1} * \cdots * \alpha_{h_n})$. With this observation, we remove the parentheses in $\pi_m(\alpha_1 * \cdots * \alpha_m)$ if they have other parentheses inside them. We denote by $\tilde{\pi}_m(\alpha_1 * \cdots * \alpha_m)$ the resulting term. $\tilde{\pi}_m(\alpha_1 * \cdots * \alpha_m)$ has the following structure:

$$\tilde{\pi}_m(\alpha_1 * \cdots * \alpha_m) = \alpha_1 * \cdots * \alpha_{k_1} * (\alpha_{k_1+1} * \cdots * \alpha_{j_1}) * \alpha_{j_1+1} * \cdots * \alpha_{k_2} * (\alpha_{k_2+1} * \cdots * \alpha_{j_2}) * \cdots$$
$$\cdots * \alpha_{k_n} * (\alpha_{k_n+1} * \cdots * \alpha_{j_n}) * \alpha_{j_n+1} * \cdots * \alpha_m. \qquad (3.96)$$

We determine here the terms (3.95) that produce (3.96). Since we cannot admit the part $(\alpha_{a_1} * \cdots * \alpha_{a_1+b})$ in (3.95) that do not appear in (3.96), the allowed terms are

$$\alpha_1 \circ \cdots \circ \alpha_{k_1} \circ (\alpha_{k_1+1} * \cdots * \alpha_{j_1}) \circ \alpha_{j_1+1} \circ \cdots \circ \alpha_{k_2} \circ (\alpha_{k_2+1} * \cdots * \alpha_{j_2}) \circ \cdots$$
$$\cdots \circ \alpha_{k_n} \circ (\alpha_{k_n+1} * \cdots * \alpha_{j_n}) \circ \alpha_{j_n+1} \circ \cdots \circ \alpha_m. \qquad (3.97)$$

and the terms obtained from changing $\circ(\alpha_{k_a+1} * \cdots * \alpha_{j_a})\circ$ in (3.97) into $\circ\alpha_{k_a+1} \circ \cdots \circ \alpha_{j_a}\circ$. Here, we omit the sign of (3.97) for brevity. If we change all the $\circ(\alpha_{k_a+1} * \cdots * \alpha_{j_a})\circ$'s into $\circ\alpha_{k_a+1} \circ \cdots \circ \alpha_{j_a}\circ$'s, we obtain $\alpha_1 \circ \cdots \circ \alpha_l$. Therefore, total number of the terms (3.95) that produce $\tilde{\pi}_m(\alpha_1 * \cdots * \alpha_m)$ is $2^n - 1 = 2^{|\tilde{\pi}_m|} - 1$. With some computation, we can see that the sign of (3.96) , coming from the term obtained from changing h of the $\circ(\alpha_{k_a+1} * \cdots * \alpha_{j_a})\circ$'s into $\circ\alpha_{k_a+1} \circ \cdots \circ \alpha_{j_a}\circ$'s, equals $(-1)^{m-1-h}$. And the number of such terms are given by $\binom{n}{h}$. Therefore, the coefficient of $\tilde{\pi}_m(\alpha_1 * \cdots * \alpha_m)$ turns out to be,

$$\sum_{h=0}^{n-1} \binom{n}{h} (-1)^{m-1-h} = (-1)^{m-1}((1-1)^n - (-1)^n) = (-1)^{m-n} = (-1)^{(m-|\tilde{\pi}_m|)}. \qquad (3.98)$$

In the case of $\pi_m(\alpha_1 * \cdots * \alpha_m)$, the situation is almost the same. The only difference is that the number of added parentheses increases by $|\pi_m| - |\tilde{\pi}_m|$. Therefore, by assumption of induction, the coefficient of $\pi_m(\alpha_1 * \cdots * \alpha_m)$ equals $(-1)^{(m-|\tilde{\pi}_m|)} \cdot (-1)^{-(|\pi_m|-|\tilde{\pi}_m|)} = (-1)^{(m-|\pi_m|)}$. Thus, (3.94) is derived for $l = m$, and the proof of Proposition 1 is completed. Q.E.D.

And the proof of the Theorem 4 is completed. Q.E.D.

Proof of Theorem 1)
From the statement of Theorem 4, we obtain the formulas in the case of $(k + 2 \leq N \leq 2k)$:

$$\gamma_0^{N,k,1}(w) = \gamma_0^{2k,k,1}(w) = k \prod_{j=1}^{k-1} (k + jw),$$
$$\gamma_0^{N,k,d}(w) = (\prod_{j=1}^{d-1} (1 + jw))^{2k-N} \gamma_0^{2k,k,d}(w) = 0, \quad (d \geq 2). \qquad (3.99)$$

But from the proof of the Corollary 1, we can determine $L_n^{N,k,d}$ only using (3.99). Therefore, we can conclude that the recursive formulas in Theorem 1 compute $L_n^{N,k,d}$ correctly. Q.E.D.

3.2 $N-k=1$ case

In this case, we had better introduce $\tilde{\psi}_\alpha(t)$,

$$\tilde{\psi}_\alpha(t) := \exp(k!\cdot\exp(t))\cdot\psi_\alpha(t), \quad (\alpha=0,1,\cdots,N-2), \tag{3.100}$$

instead of $\psi_\alpha(t)$ in (3.25) because of the following Theorem of Givental [6].

Theorem 5 (Givental)
If $N-k=1$, $\tilde{\psi}_0(t)$ satisfy the rank $N-1$ ODE:

$$\Big((\partial_t)^{N-1} - k\cdot e^t\cdot(k\partial_t+k-1)\cdots(k\partial_t+2)\cdot(k\partial_t+1)\Big)w(t)=0. \tag{3.101}$$

This theorem is equivalent to $L_m^{k+1,k,1}=\tilde{L}_m^{k+1,k,1}-k!,\ \ L_m^{k+1,k,d}=\tilde{L}_m^{k+1,k,d}\ \ (d\geq 2)$ [5].

3.3 Calabi-Yau case $(N-k=0)$

Then we turn into the case of Calabi-Yau hypersurface. To clarify the meaning of the virtual structure constants introduced in [5], we had better introduce the B-model deformation parameter x instead of t and consider the following Gauss-Manin system.

$$\begin{aligned}
&\partial_x\tilde{\psi}_{-1}(x)=\tilde{L}_{k-1}^{k,k}(e^x)\cdot\tilde{\psi}_0(x),\\
&\partial_x\tilde{\psi}_n(x)=\tilde{L}_{k-2-n}^{k,k}(e^x)\cdot\tilde{\psi}_{n+1}(x), \quad (n=0,\cdots..k-3)\\
&\partial_x\tilde{\psi}_{k-2}(x)=\tilde{L}_0^{k,k}(e^x)\cdot\tilde{\psi}_{k-1}(x).
\end{aligned} \tag{3.102}$$

We can derive the following equality from the above equations:

$$\tilde{\psi}_{k-1}(x)=\frac{1}{\tilde{L}_0^{k,k}(e^x)}(\partial_x(\frac{1}{\tilde{L}_1^{k,k}(e^x)}\cdots\partial_x(\frac{1}{\tilde{L}_{k-2}^{k,k}(e^x)}\partial_x(\frac{1}{\tilde{L}_{k-1}^{k,k}(e^x)}\partial_x\tilde{\psi}_{-1}(x)))\cdots)). \tag{3.103}$$

(3.103) motivates us to state the following theorem.

Theorem 6

$$\begin{aligned}
&\frac{1}{\tilde{L}_0^{k,k}(e^x)}(\partial_x(\frac{1}{\tilde{L}_1^{k,k}(e^x)}\cdots\partial_x(\frac{1}{\tilde{L}_{k-2}^{k,k}(e^x)}\partial_x(\frac{1}{\tilde{L}_{k-1}^{k,k}(e^x)}w(x)))\cdots))\\
&=\Big((\partial_x)^{k-1}-k\cdot e^x\cdot(k\partial_x+k-1)\cdots(k\partial_x+2)\cdot(k\partial_x+1)\Big)w(x)
\end{aligned} \tag{3.104}$$

proof) We only have to apply formally the discussion of $N-k\geq 2$ case to the $N-k=0$ case with the Gauss-Manin system (3.102). Q.E.D.

Since $\tilde{L}_n^{k,k}(e^x)=\tilde{L}_{k-1-n}^{k,k}(e^x)$, we have,

Corollary 2

$$u_j^{k,k}(x):=\tilde{L}_0^{k,k}(e^x)\int^x dx_1\tilde{L}_1^{k,k}(e^{x_1})\int^{x_1}dx_2\tilde{L}_2^{k,k}(e^{x_2})\cdots\int^{x_{j-1}}dx_j\tilde{L}_j^{k,k}(e^{x_j}). \tag{3.105}$$

Remark 3 *Representation of the Picard-Fuchs differential equation given in (3.104) can also be seen in [1]. We think that our approach via Gauss-Manin system is a kind of reduction of the method used in [1], restricted to the Kähler deformation.*

(3.105) enables us to write out $\tilde{L}_n^{k,k}(e^x)$ explicitly in terms of the solution of the Picards-Fuchs differentail equation used in the Mirror computation in [7], [13]. For example, we have:

$$\begin{aligned}
\tilde{L}_0^{k,k}(e^x) &= w_0^{k,k}(x), && (3.106)\\
\tilde{L}_1^{k,k}(e^x) &= \partial_x(x+\frac{w_1^{k,k}(x)}{w_0^{k,k}(x)}), && (3.107)\\
\tilde{L}_2^{k,k}(e^x) &= \partial_x(x+\frac{2w_1^{k,k}(x)w_0^{k,k}(x)+\partial_x w_2^{k,k}(x)w_0^{k,k}(x)-w_2^{k,k}(x)\partial_x w_0^{k,k}(x)}{2((w_0^{k,k}(x))^2+\partial_x w_1^{k,k}(x)w_0^{k,k}(x)-w_1^{k,k}(x)\partial_x w_0^{k,k}(x))}). && (3.108)
\end{aligned}$$

These results agree with the computation in [7], and they give us the proof of Theorem 2.

3.4 Extension to the General Type Hypersurfaces ($N-k<0$)

If $N-k<0$, We consider the rank $k-1$ ODE:

$$\begin{aligned}
&\Big((\partial_x)^{N-1}-k\cdot e^x\cdot(k\partial_x+k-1)\cdots(k\partial_x+2)\cdot(k\partial_x+1)\Big)w(x)\\
&\Big(1-k\cdot e^x\cdot(k\partial_x+k-1)\cdots(k\partial_x+2)\cdot(k\partial_x+1)\frac{1}{(\partial_x)^{N-1}}\Big)\partial_x^{N-1}w(x)\\
&=0. \qquad (3.109)
\end{aligned}$$

Here, we propose the B-model Gauss-Manin system associated to (3.109):

$$\partial_x\tilde{\psi}_\alpha(x)=\tilde{\psi}_{\alpha+1}(x)+\sum_{d=1}^{\infty}\exp(dx)\cdot\tilde{L}_{N-2-\alpha}^{N,k,d}\cdot\tilde{\psi}_{\alpha+1+(k-N)d}(x), \qquad (3.110)$$

where α runs through $\mathbf{Z}$. $\tilde{L}_n^{N,k,d}$ is the virtual structure constant introduced in [9]. $\tilde{L}_n^{N,k,d}$ is non-zero if $0\leq n\leq N-1+(k-N)d$ and if $d\geq 1$. All the non-vanishing $\tilde{L}_n^{N,k,d}$'s are evaluated via the recursive formulas proposed in [11]. Therefore, we have infinite number of non-vanishing virtual structure constants in this case. Straightforward application of the discussion of $N-k\geq 2$ case to this case leads us to the following theorem:

Theorem 7 *We can reconstruct the ODE (3.109) from (3.110). Conversely, we can determine $\tilde{L}_n^{N,k,d}$ by (3.109).*

Now, we explain the reconstruction process of (3.109) from (3.110). First we introduce the algebra of differential operator ∂_x.

$$\begin{aligned}
&\partial_x\cdot\frac{1}{\partial_x}=\frac{1}{\partial_x}\cdot\partial_x=1,\\
&\partial_x e^{jx}=e^{jx}(\partial_x+j),\quad \frac{1}{\partial_x}e^{jx}=e^{jx}\frac{1}{(\partial_x+j)}. \qquad (3.111)
\end{aligned}$$

Using (3.111), we can obtain the following formula:

$$\begin{aligned}
&\Big(1-k\cdot e^x\cdot(k\partial_x+k-1)\cdots(k\partial_x+2)\cdot(k\partial_x+1)\frac{1}{(\partial_x)^{N-1}}\Big)^{-1}\\
&=1+\sum_{d=1}^{\infty}e^{dx}\cdot\prod_{m=0}^{kd-1}(k\partial_x+m)\prod_{j=0}^{d-1}\frac{1}{(\partial_x+j)^N}. \qquad (3.112)
\end{aligned}$$

Looking back at (3.110), we can easily see,

$$\tilde{\psi}_j(x)=(\partial_x)^{j-N+1}\tilde{\psi}_{N-1}(x),\quad (j\geq N-1). \qquad (3.113)$$

Then using the algebras in (3.111) and (3.110), we can inductively construct the pseudo-differential operator $F_j(e^x, \partial_x)$ when $j \geq 0$,

$$(\partial_x)^{N-1+j}\tilde{\psi}_{-j}(x) = F_j(e^x, \partial_x)\tilde{\psi}_{N-1}(x). \tag{3.114}$$

Then we consider the limit $F_{\infty}(e^x, \partial_x) := \lim_{j\to\infty} F_j(e^x, \partial_x)$. Now, our assertion is the following statement:

$$\frac{1}{F_{\infty}(e^x, \partial_x)}(\partial_x)^{N-1} = (1 - k\cdot e^x \cdot (k\partial_x + k - 1)\cdots(k\partial_x + 2)\cdot(k\partial_x + 1)\frac{1}{(\partial_x)^{N-1}})(\partial_x)^{N-1}, \tag{3.115}$$

or equivalently,

$$F_{\infty}(e^x, \partial_x) = 1 + \sum_{d=1}^{\infty} e^{dx}\cdot\prod_{m=0}^{kd-1}(k\partial_x + m)\prod_{j=0}^{d-1}\frac{1}{(\partial_x + j)^N}. \tag{3.116}$$

Conversely, we can determine $\tilde{L}_n^{N,k,d}$ assuming the above equation. This process corresponds to the B-model computation in the $N = k$ case, and combining it with generalized mirror transformation, we can construct "mirror computation" to the general type hypersurface M_N^k $(N < k)$.

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