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A Level Set Approach for Computing Discontinuous Solutions of a Class of Hamilton-Jacobi Equations

Yen-Hsi Richard Tsai[*†], Yoshikazu Giga[‡], and Stanley Osher[*§]

Abstract

We introduce two types of finite difference methods to compute the L-solution [14] and the proper viscosity solution [13] recently proposed by the second author for semi-discontinuous solutions to a class of Hamilton-Jacobi equations. By regarding the graph of the solution as the zero level curve of a continuous function in one dimension higher, we can treat the corresponding level set equation using the viscosity theory introduced by Crandall and Lions [7]. However, we need to pay special attention both analytically and numerically to prevent the zero level curve from overturning so that it can be interpreted as the graph of a function. We demonstrate our Lax-Friedrichs type numerical methods for computing the L-solution using its original level set formulation. In addition, we couple our numerical methods with a singular diffusive term which is essential to computing solutions to a more general class of HJ equations that includes conservation laws. With this singular viscosity, our numerical methods do not require the divergence structure of equations and do apply to more general equations developing shocks other than conservation laws. These numerical methods are generalized to higher order accuracy using WENO Local Lax-Friedrichs methods [21]. We verify that our numerical solutions approximate the proper viscosity solutions of [13]. Finally, since the solution of scalar conservation law equations can be constructed using existing numerical techniques, we use it to verify that our numerical solution approximates the entropy solution.

1 Introduction

Nonlinear Hamilton-Jacobi Equations arise in many different fields, including mechanics, calculus of variation, geometric optics, control theory, and differential games. Because of the nonlinearity, the Cauchy problems usually have non-classical solutions due to the crossing of characteristic curves.

For scalar equations of conservation law type, there is a well known theory regarding the existence and uniqueness of a weak solution called entropy solution, using the special integral structure of equation [23]. Advanced numerical methods, e.g. [15][16][30][34], have been developed and widely used to compute approximations that converge to the correct entropy solutions.

Nevertheless, this notion of weak solution cannot be applied to many fully nonlinear equations, e.g the eikonal equation $u_t + |\nabla u| = 0$. In 1983, Crandall and Lions [7] first introduced the notion of viscosity solution for this type of equations, based on a maximum principle and the order preserving property of parabolic equations. In general, for any given Hamilton-Jacobi equation of the form

$$u_t + H(x,t,u,Du) = 0,$$

where H is a continuous function from $\Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n$, non-decreasing in u, and Ω is an open subset of $\mathbb{R}^n$, there exists a unique uniformly continuous viscosity solution if the initial data is bounded, uniformly continuous.[1] The continuity of the solution can be understood intuitively from the 1D fact that "HJ equations are the conservation laws integrated once." The viscosity solution is sometimes understood as the limit of the solutions to the equation with vanishing viscosity.

Correspondingly, Crandall and Lions in [6] proved the convergence of two approximations to the viscosity solution of equations whose Hamiltonians only depend on Du. This was generalized by Souganidis to equations with variable coefficients in [31]. Many sophisticated numerical methods have since been developed [21][24][26][27].

However, there are problems in control theory and differential games which demand discontinuous solutions. The original viscosity theory does not apply to discontinuous initial data. The notion of semi-continuous viscosity solution has been introduced first by Ishii [18, 20] using an extension of Perron's method. Because of the non-uniqueness in Ishii's result, other notions of semi-continuous solutions were proposed by various authors [2][4] with different kinds of additional properties imposed on the Hamiltonian. Some of these notions need serious restrictions on the Hamiltonians and others are implicit in the sense that the processes of taking supremum and infimum are involved. As a consequence, one cannot develop

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[1]Notice that the conservation laws do not fall into this category because the corresponding H might not be monotone in u; e g shocks may develop from smooth initial data.

numerical methods to construct approximations. For an overview of the viscosity theory and applications, see [3] and [1].

Finally, for the class of equations with Hamiltonians $H(x, u, Du)$ nondecreasing in u, M.-H. Sato and the second author [14] introduced a new notion for semi-continuous solution. This notion of solution is defined by the evolution of the zero level curve of the auxiliary level set equation which embeds the original HJ equation. It is thus called the L-solution. In this article, we will devise a Lax-Friedrichs type scheme to compute approximation of the L-solution in its original formulation (i.e. level set). We will also show that with suitable CFL condition, our schemes keep the discrete version of an important property of this class of HJ equations.

When the Hamiltonian $H(t, x, u, Du)$ is not nondecreasing in u, the solution may develop shocks in finite time even if the initial data is continuous. Recently, a new notion called the proper viscosity solution is introduced by the second author [13] to track the whole evolution. This notion is consistent with the entropy solution when the equation is a conservation law. In order to approximate the proper viscosity solution of a class of more general HJ equations, we introduce a singular diffusive term in the vertical direction to the auxiliary level set equations so that the level curves will not overturn. In the case of conservation laws, the proper viscosity solution is consistent with the entropy solution. We will show numerically that the shock solutions we obtain from the regularized level set equations satisfy the "equal area" entropy condition and thus demonstrate the validity of our regularization terms. We emphasize, that based on our numerical results, the *global* property of our singular diffusion term regularizes our nonconservative level set equations such that the entropy condition is satisfied during the time iterations.

We remark that a simple monotone Lax-Friedrichs scheme seems to produce convergent approximations of the L-solution for the first class of HJ equations in their original form, even though the scheme does not follow the original definition of the L-solution. However, for the second class of equations, it is likely that the numerical approximations obtained this way converge to the wrong weak solution. This is a well known fact for monotone schemes for conservation laws in nonconservative form. In contrast, our numerical approximations for the corresponding "nonconservative" level set equations appear to converge to the right weak solution; i.e. the proper viscosity solution and, in case of conservation laws, the entropy solution.

In the following sections, we first review briefly the previous work on using level sets as a tool to analyze and compute solution of given PDEs. We then derive the level set equation from a given HJ equation. We then devise numerical methods for the level set equations for the computation of the HJ equations solutions according to the behavior of H_u. We extend each type of our numerical schemes to higher order accuracy using the WENO schemes devised in [21].

1.1 Analysis by the level set function

Osher [25] rediscovered a method of Jacobi [5] to study the Cauchy problem of general first order nonlinear equations through the aid of the level set equations. In that paper, Osher derived from the general first order equation

$$F(x, y, u, u_x, u_y) = 0$$

a time dependent Hamilton-Jacobi equation

$$\phi_t + H(x, y, t, \phi_x, \phi_y) = 0$$

and proved that the zero level set of its solution at time t is the set $\{(x, y) : u(x, y) = t\}$. With continuous initial values, the viscosity solution theory gives the existence and uniqueness of the solution to the time-dependent Hamilton-Jacobi equations provided that H does not change sign.

In [8], Evans used the level set method described in [25] to obtain the *level surface heat equation*. He gave the geometrical interpretation to the instant "unfolding" of multi-valued initial data of the solution of linear heat equation. By considering the viscous Burgers' equation

$$u_t + u u_x = \varepsilon u_{xx}, \quad \varepsilon > 0$$

as a lower order perturbation to the heat equation, Evans provided further analysis and a geometrical explanation as to how the term εu_{xx} keeps the solution from becoming multi-valued.

Recently, M.-H. Sato and the second author proposed to characterize the semi-continuous solutions of HJ equations using a similar approach. In this paper [14], they define the L-solution and prove existence and uniqueness of the L-solution with a class of Hamiltonian. We remark that the L-solution is equivalent to the conventional viscosity solution if the hypotheses are identical.

The idea is to represent the "graph" of a semi-continuous function $u(x)$ as the zero level set of a function $\phi : \mathbb{R}^2 \to \mathbb{R}^1$ with the requirement that every level set of ϕ is the graph of some function of x. More precisely, we define the subgraph of a function u be be $\mathrm{sg}(u) := \{(x, y) \in \mathbb{R}^2 : y \leq u(x)\}$ and the curve $\Gamma(t)$ to be the upper boundary of $\mathrm{sg}(u)$. For smooth functions $u(x, t)$, $\Gamma(t)$ is simply the graph of u at time t. The numerical construction of such functions is described in the appendix.

Consider the general first order equation:

$$u_t + H(t, x, u, u_x) = 0 \tag{1}$$

4

where u is a function from $\mathbb{R} \to \mathbb{R}$. Embed $\Gamma(t)$ as the zero level set of a function $\phi : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$; i.e. $\phi(t, x, y) = 0$ for all $(x, y) \in \Gamma(t)$ for each $t \in \mathbb{R}^+$. Taking partial derivatives, we have

$$0 = \frac{d}{dt}\phi(t, x, y) = \frac{d}{dt}\phi(t, x, u(t, x)) = \phi_t + \phi_y \cdot u_t;$$

$$0 = \frac{d}{dx}\phi(t, x, y) = \frac{d}{dx}\phi(x, u(t, x)) = \phi_x + \phi_y \cdot u_x.$$

Therefore, we have formally $u_t = -\phi_t/\phi_y$ and $u_x = -\phi_x/\phi_y$, and equation (1) becomes

$$\phi_t - \phi_y H(t, x, y, -\frac{\phi_x}{\phi_y}) = 0. \tag{2}$$

Under the hypotheses described in the next section, equation (2) together with a Lipschitz continuous initial function ϕ_0 which embeds the initial data u_0 is in the class of HJ equations that is known to have continuous solution $\phi(x, y, t)$.

However, in order to interpret the zero level set $\Gamma(t)$ of $\phi(x, y, t)$ as the evolution of the "graph" of $u(x, t)$, $\Gamma(t)$ has to be a one-to-one mapping of the variable x. We will call this requirement "non-overturning". If $\phi(x, y, 0)$ is set up with $\phi_y(x, y, 0) \geq 0$ everywhere, the non-overturning requirement is equivalent to the condition $\phi_y(x, y, t) \geq 0$ for all $t \geq 0$. In devising numerical approximations, it is important to make sure that this condition is true discretely.

In the following sections, we will use H_u to denote the partial derivative of $H(x, u, u_x)$ with respect to u for the original HJ equations. Finally, the level set function ϕ is set up to be non-decreasing in y initially in the examples of this paper.

2 Model Equations

We first consider the scalar 1D equation

$$u_t + H(x, u, u_x) = 0$$

with the Hamiltonian $H(x, u, u_x)$ satisfying the following properties:

1. H is Lipschitz in all its arguments

2. $\lim_{\lambda \to 0} \lambda H(x, u, p/\lambda)$ exists.

In addition, we are concerned with the following two classes of equations: 1)Equations with $H_u \geq 0$ but with discontinuous initial data; 2)Equations such as conservation laws that do not belong in the first class.

Let us consider the following two model equations, both of which can be in either the first or the second class depending on the parameters:

- Equations that contains both terms from conservation laws and fully nonlinear first order terms:

$$u_t + uu_x + a\,u|u_x| = 0, \ a \in \mathbb{R}; \tag{3}$$

The associated level set equation reads

$$\phi_t - y \cdot (a\,\mathrm{sign}(\phi_y)\,|\phi_x| - \phi_x) = 0. \tag{4}$$

We can see that the value of a determines an important property of H_u : for $a \geq 1$, H_u will be non-decreasing. Thus the viscosity theory applies if the initial data is uniformly continuous, and we know the solution will be uniformly continuous. This falls into the first class of equations. For $0 \leq |a| < 1$, H_u changes signs according to the value of u_x. Then equation (3) belong to the second class. Notice that if $a = 0$, we have the inviscid Burgers' equation.

- Equations that prescribe the normal motion of the graph of u:

$$u_t - v(u)\sqrt{1 + u_x^2} = 0; \tag{5}$$

The corresponding level set equation is

$$\phi_t + \mathrm{sign}(\phi_y)\,v(y)|\nabla\phi| = 0. \tag{6}$$

The function v is the normal velocity of the graph of u, or the level sets of ϕ. If v ever decreases, then $H_u \leq 0$ and the equation fails to be in the first class.

The Role of $\mathrm{sign}(\phi_y)$

Let us look at the characteristics of equation (6) more carefully. The term $\mathrm{sign}(\phi_y)$ flips the direction of the characteristics whenever ϕ_y changes signs. If the characteristics on the upper part of the jump travel faster than at the lower part (i.e. $v(y)$ is increasing), the overturning will develop. With the $\mathrm{sign}(\phi_y)$ term, whenever overturning just happens, the direction of a characteristic will be reversed, making it travel backward and thus eliminate the overturning. However, this fact is not directly suitable for numerical implementation.

2.1 Geometrical Explanation of the Non-overturning Conditions

As mentioned earlier, we need to pay special attention in order to prevent the overturning of the level curves of ϕ. One equivalent criterion is to demand the minimum principle: $\phi_y(x, y, t) \geq 0$ for $t \geq 0$.

In light of the level set equation (6), we have a more geometrical requirement on the speed function v. By the method of characteristics, we know that $v(y)$ prescribes the normal velocity of the level sets of ϕ. On the vertical segments of the level sets, which correspond to jumps in u, $v(y)$ prescribes the horizontal velocity according to y. Overturning will happen if $v(y)$ is increasing, since the upper part of the jump of u moves faster than the lower part. See figure 1.

Consider the primitive function of v:

$$V(y) = \int v(s)ds.$$

The non-increasing condition of v translates to the *concavity* of V! This fact reminds us of one of the entropy conditions for conservation laws with non-concave flux function. It says that the entropy solution of the conservation law with non-convex flux f is the classical solution of the conservation law with the flux f^*, where f^* is the minimal concavification of f over the increasing jump interval. This, in turn, provides us a hint on the regularization of HJ equations (6) — we need to impose a regularization that concavifies the primitive function on the vertical segments of the level sets and nowhere else. We shall demonstrate numerically that our proposed singular diffusive regularization term does exactly that in a later part of this article.

2.2 Equations with Hamiltonian $H_u \geq 0$

We first consider the equations $u_t + H(t, x, u, u_x) = 0$ for which $H_u \geq 0$, and the corresponding level set equation equation (2) can be simplified to

$$\phi_t - \tilde{H}(t, x, y, \phi_x, \phi_y) = 0. \tag{7}$$

E.g. with the homogeneity hypothesis, the factor ϕ_y in equation (2) can be brought into the original Hamiltonian $H(x, y, \phi_x/\phi_y)$, which then transforms into a new Hamiltonian $\tilde{H}(x, y, \phi_x, \phi_y)$. The reduction from equation (5) to equation (6) is one such example.

The minimum principle

The assumption that the Hamiltonian is non-decreasing in u has an important consequence. We present here an argument about this minimum principle based on

7

Figure 1: Overturning is caused by the normal velocity which is increasing in the
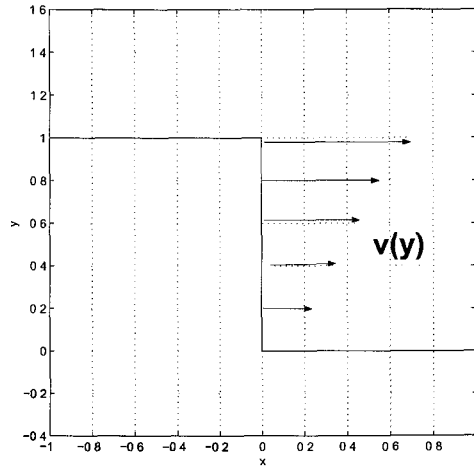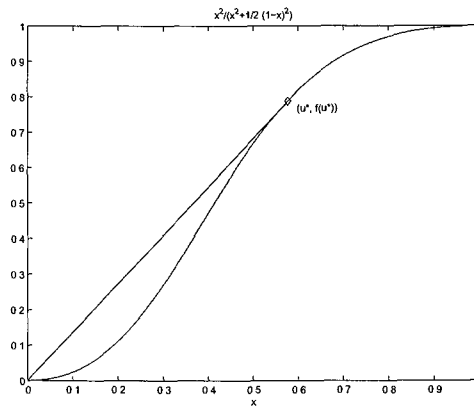y-direction.

Figure 2: The concavification of the flux in Buckley-Leverett equation

8

an argument in [14]. Consider $\phi^h(x, y, t) := \phi(x, y + h, t)$, where $h > 0$ and $\phi(x, y, t)$ is the uniformly continuous viscosity solution of equation (7) with uniformly continuous initial data $\phi_0(x, y)$. By definition, ϕ^h is the viscosity solution of

$$\phi_t - \tilde{H}(t, x, y + h, \phi_x, \phi_y) = 0$$

with initial data $\phi_0^h(x, y)$. Let v be a C^1 function, then at any local minimum of $\phi^h - v$,

$$v_t - \tilde{H}(t, x, y + h, v_x, v_y) \geq 0.$$

It is clear that if $H_y \geq 0$, then $\tilde{H}_y \geq 0$, where $\tilde{H}$ is the Hamiltonian of the corresponding level set equation. Consequently, we have

$$v_t - \tilde{H}(t, x, y, v_x, v_y) \geq v_t - \tilde{H}(t, x, y + h, v_x, v_y) \geq 0$$

at any local minimum of $\phi^h - v$ for any C^1 test function v. Thus ϕ^h is a viscosity super-solution of equation (7).

If $\phi^h(x, y, 0) - \phi(x, y, 0) \geq 0$ for all x and y, then $\phi^h(x, t) \geq \phi(x, t) \geq 0$ by the comparison principle (the reader is referred to [11][19] for the proof). This basically says that if $\phi_y(x, y, t = 0) \geq 0$ initially, then $\phi_y(x, y, t) \geq 0$ for all time! It also implies that $\{\phi = c\}$ *will remain as a graph throughout the evolution.* Therefore, we can remove the $\text{sign}(\phi_y)$ term from the derived level set equation (2) of this class of equation.

Without causing confusion, we shall continue using $H(x, y, \phi_x, \phi_y)$ in place of $\tilde{H}(x, y, \phi_x, \phi_y)$ in the following parts of this article.

The Lax-Friedrichs schemes for the level set equation

Following the methods originally conceived for HJ equations $\phi_t + H(D\phi) = 0$ in [27], see also [26], and suppressing the dependence of H on x and y, we use the Local Lax-Friedrichs (LLF) flux

$$\hat{H}^{LLF}(p^+, p^-, q^+, q^-) = H(\frac{p^+ + p^-}{2}, \frac{q^+ + q^-}{2})$$
$$-\frac{1}{2}\alpha^x(p^+, p^-)(p^+ - p^-) - \frac{1}{2}\alpha^y(q^+, q^-)(q^+ - q^-),$$

for the approximation of H. In the above scheme,

$$\alpha^x(p^+, p^-) = \max_{p \in I((p^+, p^-), C' \leq q \leq D} |H_{\phi_x}(p, q)|,$$

9

$$\alpha^y(q^+, q^-) = \max_{q \in I((q^+, q^-), A \le p \le B} |H_{\phi_y}(p, q)|,$$

$$I(a, b) = [\min(a, b), \max(a, b)],$$

and $p^{\pm}$, $q^{\pm}$ are the forward and backward approximations of ϕ_x and ϕ_y respectively.

We can use a simple Forward Euler time discretization and obtained the fully discretized scheme:

$$\phi_{i,j}^{n+1} = \phi_i^n + \Delta t\, H^{LLF}(x_i, y_j, D_x^+ \phi_{i,j}^n, D_x^- \phi_{i,j}^n) \tag{8}$$

for the level set equations with H independent of ϕ_y (after removing $\text{sign}(\phi_y)$), and

$$\phi_{i,j}^{n+1} = \phi_i^n + \Delta t\, H^{LLF}(x_i, y_j, D_x^+ \phi_{i,j}^n, D_x^- \phi_{i,j}^n, D_y^+ \phi_{i,j}^n, D_y^- \phi_{i,j}^n) \tag{9}$$

for equations such as equation (5), since the Hamiltonians depend on ϕ_y. Here, $\phi_{i,j}^n := \phi(x_i, y_j, t_n)$, and Δx, Δy, and Δt are the step size in x, y and t.

Rewrite the above schemes in the form

$$\phi_{i,j}^{n+1} = G(x_i, y_j, \phi_{i+1,j+1}^n, \phi_{i+1,j}^n, \phi_{i,j+1}^n, \phi_{i,j}^n, \phi_{i-1,j}^n, \phi_{i,j-1}^n, \phi_{i-1,j-1}^n).$$

If G is non-decreasing in all its arguments except x_i and $\Delta_y^- \phi_{i,j}^n \ge 0$ for all $i, j \in Z_d$, then

$$\begin{aligned}
\Delta_y^- \phi_{i,j}^{n+1} &= \phi_{i,j}^{n+1} - \phi_{i,j-1}^{n+1} \\
&= G(x_i, y_j, \phi_{i+1,j+1}^n, \phi_{i+1,j}^n, \phi_{i,j+1}^n, \phi_{i,j}^n, \phi_{i-1,j}^n, \phi_{i,j-1}^n, \phi_{i-1,j-1}^n) \\
&\quad - G(x_i, y_{j-1}, \phi_{i+1,j}^n, \phi_{i+1,j-1}^n, \phi_{i,j}^n, \phi_{i,j-1}^n, \phi_{i-1,j-1}^n, \phi_{i,j-2}^n, \phi_{i-1,j-2}^n) \\
&\ge 0.
\end{aligned}$$

Because of the hypothesis that $H_y \ge 0$, our Lax-Friedrichs schemes preserve the minimum principle discretely (i.e. given $\Delta_y^+ \phi_{i,j}^n \ge 0$ for all $i, j \in Z_d$, then $\Delta_y^+ \phi_{i,j}^{n+1} \ge 0$ for all $i, j \in Z_d$) if

$$\frac{\Delta t}{\Delta x} \le C \min(1/\|H_{\phi_x}\|_\infty, 1/\|H_{\phi_y}\|_\infty),$$

where $C = 1$ for equation (8) and $C = 2$ for equation (9).

Extension to higher order of accuracy

To achieve higher order accuracy and have less numerical disspation, we can discretize the spatial derivatives using WENO schemes described in [21], which essentially replace the forward/backward differencing by higher order WENO approximations. For higher order accuracy in time discretization, the TVD third order Runge-Kutta method from [30] can be used.

2.2.1 Examples

We provide here some numerical computations for some equations that belong to the class we are considering.

Constant motion along the normal

Consider the equation

$$u_t + c\sqrt{1 + u_x^2} = 0. \tag{10}$$

Given a continuous initial data, it is well-known that the following equation corresponds to motion of the graph with constant normal velocity c.

Using the notion of the L-solution, we can easily describe the motion defined by equation (10), even with piecewise continuous data. The corresponding level set equation is simply

$$\phi_t + c|\nabla\phi| = 0,$$

which describes the constant normal speed motion of each level set of ϕ. We emphasize here that since the level sets of ϕ are continuous, we can simply use the existing classical viscosity theory for the solution. Figure 3 shows the zero level curves of ϕ in different times. The reader can see that each curve is equidistant from the original curve (shown as red).

Model equation $u_t + uu_x + a\,u|u_x| = 0$

With $a \geq 1.0$, we know that this model equation retains the property that $\phi_y \geq 0$ for all time. Figure 4 show the computational result using (8) and third order WENO-LLF. The numerical solutions of this equation are computed with $a = 2.0$. Finally, we show that our Lax-Friedrichs type scheme cannot be applied to compute solutions for equation with $a < 1$. See figure5.

11

Figure 3: Numerical solution by first order LLF method for the Riemann problem for equation (10) with $u_L = 1.0$, $u_R = 0.0$, and $c = 1.0$. We plotted the zero level set at time $t = 0, 0.2, 0.4$ and 0.6.
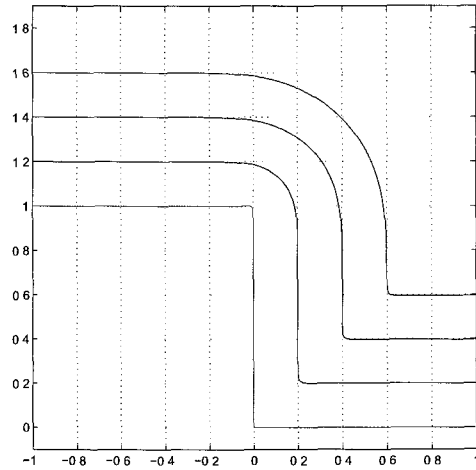

Figure 4: Numerical solution using third order WENO-LLF to the Riemann problem for equation (3) with $u_L = 0.0$, $u_R = 0.1$, and $a = 2.0$. We plotted the zero level set at time $t = 0, 0.1$, and 0.2.
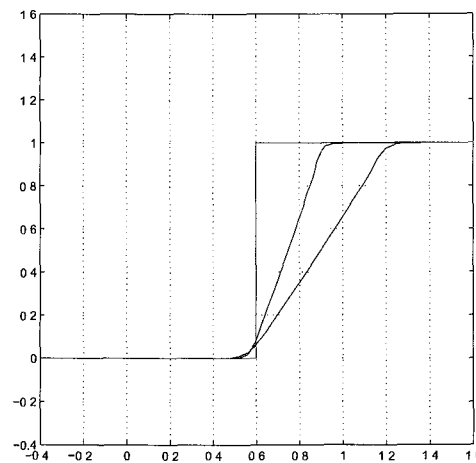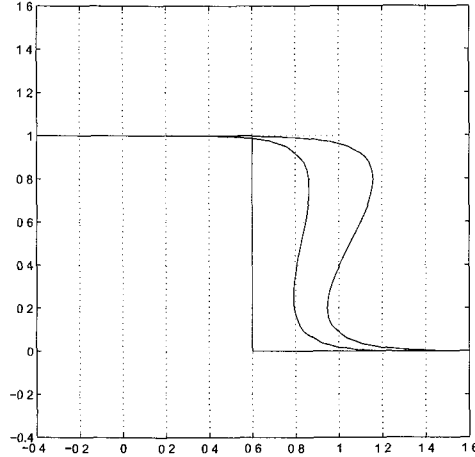
Figure 5: Incorrect (as expected) numerical solution to the Riemann problem for equation (4) with $u_L = 1.0$, $u_R = 0.0$, and $a = 0.1$. We plotted the zero level set at time $t = 0, 0.5$, and 1.0.

2.3 Singular Viscosity Regularization

Consider the model equation (4) with $|a| < 1$, and equation (5) with $v(y)$ non-decreasing. We know that it no longer has the minimum principle in ϕ_y, and "over-turning" or "folding" in its solution might develop.

Motivated by the work on a type of singular diffusion in [9, 10, 22], we will add a similar singular diffusion term in the y-direction to both our model equations:

$$M|\nabla\phi|\frac{\partial}{\partial y}(\frac{\phi_y}{|\phi_y|}).$$

We first notice that this viscosity is activated *only when* $\mathrm{sign}(\phi_y) = \phi_y/|\phi_y|$ changes signs! With M sufficiently large, this term $\partial(\mathrm{sign}(\phi_y))/\partial y$ can be shown, at least formally, to concavify the primitive of the speed function on the vertical part of the level sets [12].

We briefly describe how to find the minimum value of M. Consider the primitive function $V(y)$ of the speed function $v(y)$ of equation (5) over $[a, b]$ that is a jump of u. Let V^* be the function whose graph is the upper boundary of the convex hull of V. Let $V_M = V^* + M$. We claim that M has to be large enough such that V_M is tangent to or never crosses V^*. See figure 6 for an example with $V(y) = y^2/2$. Since the purpose of this paper is to provide the numerics, we refer the reader to the recent paper of the second author [12] for a formal reasoning.

13

Figure 6: $V(y) = y^2/2$ on $[0, 1]$. The minimum value of M should be $1/8$.
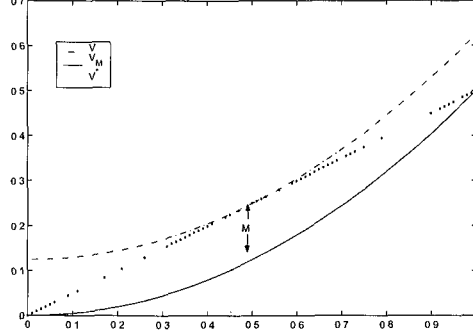
Alternatively, we describe another intuitive motivation behind this diffusion term: consider the Heaviside function $y = H(x)$ and the level set function $\phi(x, y)$ for which this is the zero level set. If we treat the zero level set of ϕ locally as a function of y wherever it is vertical, we see that the "overturning" will increase the total variation of $\{\phi = c\}$ as a function of y. This motivates the following regularization:

$$\min_{\phi} \int |\phi_y| \, dy.$$

The corresponding Euler-Lagrange derivative is

$$\frac{\partial}{\partial y} \left(\frac{\phi_y}{|\phi_y|} \right).$$

To make the diffusion term geometrical, i.e. invariant of the choice of level set function, we multiply it by $|\nabla \phi|$ and arrive at the same diffusion term. Of course, in using this argument, we have to assume that the Hamiltonian is also the Euler-Lagrange derivative of some variational integral.

Now, let us go back to our model equation with this viscosity term:

$$\phi_t - y \cdot \left(a \operatorname{sign}(\phi_y) |\phi_x| - \phi_x \right) = M |\nabla \phi| \frac{\partial}{\partial y} \left(\frac{\phi_y}{|\phi_y|} \right).$$

We use central differencing to approximate the singular diffusion term on the right hand side:

$$\sqrt{(D_x^0 \phi_{i,j})^2 + (D_y^0 \phi_{i,j})^2} \cdot \frac{\tanh(\gamma \, D_y^+ \phi_{i,j}) - \tanh(\gamma \, D_y^- \phi_{i,j})}{\Delta y},$$

14

where the signum function $\phi_y/|\phi_y|$ is approximated by $\tanh(\gamma\phi_y)$ with $\gamma = 1/\Delta y$, and

$$\tanh(\gamma\, D_y^+\, \phi_{i,j}) = \tanh(\gamma\, \frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta y})$$

is an approximation of $\phi_y/|\phi_y|$ evaluated at $(x_i, y_{j+1/2})$. Similarly $\tanh(\gamma\, D_y^-\, \phi_{i,j})$ is an approximation for $\phi_y/|\phi_y|$ at $(x_i, y_{j-1/2})$. The partial derivative ϕ_x on the left hand side is approximated by upwind differencing:

$|a| < 1$:

$$y \geq 0 : \quad \phi_x \leftarrow D_x^- \phi$$
$$y < 0 : \quad \phi_x \leftarrow D_x^+ \phi$$

$|a| \geq 1$:

$$\text{sign}(D_y^0\phi)\, a\, y \leq 0 : \quad \phi_x \leftarrow (D_-^x \phi)^+ - (D_+^x \phi)^-.$$
$$\text{sign}(D_y^0\phi)\, a\, y > 0 : \quad \phi_x \leftarrow -(D_-^x \phi)^- + (D_+^x \phi)^+.$$

Here, p^- denotes the negative part of p (with sign) and p^+ the positive part.

Because of the singular diffusion term, the stability condition becomes:

$$\frac{\Delta t}{\Delta x^3} \leq C_{M,H},$$

where $C_{M,H}$ is a constant depending on the diffusion coefficient M and the maximum values of H_{ϕ_x} and H_{ϕ_y}.

Extension to higher order accuracy

Again, we may combine the central differencing approximation of the viscosity term and the WENO-LLF scheme described in the earlier section for numerical computation. This is needed for future generalization to more complex equations or to system of equations because upwinding is no longer easy.

2.3.1   Test on the model equation: $u_t + u u_x + a\, u|u_x| = 0$

We first test our numerical scheme for the case $a = 0.1$ which cannot be handled by the Lax-Friedrichs scheme (8). Figure 7 shows that the "overturning" is prevented in contrast to the result shown in figure 5.

15

Figure 7: Numerical solution to the Riemann problem for equation (4) with $u_L = 1.0$, $u_R = 0.0$, $a = 0.1$, and $M = 0.2$. We plotted the zero level set at time $t = 0, 0.5$, and 1.0.
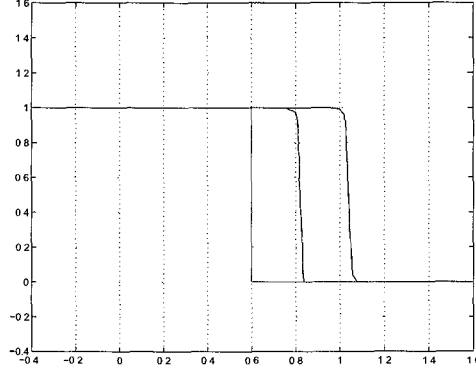
2.3.2 Tests on conservation laws

As we have mentioned earlier, equation (3) with $a = 0$ is equivalent to Burgers' equation in non-conservative form. Here we go one step further to demonstrate numerically that our regularization is equivalent to the entropy condition for conservation laws equations.

We consider the conservation laws

$$u_t + f(u)_x = 0 \tag{11}$$

with $f' \geq 0$ and its corresponding *linear* level set equation

$$\phi_t + f'(y)\phi_x = 0. \tag{12}$$

The numerical results shown in the following examples are obtained by plotting the zero contour of the numerical solution ϕ to the regularized equation:

$$\phi_t + f'(y)\phi_x = M\,|\nabla\phi|\,\frac{\partial}{\partial y}\left(\frac{\phi_y}{|\phi_y|}\right). \tag{13}$$

Burgers' Equation

With $f(u) = u^2/2$, we have the inviscid Burgers equation in non-conservative form. The corresponding level set equation becomes a linear transport equation with variable coefficient:

$$\phi_t + y\phi_x = 0.$$

16

It is then clear that the graph will overturn if u is decreasing in x.

We consider the Riemann problem $u(x) = u_L = 4.0$ for $x < 0.0$ and $u(x) = u_R = 0.0$ for $x \geq 0.0$. See figure 8. The result shown in figure 8 verifies the Rankine-Hugoniot shock speed:

$$s = \frac{[f]}{[u]} = 2.0.$$

Figure 9 shows a similar computation with $u_L = 1.0$, $u_R = 0.0$ and two different values of the diffusion coefficients ($M = 0.04$ and $M = 1.0$). We can see that overturning will develop if M is not large enough, and if it is sufficiently large, this coefficient does not affect the shock speed as predicted in [12] (the critical value for M is 0.125 in this case). We also compute the approximation obtained with no diffusion term (i.e. $M = 0$) and plot it (green curve) against the one obtained from $M = 0.2$ (blue curve), and show that the "equal-area" entropy condition is satisfied by the latter (blue curve). See figure 10. Figure 11 shows the result of a Riemann problem with two shocks and a rarefaction. We also verify that the shocks in this case travel with the right speed.

Finally, we compute solution to Burgers' equation starting with a sine curve: $\alpha \sin(\pi x) + \beta$. Figure 12 shows a first order approximation of the well-known N-wave starting with initial conditions using $\alpha = -0.8$, $\beta = 0.0$. Our diffusion term successfully keeps the vertical part from overturning. In figure 13, we obtained the numerical approximation, with $\alpha = -1.0$ and $\beta = 0.5$, using the fifth order WENO local Lax-Friedrichs in space and third order TVD Runge-Kutta in time. One can see that the excessive drop in height caused by numerical diffusion using the standard first order Lax-Friedrichs method (figure 12) is greatly reduced.

Consider the conservation law of equation 11 with initial condition $u_0(x)$. Up to shock develops, the exact solution is defined implicitly by

$$u = u_0(x - f'(u)\, t).$$

Thus, for every fixed (x, t), this can be thought of as a root finding problem in u using Newton's iterations

$$u^{\nu+1} = u^\nu - \frac{f(u^\nu)}{f'(u^\nu)},$$

where $f(u) := u_0(x - f'(u)\, t) - u$. We use this simple iterative method to find the smooth solution to the machine accuracy and compare it to the solution obtained by our level approach. Some results are provided in table 1 and table 2, in which we use a third order linear interpolation to approximate the location of the zero level curve on each grid point. In particular, table 2 shows a third order convergence of the numerical solutions in the region excluding a $5\Delta x$ neighborhood of the shock.

17

Figure 8: Numerical solution (WENO5-LLF) to the Riemann problem of Burgers' Equation with $u_L = 4.0$, $u_R = 0.0$, and $M = 2.1$. We plotted the zero level set at time $t = 0, 2.0$, and 2.5.
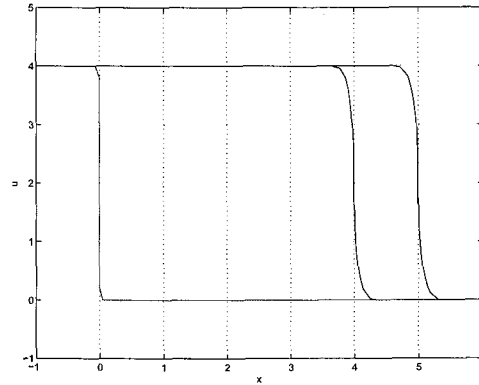

Figure 9: Numerical solution to the Riemann problem of Burgers' Equation with $u_L = 1.0$, $u_R = 0.0$. We plotted the zero level sets at time $t = 0$ and 0.5 obtained from $M = 0.04$ and 1.0.
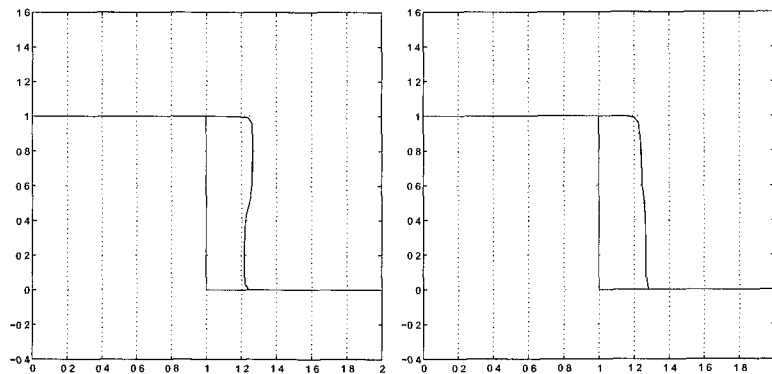

Table 1: Numerical convergence (WENO5-LLF-RK3) of Burgers' Equation with initial values $-0.8 \sin(\pi x) + 0.5$. $T = 0.2$.

	dx=2/50	dx=2/100	dx=2/200	dx=2/400
max error	5.99774e-06	3.65247e-07	3.39605e-08	3.88563e-09
rate		4.0375	3.4269	3.1276

Figure 10: Numerical solution to the Riemann problem of Burgers' Equation with $u_L = 1.0$, $u_R = 0.0$. We plotted the zero level sets at time $t = 0$ and 0.5 obtained from $M = 0.2$ and 0.0.
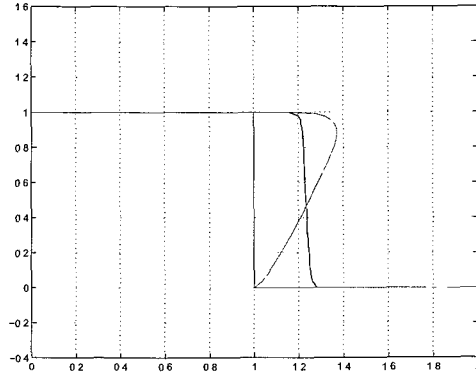

Figure 11: Numerical solution (WENO5-LLF) to the Riemann problem of the Burgers Equation with $u_L = 0.1$, $u_{M1} = 1.8$, $u_{M2} = 1.0$, $u_R = 0.5$, and $M = 0.2$. We plotted the zero level set at time $t = 0$ and 0.192.
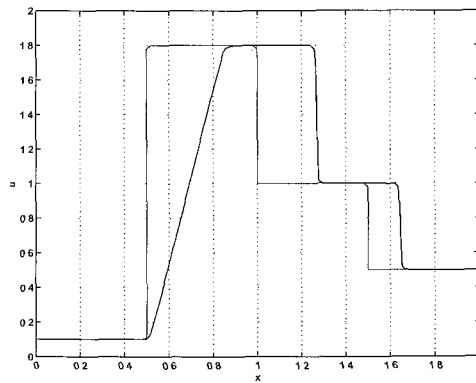

Table 2: Numerical convergence of WENO5-LLF-RK3 on Burgers' Equation with initial values $-0.8\sin(\pi x)$, $M = 0.2$, $T = 0.384$.

	dx=2/25	dx=2/50	dx=2/100
max error	0.007603086	0.0009753479	9.64295e-05
rate		2.9626	3.3384

19

Figure 12: Numerical solution to the Burgers Equation with sine wave as initial data. We plotted the zero level set at time $t = 0$ and 0.5.
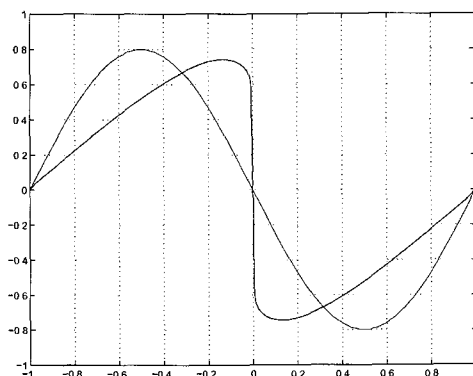

Figure 13: Numerical solution (WENO5-LLF) to the Burgers Equation with shifted sine wave as initial data. We plotted the zero level set at time $t = 0$ and 0.5.
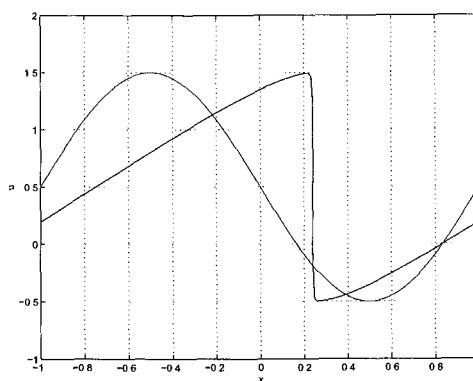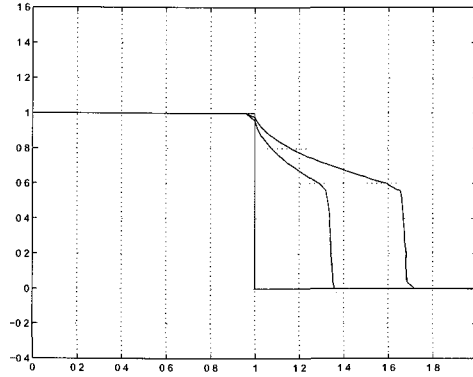

20

Figure 14: Numerical solution to the Riemann problem of the Buckley-Leverett Equation with $u_L = 1.0$, $u_R = 0.0$ and $a = 0.5$. We plotted the zero level set at time $t = 0, 0.25,$ and 0.5.

Buckley-Leverett Equation

Finally, we test our numerical method for equation (13) to substantiate our assertion that the singular diffusion term minimally concavifies the flux function f over the jump interval. We solve the Riemann problem of the conservation law

$$u_t + f(u)_x = 0$$

with

$$f(u) = \frac{u^2}{u^2 + a(1 - u)^2}, \quad a > 0, \quad u \in [0, 1].$$

and $u_L = 1.0$, $u_R = 0.0$.

The upper boundary of the convex hull of sg(f) consists of a straight line segment L from $(0, 0)$ to $(u^*, f(u^*))$ followed by $(u, f(u))$ for $u \in [u^*, 1]$, where L is a tangent line of $f(u)$. See figure 2.The slope of L is also the correct shock speed for the Riemann problem. With $a = 0.5$, a simple calculation shows that $u^* = 1/\sqrt{3} \doteq 0.57735$.

Figure 14 shows the expected rarefaction from u_L to u^* and a shock between u^* and u_R. Figure 15 shows an overlap of the solutions obtained with and without regularization. One can observe that the "equal-area" entropy condition is satisfied.

21

Figure 15: Numerical solution to the Riemann problem of the Buckley-Leverett Equation with $u_L = 1.0$, $u_R = 0.0$ and $a = 0.5$. We plotted the zero level sets at time $t = 0$ and 0.3 obtained from $M = 0.2$ and 0.0. The little fragment of contour at the lower part of the jump is due to the contour plotter.
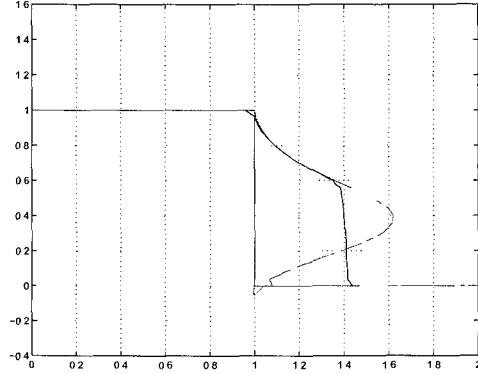
2.3.3 A two dimensional example

To show that our numerical schemes extend naturally to higher dimensions, we show our numerical solutions for the following equation

$$u_t - u\sqrt{1 + u_x^2 + u_y^2} = 0.$$

When u is continuous, this is motion in normal direction of the graph of u with speed u. The corresponding level set equation

$$\phi_t + z\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2} = 0$$

gives straightforward geometrical interpretation of the solution as level sets of ϕ. Results are shown in figure 16 and figure 17. In both figures, the initial data are represented by the blue surfaces (one cubical and the other spherical) and the solutions during later times are represented by the green and yellow surfaces. The left subfigures are computed without the regularization whereas the right ones are regularized by the singular diffusive term $M|\nabla\phi|\partial/\partial(\phi_z/|\phi_z|)$.

2.3.4 The Vanishing Viscosity Approach

Consider the Lax-Friedrichs type scheme of the following form:

$$u_i^{n+1} = u_i^n - \Delta t H\left(x_i, u_i^n, D_x^0 u_i^n\right) + c\Delta_x^+ \Delta_x^- u_i^n / 2, \tag{14}$$
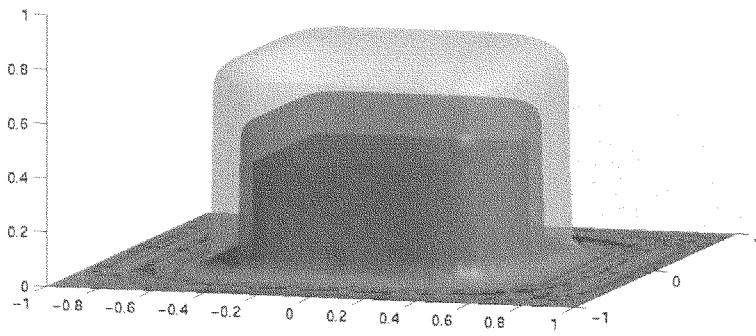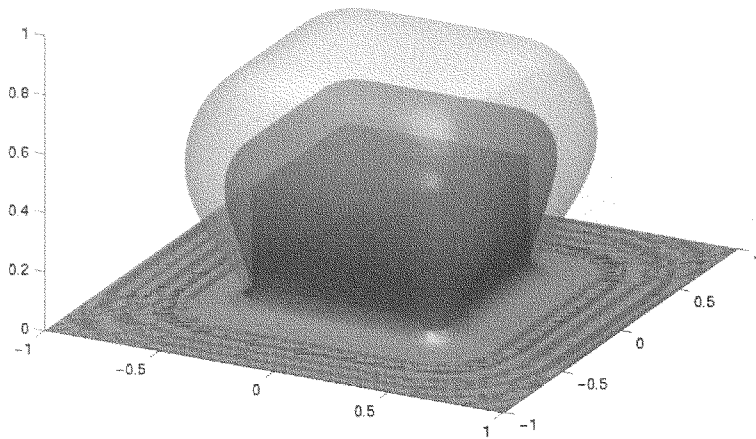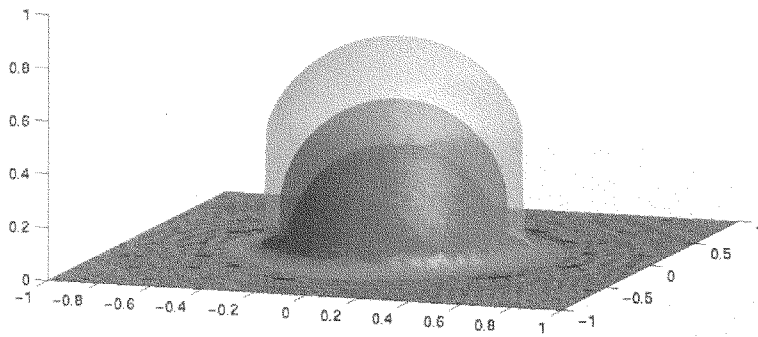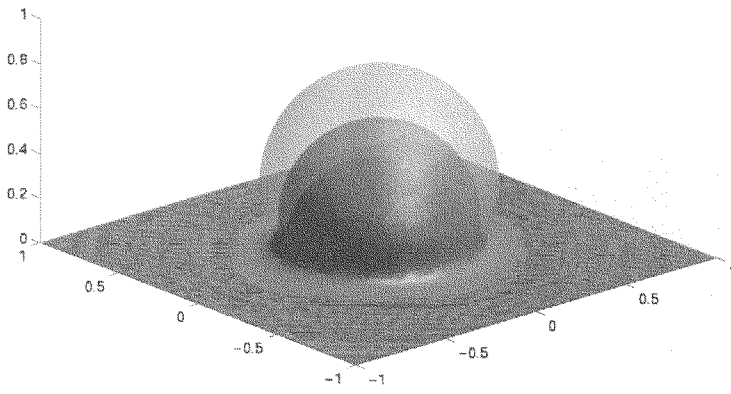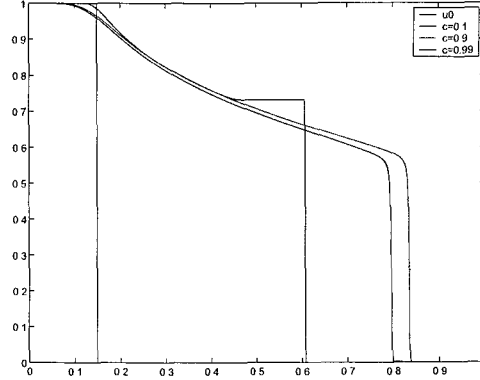
Figure 16:

Figure 17:

Figure 18: The numerical solutions of the Buckley-Leverett equation in non-conservative form obtained from the monotone Lax-Friedrichs scheme (14). The approximations are computed to $t = 0.1$ on $[0, 1]$ with $2, 500$ grid points.

where $\Delta_x^{\pm} u_i^n$ denotes the undivided forward/backward difference of u_i^n and $0 \leq c \leq 1$. With

$$\Delta t \leq \min\left(\frac{1-c}{||H_u||_\infty}, \frac{c\Delta x}{||H_{u_x}||_\infty}\right) \tag{15}$$

the scheme is monotone and seems to yield convergent approximations for the equations with $H_u \geq 0$.

However, this scheme is not suitable for the HJ equations whose solutions develop shocks. Figure 18 shows the numerical approximations using (14) with different values of c and fairly small grid size. The left most curve is the initial data. The remaining curves from left to right are obtained using $c = 0.1, 0.99$, and 0.9 respectively. One can see that the numerical solutions converge to different functions.

We maintain that our level set approach is no less efficient since we can do the computation locally around the zero level curve [28]. Also, the level set approach is more "natural" since it is a part of the theoretical notions of solutions to the HJ equations that we are concerned with.

3 Summary

In this paper, we provided two classes of finite difference methods for the computation of the semi-continuous L-solution of a class of HJ equations. By studying

25

the level set equation derived from the HJ equations, we pointed out the necessary condition for the validity of the solution defined as the zero contour line of the level set function. We have also discussed the geometrical interpretation of the motion of the solution embedded in the level set function. The remarks provide hints as how to regularize the zero level curve motion so that it can be interpreted as the graph of a function.

For the class of HJ equations with $H_y \geq 0$, we applied a straightforward Lax-Friedrichs type scheme with possibility of extension to higher order accuracy. We showed numerically that the singular diffusion term $|\nabla \phi| \, \partial(\phi_y/|\phi_y|)/\partial y$ can be applied to compute the shock solution for the class of HJ equations we considered. In particular, we numerically verified that our numerical schemes yield approximations compatible with the entropy solution of a conservation laws equation with non-convex flux. Of course, we have also shown the extension of our numerical schemes to higher order WENO-local Lax-Friedrichs schemes.

Lastly, we remark here that our numerical schemes for the derived level set equations can be computed locally around the zero level curve using the technique described in [28] for efficiency.

4   Systems of Conservation Laws

We are generalizing the result of our singular viscosity to study the solution of conservation laws system and the link to Riemann invariants. Here we briefly describe how we are approaching this problem.

Let $\vec{u} = (u, v) \in \mathbb{R}^2$, $\phi(t, x, y) : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ be the vector values level set function such that $\phi(t, x, \vec{u}(t, x)) = 0$. The system

$$\vec{u}_t + A(\vec{u})\vec{u}_x = 0$$

can be formally translated to

$$\phi_t + \phi_y A(y)\phi_y^{-1}\phi_x = 0.$$

We shall use the Riemann invariants for the 2×2 system to diagonalize $A(y)$ and desingularize the term ϕ_y^{-1}.

We propose a singular diffusion term similar to the scalar one we used. With an abuse of notation, this term can be written as

$$|\nabla_{x,y}\phi|\nabla_y \cdot (|\nabla_y\phi|^{-1}\nabla_y\phi),$$

where $\nabla_{x,y}\phi$ is the Jacobian matrix of ϕ with respect to x and y, $\nabla_y\phi = \phi_y$ is the Jacobian matrix of ϕ, and $|A| := \sqrt{AA^*}$ is the Euclidean norm of the matrix A.

26

Acknowledgment

Appendix: Initializing the Level Set Function

Since the introduction of the level set methods [26], several techniques have been developed to compute the level set function for a given curve Γ (this is called the initialization step in standard level set jargon). We point out several of such techniques for completeness of the numerics.

If the curve Γ in question is the graph of a smooth function $u(x)$, the easiest way to initialize the level set function $\phi(x, y)$ is simply

$$\phi(x, y) = y - u(x).$$

However, if $u(x)$ is only piecewise smooth, we cannot use the same approach to initialize ϕ, since the function obtained this way becomes discontinuous.

The next easy way is to use the signed distance function of the upper boundary of sg(u) as the level set function; that is, consider the upper boundary of sg(u) as a continuous curve $\Gamma \in \mathbb{R}^2$, set

$$\phi(x, y) = m(x, y) \operatorname{dist}((x, y), \Gamma),$$

where $m(x, y) = \operatorname{sgn}(y - u(x))$. We notice that the level set function constructed this way is Lipschitz continuous everywhere.

There are a variety of techniques to compute the distance function. For example, the fast marching method [17][29][33] can be used to find ϕ quickly by constructing a first order approximation of the solution of the eikonal equation

$$|\nabla \phi| = 1.$$

For a piecewise linear curve, the method describe in [32] can be used for accurate and rapid construction of the distance function. The initialization of the numerical examples presented in this paper are all constructed this way.

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27

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