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Abstract. Gonda and Gomi (T.Gonda, H.Gomi, *Ann. Glaciology,* **6** (1985), 222 224) have grown large elongated ice crystals from supersaturated vapor. Theoretically this problem may be recast in a framework similar to that used by Seeger (A.Seeger, *Philos. Mag.*, ser. 7, **44**, no 348, (1953) 1–13) for studies of planar crystals. The resulting set of equations is of Stefan type. We also include the Gibbs-Thomson relation on the crystal surface. In order to make this system tractable mathematically we assume that the Wulff crystal is a fixed cylinder. Subsequently we study a weak form of our system. We show local in time existence of solutions assuming that the initial shape is an *arbitrary* cylinder. We comment on properties of weak solutions.

1 Introduction

The goal of the present paper is to pursue a mathematical study of crystals grown from a dilute solution or from supersaturated vapor. These problems are not new, but it does not mean they have been fully solved. Among the numerous theoretical and experimental studies we mention just a few. Gonda and Gomi [GG] investigated samples of long solid prisms of ice crystal formed in the atmosphere and found mostly in Antarctica. In their laboratory they have grown specimens of elongated prisms of ice crystals. They studied morphological stability of the growth. A typical manifestation of an instability is an inclusion of an air bubble in the crystal. Their work motivates us to consider evolution of ice crystals which are deformed Wulff shapes.

The necessary theoretical background is provided by Seeger, see [Se]. He has studied the growth of planar crystals, i.e. polygons grown in dilute solution. That is he assumed the process was so slow that this justifies the quasi-steady approximation of the diffusion equation. In other words the concentration (or supersaturation) σ satisfies the equation

$$\Delta\sigma = 0 \tag{1.1a}$$

outside a crystal Ω. It is also physically reasonable to assume that the domain is unbounded, i.e (1.1a) holds in $\mathbb{R}^3 \setminus \Omega$. Moreover, we may also assume that σ has a specific value at infinity, i.e.

$$\lim_{|x|\to\infty} \sigma(x) = \sigma^\infty. \tag{1.1b}$$

The velocity of the growing crystal is determined by the normal derivative of σ at the surface,

$$\frac{\partial\sigma}{\partial\mathbf{n}} = V \tag{1.2}$$

where $\mathbf{n}$ is the outer normal.

The value of σ at the surface is coupled to the surface velocity and its curvature through the Gibbs-Thomson relation

$$-\sigma = -\mathrm{div}\ \xi - \beta V, \tag{1.3}$$

where β is the kinetic coefficient and ξ is a Cahn-Hoffman vector. A relation of this sort is quite natural, see Gurtin [G, Chapter 8].

The equation (1.2) represents mass conservation of the crystal surface and it is often called the Stefan condition. In [KIO] T. Kuroda, T. Irisawa and A. Ookawa consider the system of (1.1a), (1.1b), (1.2) with $\sigma = \beta V$ instead of (1.3) to study the morphological stability of polyhedral crystal as an extension of idea of [Se]. However, we believe the curvature term in (1.3) is important at least when the crystal is small. Also the kinetic coefficient β in (1.3) is allowed to depend on the orientation of the crystal surfaces.

Let us comment on the Cahn-Hoffman vector ξ. For smooth surfaces S and smooth energy density function γ we have

$$\xi(x) = \nabla\gamma(\mathbf{n}(x)),$$

which is a well-defined quantity, and $\gamma : \mathbb{R}^n \to \mathbb{R}$ is a 1-homogeneous function. However, for energy density functions γ which are only Lipschitz continuous and surfaces S with corners some care is necessary while defining ξ, (see [GMR] for a related study). We shall assume that γ is convex. For a convex function γ its subdifferential $\partial\gamma$ is a well-defined nonempty convex set. But in general $\partial\gamma$ is not a singleton, thus we have only that $\xi(x) \in \partial\gamma(\mathbf{n}(x))$.

Here however, we do not consider the evolution problem in its full generality, we are interested in the growth of perturbed Wulff shapes. We do so for the sake of the feasibility of present analysis and having in mind establishing further qualitative properties of solutions. We assume that the Frank diagram F_γ (see [G, Section 7.2] for a definition) is a sum of two straight cones with height ℓ, base radius ρ, having a common base. Hence the Wulff shape is a cylinder. By the term a 'perturbed Wulff shape' we mean a cylinder of different proportions. In this way, the problem becomes axis symmetrical which leads to some simplifications of the subsequent analysis.

If we suppose that the initial surface is a cylinder, then the question is under what circumstances it stays a cylinder through the evolutions. The exact conditions are to be established and this is a topic of an ongoing research. However, in order to start the discussion we consider a simplification of (1.3). Namely, we replace this condition by its averages over facets forming $S(t)$. Since the velocity V is forced to be constant on each facet S_i equation (1.3) becomes

$$-\int_{S_i} \sigma \, d\mathcal{H}^2 = \Gamma_i - \beta_i V_i \quad \text{on} \quad S_i. \tag{1.3'}$$

This paper is devoted to establishing the existence of solutions to (1.1)-(1.2) coupled to (1.3′) and augmented with an initial position of the crystal. This is achieved in §3. In Section 2 we explain our notation and we formulate the weak form of (1.1)-(1.2), (1.3′). It should be mentioned here that a formally similar planar problem was studied in [R] for polygonal interfaces. The main difference is that in [R] a two-phase problem was considered. We may expect that different methods will be effective here. We explain it momentarily. Our first step is to reduce the problem of interface evolution to a system of ordinary differential equations for the radius $R(t)$ and height $2L(t)$ of the growing cylinder $\Omega(t)$. A similar procedure is applied in [R].

To show unique existence of local in time solutions it is crucial to establish the Lipschitz continuity of a mapping

$$(R, L) \mapsto \int_{\Omega^c} \nabla f_i \cdot \nabla f_j \, dx,$$

where f_i solves $\Delta f_i = 0$ in Ω^c, $f_i(\infty) = 0$, with Neumann boundary conditions $\partial f_i/\partial \nu = 1$ on S_i, and $\partial f_i/\partial \nu = 0$ elsewhere, where S_i is either lateral, top, bottom of crystal surface $\partial\Omega$ and ν is the inner normal. The author of [R] uses Green function for this task. This was effective, because of a fixed bounded domain. This is not the case here. Instead, we develop a method to cut off contributions near the crystal surface and compare functions in variable domains.

Let us finally comment on relevant mathematical literature. Apparently, little has been published on one-phase Stefan problem (or its quasi-steady approximation) with Gibbs-Thomson relation and kinetic undercooling. Actually, if we are interested in a multi-dimensional setting, then we are aware only of papers dealing with kinetic relation alone. We have two articles in mind. Friedman–Hu, [FH], consider evolution of a free boundary which is a graph of a function over $\mathbb{R}$. Liu–Yuan, (see [LY]) deal only with classical solutions.

We also point to the fact that the Hele-Shaw problem resembles ours. One of the closer studies seem to be [DE] and [A]. The authors of [DE] consider the Hele-Shaw problem with Gibbs-Thomson relation and kinetic undercooling and the coefficients are anisotropic. They apply the technique of parabolic regularization to obtain existence of smooth interface.

On the other hand, R.Almgren in [A] considers the problem for polygonal interfaces, which move by crystalline curvature (no kinetic undercooling). He relies on complex variable methods in his studies.

These problems are rather different from ours and we will not use methods employed in the above mentioned papers.

Notation. For vectors $\mathbf{a}$, $\mathbf{b}$ in $\mathbb{R}^k$ we denote their inner product by $\mathbf{a} \cdot \mathbf{b}$.

2 Weak formulation

Let us denote an evolving crystal by $\Omega(t)$, $\Omega^c = \mathbb{R}^3 \setminus \Omega$ is its exterior, and $S(t) = \partial\Omega$ is its surface. As we have already mentioned, our basic premise here is that our growing crystal $\Omega(t)$ is a cylinder at all times. Specifically,

$$\Omega(t) = \{(x_1, x_2, z) \in \mathbb{R}^3;\ x_1^2 + x_2^2 \le R^2(t),\ |z| \le L(t)\}.$$

We shall distinguish three subsets of $S(t)$, namely

$$S_T = \{(x_1, x_2, z) :\ x_1^2 + x_2^2 \le R^2(t),\ z = L(t)\}$$
$$S_B = \{(x_1, x_2, z) :\ x_1^2 + x_2^2 \le R^2(t),\ z = -L(t)\}$$
$$S_L = \{(x_1, x_2, z) :\ x_1^2 + x_2^2 = R^2(t);\ |z| \le L(t)\}.$$

We will call them *facets*. The set of indices $\{T, B, L\}$ will be called I, but sometimes we will use numbers instead of letters, namely $L = 1$, $T = 2$, $B = 3$.

We assume that the supersaturation σ is axially symmetric i.e. $\sigma(x_1, x_2, z) = \sigma(r, z)$, where $r^2 = x_1^2 + x_2^2$, and also σ is symmetric with respect to the plane $\{z = 0\}$, i.e. $\sigma(r, z) = \sigma(r, -z)$.

We shall write (1.1)-(1.2) in a weak form in Ω^c. Since Ω^c is an unbounded domain we shall seek σ in the space $L^{1,2}(\Omega^c)$. We recall (see [HK]) that for a domain D of $\mathbb{R}^3$ the space $L^{1,2}(D)$ is defined by

$$L^{1,2}(D) = \{u \in \mathcal{D}'(D) : \ \nabla u \in L^2(D)\}.$$

We also recall that by a domain we mean an open and connected subset of $\mathbb{R}^k$, and $\mathcal{D}'(D)$ denotes the space of Schwartz' distributions.

By [HK, Theorem 5] for all $v \in L^{1,2}(\Omega^c)$ we have a unique representation

$$v = v_0 + v^\infty,$$

where $v_0 \in L^{1,2}(\Omega^c)$ and $\lim_{|x|\to\infty} v_0(x) = 0$, $v^\infty \in \mathbb{R}$. The same result implies that the space

$$H(\Omega^c) = \{u \in L^{1,2}(\Omega^c) : \ \lim_{|x|\to\infty} u(x) = 0\}$$

is a Hilbert space equipped with an inner product

$$(u, v)_{L^{1,2}} = \int_{\Omega^c} \nabla u(x) \cdot \nabla v(x) \, dx.$$

Multiplication of (1.1) by a test function $h \in H(\Omega^c)$ and integration by parts lead us to

$$0 = -\int_{\Omega^c} \nabla\sigma(x) \cdot \nabla h(x) \, dx + \int_{\partial\Omega^c} \frac{\partial\sigma}{\partial\nu}(x) h(x) \, d\mathcal{H}^2, \tag{2.1}$$

where ν is the outer normal to $\partial\Omega^c$ i.e. $\nu = -\mathbf{n}$. Hence (2.1) becomes

$$0 = -\int_{\Omega^c} \nabla\sigma(x) \cdot \nabla h(x) \, dx - \sum_{i\in I} \int_{S_i} V_i h(x) \, d\mathcal{H}^2 \tag{2.2}$$

for all $h \in H(\Omega^c)$.

We have already mentioned that (1.3) should be considered with care, especially if we want that $S(t)$ be a cylinder at all time instances. In this paper we consider (1.3′), i.e. (1.3) averaged over each facet,

$$-\!\!\!\!\!\!\int_{S_i} \sigma = -\!\!\!\!\!\!\int_{S_i} \operatorname{div} \xi - \beta |S_i| V_i, \quad i \in \{B, T, L\}, \tag{2.3}$$

where we explicitly assume that V_i is constant on S_i, $i \in \{B, T, L\}$ and since ξ is basically defined on S we consider the surface divergence $\mathrm{div}_S\, \xi$ in the formula above and subsequently on. We recall that $|S_i|$ denotes here the two-dimensional surface area of S_i.

A condition like (2.3) appears in a model of crystal growth developed by Gurtin and Matias [GM], there $-\sigma$ is temperature and (2.3) is the balance of capillary forces.

Let us stress that our assumption that $S(t)$ will always be a cylinder requires justification. In general we expect a facet breaking phenomenon. This question will be addressed in a separate paper.

Divergence operator in (1.3) or in (2.3) should be understood as a surface divergence. The problem of interpreting $-\mathrm{div}_S\, \xi$ in (1.3) would be easier if there were no supersaturation field σ. Then, V_i would be constant over S_i provided that $-\mathrm{div}_S\, \xi$ were constant. We expect that $-\mathrm{div}_S\, \xi$ is related to a curvature. Indeed, we shall see that for a choice of a Lipschitz continuous Cahn Hoffman vector field ξ, $-\mathrm{div}_S\, \xi$ is equal to the "mean crystalline curvature" which may be defined as

$$\kappa_i = -\lim_{h\to 0} \frac{\Delta E}{\Delta V}, \tag{2.4}$$

where h is the amount of motion of S_i in the direction of the outer normal to $S(t)$; ΔE is the resulting change of surface energy, and ΔV is the change of volume.

Let us now calculate κ_T, κ_L using (2.4). For this purpose we specify some data. We need to know the values $\gamma(\mathbf{n}_L)$ and $\gamma(\mathbf{n}_T)$, where $\mathbf{n}_T$ is the outer normal to the top disc of the Wulff shape (it is parallel to the rotation axis of F_γ). Vector $\mathbf{n}_L$ is perpendicular to $\mathbf{n}_T$ and it is an outer normal to the lateral part of the Wulff shape. Due to our choice of F_γ the number $\gamma(\mathbf{n}_L)$ is independent of a selection of $\mathbf{n}_L$. Let us note that $\ell = 1/\gamma(\mathbf{n}_T)$ and $\rho = 1/\gamma(\mathbf{n}_L)$.

We first deal with κ_T, let us note that $\kappa_T = \kappa_B$, the parameter h in (2.4) describes the amount of motion of S_T the direction of $\mathbf{n}_T$. We can easily see that for a cylinder of radius R and height $2L$ we have

$$\Delta E = 2\pi h R\gamma(\mathbf{n}_L),\ \ \Delta V = h\pi R^2,$$

thus

$$\kappa_T = -\lim_{h\to 0} \frac{\Delta E}{\Delta V} = -\frac{2\gamma(\mathbf{n}_L)}{R}.$$

For calculations of κ_L we note that h is the change in the radius of the cylinder thus $\Delta E = 2[(R+h)^2 - R^2]\gamma(\mathbf{n}_T)\pi + 2\pi \cdot 2L(R+h-R)\gamma(\mathbf{n}_L)$ where we take into account that the energy of both S_T and S_B changes. Finally we see

$$\Delta V = 2\pi[(R+h)^2 L - R^2].$$

Thus

$$\kappa_L = -\lim_{h\to 0}\frac{\Delta E}{\Delta V} = -\frac{\gamma(\mathbf{n}_T)}{L} - \frac{1}{R}\gamma(\mathbf{n}_L).$$

We now can make the relation between surface divergence $-\mathrm{div}_S\xi$ and κ more transparent.

Proposition 1. *If $S = \partial\Omega$, $\Omega = \{(x_1, x_2, z) : x_1^2 + x_2^2 \le R, |z| \le L\}$, then there exists a Lipschitz continuous Cahn Hoffman vector field ξ such that*

$$-div_S\xi = \kappa_i \quad i = T, L, B.$$

Proof. We set $\xi = (a\cdot x_1,\ a\cdot x_2,\ bz)$, where $a = \gamma(\mathbf{n}_L)/R$, $b = \gamma(\mathbf{n}_T)/L$. It is clear that ξ is smooth. The surface divergence of ξ, restricted to S_i, is equal $-\kappa_i$ for $i = T, B, L$ and for $\mathcal{H}^2$ a.e. all $x \in S$, $\xi(x) \in \partial\gamma(\mathbf{n}(x))$, where $\partial\gamma$ is the subdifferential of γ and finally

$$\xi\cdot\mathbf{n}_L = \max_{\zeta\in\partial\gamma(\mathbf{n}_T)} \zeta\cdot\mathbf{n}_L$$
$$\xi\cdot\mathbf{n}_T = \max_{\zeta\in\partial\gamma(\mathbf{n}_L)} \zeta\cdot\mathbf{n}_T \qquad \square$$

Thus (2.3) becomes

$$-\int_{S_i}\sigma = \Gamma_i - \beta V|S_i|, \quad i \in I \tag{2.3'}$$

where $\Gamma_i = |S_i|\kappa_i$, where $|S_i|$ denotes the surface area. Let us note that $\Gamma_T = \Gamma_B$ is a smooth function of R only, while Γ_L depends smoothly upon L as well as R.

Let us note that a solution of (1.1–1.2) and (2.3′) is given if a position of the free boundary $S(t)$ is specified and the supersaturation field σ on $\Omega^c(t)$. While the position of $S(t)$ is specified if $R(t)$ and $L(t)$ are known, the difficulty with σ is that it is defined on a variable domain. Summarizing, a *weak solution* to (1.1 1.2), (2.3′) is a triple $(R(t), L(t), \sigma(t))$ such that $R, L \in C^1([0,T))$ $\sigma(t) \in L^{1,2}(\Omega^c(t))$ for all t, and the dependence $t \mapsto \sigma(t)$ is continuous in a sense to be specified later. Moreover $V_1(t) = \frac{d}{dt}R$, $V_2(t) = V_3(t) = \frac{d}{dt}L(t)$ satisfy

$$0 = -\int_{\Omega^c}\nabla\sigma(x)\cdot\nabla h(x)\,dx - \sum_{i\in I} V_i\int_{S_i} h(x)\,d\mathcal{H}^2, \quad \forall h \in H(\Omega^c) \tag{2.5a}$$

$$-\int_{S_i}\sigma(x)\,d\mathcal{H}^2 = \Gamma_i - \beta_i V_i|S_i|, \quad i \in I, \quad R(0) = R_0, L(0) = L_0. \tag{2.5b}$$

3 Existence of weak solutions

We shall transform (2.5) to obtain a problem which is easier to handle. We shall apply the approach used in [R] for a 2-phase quasi-steady modified Stefan problem. Our problem looks like 1-phase Stefan problem, but σ is not defined in Ω, thus σ has a variable domain of definition.

Let us notice that

$$L^{1,2}(\Omega^c) \ni h \mapsto \int_{S_i} h(x)\, d\mathcal{H}^2 \in \mathbb{R}$$

is a continuous functional on $H(\Omega^c)$ thus, by Riesz' Representation Theorem there exists a unique $f_i \in H(\Omega^c)$ $i \in \{T, B, L\}$ such that

$$\int_{S_i} \varphi(x)\, d\mathcal{H}^2 = \int_{\Omega^c} \nabla\varphi(x) \cdot \nabla f_i(x)\, dx, \qquad \text{for all } \varphi \in H(\Omega^c). \tag{3.1}$$

Formally, $f_i \in H(\Omega^c)$ is the unique solution to the Neumann problem,

$$\begin{aligned} -\Delta f_i &= 0 \quad \text{in } \Omega^c \\ \tfrac{\partial f_i}{\partial \nu} &= \delta_{ij} \quad \text{on } S_j, \end{aligned}$$

where δ_{ij} is the Kronecker delta and ν is the inner normal.

Taking (3.1) into account we can write (2.5a) as

$$0 = -\int_{\Omega^c} \nabla\sigma(x) \cdot \nabla h(x)\, dx - \sum_{i\in I} V_i \int_{\Omega^c} \nabla f_i(x) \cdot \nabla h(x)\, dx \quad \text{for all } h \in H(\Omega^c). \tag{3.2}$$

Since a function σ in $L^{1,2}(\Omega^c)$ has a representation $\sigma = \sigma_0 + \sigma^\infty$, where $\sigma_0 \in H(\Omega^c)$ and σ^∞ is a real number, and $f_i \in H(\Omega^c)$ we deduce from (3.2) that

$$\sigma = -\sum_{i\in I} V_i f_j + \sigma^\infty. \tag{3.3}$$

We may insert this into (2.5b) to obtain

$$\sum_{j\in I} V_j (f_j, f_i)_{L^{1,2}} - |S_i|\sigma^\infty = \Gamma_i - \beta_i |S_i| V_i, \quad i \in I.$$

If we set $\mathcal{A} = \{(f_i, f_j)\}_{i,j\in I}$ and $\mathcal{D} = \text{diag}\ \{\beta_L|S_L|, \beta_T|S_T|, \beta_B|S_B|\}$, then we may write the system above for $\mathbf{V} = (V_L, V_T, V_B)$ in a simple form

$$\mathbf{V} = (\mathcal{A} + \mathcal{D})^{-1}\mathbf{B} \tag{3.4}$$

where

$$\mathbf{B} = (\Gamma_L + \sigma^\infty |S_L|, \Gamma_T + \sigma^\infty |S_T|, \Gamma_B + \sigma^\infty |S_B|).$$

Let us change the notation slightly and call the variable L by L_1. We introduce a dummy variable L_2. It will be equal to L. In this way we may expose the symmetric structure of (3.4). Let us also write $\mathbf{z} = (R, L_1, L_2)$, thus

$$\mathbf{V} = \frac{d\mathbf{z}}{dt} = (\frac{dR}{dt}, \frac{dL_1}{dt}, \frac{dL_2}{dt}),$$

then

$$\frac{d\mathbf{z}}{dt} = (\mathcal{A}(\mathbf{z}) + \mathcal{D}(\mathbf{z}))^{-1} B(\mathbf{z}). \tag{3.5}$$

Our object of study is this system of ODE's. In order to obtain existence and uniqueness of solutions to (3.5) it is sufficient to establish that the functions

$$\mathbf{z} \mapsto (f_i(\mathbf{z}), f_j(\mathbf{z}))_{L^{1,2}}, \quad \mathbf{z} \mapsto |S_i(\mathbf{z})|, \quad \mathbf{z} \mapsto \Gamma_i(\mathbf{z})$$

are locally Lipschitz continuous. The last two claims are easy, we shall only deal with the first one. We faced such a problem in [R], here we will proceed in a different way without using Green's function. On the way we shall derive more information on f_i's.

Let us introduce three cut-off functions $\eta_i \in C_0^\infty(\mathbb{R}^3)$, $1 \ge \eta_i \ge 0$, $i \in I$. We require that $\eta_L|_{\{r<\varepsilon\}} = 0 = \eta_L|_{\{z^2+r^2>\rho^2\}}$ for some $\varepsilon > 0$ and sufficiently large ρ, and η_L restricted to the set $\{r \in [R-\varepsilon, R+\varepsilon]\} \cap \{|z| < \frac{\rho}{2}\}$ is 1.

We need that

$$\eta_T(r,z) = \begin{cases} 0 & \text{if } z \le 0 \text{ or } r^2 + z^2 \ge \rho^2 \\ 1 & \text{if } z \in [L-\varepsilon, L+\varepsilon] \text{ and } r < \frac{\rho}{2} \end{cases}$$

finally $\eta_B(r,z) = \eta_T(r,-z)$. We also define

$$\overline{f_L}(r,z) = -\eta_L(r,z) R \ln r$$
$$\overline{f_T}(r,z) = -\eta_T z, \quad \overline{f_B}(r,z) = -\eta_B z.$$

Obviously $\overline{f_i} \in C^\infty$ and if ν is the inner normal to Ω, then

$$\frac{\partial \overline{f_i}}{\partial \nu} = \begin{cases} 1 & \text{on } S_i \\ 0 & \text{on } S_j, \quad j \ne i. \end{cases}$$

Subsequently, we define $\overline{\psi_i}$, $i \in I$ to be a unique solution in $H(\Omega^c)$ to

$$\int_{\Omega^c} (\nabla \overline{\psi_i} \cdot \nabla h - \Delta \overline{f_i} h)\, dx = 0 \quad \forall h \in H(\Omega^c). \tag{3.6}$$

Existence and uniqueness of solutions to (3.6) follows from the Riesz' representation formula provided that we check that

$$H(\Omega^c) \ni h \mapsto \int_{\Omega^c} \Delta \overline{f_i} h \, dx$$

is a continuous linear functional. Indeed, by Hölder inequality, smoothness of $\overline{f_i}$ and boundedness of support of $\Delta \overline{f_i}$ we obtain that

$$\left| \int_{\Omega^c} \Delta \overline{f_i} h \, dx \right| \leq ||\Delta \overline{f}_i||_{L^{\frac{5}{6}}} ||h||_{L^6} \leq C ||h||_{L^{1,2}},$$

where we used in the last inequality [HK, Theorem 5].

Since $\overline{\psi}_i \in H(\Omega^c)$ formally solves $\Delta \overline{\psi}_i = \Delta \overline{f}_i$ in Ω^c, with $\partial \overline{\psi}_i / \partial \nu = 0$ on $\partial \Omega^c$, we observe that

$$f_i = \overline{f}_i + \overline{\psi}_i. \tag{3.7}$$

To show this rigorously we have to check that (3.1) holds if we set $f_i = \overline{f}_i + \overline{\psi}_i$. By (3.7) we have

$$\int_{\Omega^c} \nabla f_i \cdot \nabla \varphi \, dx = \int_{\Omega^c} (\nabla \overline{f}_i + \nabla \overline{\psi}_i) \cdot \nabla \varphi \, dx, \qquad \text{for all } \varphi \in H(\Omega^c), \quad i \in I.$$

Because of smoothness of $\overline{f}_i$ we can integrate by parts, then we use the definition of $\overline{\psi}_i$

$$\begin{aligned} \int_{\Omega^c} \nabla f_i \cdot \nabla \varphi \, dx &= - \int_{\Omega^c} \Delta f_i \varphi \, dx + \int_{\partial \Omega^c} \frac{\partial \overline{f}_i}{\partial \nu} \varphi \, dx + \int_{\partial \Omega^c} \nabla \tilde{\varphi}_i \cdot \nabla \varphi \, dx \\ &= \int_{\partial \Omega^c} (\nabla \overline{f}_i \cdot \nabla \varphi - \Delta \overline{f}_i \varphi) dx + \int_{S_i} \varphi \, dS = \int_{S_i} \varphi \, dS. \qquad \square \end{aligned}$$

Subsequently, we study here how f_i's depend upon S_i. We investigate the function

$$\mathbf{z} \mapsto f_i(\mathbf{z}), \quad i \in I.$$

We need

Lemma 1. *Let us suppose that* $A_k = (a_{ij}^k)_{i,j=1}^3$, $a_{ij}^k \in L^\infty(\Omega^c)$, $A_k \geq \lambda > 0$, $k = 1, 2$, $g_1, g_2 \in L^{\frac{5}{6}}(\Omega^c) \cap L^2(\Omega^c)$ *and* $u_1, u_2 \in L^{1,2}(\Omega^c)$ *are solutions of*

$$\int_{\Omega^c} A_k(x) \nabla u_k(x) \cdot \nabla h(x) \, dx = \int_{\Omega^c} g_k(x) h(x) \, dx, \quad k = 1, 2. \tag{3.8}$$

for all $h \in L^{1,2}(\Omega^c)$. *Then*

$$||\nabla u_1 - \nabla u_2||_{L^2(\Omega^c)} \leq \frac{1}{\lambda} (C_p ||g_1 - g_2||_{L^{\frac{5}{6}}} + ||A_1 - A_2||_{L^\infty} ||\nabla u_1||_{L^2}).$$

Proof. We take difference of (3.8) for $k=1$ and $k=2$, thus

$$\int_{\Omega^c} A_1 \nabla u_1 \cdot \nabla h \, dx - \int_{\Omega^c} A_2 \nabla u_2 \cdot \nabla h \, dx = \int_{\Omega^c} (g_1 - g_2) h \, dx, \quad \text{for all } h \in H(\Omega^c).$$

We rearrange the above identity

$$\int_{\Omega^c} (A_1 - A_2) \nabla u_1 \cdot \nabla h \, dx + \int_{\Omega^c} A_2 (\nabla u_1 - \nabla u_2) \cdot \nabla h \, dx = \int_{\Omega^c} (g_1 - g_2) h \, dx.$$

We take $h = u_1 - u_2$, positivity of A_2 yields

$$\lambda ||\nabla u_1 - \nabla u_2||_{L^2}^2 \leq ||g_1 - g_2||_{L^{\frac{5}{6}}} ||u_1 - u_2||_{L^6} + ||A_1 - A_2||_{L^\infty} ||\nabla u_1||_{L^2} ||\nabla (u_1 - u_2)||_{L^2}$$

We again use the embedding $L^{1,2}(D) \hookrightarrow L^6(D)$ (see [HK, Theorem 5]), thus the above inequality becomes

$$\begin{aligned} \lambda ||\nabla u_1 & - \nabla u_2||_{L^2}^2 \\ & \leq C_p ||g_1 - g_2||_{L^{\frac{5}{6}}} ||\nabla (u_1 - u_2)||_{L^2} + ||A_1 - A_2||_{L^\infty} ||\nabla u_1||_{L^2} ||\nabla (u_1 - u_2)||_{L^2}. \end{aligned}$$

And this inequality implies our lemma.

Now we can prove the key Lemma. Its statement is similar to [R, Lemma 3]. However, the proof is completely different.

Lemma 2. *The mappings*

$$\mathbb{R}^3 \ni \mathbf{z} \mapsto (f_i(\mathbf{z}), \ f_j(\mathbf{z}))_{L^{1,2}} \in \mathbb{R}$$

for $i, j \in I$ are locally Lipschitz continuous.

Proof. Let us take $\mathbf{z}'$ close to $\mathbf{z}$. For the sake of brevity we write $f_i \equiv f_i(\mathbf{z})$, $f_i' \equiv f_i(\mathbf{z}')$. By the definition of f_i and some algebra we arrive at

$$\begin{aligned} (f_i, \ f_j)_{L^{1,2}} - (f_i', \ f_j')_{L^{1,2}} &= \left\{ \int_{\Omega^c(\mathbf{z})} \nabla \overline{f}_i \cdot \nabla \overline{f}_j \, dx - \int_{\Omega^c(\mathbf{z}')} \nabla \overline{f}_i' \cdot \nabla \overline{f}_j' \, dx \right\} \\ &+ \left\{ \int_{\Omega^c(\mathbf{z})} \nabla \overline{f}_i \cdot \nabla \overline{\psi}_j \, dx - \int_{\Omega^c(\mathbf{z}')} \nabla \overline{f}_i' \cdot \nabla \overline{\psi}_j' \, dx \right\} \\ &+ \left\{ \int_{\Omega^c(\mathbf{z})} \nabla \overline{\psi}_i \cdot \nabla \overline{f}_j \, dx - \int_{\Omega^c(\mathbf{z}')} \nabla \overline{\psi}_i' \cdot \nabla \overline{f}_j' \, dx \right\} \\ &+ \left\{ \int_{\Omega^c(\mathbf{z})} \nabla \overline{\psi}_i \cdot \nabla \overline{\psi}_j \, dx - \int_{\Omega^c(\mathbf{z}')} \nabla \overline{\psi}_i' \cdot \nabla \overline{\psi}_j' \, dx \right\} \\ &\equiv I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Our goal is to show that $|I_k| \leq C|z-z'|$, $k=1,2,3,4$. It is easy to show that

$$|I_1| \leq C|z-z'|.$$

We leave the straightforward calculations to the reader.

In order to estimate the remaining integrals we have to change variables, so that we can consider a difference of integrals over the same domain, let us introduce an affine isomorphism $H : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$H(r,z) = \left(\frac{R'}{R} r, \quad z\frac{L'}{L}\right).$$

H maps $\{r \leq R,\ |z| \leq L\}$ onto $\{r \leq R',\ |z| \leq L'\}$. Now, we write for any function u'

$$\tilde{u}(x) = u'(H(x)).$$

We note that if $y = H(x)$, then the chain formula vector yields

$$\nabla_y u' = (\nabla_x \tilde{u})^T D_y x = (\nabla_x \tilde{u})^T A \tag{3.9}$$

where we wrote $\nabla_x \tilde{u}$ as a column, and

$$A = \operatorname{diag}\left(\frac{R}{R'}, \ \frac{R}{R'}, \ \frac{L}{L'}\right) \equiv \operatorname{diag}(\alpha_1, \alpha_1, \alpha_3)$$

thus

$$\int_{\Omega^c(\mathbf{z}')} \nabla_y \overline{\psi}_i' \cdot \nabla_y \overline{f}_j'\, dy = \int_{\Omega^c(\mathbf{z})} \nabla_x \tilde{\overline{\psi}}_i \cdot A^2 \nabla_x \tilde{\overline{f}}_j \ \det\, A\, dx.$$

We also recall the equation satisfied by $\overline{\psi}_i$. Since

$$\int_{\Omega^c(\mathbf{z}')} \nabla \overline{\psi}_i' \cdot \nabla h\, dx = \int_{\Omega^c(\mathbf{z}')} \Delta \overline{f}_i' h\, dx, \qquad \text{for all } h \in H(\Omega^c(\mathbf{z}'))$$

then (3.9) yields

$$\Delta_y \overline{f}_i' = \sum_{k=1}^{2} \frac{\partial \tilde{\overline{f}}_i}{\partial x_k} \left(\frac{R}{R'}\right)^2 + \alpha_3^2 \frac{\partial^2 \tilde{\overline{f}}_i}{\partial z^2}$$

and finally $\tilde{\overline{\psi}}$ satisfies

$$\int_{\Omega^c(\mathbf{z})} A^2 \nabla_x \tilde{\overline{\psi}}_i \cdot \nabla_x \tilde{h} \ \det\, A\, dx = \int_{\Omega^c(\mathbf{z})} \sum_{k=1}^{3} \frac{\partial^2 \tilde{\overline{f}}_i}{\partial x_k^2} \alpha_k^2\, \tilde{h} \ \det\, A\, dx.$$

We are now in a position to complete estimates for I_2

$$
\begin{aligned}
I_2 &= \int_{\Omega^c(\mathbf{z})} \nabla \overline{f}_i \cdot \nabla \overline{\psi}_j \, dx - \int_{\Omega^c(\mathbf{z}')} \nabla \overline{f}'_i \cdot \nabla \overline{\psi}'_j \, dx \\
&= \int_{\Omega^c(\mathbf{z})} (\nabla \overline{f}_i \cdot \nabla \overline{\psi}_j - \nabla \tilde{\overline{f}}_i \cdot A^2 \nabla \tilde{\overline{\psi}}_j \text{ det } A) \, dx \\
&= \int_{\Omega^c(\mathbf{z})} [\nabla \overline{\psi}_j \cdot (\nabla \overline{f}_i - \nabla \tilde{\overline{f}}_i A^2 \text{ det } A) + \nabla \overline{f}_i A^2 \cdot (\nabla \overline{\psi}_j - \nabla \tilde{\overline{\psi}}_i) \text{det } A] \, dx \\
&= \int_{\Omega^c(\mathbf{z})} [\nabla \overline{\psi}_j \cdot (\nabla \overline{f}_i - \nabla \tilde{\overline{f}}_i + \nabla \tilde{\overline{f}}_i (\text{Id} - A^2 \text{det } A)) + \nabla \tilde{\overline{f}}_i A^2 \cdot (\nabla \overline{\psi}_j - \nabla \tilde{\overline{\psi}}_j) \text{ det } A] \, dx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|I_2| \leq \; & ||\nabla \overline{\psi}_j||_{L^2} (||\nabla \overline{f}_i - \nabla \tilde{\overline{f}}_i||_{L^2} + ||\nabla \tilde{\overline{f}}_i||_{L^2} ||\text{Id} - A^2 \text{ det } A||_{L^\infty}) \\
& + ||\nabla \tilde{\overline{f}}_i A^2 \text{ det } A||_{L^2} ||\nabla \overline{\psi}_j - \nabla \tilde{\overline{\psi}}_j||_{L^2}.
\end{aligned}
$$

One may directly establish that

$$
||\nabla \overline{f}_i - \nabla \tilde{\overline{f}}_i||_{L^2} \leq C|\mathbf{z} - \mathbf{z}'|. \tag{3.10}
$$

It is also clear that

$$
||\text{Id} - A^2 \text{det } A||_{L^\infty} \leq C|\mathbf{z} - \mathbf{z}'|.
$$

Because of its compact support it is obvious that $\nabla \overline{f}_i \in L^p(\Omega^c)$, for all $1 \leq p < \infty$, thus in order to estimate $||\nabla \overline{\psi}_j - \nabla \tilde{\overline{\psi}}_j||_{L^2}$ we use Lemma 1 and identity (3.6). Since A is a linear isomorphism, then $A^2 \geq \min \sigma(A^2) I > 0$, with $\sigma(A^2) > 0$, where $\sigma(A^2)$ denotes the set of eigenvalues of A^2. Thus, this yields

$$
\begin{aligned}
||\nabla \overline{\psi}_j & - \nabla \tilde{\overline{\psi}}_j||_{L^2} \\
& \leq \frac{1}{\min \sigma(A^2)} \left[C_p ||\Delta \overline{f}_j - \sum_{k=1}^{3} \frac{\partial^2 \tilde{\overline{f}}_j}{\partial x_k^2} \alpha_k^2 ||_{L^{\frac{5}{6}}} + ||\text{Id} - A^2 \text{det } A||_{L^\infty} ||\nabla \psi_j||_{L^2} \right].
\end{aligned}
$$

Next, after straightforward calculations we arrive at

$$
||\nabla \overline{\psi}_j - \nabla \tilde{\overline{\psi}}_j||_{L^2} \leq C|\mathbf{z} - \mathbf{z}'|. \tag{3.11}
$$

Thus we may conclude that

$$
|I_2| \leq c|\mathbf{z} - \mathbf{z}'|
$$

since the structure of I_3 is the same as that of I_2 we infer that

$$
|I_3| \leq c|\mathbf{z} - \mathbf{z}'|.
$$

The remaining term is I_4, proceeding as before we obtain

$$I_4 = \int_{\Omega^c} [\nabla\overline{\psi}_i \cdot (\nabla\overline{\psi}_j - A^2 \nabla\tilde{\overline{\psi}}_j \text{ det } A) + (\nabla\overline{\psi}_i - \nabla\tilde{\overline{\psi}}_i) \cdot A^2 \ \nabla\tilde{\overline{\psi}}_j \text{ det} A]\, dx.$$

Thus,

$$\begin{aligned}|I_4| \leq \ & ||\nabla\overline{\psi}_j||_{L^2}(||\nabla\overline{\psi}_j - \tilde{\overline{\psi}}_j||_{L^2} + ||\nabla\tilde{\overline{\psi}}||_{L^2}||\text{Id} - A^2 \text{ det } A||_{L^\infty}) \\ & + ||\nabla\overline{\psi}_i - \nabla\tilde{\overline{\psi}}_i||_{L^2}||A^2 \text{ det } A||_{L^\infty}||\nabla\tilde{\overline{\psi}}_j||_{L^2}\end{aligned}$$

using inequality (3.11) we infer that

$$|I_4| \leq c|\mathbf{z} - \mathbf{z}'|.$$

The Lemma follows. □

The Lemma above tells us that we actually may compare elements of function spaces defined on different domains, which are related by fixed diffeomorphism H, thus, we may adopt the notation, if $u \in L^{1,2}(\Omega^c(\mathbf{z}))$, $u' \in L^{1,2}(\Omega^c(\mathbf{z}'))$ and $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} u'(x)$, then

$$d(u, u') = ||\nabla u - A^2 \text{ det } A\nabla\tilde{u}'||_{L^2(\Omega^c(\mathbf{z}))}$$

where we keep the notation of Lemma 2. We would like to say that a function

$$t \mapsto u(t) \in L^{1,2}(\Omega^c(t))$$

is continuous (Hölder continuous). We shall adopt the following definition

$$t \mapsto u(t) \in L^{1,2}(\Omega^c(\mathbf{z}(t)) \quad \text{is continuous} \quad \text{(Hölder continuous)}$$

if the map $t \mapsto \mathbf{z}(t)$ is continuous and if

$$t \mapsto \nabla u(t)A^2(t)\text{det } A \in L^{1,2}(\Omega^c(0))$$

is continuous (Hölder continuous). In this language, due to (3.10) and (3.11) we may write

$$d(f_i(\mathbf{z}), f_i(\mathbf{z}')) \leq C|\mathbf{z} - \mathbf{z}'|. \tag{3.12}$$

we are now in a position to state and prove our existence result.

Theorem 1. *Let us suppose that an initial cylinder $S(0)$ is given. Then there exists a unique local in time solution to (2.5) such that*

$$\mathbf{z} \in C^{1,1}([0, T), \mathbb{R}^3), \ \sigma \in C^{0,1}([0, T), L^{1,2}(\Omega^c(\mathbf{z}(t))).$$

Proof. We have already reduced the problem to (3.5), i.e.

$$\frac{d\mathbf{z}}{dt} = (A(\mathbf{z}) + D(\mathbf{z}))^{-1} B(\mathbf{z}). \tag{3.13}$$

By Lemma 1 the RHS is locally Lipschitz continuous, thus the ODE has a unique local in time solution, and since the RHS of (3.13) is locally Lipschitz we obtain

$$\mathbf{z} \in C^{1,1}([0,T), \mathbb{R}^3)$$

On the other hand, we have established (see 3.3) that

$$\sigma = -\sum_{i\in I} V_i f_i + \sigma^\infty$$

thus it is easy to see that

$$d(\sigma(t), \sigma(t')) \le \sum_{i=1}^{3} |V_i(t) - V_i(t')| \; ||f_i||_{L^{1,2}(\Omega^c(t))} + \sum_{i=1}^{3} |V_i(t')| d(f_i, f_i'),$$

Lemma 1 and (3.12) yield

$$d(\sigma(t), \sigma(t')) \le C|z(t) - z(t))| \le C|t - t'| \qquad \square$$

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References

[A] R.Almgren, Crystalline Saffman-Taylor fingers, *SIAM J. Appl. Math.*, **55** (1995), 1511–1525.

[DE] K. Deckelnick and C. M. Elliott, Local and global existence result for anisotropic Hele-Shaw, *Proc. Royal Soc. Edinburgh,* **129A** (1999), 265–294.

[FH] A. Friedman, B. Hu, The Stefan problem with kinetic condition at the free boundary. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **19** (1992), no. 1, 87–111.

[GPR] Y.Giga, M.Paolini and P.Rybka, On the motion by singular interfacial energy. *Japan J. Indust. Appl. Math.* (2001), to appear.

[GG] T.Gonda, H.Gomi, Morphological instability of polyhedral ice crystals growing in air at low temperature. *Ann. Glaciology,* **6** (1985), 222–224.

[G] M.Gurtin, "Thermomechanics of Evolving Phase Boundaries in the Plane". Clarendon Press, Oxford, 1993.

[GM] M. Gurtin and J. Matias, Thermomechanics and the formulation of the Stefan problem for fully faceted interfaces. *Quart. Appl. Math.,* **53** (1995), 761–782.

[HK] P.Hajłasz, A.Kałamajska, Polynomial asymptotics and approximation of Sobolev functions. *Studia Math.* **113** (1995), no. 1, 55–64.

[KIO] T. Kuroda, T. Irisawa and A. Ookawa, Growth of a polyhedral crystal from solution and its morphological stability. *J. Crystal Growth* **42** (1977), 41–46.

[LY] Z.H. Liu, G.W. Yuan, The multi-dimension one-phase Stefan problem with kinetic condition. *J. Math. Res. Exposition* **14** (1994), no. 4, 479–486.

[R] P. Rybka, A crystalline motion: uniqueness and geometric properties. *SIAM J. Appl. Math.*, **57** (1997), pp. 53–72.

[Se] A.Seeger, Diffusion problems associated with the growth of crystals from dilute solution. *Philos. Mag.*, ser. 7, **44**, no 348, (1953) 1–13.