Geometry of higher order differential equations of finite type associated with symmetric spaces

Keizo Yamaguchi and Tomoaki Yatsui

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GEOMETRY OF HIGHER ORDER DIFFERENTIAL EQUATIONS OF FINITE TYPE ASSOCIATED WITH SYMMETRIC SPACES

KEIZO YAMAGUCHI AND TOMOAKI YATSUI

Introduction

This is a half survey article on the geometry of higher order differential equations of finite type associated with symmetric spaces.

Historically, geometric study of differential equations, especially ordinary differential equations, was initiated by Sophus Lie [Lie91]. For linear ordinary differential equations, Laguerre and Forsyth studied the differential invariants of these equations by transforming them to the canonical forms (cf. [Wil06]).

For higher order equations, after Lie, the classification of the second order ordinary differentialequations by point transformations was achieved by Tresse [Tre96] and E. Cartan [Car24] studied the case when the equation is associated with paths in projective geometry by his method of the equivalence. The third order equations were studied by S.S.Chern [Ch50], following the method of E.Cartan (cf. [SY98]). Then N.Tanaka [Tan82] studied the equivalence problem for the system of second order ordinary differential equations by point transformations and formulated this geometry in terms of the pseudo-product structures. Furthermore he constructed normal Cartan connections on these systems and utilized the connections to the normal form problem and the integration problem of these systems [Tan79] [Tan89].

For the geometrization problem for the equivalence of ordinary differential equations with some historical comments, we refer the reader to the excellent survey article [DKM99].

In this paper we adopt the point of view initiated by N.Tanaka. Let us consider a system of higher order differential equations of finite type of the following form :

$$\frac{\partial^k y^\alpha}{\partial x_{i_1}\cdots\partial x_{i_k}} = F^\alpha_{i_1\ \ i_k}(x_1,\ldots,x_n,y^1,\cdots,y^m,\ldots,p^\beta_i,\ldots,p^\beta_{j_1\ \ j_{k-1}})$$
$$(1 \leqq \alpha \leqq m, 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n),$$

where $p^\beta_{i_1\ \ i_\ell} = \frac{\partial^\ell y^\beta}{\partial x_{i_1}\ \ \partial x_{i_\ell}}$. Namely let us consider a system R of k-th order equations such that every k-th derivative is expressed in terms of the derivatives of the lower order. If we regard R as a submanifold of the k-jet space J^k with coordinates $(x_1,\ldots,x_n,y^1,\cdots,y^m,\ldots,p^\beta_i,\ldots,p^\beta_{j_1\ \ j_k})$, then R is diffeomorphic to J^{k-1}. Moreover R specifies an n-dimensional subspace $E(x)$ of $C^{k-1}(x)$ at each point x of J^{k-1}, where C^{k-1} is the **canonical differential system** on J^{k-1} (for the precise definitions, see §1). Thus R defines a differential system E on J^{k-1} such that $C^{k-1} = E \oplus F$, where $F = \mathrm{Ker}(\pi^{k-1}_{k-2})_*$ and $\pi^{k-1}_{k-2} : J^{k-1} \to J^{k-2}$ is the projection. E is completely integrable when the system R is integrable. The triplet $(J^{k-1}; E, F)$ is called the **pseudo-product structure** associated with R.

Our basic strategy to study higher order differential equations R of finite type is to utilize the 'rich' geometry of the differential system $C^{k-1} = E \oplus F$ naturally associated with R. We further pursue this approach to an important class of higher order differential equations of finite type, which is called of type $(\mathfrak{l}, S)$.

Now let us proceed to describe the contents of each section. In §1, we will explain the pseudo-product structure $(R; E, F)$ associated with a system R of higher order differential equations of finite type and give an overview on the Tanaka theory for regular differential systems. Especially we will review the **symbol algebra** $\mathfrak{m}(x) = \bigoplus_{p<0} \mathfrak{g}_p(x)$ of a regular differential system (M, D) and the notion of the (algebraic) **prolongation** $\mathfrak{g}(\mathfrak{m})$ (resp. $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$) of $\mathfrak{m}$ (resp. $(\mathfrak{m}, \mathfrak{g}_0)$), for a given fundamental graded Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$, where $\mathfrak{g}_0$ is a subalgebra of the (gradation preserving) derivation algebra $\mathfrak{g}_0(\mathfrak{m})$ of $\mathfrak{m}$. $\mathfrak{g}(\mathfrak{m})$ represents the Lie algebra of infinitesimal automorphisms of the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$, which is the local model differential system of type $\mathfrak{m}$. As an example we will calculate the symbol algebra $\mathfrak{C}^k(n, m)$ of the canonical differential system (J^k, C^k) in §1.3 and will show that this algebra has the following description:

$$\mathfrak{C}^k(n, m) = \mathfrak{C}_{-(k+1)} \oplus \mathfrak{C}_{-k} \oplus \cdots \oplus \mathfrak{C}_{-1},$$

where $\mathfrak{C}_{-(k+1)} = W$, $\mathfrak{C}_p = W \otimes S^{k+p+1}(V^*)$, $\mathfrak{C}_{-1} = V \oplus W \otimes S^k(V^*)$. Here V and W are vector spaces of dimension n and m respectively and the bracket product of $\mathfrak{C}^k(n, m) = \mathfrak{C}^k(V, W)$ is defined accordingly through the pairing between V and V^* such that V and $W \otimes S^k(V^*)$ are both abelian subspaces of $\mathfrak{C}_{-1}$. Here $S^k(V^*)$ denotes the k-th symmetric product of V^*.

Corresponding to the splitting $C^{k-1} = E \oplus F$ of the pseudo-product structure, we have the splitting in the symbol algebra $\mathfrak{C}^{k-1}(n, m)$;

$$\mathfrak{C}_{-1} = \mathfrak{e} \oplus \mathfrak{f},$$

where $\mathfrak{e} = V$, $\mathfrak{f} = W \otimes S^{k-1}(V^*)$. In §2 we first consider the prolongation $\mathfrak{g}^k(n, m)$ of $(\mathfrak{C}^{k-1}(n, m), \check{\mathfrak{g}}_0)$, where $\check{\mathfrak{g}}_0$ is the subalgebra of $\mathfrak{g}_0(\mathfrak{C}^{k-1}(n, m))$ consisting of elements which preserves both $\mathfrak{e}$ and $\mathfrak{f}$. $\mathfrak{g}^k(n, m)$ is called the **pseudo-projective GLA** (graded Lie algebra) of order k of bidegree (n, m). We will give the explicit description of these algebras in §2.1. $\mathfrak{g}^k(n, m)$ gives the Lie algebra of infinitesimal automorphisms of the (local) model k-th order differential equation R_o of finite type, where

$$R_o = \left\{ \frac{\partial^k y^\alpha}{\partial x_{i_1} \cdots \partial x_{i_k}} = 0 \quad (1 \leqq \alpha \leqq m, 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n) \right\}.$$

Generalizing the structure of $\mathfrak{g}^k(n, m)$, we will now introduce the important class of pseudo-product GLA of the irreducible type. Namely, starting from a reductive GLA $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ and a faithful irreducible $\mathfrak{l}$-module S, we define the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$ as follows: Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a finite dimensional **reductive GLA** of the first kind such that

(1) The ideal $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ of $\mathfrak{l}$ is a simple Lie algebra.
(2) The center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ is contained in $\mathfrak{l}_0$.

Let S be a finite dimensional **faithful irreducible $\mathfrak{l}$-module**. We put

$$S_{-1} = \{ s \in S \mid \mathfrak{l}_1 \cdot s = 0 \}$$

and

$$S_p = \mathrm{ad}(\mathfrak{l}_{-1})^{-p-1} S_{-1} \quad \text{for } p < 0$$

We form the semi-direct product $\mathfrak{g}$ of $\mathfrak{l}$ by S, and put

$$\mathfrak{g} = S \oplus \mathfrak{l}, \qquad [S, S] = 0$$

$$\mathfrak{g}_k = \mathfrak{l}_k \ (k \geqq 0), \quad \mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1},$$

$$\mathfrak{g}_p = S_p \ (p < -1).$$

Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ enjoys the following properties (Lemma 2.1);

(1) $S = \bigoplus_{p=-1}^{-\mu} S_p$, *where* $S_{-\mu} = \{ s \in S \mid [\mathfrak{l}_{-1}, s] = 0 \}$.
(2) $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is generated by $\mathfrak{g}_{-1}$.

(3) *S_p is naturally embedded as a subspace of $W \otimes S^{\mu+p}(\mathfrak{l}^*_{-1})$ through the bracket operation in $\mathfrak{m}$, where $W = S_{-\mu}$.*

Thus $\mathfrak{m}$ is a graded subalgebra of $\mathfrak{C}^{\mu-1}(V, W)$, which has the splitting $\mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1}$, where $V = \mathfrak{l}_{-1}$ and $W = S_{-\mu}$. Hence $\mathfrak{m}$ is a symbol algebra of μ-th order differential equations of finite type, which is called the typical symbol of type $(\mathfrak{l}, S)$.

This class of higher order (linear) differential equations of finite type were first appeared in the work of Y.Se-ashi [Sea88], who discussed the linear equivalence of this class of equations and gave the complete system of differential invariants of these equations, generalizing the classical theory of Laguerre-Forsyth for linear ordinary differential equations.

We will ask the following questions for the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$:

(A) *When is $\mathfrak{g}$ the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{g}_0)$?*

(B) *Find the fundamental invariants for equations of type $\mathfrak{m}$.*

Utilizing the Tanaka-Morimoto theory of normal Cartan connections [Tan79], [Mor93], these questions will be answered by calculating the first and second **generalized Spencer cohomology spaces**. The complete answer for (A) will be given as Theorem 5.2 in §5 and the problem (B) will be discussed in §6 and §7.

In §3, we will recall the construction of the **model equations** for the typical symbol of type $(\mathfrak{l}, S)$, following [Sea88]. For simplicity, let us explain this construction here in holomorphic category. Assuming the condition $H^1(\mathfrak{G})_{0,0} = 0$ (see §4.5), we see that the Lie algebra $\mathfrak{l}$ coincides with the Lie algebra of infinitesimal (linear) automorphisms of the local model equations of type $\mathfrak{m}$ (see the discussion in §3.1). Regarding $\mathfrak{l}$ as a subalgebra of $\mathfrak{gl}(S)$, let L and L' be the Lie subgroup of $GL(S)$ with Lie algebra $\mathfrak{l}$ and $\mathfrak{l}'$ respectively, where $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Moreover let $\hat{L}$ be the Lie subgroup of $GL(S)$ with Lie algebra $\hat{\mathfrak{l}}$ and put $\hat{L}' = L' \cap \hat{L}$. Then $\hat{L}'$ is a parabolic subgroup of $\hat{L}$ and $M = L/L' = \hat{L}/\hat{L}'$ is an irreducible compact hermitian symmetric space (cf. §3.2).

Since L' preserves the filtration $\{S^q\}_{q<0}$ of S, where $S^q = \bigoplus_{p=-1}^{q} S_p$, we get the representation ρ_W of L':

$$\rho_W : L' \to GL(W),$$

through the projection $\pi_0 : S = \bigoplus_{p=-1}^{-\mu} S_p \to S_{-\mu} = W$.

Let E_S be the vector bundle over $M = L/L'$ associated with the representation $\rho_W : L' \to GL(W)$. As is well known, each $s \in S$ defines a global section σ_s of the vector bundle E_S (see the discusssion in §3.1).

Let $J^\mu(E_S)$ be the bundle of μ-jets of E_S. At each point $x \in M = L/L'$, let $(R_S)_x$ be the subspace of $J^\mu_x(E_S)$ defined by

$$(R_S)_x = \{\, j^\mu_x(\sigma_s) \mid s \in S \,\}.$$

where $j^\mu_x(\sigma_s)$ is the μ-jet at x of the section σ_s. Then the model (linear) equation R_S for the typical symbol of type $(\mathfrak{l}, S)$ is defined as the subbundle of $J^\mu(E_S)$ by

$$R_S = \bigcup_{x\in M} (R_S)_x.$$

R_S is the system of differential equations of finite type which characterizes global sections of E_S locally.

In §3.2, we will discuss the Plücker embedding equations as our examples of model equations. We assume here that $\mathfrak{l} = \hat{\mathfrak{l}} \oplus \mathfrak{z}(\mathfrak{l})$. Then, a little generally, an equivariant projective embedding of the model space $M = L/L' = \hat{L}/\hat{L}'$ can be obtained from an irreducible representation of $\hat{L}$ as follows: Let $\tau : \hat{L} \to GL(T)$ be an irreducible representation of $\hat{L}$ with the highest weight Λ. Let t_Λ be a maximal vector in T of the highest weight Λ. Then a stabilizer of the line $[t_\Lambda]$ spanned by t_Λ in T is a parabolic subgroup of $\hat{L}$. When this stabilizer coincides with $\hat{L}'$, we obtain an equivariant projective embedding of $M = \hat{L}/\hat{L}'$ by taking the $\hat{L}$-orbit passing through $[t_\Lambda]$ in the projective space $P(T)$. In this case, it can be shown that this embedding can be obtained by global sections of the line bundle F which is constructed from the dual representation $\rho = \tau^*$ of $\hat{L}$ on $S = T^*$ (see the discussion in §3.2). Then the model equation R_S for the typical symbol of type $(\mathfrak{l}, S)$, which characterizes global sections of F locally, can be called the embedding equation for M. The case when $\mathfrak{l} = \mathfrak{sl}(\ell+1, \mathbb{C})$ and $S = \bigwedge^{\ell-k+1} \mathbb{C}^{\ell+1}$ corresponds to the **Plücker embedding equations** for the Grassmann manifold $M = \mathrm{Gr}(k, \ell+1)$.

From §4, we will study the cohomology group $H^*(\mathfrak{G}) = H^*(\mathfrak{m}, \mathfrak{g})$ associated with the adjoint representation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ on $\mathfrak{g}$ for the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$. Namely we will study the cohomology space of the cochain complex $C^*(\mathfrak{G}) = \bigoplus C^p(\mathfrak{G})$ with the coboundary operator $\partial : C^p(\mathfrak{G}) \longrightarrow C^{p+1}(\mathfrak{G})$ given by

$$\partial^p \omega(x_1, \ldots, x_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} [x_i, \omega(x_1, \ldots, \hat{x}_i, \ldots, x_{p+1})] + \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{p+1}),$$

where $\omega \in C^p(\mathfrak{G}) = \mathrm{Hom}(\bigwedge^p \mathfrak{m}, \mathfrak{g})$ and $x_i \in \mathfrak{m}$. Moreover we put $\mathfrak{b}_{-1} = S$, $\mathfrak{b}_0 = \mathfrak{l}$ and $\mathfrak{b}_p = 0$ $(p \neq -1, 0)$. We utilize the bigradation $(\mathfrak{g}_{p,q})_{p,q\in\mathbb{Z}}$ of $\mathfrak{g} = S \oplus \mathfrak{l}$ given by $\mathfrak{g}_{p,q} = \mathfrak{g}_p \cap \mathfrak{b}_q$. Then $C^p(\mathfrak{G})$ has the following decomposition : $C^p(\mathfrak{G}) = \bigoplus_{r,s} C^p(\mathfrak{G})_{r,s}$, where

$$C^p(\mathfrak{G})_{r,s} = \left\{ \omega \in C^p(\mathfrak{G}) \;\middle|\; \begin{array}{c} \omega(\mathfrak{g}_{i_1,j_1} \wedge \cdots \wedge \mathfrak{g}_{i_p,j_p}) \subset \mathfrak{g}_{i_1+ \cdots +i_p+r, j_1+\cdots+j_p+s} \\ \text{for all } \; i_1, \ldots, i_p, j_1, \ldots, j_p \end{array} \right\}.$$

In §4.3, we prepare the fundamental theorem (Theorem 4.1) to calculate the cohomology space $H^p(\mathfrak{G})_{r,s}$ by means of Kostant's theorem on Lie algebra cohomology.

Now let us describe $(\mathfrak{l}, S)$ in detail, utilizing the structure theory of semi-simple Lie algebras over $\mathbb{C}$. First we decompose $\mathfrak{l} = \hat{\mathfrak{l}} \oplus \mathfrak{z}_\mathfrak{l}(\hat{\mathfrak{l}})$, $\mathfrak{l}_0 = \hat{\mathfrak{l}}_0 \oplus \mathfrak{z}_\mathfrak{l}(\hat{\mathfrak{l}})$. Since S is a faithful irreducible $\mathfrak{l}$-module, there is an irreducible $\hat{\mathfrak{l}}$-module T and $\mathfrak{z}_\mathfrak{l}(\hat{\mathfrak{l}})$-module U such that S is isomorphic to $T \otimes U$ as a $\mathfrak{l}$-module. Then we impose the following condition for $(\mathfrak{l}, S)$: $\mathfrak{z}_\mathfrak{l}(\hat{\mathfrak{l}})$ *is isomorphic to* $\mathfrak{gl}(U)$. We interprete this condition as the condition $H^1(\mathfrak{G})_{0,0} = 0$ for the first cohomology in Lemma 4.5.

Fixing a Cartan subalgebra $\mathfrak{h}$ (the set Φ of roots relative to $\mathfrak{h}$) and a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, we have the root space decomposition of the simple Lie algebra $\hat{\mathfrak{l}}$. Then the gradation $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus \hat{\mathfrak{l}}_0 \oplus \mathfrak{l}_1$ can be described as (X_ℓ, Δ_1) (see §4.4). Here X_ℓ stands for the Dynkin diagram of $\hat{\mathfrak{l}}$, $\Delta_1 = \{\alpha_{i_o}\} \subset \Delta$ and α_{i_o} satisfies $m_{i_o}(\theta) = 1$, where θ is the highest root and m_i is a $\mathbb{Z}$-valued function on Φ defined by $m_i(\alpha) = k_i$ for $\alpha = \sum_{j=1}^{\ell} k_j \alpha_j \in \Phi$. Namely the gradation $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus \hat{\mathfrak{l}}_0 \oplus \mathfrak{l}_1$ is given as the decomposition by root spaces according to the height $m_{i_o}(\alpha)$ of each root α, for suitable choices of $\mathfrak{h}$ and Δ. Thus $(\mathfrak{l}, S)$ can be described by the triplet $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ as follows : $(\mathcal{D}(\mathfrak{l}), \Delta_1)$ is of type $(X_\ell \times A_n, \{\alpha_i\})$ and S is an irreducible $D(\mathfrak{l})$-module with highest weight $\Xi = \chi + \pi_1$ when $\dim U > 1$ and $(\mathcal{D}(\mathfrak{l}), \Delta_1)$ is of type $(X_\ell, \{\alpha_i\})$ and S is an irreducible $\hat{\mathfrak{l}}$-module with highest weight $\Xi = \chi$ when $\dim U = 1$. Here π_1 is the fundamental weight corresponding to the identity representation on U of $\mathfrak{z}_\mathfrak{l}(\hat{\mathfrak{l}}) = \mathfrak{gl}(U)$ of type A_n and χ is a dominant integral weight of $\hat{\mathfrak{l}}$ of type X_ℓ.

In §5, we will calculate the first cohomology $H^1(\mathfrak{G})$ and give the answer for the question (A) raised in §3 in Theorem 5.2 : *For a pseudo-product GLA* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *of type* $(\mathfrak{l}, S)$ *satisfying the condition* $H^1(\mathfrak{G})_{0,0} = 0$, $\mathfrak{g}$ *is the prolongation of* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *except for three cases.* Let $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ be the prolongation of $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$. Then the three exceptional cases correspond to cases :

(*a*) $\dim \check{\mathfrak{b}} < \infty$ and $\check{\mathfrak{b}}_1 \neq 0$,
(*b*) $\dim \check{\mathfrak{b}} = \infty$,
(*c*) $\mathfrak{g}$ is a pseudo-projective GLA.

In case (*a*), $\check{\mathfrak{b}} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \check{\mathfrak{b}}_1$ becomes a simple GLA containing $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$ as a parabolic subalgebra. Theorem 5.2 lists up all these exceptions explicitly in terms of the data $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$.

We will calculate the second cohomology $H^2(\mathfrak{G})$ in §6. Especially we will enumerate the cases when $H^2(\mathfrak{G})_r \neq 0$ $(r \geqq 1)$ for the above exceptional cases (*a*) and (*b*). Finally in §7, we will show the rigidity theorem for the Plücker embedding equations for $M = \mathrm{Gr}(k, \ell + 1)$, when $k = 2$ $(\ell \geqq 4)$ or $k = 3$ $(\ell \geqq 6)$.

1. Differential Systems and Pseudo-Product Structures

1.1. Differential Equations of Finite Type. Let us consider a higher order ordinary differential equation

$$y^{(k)} = F(x, y, y', \dots, y^{(k-1)}),$$

or more generally, a system of higher order differential equations of finite type

$$\frac{\partial^k y^\alpha}{\partial x_{i_1} \cdots \partial x_{i_k}} = F^\alpha_{i_1 \cdots i_k}(x_1, \dots, x_n, y^1, \cdots, y^m, \dots, p^\beta_i, \dots, p^\beta_{j_1 \cdots j_{k-1}})$$
$$(1 \leqq \alpha \leqq m, 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n),$$

where $p^\beta_{i_1 \cdots i_\ell} = \frac{\partial^\ell y^\beta}{\partial x_{i_1} \cdots \partial x_{i_\ell}}$. These equations define a submanifold R in k-jets space J^k such that the restriction p to R of the bundle projection $\pi^k_{k-1} : J^k \to J^{k-1}$ gives a diffeomorphism ;

$$p : R \to J^{k-1}; \text{diffeomorphism} \tag{1.1}$$

On J^k, we have the Contact (differential) system C^k defined by

$$C^k = \{\varpi^\alpha = \varpi^\alpha_i = \cdots = \varpi^\alpha_{i_1 \cdots i_{k-1}} = 0\},$$

where

$$\begin{cases} \varpi^\alpha = d y^\alpha - \sum_{i=1}^n p^\alpha_i d x_i, & (1 \leqq \alpha \leqq m) \\ \varpi^\alpha_i = d p^\alpha_i - \sum_{j=1}^n p^\alpha_{ij} d x_j, & (1 \leqq \alpha \leqq m, 1 \leqq i \leqq n) \\ \cdots\cdots\cdots\cdots, & \\ \varpi^\alpha_{i_1 \cdots i_{k-1}} = d p^\alpha_{i_1 \cdots i_{k-1}} - \sum_{j=1}^n p^\alpha_{i_1 \cdots i_{k-1} j} d x_j & \\ \qquad (1 \leqq \alpha \leqq m, 1 \leqq i_1 \leqq \cdots \leqq i_{k-1} \leqq n). & \end{cases} \tag{1.2}$$

Then C^k gives a foliation on R when R is integrable. Namely the restriction E' of C^k to R is completely integrable.

Thus, through the diffeomorphism (1.1), R defines a completely integrable differential system $E = p_*(E')$ on J^{k-1} such that

$$C^{k-1} = E \oplus F, \quad F = \mathrm{Ker}(\pi^{k-1}_{k-2})_*$$

where $\pi^{k-1}_{k-2} : J^{k-1} \to J^{k-2}$ is the bundle projection.

To treat with this situation, N.Tanaka ([Tan85]) introduced the notion of pseudo-product manifolds as follows.

Pseudo-Product Manifolds $(R; E, F)$

(1) E and F are differential systems on a manifold R.

(2) $E \cap F = 0$, and both E and F are completely integrable.

(3) $D = E \oplus F$ is non-degenerate.

(4) The full derived systems of D coincides with $T(R)$

In fact he studied ([Tan82, 89]), in this setting, the geometry of systems of second order ordinary differential equations in depth, utilizing his theory of Cartan connections associated with simple graded Lie algebras $\mathfrak{sl}(m+2, \mathbb{R}) = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$.

More generally let R be an involutive system of k-th order differential equations of finite type. Let $R^{(1)} \subset J^{k+1}$ be the first prolongation of R (cf. [Yam82] for a precise definition of involutive systems). The condition "R is involutive and of finite type " implies that $p^{(1)} : R^{(1)} \to R$ is a diffeomorphism (cf. Lemma 7.3 [Yam82]). This implies that $\operatorname{Ker} p_* = \{0\}$ and E is completely integrable, where $p = \pi^k_{k-1}|_R$ and $E = C^k|_R$. Then, as in the above example, by putting $F = \operatorname{Ker}(\pi)_*$, $(R; E, F)$ enjoys the properties (1) and (2) above, where $\pi = \pi^{k-1}_{k-2} \circ p$. Here $D = E \oplus F$ is the pull back of the canonical system C^{k-1} by p_*. We call $(R; E, F)$ the pseudo-product structure associated with R in the broader sense.

Our basic strategy to study (involutive) higher order differential equations R of finite type is to utilize the 'rich' geometry of the differential system $D = E \oplus F$ naturally associated with R.

1.2. **Geometry of Differential Systems (Tanaka Theory).** We summarize here the basic notion for (linear) differential systems following [Tan70] and [Yam93].

1.2.1. *Derived Systems and Characteristic Systems.* For a manifold M of dimension d, a subbundle $D \subset T(M)$ of rank r $(s + r = d)$ is called a **differential system** of rank r (or codimension s).

$$D = \{\, \omega_1 = \cdots = \omega_s = 0 \,\}.$$

For two differential systems (M, D) and $(\hat{M}, \hat{D})$, a diffeomorphism φ of M onto $\hat{M}$ is called an **isomorphism** of (M, D) onto $(\hat{M}, \hat{D})$ if the differential map φ_* of φ sends D onto $\hat{D}$.

By the Frobenius theorem, we know that D is **completely integrable** if and only if

$$d\omega_i \equiv 0 \pmod{\omega_1, \ldots, \omega_s} \qquad \text{for } i = 1, \ldots, s,$$

or equivalently, if and only if

$$[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}.$$

where $\mathcal{D} = \Gamma(D)$ denotes the space of sections of D.

Thus, for a non-integrable differential system D, the **derived system** ∂D of D is defined, in terms of sections, by

$$\partial\mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

The **Cauchy characteristic system** $\operatorname{Ch}(D)$ of (M, D) is defined at each point $x \in M$ by

$$\operatorname{Ch}(D)(x) = \left\{ X \in D(x) \;\middle|\; \begin{array}{l} X \rfloor d\omega_i \equiv 0 \pmod{\omega_1, \ldots, \omega_s} \\ \qquad\qquad\qquad \text{for } i = 1, \ldots, s \end{array} \right\},$$

Then $\operatorname{Ch}(D)$ is always completely integrable when it is a subbundle (i.e. has constant rank) (cf. [Yam82]).

Moreover higher derived systems $\partial^k D$ are usually defined successively by

$$\partial^k D = \partial(\partial^{k-1} D),$$

where we put $\partial^0 D = D$ for convention.

On the other hand we define the k-th weak derived system $\partial^{(k)} D$ of D inductively by

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}],$$

where $\partial^{(0)}D = D$ and $\partial^{(k)}\mathcal{D}$ denotes the space of sections of $\partial^{(k)}D$. This notion is one of the key point in the Tanaka theory ([Tan70]).

A differential system (M, D) is called **regular**, if $D^{-(k+1)} = \partial^{(k)}D$ are subbundles of $T(M)$ for every integer $k \geqq 1$. For a regular differential system (M, D), we have ([Tan70], Proposition 1.1)

(S1) *There exists a unique integer* $\mu > 0$ *such that, for all* $k \geqq \mu$,

$$D^{-k} = \cdots = D^{-\mu} \supsetneqq D^{-\mu+1} \supsetneqq \cdots \supsetneqq D^{-2} \supsetneqq D^{-1} = D,$$

(S2) $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$ *for all* $p, q < 0$.

where $\mathcal{D}^p$ denotes the space of sections of D^p. (S2) can be checked easily by induction on q. Thus $D^{-\mu}$ is the smallest completely integrable differential system, which contains $D = D^{-1}$.

1.2.2. *Symbol Algebras.* From now on, we will consider **regular** differential systems (M, D) such that $T(M) = D^{-\mu}$. As a first invariant for non-integrable differential system, the **symbol algebra** $\mathfrak{m}(x)$ of (M, D) at x is defined as follows ([Tan70]);

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x),$$

where $\mathfrak{g}_{-1}(x) = D^{-1}(x)$, $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$ $(p < -1)$. Let ϖ_p be the projection of $D^p(x)$ onto $\mathfrak{g}_p(x)$. Then, for $X \in \mathfrak{g}_p(x)$ and $Y \in \mathfrak{g}_q(x)$, the bracket product $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is defined by

$$[X, Y] = \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x),$$

where $\tilde{X}$ and $\tilde{Y}$ are any element of $\mathcal{D}^p$ and $\mathcal{D}^q$ respectively such that $\varpi_p(\tilde{X}_x) = X$ and $\varpi_q(\tilde{Y}_x) = Y$.

Endowed with this bracket operation, by $(S2)$ above, $\mathfrak{m}(x)$ becomes a nilpotent graded Lie algebra such that $\dim \mathfrak{m}(x) = \dim M$ and satisfies

$$\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)] \qquad \text{for } p < -1.$$

Furthermore, let $\mathfrak{m}$ be a fundamental graded Lie algebra (**FGLA**) of μ-th kind, that is,

$$\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$$

is a nilpotent graded Lie algebra such that

$$\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \qquad \text{for } p < -1.$$

Then (M, D) is called of type $\mathfrak{m}$ if the symbol algebra $\mathfrak{m}(x)$ is isomorphic with $\mathfrak{m}$ at each $x \in M$.

Conversely, given a FGLA $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$, we can construct a model differential system of type $\mathfrak{m}$ as follows: Let $M(\mathfrak{m})$ be the simply connected Lie group with Lie algebra $\mathfrak{m}$. Identifying $\mathfrak{m}$ with the Lie algebra of left invariant vector fields on $M(\mathfrak{m})$, $\mathfrak{g}_{-1}$ defines a left invariant subbundle $D_{\mathfrak{m}}$ of $T(M(\mathfrak{m}))$. By definition of symbol algebras, it is easy to see that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is a regular differential system of type $\mathfrak{m}$. $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is called the **standard differential system** of type $\mathfrak{m}$. The Lie algebra $\mathfrak{g}(\mathfrak{m})$ of all infinitesimal automorphisms of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ can be calculated algebraically as the (algebraic) **prolongation** of $\mathfrak{m}$ ([Tan70], cf. [Yam93]).

1.2.3. *Prolongation of* $(\mathfrak{m}, \mathfrak{g}_0)$. Here we recall some basic facts on the algebraic prolongation $\mathfrak{g}(\mathfrak{m})$ of a FGLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ (see §2 of [Yam93]).

$\mathfrak{g}(\mathfrak{m})$ is first characterized as the graded Lie algebra which satisfies the following conditions:

(1) $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$, where $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$.
(2) For $k \geqq 0$, if $X \in \mathfrak{g}_k(\mathfrak{m})$ and $[X, \mathfrak{m}] = \{0\}$, then $X = 0$.
(3) $\mathfrak{g}(\mathfrak{m})$ is maximum among graded algebras satisfying conditions (1) and (2) above.

More precisely, we can define $\mathfrak{g}_k(\mathfrak{m})$ as follows: First we decompose $\bigwedge^2 \mathfrak{m}^* = \bigoplus_{j<-1} \wedge_j^2 \mathfrak{m}^*$ according to the gradation $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$, where

$$\wedge_j^2 \mathfrak{m}^* = \bigoplus_{p+q=j} \mathfrak{g}_p^* \wedge \mathfrak{g}_q^*.$$

Putting $C_k^1 = \bigoplus_{p<0} \mathfrak{g}_{p+k} \otimes \mathfrak{g}_p^*$ and $C_k^2 = \bigoplus_{j<-1} \mathfrak{g}_{j+k} \otimes \wedge_j^2 \mathfrak{m}^*$, we can define $\mathfrak{g}_k = \mathfrak{g}_k(\mathfrak{m})$ for $k \geqq 0$ inductively by the following exact sequence;

$$0 \to \mathfrak{g}_k \to C_k^1 \xrightarrow{\partial} C_k^2,$$

where the coboundary operator $\partial : C_k^1 \to C_k^2$ is given by

$$(\partial p)(X, Y) = [X, p(Y)] - [Y, p(X)] - p([X, Y]).$$

Thus $\mathfrak{g}_0(\mathfrak{m})$ is the (gradation preserving) derivation algebra of $\mathfrak{m}$. Moreover, for $u \in \mathfrak{g}_k(\mathfrak{m})$ and $v \in \mathfrak{g}_\ell(\mathfrak{m})$ $(k, \ell \geqq 0)$, by induction on the integer $k + \ell \geqq 0$, we can define $[u, v] \in \mathfrak{g}_{k+\ell} \subset C_{k+\ell}^1$ by

$$[u, v](X) = [[u, X], v] + [u, [v, X]] \qquad \text{for } X \in \mathfrak{m}.$$

With this bracket product, $\mathfrak{g}(\mathfrak{m})$ becomes a graded Lie algebra.

Now let $\mathfrak{g}_0$ be a subalgebra of $\mathfrak{g}(\mathfrak{m})$. We define a subspace $\mathfrak{g}_k$ of $\mathfrak{g}_k(\mathfrak{m})$ for $k \geqq 1$ inductively by

$$\mathfrak{g}_k = \{\, u \in \mathfrak{g}_k(\mathfrak{m}) \mid [u, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{k-1} \,\}.$$

Then, putting

$$\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \bigoplus_{k \geqq 0} \mathfrak{g}_k,$$

we see, with the generating condition of $\mathfrak{m}$, that $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is a graded subalgebra of $\mathfrak{g}(\mathfrak{m})$. $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is called the **prolongation** of $(\mathfrak{m}, \mathfrak{g}_0)$.

By utilizing the above definition of the algebraic prolongation, we will consider the following situation: Let $\mathfrak{h} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{h}_p$ be a graded Lie algebra such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{h}_p$ is a FGLA. To check whether $\mathfrak{h}$ is the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{h}_0)$, we consider the Lie algebra cohomology $H^p(\mathfrak{m}, \mathfrak{h})$ associated with the adjoint representation $ad : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{h})$. According to the gradation of $\mathfrak{h}$, this cohomology space has a bigradation (for the precise definition see §4);

$$H^p(\mathfrak{m}, \mathfrak{h}) = \bigoplus_r H^p(\mathfrak{m}, \mathfrak{h})_r$$

With this cohomology group, we will utilize the following criterion in §4.

Lemma A (Lemma 2.1 [Yam93]). *Let* $\mathfrak{h} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{h}_p$ *be a graded Lie algebra such that* $\mathfrak{h}_p = [\mathfrak{h}_{p+1}, \mathfrak{h}_{-1}]$ *for* $p < -1$. *Then* $\mathfrak{h}$ *is the prolongation of* $\mathfrak{m}$ (*resp. of* $(\mathfrak{m}, \mathfrak{h}_0)$) *if and only if the following two conditions hold*:

(1) *For* $k \geqq 0$, *if* $X \in \mathfrak{h}_k$ *and* $[X, \mathfrak{m}] = 0$, *then* $X = 0$.
(2) $H^1(\mathfrak{m}, \mathfrak{h})_r = \{0\}$ *for* $r \geqq 0$ (*resp.* $r \geqq 1$).

1.3. **Symbol Algebra of** (J^k, C^k)**.** As an example, we will now calculate the symbol algebra of the canonical differential system (J^k, C^k). First recall that C^k is defined by 1-forms (1.2) on a coordinate system $U;(x^i, y^\alpha, p_i^\alpha, \cdots, p^\alpha_{i_1 \cdots i_k})$ of J^k ;

$$C^k = \{\varpi^\alpha = \varpi_i^\alpha = \cdots = \varpi^\alpha_{i_1 \cdots i_{k-1}} = 0\}$$

Then we have a following coframe

$$\{\varpi^\alpha, \ldots, \varpi^\alpha_{i_1 \cdots i_\ell}, \ldots, d\,x_i, d\,p^\alpha_{i_1 \cdots i_k}\},$$

at each point in U. Now let us take the dual frame of this coframe;

$$\left\{\frac{\partial}{\partial y^\alpha}, \ldots, \frac{\partial}{\partial p^\alpha_{i_1 \cdots i_\ell}}, \ldots, \frac{d}{d\,x_i}, \frac{\partial}{\partial p^\alpha_{i_1 \cdots i_k}}\right\}$$

where

$$\frac{d}{d\,x_i} = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} p_i^\alpha \frac{\partial}{\partial y^\alpha} + \sum_{\alpha=1}^{m} \sum_{j_1 \leqq \cdots \leqq j_\ell} p^\alpha_{i j_1 \cdots j_\ell} \frac{\partial}{\partial p^\alpha_{j_1 \cdots j_\ell}}$$

We have

$$\left[\frac{\partial}{\partial p_j^\alpha}, \frac{d}{d\,x_i}\right] = \delta^i_j \frac{\partial}{\partial y^\alpha}, \quad \left[\frac{\partial}{\partial p^\alpha_{j_1 \cdots j_\ell}}, \frac{d}{d\,x_i}\right] = \sum_{k=1}^{\ell} \delta^i_{j_k} \frac{\partial}{\partial p^\alpha_{j_1 \cdots \hat{j}_k \cdots j_\ell}}$$

Then we see that (J^k, C^k) is a regular differential system of type $\mathfrak{C}^k(n, m)$:

$$\mathfrak{C}^k(n, m) = \mathfrak{C}_{-(k+1)} \oplus \mathfrak{C}_{-k} \oplus \cdots \oplus \mathfrak{C}_{-1},$$

where $\mathfrak{C}_{-(k+1)} = W$, $\mathfrak{C}_p = W \otimes S^{k+p+1}(V^*)$, $\mathfrak{C}_{-1} = V \oplus W \otimes S^k(V^*)$.

Here V and W are vector spaces of dimension n and m respectively and the bracket product of $\mathfrak{C}^k(n, m) = \mathfrak{C}^k(V, W)$ is defined accordingly through the pairing between V and V^* such that V and $W \otimes S^k(V^*)$ are both abelian subspaces of $\mathfrak{C}_{-1}$. Namely

$$\begin{aligned} &[W, V] = \{0\}, \qquad [V, V] = \{0\}, \\ &[W \otimes S^r(V^*), W \otimes S^s(V^*)] = \{0\} \quad (r, s = 0. \cdots, k), \\ &[W \otimes S^r(V^*), V] = W \otimes (i(V)(S^r(V^*))) \quad (r = 1, \cdots, k), \end{aligned}$$

i.e., $[w \otimes s, v] = w \otimes (i(v)s)$ for $v \in V$, $w \in W$ and $s \in S^r(V^*)$, where $i(v)$ denotes the interior multiplication.

The subspace $W \otimes S^k(V^*)$ of $\mathfrak{C}_{-1}$ corresponds to the subbundle $\mathrm{Ker}(\pi^k_{k-1})_* = \mathrm{Ch}\,(\partial C^k)$ of C^k and the identification $\mathrm{Ker}(\pi^k_{k-1})_* \to W \otimes S^k(V^*)$ corresponds to the fundamental identification of the jet bundle theory. For the geometry of the higher order contact system (J^k, C^k), we refer the reader to [Yam82].

2. Pseudo-product GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$

We now discuss the prolongation of symbol algebras of the pseudo-product structures associated with higher order differential equations of finite type. Moreover we will generalize this algebra to the notion of the pseudo-product GLA (graded Lie algebras) of irreducible type and introduce the pseudo-product GLA of type $(\mathfrak{l}, S)$.

2.1. **Pseudo-projective GLA of order k of bidegree (n,m).** For a k-th order differential equation R of finite type given in §1.1, we have the pseudo-product structure $(R;E,F)$. Corresponding to the splitting $D = E \oplus F$, we have the splitting in the symbol algebra of the regular differential system $(R,D) \cong (J^{k-1}, C^{k-1})$ of type $\mathfrak{C}^{k-1}(n,m)$;

$$\mathfrak{C}_{-1} = \mathfrak{e} \oplus \mathfrak{f},$$

where $\mathfrak{e} = V, \mathfrak{f} = W \otimes S^{k-1}(V^*)$. At each point $x \in R$, $\mathfrak{e}$ corresponds to $E'(x)$ (the point in $R^{(1)}$ over x) and $\mathfrak{f}$ corresponds to $\mathrm{Ker}(\pi^{k-1}_{k-2})_*(p(x))$.

Now we put

$$\check{\mathfrak{g}}_0 = \{X \in \mathfrak{g}_0(\mathfrak{C}^{k-1}(n,m)) \mid [X,\mathfrak{e}] \subset \mathfrak{e}, [X,\mathfrak{f}] \subset \mathfrak{f}\}$$

and consider the (algebraic) prolongation $\mathfrak{g}^k(n,m)$ of $(\mathfrak{C}^{k-1}(n,m), \check{\mathfrak{g}}_0)$, which is called the pseudo-projective GLA of order k of bidegree (n,m) ([Tan89]).

Let $\check{G}_0 \subset GL(\mathfrak{C}^{k-1}(n,m))$ be the (gradation preserving) automorphism group of $\mathfrak{C}^{k-1}(n,m)$ which also preserve the splitting $\mathfrak{C}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$. Then $\check{G}_0$ is the Lie subgroup of $GL(\mathfrak{C}^{k-1}(n,m))$ with Lie algebra $\check{\mathfrak{g}}_0$. The pseudo-product structure on a k-th order differential equation R of finite type given in §1, which is called the pseudo-projective system of order k of bidegree (n,m) in [Tan89], can be formulated as the $\check{G}_0^\sharp$-structure over a regular differential system of type $\mathfrak{C}^{k-1}(n,m)$ ([Tan70], [Tan89], [DKM99]). Thus the prolongation $\mathfrak{g}^k(n,m)$ of $(\mathfrak{C}^{k-1}(n,m), \check{\mathfrak{g}}_0)$ represents the Lie algebra of infinitesimal automorphisms of the (local) model k-th order differential equation R_o of finite type, where

$$R_o = \left\{ \frac{\partial^k y^\alpha}{\partial x_{i_1} \cdots \partial x_{i_k}} = 0 \quad (1 \leqq \alpha \leqq m, 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n) \right\}.$$

The isomorphism ϕ of the pseudo-product structure on R preserves the differential system $D = E \oplus F$, which is equivalent to the canonical system C^{k-1} on J^{k-1}. Hence, by Bäcklund's Theorem (cf. [Yam83]), ϕ is the lift of a point transformation on J^0 when $m \geqq 2$ and $k \geqq 2$ and is the lift of a contact transformation on J^1 when $m = 1$ and $k \geqq 3$. When $(m,k) = (1,2)$, ϕ is the lift of the point transformation on J^0, since ϕ preserves both D and $F = \mathrm{Ker}(\pi^1_0)_*$. Thus the equivalence of the pseudo-product structure on R is the equivalence of the k-th order equation under point or contact transformations. To settle the equivalence problem for the pseudo-projective systems of order k of bidegree (n,m), N.Tanaka constructed normal Cartan connections of type $\mathfrak{g}^k(n,m)$ ([Tan79], [Tan82], [Tan89]).

It is well known that $\mathfrak{g}^k(n,m)$ $(k \geqq 2)$ has the following structure ([Tan89], [Yam93], [DKM99]);

(1) $k = 2$ $\quad$ $\mathfrak{g}^2(n,m)$ is isomorphic to $\mathfrak{sl}(m+n+1,\mathbb{R})$ and has the following gradation:

$$\mathfrak{sl}(m+n+1,\mathbb{R}) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where the gradation is given by subdividing matrices as follows;

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \xi & 0 & 0 \end{pmatrix} \middle| \quad \xi \in W \cong \mathbb{R}^m \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & A & 0 \end{pmatrix} \middle| \; x \in V \cong \mathbb{R}^n, \; A \in M(m,n) = W \otimes V^* \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \middle| \; \begin{array}{l} a \in \mathbb{R}, \; B \in \mathfrak{gl}(V), \; C \in \mathfrak{gl}(W), \\ a + \mathrm{tr} B + \mathrm{tr} C = 0 \end{array} \right\},$$

$$\mathfrak{g}_1 = \{ {}^tX \mid X \in \mathfrak{g}_{-1} \}, \qquad \mathfrak{g}_2 = \{ {}^tX \mid X \in \mathfrak{g}_{-2} \},$$

where $V = M(n,1)$, $W = M(m,1)$ and $M(a,b)$ denotes the set of $a \times b$ matrices.

(2) $k = 3$ and $m = 1$ $\quad$ $\mathfrak{g}^3(n,1)$ is isomorphic to $\mathfrak{sp}(n+1,\mathbb{R})$ and has the following gradation:

$$\mathfrak{sp}(n+1,\mathbb{R}) = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3.$$

First we describe

$$\mathfrak{sp}(n+1,\mathbb{R}) = \{ X \in \mathfrak{gl}(2n+2,\mathbb{R}) \mid {}^tXJ + JX = 0 \},$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(2n+2,\mathbb{R}), \qquad I_n = (\delta_{ij}) \in \mathfrak{gl}(n,\mathbb{R}).$$

Here $I_n \in \mathfrak{gl}(n,\mathbb{R})$ is the unit matrix and the gradation is given again by subdividing matrices as follows;

$$\mathfrak{g}_{-3} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2a & 0 & 0 & 0 \end{pmatrix} \middle| \; a \in \mathbb{R} \right\},$$

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 \\ 0 & {}^t\xi & 0 & 0 \end{pmatrix} \middle| \; \xi \in \mathbb{R}^n \cong V^* \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & -{}^tx & 0 \end{pmatrix} \middle| \; x \in \mathbb{R}^n = V, \; A \in \mathrm{Sym}(n) \cong S^2(V^*) \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & -{}^tB & 0 \\ 0 & 0 & 0 & -b \end{pmatrix} \middle| \; b \in \mathbb{R}, \; B \in \mathfrak{gl}(V), \right\}$$

$$\mathfrak{g}_k = \{ {}^tX \mid X \in \mathfrak{g}_{-k} \} \; (k = 1,2,3),$$

where $\mathrm{Sym}(n) = \{ A \in \mathfrak{gl}(n,\mathbb{R}) \mid {}^tA = A \}$ is the space of symmetric matrices.

(3) otherwise $\quad$ For vector spaces V and W of dimension n and m respectively, $\mathfrak{g}^k(n,m) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ has the following description:

$$\mathfrak{g}_k = \{0\} \quad (k \geqq 2), \qquad \mathfrak{g}_1 = V^*, \qquad \mathfrak{g}_0 = \mathfrak{gl}(V) \oplus \mathfrak{gl}(W),$$

$$\mathfrak{g}_{-1} = V \oplus W \otimes S^{k-1}(V^*), \quad \mathfrak{g}_p = W \otimes S^{k+p}(V^*) \quad (p < -1).$$

Here the bracket product in $\mathfrak{g}^k(n,m)$ is given through the natural tensor operations.

For the proof of these facts ((1) and (2)), see e.g., Theorem 5.3 in [Yam93]. We refer the reader to [DKM99] for the description of these algebras as the Lie algebras of infinitesimal automorphisms (polynomial vector fields) of the (local) model equations. We will also give the proof of (3) in §5 by calculating the first generalized Spencer cohomology. For this, we observe the following points: We put

$$\begin{aligned} \mathfrak{l} &= V \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = (V \oplus \mathfrak{gl}\,(V) \oplus V^*) \oplus \mathfrak{gl}\,(W) \\ &\cong \mathfrak{sl}\,(\hat{V}) \oplus \mathfrak{gl}\,(W), \\ S &= W \otimes S^{k-1}(\hat{V}^*), \qquad \hat{V} = \mathbb{R} \oplus V. \end{aligned} \tag{2.1}$$

where the gradation of the first kind; $\mathfrak{sl}\,(\hat{V}) = V \oplus \mathfrak{gl}\,(V) \oplus V^*$ is given by subdividing matrices corresponding to the decomposition $\hat{V} = \mathbb{R} \oplus V$.
Then

$$S^{k-1}(\hat{V}^*) \cong \bigoplus_{\ell=0}^{k-1} S^{\ell}(V^*),$$

and S is a faithful irreducible $\mathfrak{l}$-module such that $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is a reductive graded Lie algebras, where $\mathfrak{l}_{-1} = V$, $\mathfrak{l}_0 = \mathfrak{g}_0$, $\mathfrak{l}_1 = \mathfrak{g}_1$. Moreover $\mathfrak{g}^k(n,m) \cong S \oplus \mathfrak{l}$ is the semi-direct product of $\mathfrak{l}$ by S.

2.2. **Pseudo-product GLA of type** $(\mathfrak{l}, S)$**.** Generalizing the pseudo-projective GLA of order k of bidegree (n,m), we will now give the notion of the pseudo-product GLA of type $(\mathfrak{l}, S)$.

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a (transitive) graded Lie algebra (GLA) over the field $\mathbb{K}$ such that the negative part $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is a FGLA, where $\mathbb{K}$ is the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let $\mathfrak{e}$ and $\mathfrak{f}$ be subspaces of $\mathfrak{g}_{-1}$. Then the system $\mathfrak{G} = (\mathfrak{g}, (\mathfrak{g}_p)_{p\in\mathbb{Z}}, \mathfrak{e}, \mathfrak{f})$ is called a pseudo-product GLA (PPGLA) of irreducible type if the following conditions hold:

(1) $\mathfrak{g}$ is transitive, i.e., for each $k \geqq 0$, if $X \in \mathfrak{g}_k$ and $[X, \mathfrak{g}_{-1}] = 0$, then $X = 0$.
(2) $\mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$, $\quad [\mathfrak{e}, \mathfrak{e}] = [\mathfrak{f}, \mathfrak{f}] = 0$.
(3) $[\mathfrak{g}_0, \mathfrak{e}] \subset \mathfrak{e}$ and $[\mathfrak{g}_0, \mathfrak{f}] \subset \mathfrak{f}$.
(4) $\mathfrak{g}_{-2} \neq 0$ and the $\mathfrak{g}_0$-modules $\mathfrak{e}$ and $\mathfrak{f}$ are irreducible.

It is known that $\mathfrak{g}$ becomes finite dimensional under these conditions (see [Tan85], [Yat88]).

As a typical example, starting from a reductive GLA $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ and a faithful irreducible $\mathfrak{l}$-module S, we define the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$ as follows: Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a finite dimensional **reductive GLA** of the first kind such that

(1) The ideal $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ of $\mathfrak{l}$ is a simple Lie algebra.
(2) The center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ is contained in $\mathfrak{l}_0$.

Let S be a finite dimensional **faithful irreducible** $\mathfrak{l}$**-module**. We put

$$S_{-1} = \{s \in S \mid \mathfrak{l}_1 \cdot s = 0\}$$

and

$$S_p = \mathrm{ad}(\mathfrak{l}_{-1})^{-p-1} S_{-1} \quad \text{for } p < 0$$

We form the semi-direct product $\mathfrak{g}$ of $\mathfrak{l}$ by S, and put

$$\mathfrak{g} = S \oplus \mathfrak{l}, \qquad [S, S] = 0$$

$$\mathfrak{g}_k = \mathfrak{l}_k \ (k \geqq 0), \quad \mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1},$$

$$\mathfrak{g}_p = S_p \ (p < -1).$$

Namely $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathfrak{A}(S) = S \oplus \mathfrak{gl}\,(S)$ of infinitesimal affine transformations of S. Then we have

Lemma 2.1. *Notations being as above,*

(1) $S = \bigoplus_{p=-1}^{-\mu} S_p$, *where* $S_{-\mu} = \{s \in S \mid [\mathfrak{l}_{-1}, s] = 0\}$.

(2) $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *is generated by* $\mathfrak{g}_{-1}$.

(3) $[S_p, \mathfrak{l}_1] = S_{p+1}$ *for* $p < -1$.

(4) S_p *is naturally embedded as a subspace of* $W \otimes S^{\mu+p}(\mathfrak{l}_{-1}^*)$ *through the bracket operation in* $\mathfrak{m}$, *where* $W = S_{-\mu}$.

(5) $S_{-1}, S_{-\mu}$ *are irreducible* $\mathfrak{l}_0$ *-modules.*

Proof. We have the characteristic element $Z \in \hat{\mathfrak{l}}_0$ (Lemma 4.1.1 [Sea88]), which defines the gradation of $\mathfrak{l}$:

$$\mathfrak{l}_p = \{X \in \mathfrak{l} \mid [Z, X] = pX\} \qquad \text{for} \quad p = -1, 0, 1.$$

Since ad(Z) is a semi-simple endomorphism with eigenvalues $-1, 0, 1$. ad(Z) is a semi-simple endomorphism of S (Corollary 6.4 [Hum72]) with real eigenvalues (see the argument in §4.5). Moreover, for the eigenspaces $S_{(\lambda)} = \{s \in S \mid [Z, s] = \lambda s\}$, by the Jacobi identity, we have

$$[\mathfrak{l}_p, S_{(\lambda)}] \subset S_{(\lambda+p)} \qquad \text{for} \quad p = -1, 0, 1.$$

For each eigenvalue λ of ad(Z), we consider the following subspaces $S^1(\lambda)$ and $S^{-1}(\lambda)$ of S:

$$S^1(\lambda) = \bigoplus_{k \geqq 0} S^1_{(\lambda+k)}(\lambda), \qquad S^{-1}(\lambda) = \bigoplus_{k \geqq 0} S^{-1}_{(\lambda-k)}(\lambda),$$

where

$$S^1_{(\lambda)}(\lambda) = \{s \in S_{(\lambda)} \mid [\mathfrak{l}_{-1}, s] = 0\}, \; S^{-1}_{(\lambda)}(\lambda) = \{s \in S_{(\lambda)} \mid [\mathfrak{l}_1, s] = 0\},$$

$$S^1_{(\lambda+k)}(\lambda) = [\mathfrak{l}_1, S^1_{(\lambda+k-1)}(\lambda)] \subset S_{(\lambda+k)} \quad \text{for } k \geqq 1$$

$$S^{-1}_{(\lambda-k)}(\lambda) = [\mathfrak{l}_{-1}, S^{-1}_{(\lambda-k+1)}(\lambda)] \subset S_{(\lambda-k)} \quad \text{for } k \geqq 1$$

Then $S^1_{(\lambda)}(\lambda)$ and $S^{-1}_{(\lambda)}(\lambda)$ are $\mathfrak{l}_0$-invariant subspaces of $S_{(\lambda)}$. One can easily check that $S^1_{(\lambda+k)}(\lambda)$ and $S^{-1}_{(\lambda-k)}(\lambda)$ are $\mathfrak{l}_0$-invariant and

$$[\mathfrak{l}_{-1}, S^1_{(\lambda+k)}(\lambda)] \subset S^1_{(\lambda+k-1)}(\lambda), \qquad [\mathfrak{l}_1, S^{-1}_{(\lambda-k)}(\lambda)] \subset S^{-1}_{(\lambda-k+1)}(\lambda)$$

by induction on $k \geqq 0$. Thus $S^1(\lambda)$ and $S^{-1}(\lambda)$ are $\mathfrak{l}$-submodule of S for each λ.

Let λ_0 and λ_1 be the minimum and maximum eigenvalue of ad(Z). We have

$$S^1_{(\lambda_0)}(\lambda_0) = S_{(\lambda_0)}, \qquad S^{-1}_{(\lambda_1)}(\lambda_1) = S_{(\lambda_1)},$$

and $S^1(\lambda)$ and $S^{-1}(\lambda)$ are both proper subspaces of S for an intermediate eigenvalue λ. Then, since S is an irreducible $\mathfrak{l}$-module, we get

$$S^1(\lambda_0) = S^{-1}(\lambda_1) = S \qquad \text{and} \qquad S^1(\lambda) = S^{-1}(\lambda) = 0 \quad \text{otherwise.}$$

Especially, from $S^1_{(\lambda)}(\lambda) = S^{-1}_{(\lambda)}(\lambda) = 0$, we get

$$S_{(\lambda_1)} = \{s \in S \mid [\mathfrak{l}_1, s] = 0\} = S_{-1}, \qquad S_{(\lambda_0)} = \{s \in S \mid [\mathfrak{l}_{-1}, s] = 0\}.$$

Hence, from $S^{-1}(\lambda_1) = S$, we obtain (1) and (2). Moreover, from $S^1(\lambda_0) = S$, we get (3).

Now we put $V = \mathfrak{l}_{-1}$ and $W = S_{-\mu}$. Then we have a linear map ι_r of $S_{r-\mu}$ into $W \otimes S^r(V^*)$ $(r = 1, \ldots, \mu - 1)$ defined by

$$\iota_r(s)(X_1, \ldots, X_r) = [[\ldots[s, X_1], \ldots], X_r] \in W \quad \text{for } s \in S_{r-\mu}, X_i \in V.$$

Since $\mathfrak{l}_{-1}$ is abelian, ι_r is well-defined and is injective by (1). Thus we get (4).

Starting from a $\mathfrak{l}_0$-submodule $T_{(\lambda_1)}$ in S_{-1}, similarly as above, we can form the $\mathfrak{l}$-submodule $T^{-1}(\lambda_1)$ of S by putting;

$$T^{-1}(\lambda_1) = \bigoplus_{k \geqq 0} T_{(\lambda_1 - k)},$$

where $T_{(\lambda_1-k)} = [\mathfrak{l}_{-1}, T_{(\lambda_1-k+1)}] \subset S_{-(k+1)}$ for $k \geqq 1$. Hence we get $T^{-1}(\lambda_1) = S$ or 0, which implies $T_{(\lambda_1)} = S_{-1}$ or 0. Thus S_{-1} is an irreducible $\mathfrak{l}_0$-module. By the similar argument, we see that $S_{-\mu}$ is an irreducible $\mathfrak{l}_0$-module, which completes the proof of Lemma. □

Thus $\mathfrak{m}$ is a graded subalgebra of $\mathfrak{C}^{\mu-1}(V, W)$, which has the splitting $\mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1}$, where $V = \mathfrak{l}_{-1}$ and $W = S_{-\mu}$. Hence $\mathfrak{m}$ is a symbol algebra of μ-th order differential equations of finite type, which is called the typical symbol of type $(\mathfrak{l}, S)$. Moreover the system $\mathfrak{G} = (\mathfrak{g}, (\mathfrak{g}_p)_{p\in\mathbb{Z}}, \mathfrak{l}_{-1}, S_{-1})$ becomes a PPGLA of irreducible type, which is called the pseudo-product GLA of type $(\mathfrak{l}, S)$.

This class of higher order (linear) differential equations of finite type were first appeared in the work of Y.Se-ashi [Sea88]. We will construct the model (linear) equations for each PPGLA $\mathfrak{G}$ of type $(\mathfrak{l}, S)$ in §3, following [Sea88]. Moreover we remark that the PPGLA $\mathfrak{G}$ of type $(\mathfrak{l}, S)$ also naturally appeared in the classification of PPGLA's of irreducible type under mild conditions in [Yat92].

In this paper we will consider only pseudo-product graded Lie algebras $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$. We recall that there exist an anti-linear involution τ of $\mathfrak{l}$ and an hermitian inner product $(\cdot \mid \cdot)$ of $\mathfrak{g}$ having the following properties:

(i) $\tau(\mathfrak{l}_p) = \mathfrak{l}_{-p}$;
(ii) $(\mathfrak{g}_p \mid \mathfrak{g}_q) = 0$ for $p \neq q$;
(iii) $([x, s] \mid s') + (s \mid [\tau(x), s']) = 0$ for all $x \in \mathfrak{l}$ and $s, s' \in S$.

Thus the PPGLA $\mathfrak{G}$ of type $(\mathfrak{l}, S)$ satisfies the criterion (Proposition 3.10.1) in [Mor93] . Hence, when $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{g}_0)$, for the equivalence of the pseudo-product structure associated with $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$, we can utilize the Morimoto's theory of normal Cartan connections [Mor93]. Especially we can utilize the harmonic theory for the curvature of the normal Cartan connections (cf. [DKM99]). Namely, regarding the curvature as $C^2(\mathfrak{m}, \mathfrak{g})$-valued function, its harmonic parts constitute the fundamental system of invariants of the connection. In particular the curvature vanishes if and only if its harmonic part vanishes (Theorem 3 [DKM99]).

We will ask the following questions for the PPGLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$ in the subsequent sections:

(A) *When is* $\mathfrak{g}$ *the prolongation of* $\mathfrak{m}$ *or* $(\mathfrak{m}, \mathfrak{g}_0)$?
(B) *Find the fundamental invariants for equations of type* $\mathfrak{m}$.

Utilizing the Tanaka-Morimoto theory of normal Cartan connections, these questions will be answered by calculating the first and second (generalized Spencer) cohomology spaces. The complete answer for (A) will be given in §5 and the problem (B) will be discussed in §6 and §7.

In the rest of this section, as an example to obtain the local model equations, we will realize the negative part $\mathfrak{m}$ of a pseudo-product GLA of type $(\mathfrak{l}, S)$ as the subalgebra of $\mathfrak{C}^{\mu-1}(n, m)$, which is called the typical symbol of type $(\mathfrak{l}, S)$ or $(\mathfrak{l}, \rho)$ in [SYY97]. Here $\mathfrak{l} = \mathfrak{sl}(\ell + 1, \mathbb{K})$ is endowed with the gradation given by ;

$$\mathfrak{sl}(\ell + 1, \mathbb{K}) = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1, \tag{2.2}$$

where

$$\mathfrak{l}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C \in M(p, k) \right\}, \ \mathfrak{l}_1 = \left\{ \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \mid D \in M(k, p) \right\},$$

$$\mathfrak{l}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in \mathfrak{gl}(k, \mathbb{K}),\ B \in \mathfrak{gl}(p, \mathbb{K}) \text{ and } \mathrm{tr}A + \mathrm{tr}B = 0 \right\}.$$

Here $p = \ell - k + 1$ and $M(a, b)$ denotes the set of $a \times b$ matrices.

And $S = \bigwedge^{\ell-k+1} \mathbb{K}^{\ell+1}$ is the faithful irreducible $\mathfrak{l}$-module given by the following exterior representation $\rho = \rho_0$:

$$\rho_0 : \mathfrak{sl}\,(\ell + 1, \mathbb{K}) \to \mathfrak{gl}\,(\bigwedge^{\ell-k+1} \mathbb{K}^{\ell+1}),$$

where

$$\rho_0(X)(v_1 \wedge \cdots \wedge v_{\ell-k+1}) = \sum_{i=1}^{\ell-k+1} v_1 \wedge \cdots \wedge X(v_i) \wedge \cdots \wedge v_{\ell-k+1}$$

for $X \in \mathfrak{sl}\,(\ell + 1, \mathbb{K})$ and $v_i \in \mathbb{K}^{\ell+1}$ $(i = 1, 2, \ldots, \ell - k + 1)$.

Let $\{e_1, \ldots, e_{\ell+1}\}$ be the natural basis of $\mathbb{K}^{\ell+1}$. Then $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is the isotropy (stabilizer) algebra of the line $[e_1 \wedge \cdots \wedge e_k]$ in $\bigwedge^k \mathbb{K}^{\ell+1}$. We denote by $E_{ab} \in \mathfrak{gl}\,(\ell+1, \mathbb{K})$ $(1 \leqq a, b \leqq \ell+1)$ the matrix whose (a, b)-component is 1 and all of whose other components are 0. From (2.2), we have the following basis for $V = \mathfrak{l}_{-1}$ and $\mathfrak{l}_1$:

$$V = \mathfrak{l}_{-1} = \langle E_{pi} \mid 1 \leqq i \leqq k,\ k + 1 \leqq p \leqq \ell + 1 \rangle$$
$$\mathfrak{l}_1 = \langle E_{ip} \mid 1 \leqq i \leqq k,\ k + 1 \leqq p \leqq \ell + 1 \rangle$$

Since $E_{pi}(e_j) = \delta_{ij} e_p$ for $1 \leqq j \leqq k$ and, $E_{pi}(e_q) = 0$ for $k + 1 \leqq q \leqq \ell + 1$, we have from Lemma 2.1 (1)

$$W = S_{-\mu} = \langle e_{k+1} \wedge \cdots \wedge e_{\ell+1} \rangle.$$

Hence we have $m = 1$ and $n = k(\ell - k + 1)$. For $1 \leqq i_1 < \cdots < i_r \leqq k$ and $k + 1 \leqq p_1 < \cdots < p_r \leqq \ell + 1$, we put

$$e(p_1, \ldots, p_r) = e_{k+1} \wedge \cdots \wedge \widehat{e_{p_1}} \wedge \cdots \wedge \widehat{e_{p_r}} \wedge \cdots \wedge e_{\ell+1} \in \bigwedge^{\ell-k-r+1} \mathbb{K}^{\ell+1},$$

and consider the following element of S:

$$s(i_1, \ldots, i_r, p_1, \ldots, p_r) = e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge e(p_1, \ldots, p_r) \in S = \bigwedge^{\ell-k+1} \mathbb{K}^{\ell+1}.$$

Then, from Lemma 2.1 (3) and $E_{ip}(e_j) = 0$, $E_{ip}(e_q) = \delta_{pq} e_i$ for $1 \leqq j \leqq k$, $k + 1 \leqq q \leqq \ell + 1$, we get

$$S_{r-\mu} = \langle s(i_1, \ldots, i_r, p_1, \ldots, p_r) \mid 1 \leqq i_1 < \cdots < i_r \leqq k, k + 1 \leqq p_1 < \cdots < p_r \leqq \ell + 1 \rangle,$$

for $r = 1, 2, \ldots, p_0 - 1$ and

$$S_{r-\mu} = \{0\},$$

for $r \geqq p_0 = \min\{k + 1, \ell - k + 2\}$. Thus we have $\mu = p_0$. Moreover, for $X = \sum_{ip} X_{ip} E_{pi} \in V$, we have

$$\begin{aligned}
&\iota_r(s(i_1, \ldots, i_r, p_1, \ldots, p_r))(X, \ldots, X) \\
=&r!(-1)^r X(e_{i_1}) \wedge \cdots \wedge X(e_{i_r}) \wedge e(p_1, \ldots, p_r) \\
=&r!(-1)^r (\sum_{\sigma} \operatorname{sgn} \sigma X_{i_1 p_{\sigma(1)}} \cdots X_{i_r p_{\sigma(r)}}) e_{p_1} \wedge \cdots \wedge e_{p_r} \wedge e(p_1, \ldots, p_r).
\end{aligned}$$

Thus, by fixing a basis of W and identifying SV^* with the ring of polynomials on V, we see that $S_{1-\mu} = V^*$ and $S_{r-\mu} \subset S^r V^*$ is spanned by the minor determinants of degree r of the matrix (X_{ip}), which are the linear coordinates of V.

Moreover it is known that $S_{r-\mu} \subset S^r(V^*)$ is equal to the $(r-2)$-th prolongation $p^{(r-2)}(S_{2-\mu})$ of $S_{2-\mu} \subset S^2(V^*)$ (Lemma 3.1 [SYY97]). Hence the local model equation in this case is given as the prolongation of the following second order equations:

$$\frac{\partial^2 y}{\partial x_{ip}\partial x_{jq}} + \frac{\partial^2 y}{\partial x_{iq}\partial x_{jp}} = 0, \qquad (1 \leqq i < j \leqq k, k+1 \leqq p < q \leqq \ell + 1).$$

3. Se-ashi's Theory for Linear Equations of Finite Type

We here recall some relevant facts from Se-ashi's theory [Sea88] for the equivalence of higher order linear differential equations of finite type. Especially we will recall the construction of the model equations for the typical symbol of type $(\mathfrak{l}, S)$ and his "Rigidity Theorem". We will also discuss the Plücker embedding equations as our examples of model equations, following [SYY97] rather closely.

3.1. **Model equations for the typical symbol of type** $(\mathfrak{l}, S)$**.** Starting from the typical symbol $\mathfrak{m} = S \oplus \mathfrak{l}_{-1}$ of type $(\mathfrak{l}, S)$ in §2.2, where $S = \bigoplus_{p=-\mu}^{-1} S_p \subset \bigoplus_{r=0}^{\mu-1} W \otimes S^r V^*$, $V = \mathfrak{l}_{-1}$, $W = S_{-\mu}$, we now explain a recipe to construct an integrable system of linear differential equations of finite type of order μ modeled after $\mathfrak{m}$.

The construction of the model system R_S is preceded by the consideration of the Lie algebra $\mathfrak{a}$ of infinitesimal bundle automorphisms of the constant coefficient differential equations modeled after $S = \bigoplus_{r=0}^{\mu-1} S_{r-\mu} \subset \bigoplus_{r=0}^{\mu} W \otimes S^r V^*$.

Let $E_0 = V \times W$ be the trivial vector bundle over the vector space V. Let $J^\mu(E_0)$ be the bundle of μ-jets of E_0. Then the fibre $J^\mu_0(E_0)$ of $J^\mu(E_0)$ at the origin $0 \in V$ is identified with $\bigoplus_{r=0}^{\mu} W \otimes S^r V^*$, where $W \otimes S^r V^*$ can be regarded as the set of W-valued homogeneous polynomials of degree r on V. Thus, starting from the typical symbol $S = \bigoplus_{r=0}^{\mu-1} S_{r-\mu} \subset \bigoplus_{r=0}^{\mu} W \otimes S^r V^*$, our first (local) model is the constant coefficient differential equations given as the subbundle $\hat{R}_S = V \times S$ of $J^\mu(E_0)$, whose solutions consist of W-valued polynomials contained in $S \subset W \otimes S V^*$.

Let us consider an infinitesimal bundle automorphism of E_0 preserving $\hat{R}_S$. An infinitesimal bundle automorphism of E_0 has a form

$$\sum_i \xi^i(x)\frac{\partial}{\partial x^i} + \sum_{\alpha,\beta} A_{\alpha\beta}(x) y^\beta \frac{\partial}{\partial y^\alpha},$$

where (x^i) and (y^α) are linear coordinates of V and W, respectively. Thus the Lie algebra $\tilde{\mathfrak{a}}$ of (formal) infinitesimal bundle automorphismsof E_0 can be expressed as a graded Lie algebra $\tilde{\mathfrak{a}} = \bigoplus_{r \geqq -1} \tilde{\mathfrak{a}}_r$ by putting

$$\tilde{\mathfrak{a}}_r = S^{r+1}V^* \otimes V \oplus S^r V^* \otimes \mathfrak{gl}(W),$$

where $\tilde{\mathfrak{a}}_{-1} = V$ corresponds to constant coefficient vector fields on V. The bracket operation in $\tilde{\mathfrak{a}}$ is given by

$$\begin{aligned}
[f \otimes v, g \otimes w] &= -f(i(v)g) \otimes w + g(i(w)f) \otimes v, \\
[f \otimes A, g \otimes w] &= g(i(w)f) \otimes A, \\
[f \otimes A, g \otimes B] &= fg \otimes [A, B],
\end{aligned}$$

where $f, g \in S V^*$, $v, w \in V$ and $A, B \in \mathfrak{gl}(W)$; $i(v)$ denotes the inner multiplication. The Lie algebra $\tilde{\mathfrak{a}}$ acts naturally on the space $S V^* \otimes W$ which is regarded as the space of cross sections of E_0:

$$(f \otimes v + g \otimes A)(h \otimes w) = -f(i(v)h) \otimes w + gh \otimes A(w),$$

where $f, g, h \in S V^*$, $v, w \in V$ and $A \in \mathfrak{gl}(W)$.

Then the Lie algebra $\mathfrak{a}$ of infinitesimal automorphisms of $\hat{R}_S$ is given by

$$\mathfrak{a} = \{ X \in \tilde{\mathfrak{a}} \mid X(S) \subset S \}.$$

$\mathfrak{a}$ is a graded subalgebra of $\tilde{\mathfrak{a}} = \bigoplus_{r \geqq -1} \tilde{\mathfrak{a}}_r$, i.e., $\mathfrak{a} = \bigoplus_{r \geqq -1} \mathfrak{a}_r$, where $\mathfrak{a}_r = \mathfrak{a} \cap \tilde{\mathfrak{a}}_r$. The Lie algebra $\mathfrak{gl}(S)$ has also the gradation given by

$$\mathfrak{gl}(S)_r = \{ X \in \mathfrak{gl}(S) \mid X(S_p) \subset S_{p+r} \quad \text{for any} p \}.$$

Referring the action above we have a restriction homomorphism: $\mathfrak{a} \to \mathfrak{gl}(S)$, which sends $\mathfrak{a}_r$ into $\mathfrak{gl}(S)_r$. Assume here the following two conditions for S, which are satisfied by the typical symbol of type $(\mathfrak{l}, S)$:

(A1) The action of $\tilde{\mathfrak{a}}_{-1} = V$ leave S invariant.
(A2) The action of $\tilde{\mathfrak{a}}_{-1} = V$ on S is faithful.

Then this homomorphism turns out to be injective and we can characterize $\mathfrak{a}_r$ as a subspace of $\mathfrak{gl}(S)_r$ as follows (Proposition 2.2.2 [Sea88]):

$$\text{(3.1)} \qquad \mathfrak{a}_{-1} = V, \qquad \mathfrak{a}_r = \{ X \in \mathfrak{gl}(S)_r \mid [\mathfrak{a}_{-1}, X] \subset \mathfrak{a}_{r-1} \} \quad (r \geqq 0).$$

Put $\tilde{\mathfrak{n}}_r = S^r V^* \otimes \mathfrak{gl}(W) \subset \tilde{\mathfrak{a}}_r$. Then $\tilde{\mathfrak{n}} = \bigoplus_{r \geqq 0} \tilde{\mathfrak{n}}_r$ is an ideal of $\tilde{\mathfrak{a}}$ and $\mathfrak{n} = \tilde{\mathfrak{n}} \cap \mathfrak{a}$ is an ideal of $\mathfrak{a}$. We can see

$$\text{(3.2)} \qquad \mathfrak{n}_r = \{ X \in \mathfrak{gl}(S)_r \mid [\mathfrak{a}_{-1}, X] \subset \mathfrak{n}_{r-1} \} \qquad (r \geqq 0),$$

where we put $\mathfrak{n}_{-1} = \{0\}$ for convention.

In the case of the typical symbol of type $(\mathfrak{l}, S)$, we have the following: Since S is a faithful $\mathfrak{l}$-module, $\mathfrak{l}$ is a subalgebra of $\mathfrak{gl}(S)$. We have $\mathfrak{a}_{-1} = \mathfrak{l}_{-1}$ and it follows from (3.1) that $\mathfrak{a}_0 = \check{\mathfrak{g}}_0$, where $\check{\mathfrak{g}}_0$ is the Lie algebra of derivations of $\mathfrak{m} = S \oplus \mathfrak{l}_{-1}$ such that $D(S_p) \subset S_p = \mathfrak{g}_p$ $(p < -1)$, $D(\mathfrak{l}_{-1}) \subset \mathfrak{l}_{-1}$ and $D(S_{-1}) \subset S_{-1}$.

Let $\mathfrak{z}_S(\mathfrak{l})$ denote the centralizer of $\mathfrak{l}$ in $\mathfrak{gl}(S)$ and $\mathfrak{a}^\perp$ the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{gl}(S)$ with respect to thenon-degenerate bilinear form tr given by $\mathrm{tr}(X, Y) = \mathrm{trace}\, XY$ for $X, Y \in \mathfrak{gl}(S)$. Then, from (3.1) and (3.2), we have (Proposition 4.4.1 [Sea88])

$$\text{(3.3)} \qquad \begin{aligned} \mathfrak{a} &= [\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{z}_S(\mathfrak{l}), \qquad \mathfrak{z}_S(\mathfrak{l}) \subset \mathfrak{n}, \\ \mathfrak{gl}(S) &= [\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{z}_S(\mathfrak{l}) \oplus \mathfrak{a}^\perp \quad \text{(tr-orthogonal)}. \end{aligned}$$

We here note that our assumption on $\mathfrak{l}$ is a little different from that in [Sea88]. We will discuss the condition for $\mathfrak{a} = \mathfrak{l}$ in §4.5.

Now the model equation R_S is constructed as follows: We filtrate the space S by subspaces $S^q = \bigoplus_{p=-1}^{q} S_p$. Notice that the group $GL(V) \times GL(W)$ acts on $\tilde{\mathfrak{a}}$ by the adjoint action: for $a \in GL(V) \times GL(W)$ and $X \in \tilde{\mathfrak{a}}$, the action is $(aX)(s) = (a \cdot X \cdot a^{-1})(s)$ for $s \in S$. Let us define groups

$$A_0 = \{ a \in GL(V) \times GL(W) \mid a(S) \subset S \},$$
$$GL^{(0)}(S) = \{ g \in GL(S) \mid g(S^q) \subset S^q \quad \text{for any q} \}.$$

Let $\widetilde{A}$ be the analytic subgroup of $GL(S)$ with Lie algebra $\mathfrak{a} \subset \mathfrak{gl}(S)$ and put

$$A = \widetilde{A} \cdot A_0,$$
$$A' = A \cap GL^{(0)}(S).$$

We see that the groups A_0 and A' are Lie subgroups of $GL(S)$ with Lie algebras $\mathfrak{a}_0$ and $\mathfrak{a}' = \bigoplus_{r \geqq 0} \mathfrak{a}_r$ respectively. Since A' preserves the filtration $\{S^q\}_{q<0}$ of S, we get the representation ρ_W of A':

$$\rho_W : A' \to GL(W),$$

through the projection $\pi_0 : S = \bigoplus_{p=-\mu}^{-1} S_p \to S_{-\mu} = W$.

Let E_S be the vector bundle over $M = A/A'$ associated with the representation $\rho_W : A' \to GL(W)$; A' acts on $A \times W$ on the right by

$$(a, w)a' = (aa', \rho_W(a')^{-1}(w)),$$

for $a \in A, w \in W$ and $a' \in A'$. Then E_S is the vector bundle over $M = A/A'$ defined by $E_S = A \times W/A'$.

As is well known, the space $\Gamma(E_S)$ of global sections of E_S is identified with the space $\mathcal{F}(A, W)_{A'}$ of all W-valued functions f on A satisfying

$$f(aa') = \rho_W(a')^{-1} f(a),$$

for $a \in A$ and $a' \in A'$, via the correspondence $f \in \mathcal{F}(A, W)_{A'} \mapsto \sigma_f \in \Gamma(E_S)$ given by

$$\sigma_f(\pi_1(a)) = \pi_2(a, f(a)),$$

where $\pi_1 : A \to M = A/A'$ and $\pi_2 : A \times W \to E_S$ denote the natural projections. Then each $s \in S$ defines an element $\sigma_s \in \Gamma(E_S)$ via the above correspondence by

$$f_s(a) = \pi_0(\rho(a^{-1})s)$$

for $a \in A$.

At each point $x \in M = A/A'$, let $(R_S)_x$ be the subspace of $J^\mu_x(E_S)$ defined by

$$(R_S)_x = \{ j^\mu_x(\sigma_s) \mid s \in S \}.$$

where $j^\mu_x(\sigma_s)$ is the μ-jet at x of the section σ_s. Let R_S be the subbundle of $J^\mu(E_S)$ defined by

$$R_S = \bigcup_{x \in M} (R_S)_x.$$

Then we have

Proposition A (Proposition 2.4.1 [Sea88]). *R_S is an integrable system of linear differential equations of finite type of order μ of type S and every local solution of R_S is a restriction of σ_s for some $s \in S$.*

We call R_S the system of equations *modeled after* S. R_S is the system of differential equations which characterizes global sections of E_S given by elements of S even locally. By the construction, it follows that R_S is locally isomorphic with the constant coefficient differential equations $\hat{R}_S$.

Here the condition for R_S to be a system of differential equations of type S means that the total symbol S_x of R_S is isomorphic with $S = \bigoplus_{r=0}^{\mu-1} S_{r-\mu} \subset \bigoplus_{r=0}^{\mu} W \otimes S^r V^*$ at each $x \in M$. For the precise definition in terms of the jet bundle theory, we refer the reader to §2 of [Sea88] or §2.1 of [SYY97].

What is important for us here is that, as a submanifold of $J^\mu(E_S)$, R_S has the pseudo-product structure induced from canonical systems C^μ and $C^{\mu-1}$ as in §1.1. Then, when S is the typical symbol of type $(\mathfrak{l}, S)$, the above condition is equivalent to say that the symbol algebra of the pseudo-product structure is isomorphic with $\mathfrak{m} = S \oplus \mathfrak{l}_{-1}$ at each $v \in R_S$ (cf. [Yam82]), which follows from the fact that R_S is locally isomorphic with $\hat{R}_S$. R_S is our (global) model equations for the pseudo-product structures associated with the PPGLA $\mathfrak{G}$ of type $(\mathfrak{l}, S)$.

Moreover it follows from (3.3) that $A/A' = \hat{L}/\hat{L}'$, where $\hat{L}$ is the Lie subgroup of A with Lie algebra $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$. Especially, in the case of pseudo-projective GLA $\mathfrak{g}^k(n, m)$ of order k of bidegree (n, m), we see from (2.1) that the model space M coincides with the projective space $\mathbb{P}^n$ and it is known ([Tan89]) that the vector bundle E_S is the tensor product $\bar{W} \otimes H^{k-1}$ of the trivial bundle $\bar{W} = \mathbb{P}^n \times W$ with the $(k-1)$-th power of the hyperplane bundle H over $\mathbb{P}^n$ (see the discussion in §3.2).

Y.Se-ashi developed in [Sea88] the theory for the linear equivalence of integrable higher order differential equations R of finite type with the typical symbol of type $(\mathfrak{l}, S)$. He gave the complete system of differential invariants of R and interpreted these invariants in terms of Cartan connections constructed over R. Utilizing these invariants he showed the following **"Rigidity Theorem"** in the linear equivalence of these equations.

Theorem A (cf. Corollary 3 [SYY97]). *Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra over $\mathbb{C}$ and let $M = L/L'$ be the model space associated with $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Let S be a faithful irreducible $\mathfrak{l}$-module. Then, except when $M = \mathbb{P}^n$ or Q^n, every integrable system R of differential equations of type S is locally isomorphic with the model system R_S of type $(\mathfrak{l}, S)$, where $\mathbb{P}^n$ is the projective space and Q^n is the hyperquadric in $\mathbb{P}^{n+1}$.*

We will discuss the invariants of the pseudo-product structure on these equations in subsequent sections.

3.2. **Plücker embedding equations.** In order to discuss the Plücker embedding equations, a little generally, we will consider here projective embedding of hermitian symmetric spaces, following §1 in [SYY97].

Group-theoretically, a compact irreducible hermitian symmetric space M corresponds to a simple graded Lie algebra over $\mathbb{C}$ of the first kind as follows: Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra of the first kind, i.e.,

(1) $\mathfrak{l}$ is a simple Lie algebra over $\mathbb{C}$.

(2) $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is a vector space direct sum such that $\mathfrak{l}_{-1} \neq \{0\}$.

(3) $[\mathfrak{l}_p, \mathfrak{l}_q] \subset \mathfrak{l}_{p+q}$, where $\mathfrak{l}_p = \{0\}$ for $|p| \geqq 2$.

Let L be the simply connected Lie group with Lie algebra $\mathfrak{l}$ and L' be the analytic subgroup of L with Lie algebra $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Then $M = L/L'$ is a compact (irreducible) hermitian symmetric space and every compact irreducible hermitian symmetric space is obtained in this manner from a simple graded Lie algebra of the first kind. M is called the **model space** associated with $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. For example, when $M = \mathrm{Gr}(k, \ell+1)$ is the Grassmann manifold of k-dimensional subspaces in $\mathbb{C}^{\ell+1}$, we have $\mathfrak{l} = \mathfrak{sl}(\ell+1, \mathbb{C})$ and the gradation $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is given by subdividing matrices as (2.2) in §2.2. As the extreme case $k = 1$, we have the projective space $M = \mathbb{P}^\ell = P(\hat{V})$. In this case $\mathfrak{l} = \mathfrak{sl}(\ell+1, \mathbb{C}) = \mathfrak{sl}(\hat{V})$ and the gradation is given as in (2.1) in §2.1.

An equivariant projective embedding of the model space $M = L/L'$ can be obtained from an irreducible representation of L as follows: Let $\tau : L \to GL(T)$ be an irreducible representation of L with the highest weight Λ. Let t_Λ be a maximal vector in T of the highest weight Λ. Then a stabilizer of the line $[t_\Lambda]$ spanned by t_Λ in T is a parabolic subgroup of L. When this stabilizer coincides with L', we obtain an equivariant projective embedding of $M = L/L'$ by taking the L-orbit passing through $[t_\Lambda]$ in the projective space $P(T)$ consisting of all lines in T passing through the origin. For example, when $M = \mathrm{Gr}(k, \ell+1)$, we take the exterior representation τ_0 of $L = SL(\ell+1, \mathbb{C})$ on $T = \bigwedge^k \mathbb{C}^{\ell+1}$:

$$\tau_0 : SL(\ell+1, \mathbb{C}) \to GL(\bigwedge^k \mathbb{C}^{\ell+1}),$$

where $\tau_0(a)(v_1 \wedge \cdots \wedge v_k) = a(v_1) \wedge \cdots \wedge a(v_k)$ for $a \in SL(\ell+1, \mathbb{C})$ and $v_i \in \mathbb{C}^{\ell+1}$ $(i = 1, 2, \ldots, k)$. Let $\{e_1, \ldots, e_{\ell+1}\}$ be the natural basis of $\mathbb{C}^{\ell+1}$. Then τ_0 is an irreducible representation of $SL(\ell+1, \mathbb{C})$ with the maximal vector $e_1 \wedge \cdots \wedge e_k$ for a suitable choice of a Cartan subalgebra $\mathfrak{h}$ and a simple root system Δ of $\mathfrak{sl}(\ell+1,\mathbb{C})$. τ_0 is the irreducible representation of $SL(\ell+1,\mathbb{C})$ with the highest weight ϖ_k, where $\{\varpi_1,\ldots,\varpi_\ell\}$ is the set of fundamental dominant weight relative to Δ (see §4.5). From (2.2), we see that the stabilizer of the line $[e_1\wedge\cdots\wedge e_k]$ coincides with L'. Thus we see that the Plücker embedding of $\mathrm{Gr}(k,\ell+1)$ is obtained from the irreducible representation τ_0 of $SL(\ell+1,\mathbb{C})$.

As other examples, we take the symmetric representation ν_k of $L = SL(\hat{V})$ on $T = S^k(\hat{V})$, where $S^k(\hat{V})$ denotes the k-th symmetric product of $\hat{V}$:

$$\nu_k : SL(\hat{V}) \to GL(S^k(\hat{V})),$$

where $\nu_k(a)(v_1 \circledcirc \cdots \circledcirc v_k) = a(v_1)\circledcirc\cdots\circledcirc a(v_k)$ for $a \in SL(\hat{V})$ and $v_i \in \hat{V}$ $(i = 1,2,\ldots,k)$ and $\circledcirc$ is the symmetric product. Let us take a highest weight vector $v_o \in \hat{V}$ of the identity representation ν_1. Here $\hat{V} = \langle\{v_o\}\rangle \oplus V$ in the notation of (2.1). Then $v_o^k = v_0 \circledcirc\cdots\circledcirc v_o$ is the highest weight vector of ν_k. ν_k is the irreducible representation of $SL(\hat{V})$ with the highest weight $k\varpi_1$. From (2.1), we see that the stabilizer of the line $[v_o^k]$ coincides with L'. Thus we obtain projective embeddings of $P(\hat{V})$ from irreducible representations ν_k. Here we note that the line bundle over $P(\hat{V}) = L/L'$ obtained from the representation of L' on $W_0 = \langle\{v_o\}\rangle$ is isomorphic with the universal bundle U over $P(\hat{V})$. Hence the line bundle over $P(\hat{V})$ obtained from the representation of L' on $W_k = \langle\{v_o^k\}\rangle$ is isomorphic with the k-th power U^k of U.

Next, for an irreducible representation $\tau : L \to GL(T)$, we will construct a (positive) line bundle F over M such that the above orbit is obtained as an embedding of M by global sections of F. To construct F, let us take the dual representation $\rho : L \to GL(S)$ of τ, i.e., $S = T^*$ is the dual space of T and $\rho = \tau^*$ is defined by

$$\langle \rho(g)(\xi), t\rangle = \langle \xi, \tau(g^{-1})(t)\rangle,$$

for $g \in L, t \in T, \xi \in T^*$ and $\langle\ ,\ \rangle$ is the canonical pairing between T^* and T. Then, when τ is an irreducible representation with the highest weight Λ (for a fixed choice of a Cartan subalgebra and a simple root system of $\mathfrak{l}$), ρ is the irreducible representation with the lowest weight $-\Lambda$. Let us take a basis $\{t_1,\ldots,t_r\}$ of T consisting of weight vectors of τ such that $t_1 = t_\Lambda$. Then the dual basis $\{s_1,\ldots,s_r\}$ of $\{t_1,\ldots,t_r\}$ in $S = T^*$ consists of weight vectors of ρ and s_1 is a weight vector corresponding to $-\Lambda$. Let W and W' be the subspaces of S spanned by a vector s_1 and by vectors $s_2,\ldots,s_r$, respectively. Then, since s_1 is a lowest weight vector, we have $W = S_{-\mu}$ in the notation of Lemma 2.1. Since L' is the stabilizer of the line $[t_1]$, W' is preserved by L'. Hence we get the representation ρ_W of L':

$$\rho_W : L' \to GL(W),$$

through the projection $\pi_0 : S = W \oplus W' \to W$.

Relative to the representation ρ_W, L' acts on $L\times W$ on the right by

$$(g,w)g' = (gg', \rho_W(g')^{-1}(w)),$$

for $g \in L, w \in W$ and $g' \in L'$. Then $F = L\times W/L'$ is the line bundle over $M = L/L'$.

As in §3.1, the space $\Gamma(F)$ of global sections of F is identified with the space $\mathcal{F}(L,W)_{L'}$ of all W-valued functions f on L satisfying

$$f(gg') = \rho_W(g')^{-1}f(g),$$

for $g \in L$ and $g' \in L'$. Then each $s \in S$ defines an element $\sigma_s \in \Gamma(F)$ via this correspondence by

$$f_s(g) = \pi_0(\rho(g^{-1})s)$$

for $g \in L$.

Now let us check that global sections of F give the desired embedding of M into $P(T)$. We utilize the above basis $\{t_1, \ldots, t_r\}$ and $\{s_1, \ldots, s_r\}$ of T and $S = T^*$. Let us consider a map $\hat{\varphi}$ of L into T defined by

$$\hat{\varphi}(g) = \sum_{i=1}^{r} \langle f_{s_i}(g), t_1 \rangle t_i \tag{3.4}$$

for $g \in L$. Then, from $\langle f_{s_i}(g), t_1 \rangle = \langle \rho(g^{-1}) s_i, t_1 \rangle$, $\hat{\varphi}$ induces a map φ of M into $P(T)$ satisfying the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\hat{\varphi}} & T \backslash \{0\} \\ \downarrow & & \downarrow \\ M = L/L' & \xrightarrow{\varphi} & P(T). \end{array}$$

For $g \in L$, if we represent $\tau(g)$ as a matrix A with respect to the basis $\{t_1, \ldots, t_r\}$, $\rho(g^{-1})$ is represented by the transposed matrix ${}^tA$ of A with respect to the basis $\{s_1, \ldots, s_r\}$.From (3.4), $\hat{\varphi}(g)$ corresponds to the first row vector of ${}^tA$. Hence we obtain

$$\hat{\varphi}(g) = \tau(g)(t_1).$$

Thus the image of φ coincides with the L-orbit passing through $[t_1]$ in $P(T)$.

In particular we see that, for the model equation R_S associated with the pseudo-projective GLA $\mathfrak{g}^k(n, 1)$, the line bundle E_S is isomorphic with the $(k-1)$-th power H^{k-1} of the hyperplane bundle H over $P(\hat{V})$, which is dual to the line bundle over $P(\hat{V})$ obtained from the representation of L' on $W_{k-1} = \langle \{v_o^{k-1}\} \rangle$.

Furthermore we see that the Plücker embedding of $\mathrm{Gr}(k, \ell + 1)$ into $P(\bigwedge^k \mathbb{C}^{\ell+1})$ is obtained by global sections of the line bundle F, which is constructed from the irreducible representation ρ_0 of $SL(\ell + 1, \mathbb{C})$ on $S = \bigwedge^{\ell-k+1} \mathbb{C}^{\ell+1}$. Here ρ_0 is the dual representation of τ_0 on $T = \bigwedge^k \mathbb{C}^{\ell+1}$. Let $R^{\rho_0} = R_S$ be the system of equations modeled after $S = \bigwedge^{\ell-k+1} \mathbb{C}^{\ell+1}$ constructed in §3.1. Then, by Proposition A, R^{ρ_0} is the system of equations of finite type, whose local solution is the restriction of a global section of F and whose projective solution coincides with the Plücker embedding of $\mathrm{Gr}(k, \ell + 1)$ (cf. §1 of [SYY97]). Thus R^{ρ_0} can be called the **Plücker embedding equation**. Theorem A in §3.1 states the rigidity for these equations in the linear equivalence. For the application of these facts to a problem of the hypergeometric systems, we refer the reader to [SYY97]. We will discuss the rigidity property of these equations in the contact equivalence in §7.

In fact, the symbol algebra $\mathfrak{m}$ of R^{ρ_0} is already calculated in the last paragraph in §2.2 and we see that R^{ρ_0} is a system of order $\mu = \min\{k + 1, \ell - k + 2\}$ such that $S_{-\mu} = \mathbb{C}$ and $S_{1-\mu} = V^*$. Namely the system R^{ρ_0} has no equation of the first order. Then, since the symbol algebra $\mathfrak{m}$ of R^{ρ_0} is generated by $\mathfrak{g}_{-1}$ (Lemma 2.1.(2)), it follows from Corollary 5.4 [Yam82] that the equivalence of the pseudo-product structure on the equation R of type $\mathfrak{m}$ is the equivalence of the μ-th order equation under contact transformations.

4. Generalized Spencer Cohomology

From this section, we assume that the ground field is the field $\mathbb{C}$ of complex numbers for the sake of simplicity. For the discussion over $\mathbb{R}$, the corresponding results will be obtained easily through the argument of complexification as in §3.2 in [Yam93]. We use the following notation: For a graded vector space $V = \bigoplus_{p \in \mathbb{Z}} V_p$, we put $V_{\leqq k} = \bigoplus_{p \leqq k} V_p$ and $V_{\geqq k} = \bigoplus_{p \geqq k} V_p$. In particular, we set $V_- = V_{\leqq -1}$ and $V_+ = V_{\geqq 1}$. For a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $\mathrm{Der}(\mathfrak{g})$ denotes the Lie algebra of all derivations of $\mathfrak{g}$, $\mathcal{D}(\mathfrak{g})$ denotes the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$

of $\mathfrak{g}$, $\mathfrak{z}(\mathfrak{g})$ denotes the center of $\mathfrak{g}$, and $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ denotes the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. For a $\mathfrak{g}$-module M, we denote by $\mathrm{ch}_{\mathfrak{g}}(M)$ the isomorphism class of M.

4.1. **Cohomology of Lie algebras.** Let $\mathfrak{a} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{a}_p$ be a finite dimensional GLA and $V = \bigoplus_{p\in\mathbb{Z}} V_p$ be a graded $\mathfrak{a}$-module. (i.e., V is a vector space with a gradation such that $\mathfrak{a}_p \cdot V_q \subset V_{p+q}$.) Then we have a cohomology space $H^p(\mathfrak{a}, V)$ associated with the cochain complex $(C^p(\mathfrak{a}, V), \partial)$, where $C^p(\mathfrak{a}, V) = \mathrm{Hom}(\bigwedge^p \mathfrak{a}, V)$ and the coboundary operator $\partial^p : C^p(\mathfrak{a}, V) \to C^{p+1}(\mathfrak{a}, V)$ is defined by

$$\begin{aligned}\partial^p \omega(x_1, \dots, x_{p+1}) = &\sum_{i=1}^{p+1}(-1)^{i+1} x_i \cdot \omega(x_1, \dots, \hat{x}_i, \dots, x_{p+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{p+1}),\end{aligned}$$

where $x_i \in \mathfrak{a}$ and $\omega \in C^p(\mathfrak{a}, V)$. Since both $\mathfrak{a}$ and V are graded, we have the natural gradation: $C^p(\mathfrak{a}, V) = \bigoplus_r C^p(\mathfrak{a}, V)_r$, where

$$C^p(\mathfrak{a}, V)_r = \{\, \omega \in C^p(\mathfrak{a}, V) \mid \omega(\mathfrak{a}_{i_1} \wedge \cdots \wedge \mathfrak{a}_{i_p}) \subset V_{i_1+ \quad +i_p+r} \,\}.$$

It is easy to see that $C^*(\mathfrak{a}, V)_r = \bigoplus C^p(\mathfrak{a}, V)_r$ is a subcomplex of $C^*(\mathfrak{a}, V)$, whose cohomology space (resp. p-th cohomology space) will be denoted by $H(\mathfrak{a}, V)_r$ (resp. $H^p(\mathfrak{a}, V)_r$). Then we have

$$H^*(\mathfrak{a}, V) = \bigoplus_r H^*(\mathfrak{a}, V)_r = \bigoplus_p H^p(\mathfrak{a}, V) = \bigoplus_{p,r} H^p(\mathfrak{a}, V)_r.$$

4.2. **Generalized Spencer cohomology $H^*(\mathfrak{G})$ and $H^*(\mathfrak{b}_-, \mathfrak{g})$.** Let $\mathfrak{G} = (\mathfrak{g}, (\mathfrak{g}_p)_{p\in\mathbb{Z}}, \mathfrak{l}_{-1}, S_{-1})$ be a PPGLA of type $(\mathfrak{l}, S)$. We set $\mathfrak{b}_{-1} = S$, $\mathfrak{b}_0 = \mathfrak{l}$ and $\mathfrak{b}_p = 0$ $(p \neq -1, 0)$; then $\mathfrak{g}$ has a bigradation $(\mathfrak{g}_{p,q})_{p,q\in\mathbb{Z}}$, where $\mathfrak{g}_{p,q} = \mathfrak{g}_p \cap \mathfrak{b}_q$. We have the cohomology group $H^*(\mathfrak{G}) = H^*(\mathfrak{m}, \mathfrak{g})$ associated with the adjoint representation of $\mathfrak{m} = \mathfrak{g}_-$ on $\mathfrak{g}$, that is, the cohomology space of the cochain complex $C^*(\mathfrak{G}) = \bigoplus C^p(\mathfrak{G})$ with the coboundary operator $\partial : C^p(\mathfrak{G}) \longrightarrow C^{p+1}(\mathfrak{G})$, where $C^p(\mathfrak{G}) = \mathrm{Hom}(\bigwedge^p \mathfrak{g}_-, \mathfrak{g})$. We put

$$C^p(\mathfrak{G})_{r,s} = \left\{ \omega \in C^p(\mathfrak{G}) \,\middle|\, \begin{array}{c} \omega(\mathfrak{g}_{i_1,j_1} \wedge \cdots \wedge \mathfrak{g}_{i_p,j_p}) \subset \mathfrak{g}_{i_1+ \quad +i_p+r, j_1+ \quad +j_p+s} \\ \text{for all} \quad i_1, \dots, i_p, j_1, \dots, j_p \end{array} \right\}.$$

As is easily seen, $C^*(\mathfrak{G})_{r,s} = \bigoplus_p C^p(\mathfrak{G})_{r,s}$ is a subcomplex of $C^*(\mathfrak{G})$. Denoting its cohomology space by $H(\mathfrak{G})_{r,s} = \bigoplus H^p(\mathfrak{G})_{r,s}$, we obtain the direct sum decomposition

$$H^*(\mathfrak{G}) = \bigoplus_{p,r,s} H^p(\mathfrak{G})_{r,s}.$$

The cohomology space, endowed with this tri-gradation, is called the generalized Spencer cohomology space of the PPGLA $\mathfrak{G}$ of type $(\mathfrak{l}, S)$. Note that $H^1(\mathfrak{G})_{0,0} = 0$ if and only if $\check{\mathfrak{g}}_0 = \mathfrak{g}_0$, where $\check{\mathfrak{g}}_0$ is the Lie algebra of derivations of $\mathfrak{m}$ such that $D(\mathfrak{g}_p) \subset \mathfrak{g}_p$ $(p < 0)$, $D(\mathfrak{l}_{-1}) \subset \mathfrak{l}_{-1}$ and $D(S_{-1}) \subset S_{-1}$.

Furthermore we have the cohomology space $H^*(\mathfrak{b}_-, \mathfrak{g})$ associated with the adjoint representation of $\mathfrak{b}_-$ on $\mathfrak{g}$, that is, the cohomology space of the cochain complex $C^*(\mathfrak{b}_-, \mathfrak{g}) = \bigoplus C^p(\mathfrak{b}_-, \mathfrak{g})$ with the coboundary operator $\delta^p : C^p(\mathfrak{b}_-, \mathfrak{g}) \to C^{p+1}(\mathfrak{b}_-, \mathfrak{g})$, where $C^p(\mathfrak{b}_-, \mathfrak{g}) = \mathrm{Hom}(\bigwedge^p \mathfrak{b}_-, \mathfrak{g})$. Let $C^p(\mathfrak{b}_-, \mathfrak{g})_s$ be the subspace of $C^p(\mathfrak{b}_-, \mathfrak{g})$ consisting of all the elements $\omega \in C^p(\mathfrak{b}_-, \mathfrak{g})$ such that $\omega(\mathfrak{b}_{j_1} \wedge \cdots \wedge \mathfrak{b}_{j_p}) \subset \mathfrak{b}_{j_1+ \quad +j_p+s}$ for all $j_1, \dots, j_p < 0$. $C^*(\mathfrak{b}_-, \mathfrak{g})_s = \bigoplus_p C^p(\mathfrak{b}_-, \mathfrak{g})_s$ is a subcomplex of $C^*(\mathfrak{b}_-, \mathfrak{g})$. Denoting its cohomology space by $H^*(\mathfrak{b}_-, \mathfrak{g})_s = \bigoplus H^p(\mathfrak{b}_-, \mathfrak{g})_s$, we obtain the direct sum decomposition

$$H^*(\mathfrak{b}_-, \mathfrak{g}) = \bigoplus_{p,s} H^p(\mathfrak{b}_-, \mathfrak{g})_s.$$

The cohomology space, endowed with this bi-gradation, is called the Spencer cohomology of the GLA $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$.

4.3. **Calculation of the cohomology $H^p(\mathfrak{G})_{r,s}$.** Here we prepare the fundamental theorem to calculate the cohomology space $H^p(\mathfrak{G})_{r,s}$ by means of Kostant's theorem.

Since $H^i(\mathfrak{b}_-, \mathfrak{g})_s$ naturally has a graded $\mathfrak{l}$-module structure, we obtain the following direct sum decomposition

$$H^p(\mathfrak{l}_-, H^i(\mathfrak{b}_-, \mathfrak{g})_s) = \bigoplus_r H^p(\mathfrak{l}_-, H^i(\mathfrak{b}_-, \mathfrak{g})_s)_r.$$

For $i = s, s+1$, we set

$$C_s^{p,i} = \mathrm{Hom}(\bigwedge^{p-i} \mathfrak{l}_- \otimes \bigwedge^{i} \mathfrak{b}_{-1}, \mathfrak{b}_{s-i}),$$

and $C_s^p = C_s^{p,s} \oplus C_s^{p,s+1}$. For $\omega \in C_s^{p,i}$, we denote by $\partial_{j,i}^p \omega$ the $C_s^{p,j}$-component of $\partial^p \omega$. Then $\partial_{j,i}$ is a $\mathfrak{g}_0$-module homomorphism of $C_s^{p,i}$ into $C_s^{p+1,j}$.

Lemma 4.1. *Under the above assumption, we have*

(1) $\partial_{j,i}^p = 0$ *for* $j \neq i, i+1$.

(2) $\partial_{i,i}^{p+1} \partial_{i,i}^p = 0$, $\partial_{i+2,i+1}^{p+1} \partial_{i+1,i}^p = 0$.

(3) $\partial_{i+1,i}^{p+1} \partial_{i,i}^p + \partial_{i+1,i+1}^{p+1} \partial_{i+1,i}^p = 0$.

Proof. Let $\omega \in C_s^{p,i}$. Then, for $x_1, \dots, x_q \in \mathfrak{l}_-$ $m_1, \dots, m_{k+i} \in \mathfrak{b}_-$ $(k \in \mathbb{Z})$,

$$\begin{aligned}
&\partial\omega(x_1, \dots, x_q, m_1, \dots, m_{k+i}) \\
&\quad = \sum_{a=1}^{q} (-1)^{a+1} [x_a, \omega(x_1, \dots, \hat{x}_a, \dots, x_q, m_1, \dots, m_{k+i})] \\
&\quad + \sum_{a=1}^{i+k} (-1)^{q+a-1} [m_a, \omega(x_1, \dots, x_q, m_1, \dots, \hat{m}_a, \dots, m_{k+i})] \\
&\qquad + \sum_{a,b} (-1)^{a+q+b} \omega([x_a, m_b], x_1, \dots, \hat{x}_a, \dots, x_q, m_1, \dots, \hat{m}_b, \dots, m_{k+i})]
\end{aligned}$$

where $q = p - i + k + 1$. If $k \neq 0, 1$, then

$$\partial\omega(x_1, \dots, x_{p-i-k+1}, m_1, \dots, m_{k+i}) = 0.$$

Thus $\partial_{j,i}^p = 0$ for $j \neq i, i+1$, which proves (1). Moreover we see that

$$\begin{aligned}
0 = \partial^{p+1}\partial^p\omega &= \partial^{p+1}(\partial_{ii}^p\omega + \partial_{i+1,i}^p\omega) \\
&= \partial_{i,i}^{p+1}\partial_{i,i}^p\omega + \partial_{i+1,i}^{p+1}\partial_{ii}^p\omega + \partial_{i+1,i+1}^{p+1}\partial_{i+1,i}^p\omega + \partial_{i+2,i+1}^{p+1}\partial_{i+1,i}^p\omega.
\end{aligned}$$

This proves (2) and (3). □

We define a linear mapping $\phi_{p,i}$ of $C_s^{p,i}$ onto $C^{p-i}(\mathfrak{l}_-, C^i(\mathfrak{b}_-, \mathfrak{g})_s)$ $(i = s, s+1)$ as follows:

$$\phi_{p,i}(\omega)(x_1, \dots, x_{p-i})(m_1, \dots, m_i) = \omega(x_1, \dots, x_{p-i}, m_1, \dots, m_i).$$

Then $\phi_{p,i}$ is a $\mathfrak{g}_0$-module isomorphism.

From the proof of Lemma 4.1, we have

$$\partial_{i,i}\omega(x_1,\dots,x_q,m_1,\dots,m_i) = \sum_{a=1}^{q}(-1)^{a+1}[x_a,\omega(x_1,\dots,\hat{x}_a,\dots,x_q,m_1,\dots,m_i)] + \sum_{a,b}(-1)^{a+q+b}\omega([x_a,m_b],x_1,\dots,\hat{x}_a,\dots,x_q,m_1,\dots,\hat{m}_b,\dots,m_i)],$$

where $\omega\in C_s^{p,i}$ and $q=p-i+1$.

$$\partial_{i+1,i}\omega(x_1,\dots,x_q,m_1,\dots,m_{i+1}) = \sum_{a=1}^{i+1}(-1)^{r+a-1}[m_a,\omega(x_1,\dots,x_r,m_1,\dots,\hat{m}_a,\dots,m_{i+1})],$$

where $\omega\in C_s^{p,i}$ and $r=p-i+2$. Hence we obtain the following lemma.

Lemma 4.2. *Let $\omega\in C_s^{p,i}$ $(i=s,s+1)$. Then:*

(1) *$\phi_{p+1,i}(\partial^p_{i,i}\omega)=\rho(\phi_{p,i}(\omega))$, where ρ is the coboundary operator of $C^*(\mathfrak{l}_-,C^i(\mathfrak{b}_-,\mathfrak{g})_s)$.*

(2) *For $x_1,\dots,x_{p-i}\in\mathfrak{l}_{-1}$, we have*

$$\phi_{p+1,i}(\partial^p_{i+1,i}\omega)(x_1,\dots,x_{p-i})=(-1)^{p-i}\delta^i(\phi_{p,i}(\omega)(x_1,\dots,x_{p-i})).$$

We recall that there exist an anti-linear involution τ of $\mathfrak{l}$ and an hermitian inner product $(\cdot\,|\,\cdot)$ of $\mathfrak{g}$ having the following properties:

(i) $\tau(\mathfrak{l}_p)=\mathfrak{l}_{-p}$;

(ii) $(\mathfrak{b}_p\,|\,\mathfrak{b}_q)=0$ for $p\neq q$ and, $(\mathfrak{g}_p\,|\,\mathfrak{g}_q)=0$ for $p\neq q$;

(iii) $([x,s]\,|\,s')+(s\,|\,[\tau(x),s'])=0$ for all $x\in\mathfrak{l}$ and $s,s'\in\mathfrak{b}_{-1}$.

The inner product $(\cdot\,|\,\cdot)$ induces inner products on $\mathfrak{b}^*_{-1}$ and $C^*(\mathfrak{b}_-,\mathfrak{g})$, which are denoted by the same symbol. Namely, for $\omega=\alpha\otimes v_1^*\wedge\cdots\wedge v_p^*$, $\omega'=\beta\otimes w_1^*\wedge\cdots\wedge w_p^*$, we see that

$$(\omega\,|\,\omega')=(\alpha\,|\,\beta)\det(v_i^*\,|\,w_j^*),$$

where $\alpha,\beta\in\mathfrak{g}$, $v_i^*,w_i^*\in\mathfrak{b}^*_{-1}$.

Now we denote by δ^* the adjoint operator of δ with respect to the inner product $(\cdot\,|\,\cdot)$,i.e.,

$$(\delta\omega\,|\,\omega')=(\omega\,|\,\delta^*\omega')\quad\text{for}\quad\omega,\omega'\in C^*(\mathfrak{b}_-,\mathfrak{g}).$$

We see that δ^* is a homomorphism as a $\mathfrak{b}_0$-module and $\delta^*(C^p(\mathfrak{b}_-,\mathfrak{g})_s)\subset C^{p-1}(\mathfrak{b}_-,\mathfrak{g})_s$. As usual, the operator $\Delta=\delta\delta^*+\delta^*\delta$ is called the Laplacian. We set $\mathcal{H}^p(\mathfrak{b}_-,\mathfrak{g})_s=\{\,\omega\in C^p(\mathfrak{b}_-,\mathfrak{g})_s\mid\Delta\omega=0\,\}$; then

$$C^p(\mathfrak{b}_-,\mathfrak{g})_s=\mathcal{H}^p(\mathfrak{b}_-,\mathfrak{g})_s\oplus\Delta(C^p(\mathfrak{b}_-,\mathfrak{g})_s).$$

We put $\mathcal{D}^p(\mathfrak{b}_-,\mathfrak{g})_s=\Delta(C^p(\mathfrak{b}_-,\mathfrak{g})_s)$. Then $\mathcal{H}^p(\mathfrak{b}_-,\mathfrak{g})_s$ is isomorphic to $H^p(\mathfrak{b}_-,\mathfrak{g})_s$ as an $\mathfrak{l}$-module and $\mathcal{D}^p(\mathfrak{b}_-,\mathfrak{g})_s=\delta^2(C_s^p)\oplus(\operatorname{Ker}\delta|C_s^p)^\perp$. We put

$$\mathcal{H}_s^{p,i}=\phi_{p,i}^{-1}(\operatorname{Hom}(\textstyle\bigwedge^{p-i}\mathfrak{l}_-,\mathcal{H}^i(\mathfrak{b}_-,\mathfrak{g})_s)),$$

$$\mathcal{D}_s^{p,i}=\phi_{p,i}^{-1}(\operatorname{Hom}(\textstyle\bigwedge^{p-i}\mathfrak{l}_-,\mathcal{D}^i(\mathfrak{b}_-,\mathfrak{g})_s)),$$

$$\mathcal{H}_s^p=\bigoplus_i\mathcal{H}_s^{p,i},\quad\mathcal{D}_s^p=\bigoplus_i\mathcal{D}_s^{p,i}.$$

Then $C_s^{p,i}=\mathcal{H}_s^{p,i}\oplus\mathcal{D}_s^{p,i}$ and we obtain the following exact sequence:

(4.1) $$\mathcal{D}_s^{p-1,i-1}\xrightarrow{\partial_{i,i-1}}\mathcal{D}_s^{p,i}\xrightarrow{\partial_{i+1,i}}\mathcal{D}_s^{p+1,i+1}.$$

From Lemma 4.2, we obtain the following lemma.

Lemma 4.3. (1) $\partial^p_{i,i}(\mathcal{H}^{p,i}_s) \subset \mathcal{H}^{p+1,i}_s$, $\partial^p_{i+1,i}(\mathcal{H}^{p,i}_s) = 0$.
(2) $\partial^p_{i,i}(\mathcal{D}^{p,i}_s) \subset \mathcal{D}^{p+1,i}_s$, $\partial^p_{i+1,i}(\mathcal{D}^{p,i}_s) \subset \mathcal{D}^{p+1,i+1}_s$.

Lemma 4.4. *Let $\omega \in \operatorname{Ker}\partial^p \cap C^p_s$. We decompose ω as follows: $\omega = \omega' + \omega''$, where $\omega' \in \mathcal{H}^p_s$ and $\omega'' \in \mathcal{D}^p_s$. Then $\omega'' \in \operatorname{Im}\partial^{p-1}$.*

Proof. We decompose ω'' as follows: $\omega'' = \omega''_s + \omega''_{s+1}$, where $\omega''_s \in \mathcal{D}^{p,s}_s$ and $\omega''_{s+1} \in \mathcal{D}^{p,s+1}_s$. Since $\partial\omega = 0$, we have $\partial^p_{s+2,s+1}\omega''_{s+1} = 0$ and $\partial^p_{s+1,s+1}\omega''_{s+1} + \partial^p_{s+1,s}\omega''_s = 0$. By (4.1), there is an element $c_s \in \mathcal{D}^{p-1,s}_s$ such that $\partial^{p-1}_{s+1,s}c_s = \omega''_{s+1}$. Furthermore $\omega''_s - \partial^{p-1}_{s,s}c_s \in \mathcal{D}^{p,s}_s$ and

$$\begin{aligned}\partial^p_{s+1,s}(\omega''_s - \partial^{p-1}_{s,s}c_s) &= \partial^p_{s+1,s}\omega''_s - \partial^p_{s+1,s}\partial^{p-1}_{s,s}c_s \\ &= \partial^p_{s+1,s}\omega''_s + \partial^p_{s+1,s+1}\partial^{p-1}_{s+1,s}c_s \\ &= \partial^p_{s+1,s}\omega''_s + \partial^p_{s+1,s+1}\omega''_{s+1} = 0.\end{aligned}$$

Since $\partial^p_{s+1,s}|\mathcal{D}^{p,s}_s$ is injective, we have $\omega''_s = \partial^{p-1}_{s,s}c_s$. Thus $\omega'' = \omega''_s + \omega''_{s+1} = (\partial^{p-1}_{s,s} + \partial^{p-1}_{s+1,s})c_s = \partial c_s$. □

We are ready to state the following fundamental theorem.

Theorem 4.1. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$. Then $H^p(\mathfrak{G})_{r,s}$ is isomorphic to $\bigoplus_{i=0}^p H^{p-i}(\mathfrak{l}_-, H^i(\mathfrak{b}_-, \mathfrak{g})_s)_r$ as a $\mathfrak{g}_0$-module.*

Proof. We define a linear mapping

$$\Phi : C^p(\mathfrak{G})_{r,s} \to \bigoplus_{i=0}^p C^{p-i}(\mathfrak{l}_-, \mathcal{H}^i(\mathfrak{b}_-, \mathfrak{g})_s)_r$$

as follows: $\Phi(\omega) = \sum_{i=0}^p \phi_{p,i}(\omega'_i)$, where $\omega = \sum_{i=0}^p \omega'_i + \omega'' \in C^p(\mathfrak{G})_{r,s}$, $\omega'_i \in \mathcal{H}^{p,i}_s$, $\omega'' \in \mathcal{D}^p_s$. By Lemma 4.2, $\Phi\partial = \partial\Phi$ and hence Φ induces a $\mathfrak{g}_0$-module homomorphism

$$\Phi^* : H^p(\mathfrak{G})_{r,s} \to \bigoplus_{i=0}^p H^{p-i}(\mathfrak{l}_-, \mathcal{H}^i(\mathfrak{b}_-, \mathfrak{g})_s)_r.$$

By Lemma 4.4, Φ^* is an isomorphism. □

4.4. **Gradations of semisimple Lie algebras and Kostant's theorem.** Here we recall some basic facts on gradations of semisimple Lie algebras following [Yam93] and state Kostant's theorem on Lie algebra cohomology, which is our basic tool in the discussion of subsequent sections.

Let $\mathfrak{s}$ be a complex semisimple Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$ and the set Φ of roots of $\mathfrak{s}$ relative to $\mathfrak{h}$. Let us fix a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of Φ. For a subset Δ_1 of Δ and an integer p, we put

$$\Phi_p = \{\, \alpha \in \Phi \mid \sum_{\alpha_i \in \Delta_1} m_i(\alpha) = p \,\},$$

where m_i is a $\mathbb{Z}$-valued function on Φ defined by $m_i(\sum_{j=1}^\ell k_j\alpha_j) = k_i$. Then we can construct a gradation $(\mathfrak{s}_p)_{p\in\mathbb{Z}}$ of $\mathfrak{s}$ as follows:

$$\mathfrak{s}_0 = \mathfrak{h} \oplus \sum_{\alpha\in\Phi_0} \mathfrak{s}^\alpha, \quad \mathfrak{s}_p = \sum_{\alpha\in\Phi_p} \mathfrak{s}^\alpha \quad (p \neq 0),$$

where $\mathfrak{s}^\alpha$ is the root space corresponding to a root α of $\mathfrak{s}$. Then $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ is a GLA such that $\mathfrak{s}_-$ is generated by $\mathfrak{s}_{-1}$. If $\mathfrak{s}$ is of type X_ℓ, then the GLA $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ is said to be of type (X_ℓ, Δ_1), where X_ℓ stands for the Dynkin diagram of $\mathfrak{s}$.

Conversely, let $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ be a simple GLA such that $\mathfrak{s}_-$ is generated by $\mathfrak{s}_{-1}$. Assume that $\mathfrak{s}$ is of type X_ℓ. Then $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ is isomorphic to a simple GLA of type (X_ℓ, Δ_1), for some $\Delta_1 \subset \Delta$. Let $\mathfrak{t} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{t}_p$ be an another simple GLA of type (X_ℓ, Δ_1'). Then $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ is isomorphic to $\mathfrak{t} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{t}_p$ if and only if there exists a diagram automorphism ϕ of X_ℓ such that $\phi(\Delta_1) = \Delta_1'$ (Theorem 3.12 [Yam93]).

Let W be the Weyl group of $\mathfrak{s}$, Φ_+ the set of positive roots. Moreover we put

$$T_w = w\Phi_- \cap \Phi_+ \quad (\text{where } \Phi_- = \Phi \backslash \Phi_+),$$

$$W_1^j = \{w \in W \mid \#(T_w) = j, \quad T_w \subset \Phi(\mathfrak{s}_+)\},$$

where $\Phi(\mathfrak{s}_+) = \{\alpha \in \Phi \mid \mathfrak{s}^\alpha \subset \mathfrak{s}_+\}$. For an antidominant integral weight ω of $\mathfrak{s}$ (resp. $\mathfrak{s}_0$) we denote the irreducible $\mathfrak{s}$ (resp. $\mathfrak{s}_0$)-module with lowest weight ω by $M(\omega)$ (resp. $m(\omega)$). Then we have the following theorem due to Kostant.

Theorem B (Theorem 5.14 [Kos61]). *Let $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ be a simple GLA of type (X_ℓ, Δ_1) and $M(\omega)$ be an irreducible $\mathfrak{s}$-module with the lowest weight ω. Then*

$$\mathrm{ch}_{\mathfrak{s}_0}(H^j(\mathfrak{s}_-, M(\omega))) = \sum_{w\in W_1^j} \mathrm{ch}_{\mathfrak{s}_0}(m(w(\omega - \rho) + \rho)),$$

where ρ is the half sum of positive roots.

4.5. **Parameterization of pseudo-product GLA of type** $(\mathfrak{l}, S)$. Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$. We set $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ and $\mathfrak{u} = \mathcal{D}(\mathfrak{z}_\mathfrak{l}(\hat{\mathfrak{l}}))$; then $\mathfrak{l} = \hat{\mathfrak{l}} \oplus \mathfrak{u} \oplus \mathfrak{z}(\mathfrak{l})$ and $\hat{\mathfrak{l}} = \bigoplus_{p\in\mathbb{Z}} \hat{\mathfrak{l}}_p$ is a simple GLA. Let $(\mathfrak{b}_q)_{q\in\mathbb{Z}}$ be as in §4.2.

Let us take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ such that $\mathfrak{h} \subset \mathfrak{l}_0$. Then $\mathfrak{h} \cap \hat{\mathfrak{l}}$ (resp. $\mathfrak{h} \cap \mathfrak{u}$) is a Cartan subalgebra of $\hat{\mathfrak{l}}$ (resp. $\mathfrak{u}$). Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ (resp. $\Delta' = \{\beta_1, \ldots, \beta_m\}$) be a simple root system of $(\hat{\mathfrak{l}}, \mathfrak{h} \cap \hat{\mathfrak{l}})$ (resp. $(\mathfrak{u}, \mathfrak{h} \cap \mathfrak{u})$) such that $\alpha(Z) \geqq 0$ for all $\alpha \in \Delta$, where Z is the characteristic element of the GLA $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$. We assume that $\hat{\mathfrak{l}}$ is a simple Lie algebra of type X_ℓ. We set $\Delta_1 = \{\alpha \in \Delta \mid \alpha(Z) = 1\}$. It is well known that the pair (X_ℓ, Δ_1) is one of the following type (up to a diagram automorphism) (cf. §3 in [Yam93]):

$$(A_\ell, \{\alpha_i\})\ (1 \leqq i \leqq [(\ell+1)/2]),\ (B_\ell, \{\alpha_1\})\ (\ell \geqq 3),\ (C_\ell, \{\alpha_\ell\})\ (\ell \geqq 2),$$

$$(D_\ell, \{\alpha_1\})\ (\ell \geqq 4),\ (D_\ell, \{\alpha_{\ell-1}\})\ (\ell \geqq 5),\ (E_6, \{\alpha_1\}),\ (E_7, \{\alpha_7\}).$$

We denote by $\{\varpi_1, \ldots, \varpi_\ell\}$ (resp. $\{\pi_1, \ldots, \pi_n\}$) the set of fundamental weights relative to Δ (resp. Δ'). Since S is a faithful $\mathfrak{l}$-module, we have $\dim \mathfrak{z}(\mathfrak{l}) \leqq 1$. Assume that $\mathfrak{z}(\mathfrak{l}) \neq \{0\}$. Let σ be the element of $\mathfrak{z}(\mathfrak{l})^*$ such that $\sigma(J) = 1$, where J is the characteristic element of the GLA $\mathfrak{g} = \bigoplus_{q\in\mathbb{Z}} \mathfrak{b}_q$. Namely $J = -id_S \in \mathfrak{z}(\mathfrak{l}) \subset \mathfrak{b}_0 = \mathfrak{l}$ as the element of $\mathfrak{gl}(S)$. There is an irreducible $\hat{\mathfrak{l}}$-module T (resp. $\mathfrak{z}_\mathfrak{l}(\hat{\mathfrak{l}})$-module U) with highest weight χ (resp. $\eta - \sigma$) such that $S = \mathfrak{b}_{-1}$ is isomorphic to $T \otimes U$ as an $\mathfrak{l}$-module, where η is a weight of $\mathfrak{u}$. Then we have

Lemma 4.5. $H^1(\mathfrak{G})_{0,0} = 0$ *if and only if $\mathfrak{z}_\mathfrak{l}(\hat{\mathfrak{l}})$ is isomorphic to $\mathfrak{gl}(U)$ and $\eta = \pi_1$. Especially, when $\mathcal{D}(\mathfrak{l}) = \hat{\mathfrak{l}}$, $H^1(\mathfrak{G})_{0,0} = 0$ if and only if $\mathfrak{l} = \hat{\mathfrak{l}} \oplus \mathfrak{z}(\mathfrak{l})$, where $\mathfrak{z}(\mathfrak{l}) = \langle J \rangle$.*

Proof. We first remark that $H^1(\mathfrak{G})_{0,0} = 0$ if and only if $\check{\mathfrak{g}}_0 = \mathfrak{g}_0$ and that

$$\mathfrak{g}_0 = [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{z}_\mathfrak{l}(\hat{\mathfrak{l}}), \qquad \check{\mathfrak{g}}_0 = [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{z}_{\check{\mathfrak{g}}_0}(\hat{\mathfrak{l}}).$$

For $\varphi \in \mathfrak{gl}(U)$, we define $D_\varphi \in \mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g}_-)$ as follows:

$$D_\varphi(\mathfrak{l}_{-1}) = 0, \quad D_\varphi(t \otimes u) = t \otimes \varphi(u) \text{ for } t \in T \text{ and } u \in U.$$

Then $D_\varphi \in \check{\mathfrak{g}}_0$ and the mapping $\mathfrak{gl}(U) \ni \varphi \mapsto D_\varphi \in \check{\mathfrak{g}}_0$ is injective. By Schur's lemma, this mapping is also surjective. This proves our assertion. □

Thus, when $H^1(\mathfrak{G})_{0,0} = 0$, the semisimple GLA $\mathcal{D}(\mathfrak{l})$ is of type $(X_\ell \times A_n, \{\alpha_i\})$ and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight $\Xi = \chi + \pi_1$ when $\dim U > 1$ and $\mathcal{D}(\mathfrak{l})$ is of type $(X_\ell, \{\alpha_i\})$ and S is an irreducible $\hat{\mathfrak{l}}$-module with highest weight χ, when $\mathcal{D}(\mathfrak{l}) = \hat{\mathfrak{l}}$ (i.e., when $\dim U = 1$). We will impose the condition $H^1(\mathfrak{G})_{0,0} = 0$ on $\mathfrak{G}$ in the rest of this paper.

Now let us consider the characteristic element E of the gradation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. Since S is the irreducible $\mathfrak{l}$-module with highest weight $\Xi = \chi + \pi_1 - \sigma$, and $Z \in \mathfrak{h} \cap \hat{\mathfrak{l}}$ is defined by $\alpha_{i_o}(Z) = 1$ and $\alpha_i(Z) = 0$ $(i \neq i_o)$ for $\Delta_1 = \{\alpha_{i_o}\} \subset \Delta = \{\alpha_1, \dots, \alpha_\ell\}$, we see that the semi-simple endomorphism $\mathrm{ad}(Z)$ has consecutive eigenvalues in S of the following form:

$$\lambda_1 - k \qquad \text{for } k = 0, \dots, \mu - 1.$$

where $\lambda_1 = \chi(Z)$ and $\lambda_0 = \lambda_1 - \mu + 1$ is the minimum eigenvalue (see Lemma 2.1). Thus the characteristic element E of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is given by

$$E = Z + (\lambda_1 + 1)J \in \hat{\mathfrak{l}}_0 \oplus \mathfrak{z}(\mathfrak{l}) \subset \mathfrak{l}_0 = \mathfrak{g}_0.$$

Then we note that, utilizing the characteristic elements E and J, the decomposition of $\mathfrak{l}_0$- modules $C^p(\mathfrak{G})$ and $C^p(\mathfrak{b}_-, \mathfrak{g})$:

$$C^p(\mathfrak{G}) = \bigoplus_r C^p(\mathfrak{G})_r = \bigoplus_s C^p(\mathfrak{G})_s, \quad C^p(\mathfrak{b}_-, \mathfrak{g}) = \bigoplus_s C^p(\mathfrak{b}_-, \mathfrak{g})_s,$$

are given as the eigenspace decomposition of the action of E and J respectively, where r and s denote eigenvalues of E and J respectively. Moreover the decomposition $C^p(\mathfrak{G}) = \bigoplus_{r,s} C^p(\mathfrak{G})_{r,s}$ given in §4.2 is the simultaneous eigenspace decomposition of commuting semi-simple elements E and J.

Let Λ be a dominant integral weight of $\mathcal{D}(\mathfrak{l})$. We denote by $L^s(\Lambda)$ the irreducible $\mathfrak{l}$-module with highest weight $\Lambda + s\sigma$. Now we apply the notation of §4.4 to the case when $\mathfrak{s} = \mathcal{D}(\mathfrak{l})$. For $w \in W$, we set

$$\xi^s_w(\Lambda) = w(w_0(\Lambda) - \rho) + \rho + s\sigma,$$

where w_0 is the element of W such that $w_0(\Delta) = -\Delta$, which sends the highest weight to the lowest weight. Since $S = \mathfrak{b}_{-1}$ is an irreducible $\mathfrak{l}$-module with highest weight $\Xi = \chi + \pi_1 - \sigma$, we have $\langle \chi - \sigma, E\rangle = -1$ and hence $\langle \xi^s_w(\Lambda), J\rangle = s$ and

$$\begin{aligned}\langle \xi^s_w(\Lambda), E\rangle &= \langle w(w_0(\Lambda) - \rho) + \rho, E\rangle + s(1 + \langle \chi, E\rangle)\\ &= \langle w(w_0(\Lambda) - \rho) + \rho, Z\rangle + s(\lambda_1 + 1).\end{aligned}$$

By Kostant's Theorem (Theorem B), $H^p(\mathfrak{l}_-, L^s(\Lambda))_r \neq 0$ if and only if $\langle \xi^s_w(\Lambda), E\rangle = r$ for some $w \in W^p_1$.

5. First cohomology of pseudo-product graded Lie algebras

Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ satisfying the condition $H^1(\mathfrak{G})_{0,0} = 0$. We use the same notation as in §4.5. By Theorem 4.1,

$$H^1(\mathfrak{G})_{r,s} \cong H^1(\mathfrak{l}_-, H^0(\mathfrak{b}_-, \mathfrak{g})_s)_r \oplus H^0(\mathfrak{l}_-, H^1(\mathfrak{b}_-, \mathfrak{g})_s)_r. \tag{5.1}$$

In particular, we see that $H^1(\mathfrak{G})_{r,s} = 0$ for $s \leqq -2$ or $s \geqq 2$. First of all, we consider $H^1(\mathfrak{G})_{r,-1}$. By (5.1),

$$H^1(\mathfrak{G})_{r,-1} \cong H^1(\mathfrak{l}_-, \mathfrak{b}_{-1})_r.$$

By Kostant's Theorem (Theorem B), we see that

(i) $H^1(\mathfrak{G})_{r,-1} = 0$ for $r \geqq 1$;

(ii) $H^1(\mathfrak{G})_{0,-1} \neq 0$ if and only if $\mathcal{D}(\mathfrak{l})$ is of type $(A_\ell \times A_n, \{\alpha_1\})$ and S is an irreducible $\mathfrak{l}$-module with highest weight $k\varpi_\ell + \pi_1$. (cf. [Yat92, pp.323–324].)

Secondly we consider $H^1(\mathfrak{G})_{r,0}$. Clearly

$$H^1(\mathfrak{G})_{r,0} \cong H^0(\mathfrak{l}_-, H^1(\mathfrak{b}_-, \mathfrak{g})_0)_r.$$

Let Λ be the highest weight of an irreducible component of the $\mathfrak{l}$-module $H^1(\mathfrak{b}_-, \mathfrak{g})_0$. Then $\langle \xi_1^0(\Lambda), E\rangle = \langle w_0(\Lambda), \varpi_i^\vee\rangle$. If $w_0(\Lambda) = \sum_{i=1}^{\ell} c_i\alpha_i$ ($c_i \in \mathbb{R}$), then $c_i \leqq 0$ for all i, so $\langle \xi_1^0(\Lambda), E\rangle \leqq 0$. By Kostant's Theorem (Theorem B), we obtain that $H^1(\mathfrak{G})_{r,0} = 0$ for $r \geqq 1$.

Thirdly, we consider $H^1(\mathfrak{G})_{r,1}$. Let $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ be the prolongation of $\bigoplus_{p\leq 0} \mathfrak{b}_p$. Clearly

$$H^1(\mathfrak{G})_{r,1} \cong H^0(\mathfrak{l}_-, \check{\mathfrak{b}}_1)_r.$$

If $\dim \check{\mathfrak{b}} < \infty$ and $\check{\mathfrak{b}}_1 \neq 0$, then $\check{\mathfrak{b}}$ is simple and the prolongation of $\mathfrak{G}_-$ is a simple PPGLA. Hence $H^1(\mathfrak{G})_{1,1} \cong S^*_{-1}$ and $H^1(\mathfrak{G})_{r,1} = 0$ ($r \neq 1$). If $\dim \check{\mathfrak{b}} = \infty$, then, by the theorem of Kobayashi and Nagano [KN65], $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ is isomorphic to one of $W(n;\mathbf{1})$ and $CH(m;\mathbf{1};2)$, where $n = 2m = \dim \mathfrak{b}_{-1}$. (For the definition of $W(n;\mathbf{1})$ and $CH(m;\mathbf{1};2)$, see [Yat92].) If $\check{\mathfrak{b}} \cong W(n;\mathbf{1})$, then $\mathfrak{l}$ is of type $(A_\ell, \{\alpha_i\})$ and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight ϖ_1. Hence $H^1(\mathfrak{G})_{0,1} \cong S_{-2} \otimes S^2(S^*_{-1})$, $H^1(\mathfrak{G})_{1,1} \cong S^*_{-1}$, and $H^1(\mathfrak{G})_{r,1} = 0$ ($r \neq 0, 1$). If $\check{\mathfrak{b}} \cong CH(m;\mathbf{1};2)$, then $\mathcal{D}(\mathfrak{l})$ is of type $(C_\ell, \{\alpha_\ell\})$ and S is an irreducible $\mathfrak{l}$-module with highest weight ϖ_1. Hence $H^1(\mathfrak{G})_{0,1} \cong L^{-2}(0) \otimes S^3(S^*_{-1})$, and $H^1(\mathfrak{G})_{r,1} = 0$ ($r \neq 0$).

We summarize the above results in the following theorem.

Theorem 5.1. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ satisfying the condition $H^1(\mathfrak{G})_{0,0} = 0$. Let $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ be the prolongation of $\bigoplus_{p\leq 0} \mathfrak{b}_p$. Then*

(1) *$H^1(\mathfrak{G})_{r,s} = 0$ for $r \geqq 0$, $s \neq -1, 1$.*
(2) *$H^1(\mathfrak{G})_{r,-1} = 0$ for $r \geqq 1$.*
(3) *$H^1(\mathfrak{G})_{0,-1} \neq 0$ if and only if $\mathcal{D}(\mathfrak{l})$ is of type $(A_\ell \times A_n, \{\alpha_1\})$ ($n \geqq 1$) or $(A_\ell, \{\alpha_1\})$, and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight $k\varpi_\ell + \pi_1$. In this case,*

$$H^1(\mathfrak{G})_{0,-1} \cong S_{-\mu} \otimes S^\mu(\mathfrak{l}^*_{-1}).$$

(4) *If $\check{\mathfrak{b}}_1 = 0$, then $H^1(\mathfrak{G})_{r,1} = 0$ for all r.*
(5) *If $\dim \check{\mathfrak{b}} < \infty$ and $\check{\mathfrak{b}}_1 \neq 0$, then*

$$H^1(\mathfrak{G})_{r,1} \cong \begin{cases} S^*_{-1} & \text{if } r = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) *If $\dim \check{\mathfrak{b}} = \infty$, then $\mathfrak{G}$ is isomorphic to one of PPGLAs of the following types:*
   (a) *$\mathcal{D}(\mathfrak{l})$ is of type $(A_\ell, \{\alpha_i\})$ ($1 \leqq i \leqq \ell$) and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight ϖ_1.*
   (b) *$\mathcal{D}(\mathfrak{l})$ is of type $(C_\ell, \{\alpha_\ell\})$ and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight ϖ_1.*

   In these cases, we have
   (i) *If $\mathcal{D}(\mathfrak{l})$ is of type $(A_\ell, \{\alpha_i\})$ ($1 \leqq i \leqq \ell$) and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight ϖ_1, then*

$$H^1(\mathfrak{G})_{r,1} \cong \begin{cases} S_{-2} \otimes S^2(S^*_{-1}) & \text{if } r = 0 \\ S^*_{-1} & \text{if } r = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If $\mathcal{D}(\mathfrak{l})$ is of type $(C_\ell, \{\alpha_\ell\})$ $(\ell \geqq 2)$ and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight ϖ_1, then*

$$H^1(\mathfrak{G})_{r,1} \cong \begin{cases} L^{-2}(0) \otimes S^3(S^*_{-1}) & \text{if } \ r = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 5.1. *Let $\mathcal{G} = \bigoplus_{p \in \mathbb{Z}} \mathcal{G}_p$ be the prolongation of $\mathfrak{g}_-$. Then $\dim \mathcal{G} = \infty$ if and only if $H^1(\mathfrak{G})_{0,s} \neq 0$ for some $s \neq 0$.*

If $\dim \check{\mathfrak{b}} < \infty$ and $\check{\mathfrak{b}}_1 \neq 0$, then $H^1(\mathfrak{G})_{1,1} \neq 0$. Now we classify pseudo-product GLAs $\mathfrak{G}$ of type $(\mathfrak{l}, S)$ such that $H^1(\mathfrak{G})_{1,1} \neq 0$. Let $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ be the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$. The condition $H^1(\mathfrak{G})_{1,1} \neq 0$ implies $\mathfrak{g}_1 \neq \mathfrak{s}_1$. Then, by Theorem 3.2 [Yat88], $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ is a simple GLA such that $\mathfrak{g}_p = \mathfrak{s}_p$ for all $p \leqq 0$. Since the $\mathfrak{g}_0$-module $\mathfrak{s}_0$ is completely reducible, there exists a $\mathfrak{g}_0$-submodule $\mathfrak{s}_1^{(1)}$ of $\mathfrak{s}_1$ such that $\mathfrak{s}_1 = \mathfrak{g}_1 \oplus \mathfrak{s}_1^{(1)}$. Since $\mathfrak{s}_1$ is contragredient to $\mathfrak{s}_{-1}$ as a $\mathfrak{s}_0$-module, the $\mathfrak{g}_0$-module $\mathfrak{s}_1^{(1)}$ is irreducible. Also $\mathfrak{s}_1^{(1)}$ is contragredient to S_{-1} as a $\mathfrak{g}_0$-module. For $q \in \mathbb{Z}$, we put $\mathfrak{c}_q = \{ x \in \mathfrak{s} \mid [J, x] = qx \}$; then $(\mathfrak{c}_q)_{q \in \mathbb{Z}}$ gives a gradation of $\mathfrak{s}$. Note that each $\mathfrak{c}_q$ is $\operatorname{ad} E$-stable. Since $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$ and $\mathfrak{s}_{\geqq 1}$ is generated by $\mathfrak{s}_1$, we have $\mathfrak{c}_{-1} = S$, $\mathfrak{c}_0 = \mathfrak{l}$ and $\mathfrak{s}_1^{(1)} = \mathfrak{c}_1 \cap \mathfrak{s}_1$. Since $\mathfrak{s}$ is simple, we get $\mathfrak{s} = \mathfrak{c}_{-1} \oplus \mathfrak{c}_0 \oplus \mathfrak{c}_1$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{s}$ such that $E \in \mathfrak{h}$; then $J \in \mathfrak{h} \subset \mathfrak{s}_0$. Let Φ be the root system of $(\mathfrak{s}, \mathfrak{h})$. There exists a simple root system $\Sigma = \{\gamma_1, \dots, \gamma_{\ell+n+1}\}$ of $(\mathfrak{s}, \mathfrak{h}, \Phi)$ such that $\gamma_j(E) \geqq 0$ for all j, where $\ell = \operatorname{rank} \hat{\mathfrak{l}}$ and $n = \operatorname{rank} \mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}})$. We assume that $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ is of type $(Y_{\ell+n+1}, \Sigma_1)$, $\Delta \subset \Sigma$ and $\Delta_1 \subset \Sigma_1$. For $\gamma_i \in \Sigma_1$, we denote by $\mathfrak{s}_{-1}^{(i)}$ the $\mathfrak{g}_0$ submodule of $\mathfrak{s}_{-1}$ generated by $\mathfrak{s}_{-\gamma_i}$. $\mathfrak{s}_{-1}^{(i)}$ is an irreducible $\mathfrak{s}_0$-submodule of $\mathfrak{s}_{-1}$ with highest weight $-\gamma_i$ (cf. [VOG90, Ch.2, 3.5]). Since $\mathfrak{l}_{-1}$ and S_{-1} is not isomorphic to each other as a $\mathfrak{g}_0$-module, there exist $\gamma_{i_1}, \gamma_{i_2} \in \Sigma_1$ satisfying (i) $\mathfrak{l}_{-1} = \mathfrak{s}_{-1}^{(i_1)}$ and $S_{-1} = \mathfrak{s}_{-1}^{(i_2)}$ or (ii) $\mathfrak{l}_{-1} = \mathfrak{s}_{-1}^{(i_2)}$ and $S_{-1} = \mathfrak{s}_{-1}^{(i_1)}$. In particular, Σ_1 consists of two elements γ_{i_1} and γ_{i_2}. We may assume the case (i). Thus the GLA $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ is of type $(Y_{\ell+n+1}, \{\gamma_{i_1}, \gamma_{i_2}\})$ and the GLA $\mathfrak{s} = \mathfrak{c}_{-1} \oplus \mathfrak{c}_0 \oplus \mathfrak{c}_1$ is of type $(Y_{\ell+n+1}, \{\gamma_{i_2}\})$. For each $\gamma_i \in \Sigma_1$, we set $\Phi_p^{(i)} = \{\alpha \in \Phi \mid m_i(\alpha) = p \text{ and } m_j(\alpha) = 0 \text{ for } j \in I \setminus \{i\} \}$. Moreover we set $\Phi^{(i)} = \bigcup_{p \in \mathbb{Z}} \Phi_p^{(i)}$, $\Phi_+^{(i)} = \Phi^{(i)} \cap \Phi_+$ and $\mathfrak{s}^{(i)} = \sum_{\alpha \in \Phi^{(i)}} \mathfrak{s}^\alpha + \mathfrak{h}$. Since $\Phi^{(i)} = -\Phi^{(i)}$, we know that $\mathfrak{s}^{(i)}$ is a reductive graded subalgebra of $\mathfrak{s}$ (cf. [Bou75, Ch.8, §3, no.4, Prop.2]), which we write $\mathfrak{s}^{(i)} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p^{(i)}$. Then $\mathfrak{s}^{(i)} = \mathcal{D}(\mathfrak{s}^{(i)}) \oplus \mathfrak{z}(\mathfrak{s}^{(i)})$, $\mathfrak{s}_0^{(i)} = \mathfrak{s}_0$, $\mathfrak{s}_{-1}^{(i)} = \sum_{\alpha \in \Phi_{-1}^{(i)}} \mathfrak{s}^\alpha$ and $\mathfrak{s}_{-1} = \bigoplus_{i \in I} \mathfrak{s}_{-1}^{(i)}$. Since $[\mathfrak{s}_{-1}^{(i)}, \mathfrak{s}_{-1}^{(i)}] = 0$, $\Phi_p^{(i)} = \emptyset$ for $|p| > 2$. Clearly $\Phi^{(i)}$ is a root system of $(\mathfrak{s}^{(i)}, \mathfrak{h})$. We set $\Sigma^{(i)} = \Sigma \cap \Sigma^{(i)}$. Since $\langle \alpha, h \rangle = 0$ for all $\alpha \in \Phi^{(i)}$ and $h \in \mathfrak{z}(\mathfrak{s}^{(i)})$, we see that $\Sigma^{(i)}$ is a system of simple roots of $\Phi^{(i)}$ (cf. [Bou75, Ch.8, §1, no.7, Cor.3 to Prop.20]). Here we extend the definition of a root system and a system of simple roots, etc. to the cases of reductive Lie algebras (cf. Ch.8, §2 in [Bou75]). Thus the derived subalgebra $\mathcal{D}(\mathfrak{s}^{(i)})$ of $\mathfrak{s}^{(i)}$ is a semisimple Lie algebra whose Dynkin diagram is the subdiagram of $Y_{\ell+n+1}$ consisting of the vertices $(\{1, \dots, \ell+n+1\} \setminus I) \cup \{i\}$. In particular, $\hat{\mathfrak{s}}^{(i)}$ is a simple Lie algebra whose Dynkin diagram $Y_{\ell+n+1}^{(i)}$ is the connected component containing $\{i\}$ of the diagram of $\mathcal{D}(\mathfrak{s}^{(i)})$. Let θ (resp.$\theta^{(i)}$) be the highest root of $\mathfrak{s}$ (resp.$\mathfrak{s}^{(i)}$). Then, for $\alpha \in \Phi_+$ and $\beta \in \Phi_+^{(i)}$, we have $m_j(\alpha) \leqq m_j(\theta)$ for all j and $m_k(\beta) \leqq m_k(\theta^{(i)})$ for all k, so $\theta \in \Phi_\mu$ and $\theta^{(i)} \in \Phi_{\mu_i}$, where μ (resp.μ_i) is the depth of $\mathfrak{s}$ (resp.$\mathfrak{s}^{(i)}$). Thus $\mu = \sum_{i \in I} m_i(\theta)$ and $\mu_i = m_i(\theta^{(i)})$. Since $[\mathfrak{l}_{-1}, \mathfrak{l}_{-1}] = [S_{-1}, S_{-1}] = 0$, we see that $\mu_i = 1$ for all i. Also since $[\mathfrak{c}_{-1}, \mathfrak{c}_{-1}] = 0$, $m_{i_2}(\theta) = 1$. Hence $(Y_{\ell+n+1}, \Sigma_1)$ is one of the following types:

$(A_{\ell+n+1}, \{\gamma_i, \gamma_{\ell+1}\})\ (n \geqq 0, 1 \leqq i \leqq \ell), (B_{\ell+1}, \{\gamma_1, \gamma_2\})\ (\ell \geqq 2),$
$(C_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})(\ell \geqq 1, 1 \leqq i \leqq \ell),\ (D_{\ell+1}, \{\gamma_1, \gamma_2\})\ (\ell \geqq 3),$
$(D_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})\ (\ell \geqq 3, 1 \leqq i \leqq \ell),$
$(E_6, \{\gamma_1, \gamma_6\}),\ (E_6, \{\gamma_1, \gamma_2\}),\ (E_6, \{\gamma_1, \gamma_3\}),$
$(E_7, \{\gamma_1, \gamma_7\}),\ (E_7, \{\gamma_6, \gamma_7\}).$

Summarizing the above discussion, we obtain the following answer for our problem (1) cited in §2.

Theorem 5.2. *Let $\mathfrak{G}$ be a pseudo-product GLA of type $(\mathfrak{l}, S)$ satisfying the condition $H^1(\mathfrak{G})_{0,0} = 0$. Let $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ be the prolongation of $\bigoplus_{p\leq 0} \mathfrak{b}_p$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$ except for the following three cases.*

(a) $\dim \check{\mathfrak{b}} < \infty$ *and* $\check{\mathfrak{b}}_1 \neq 0$ ($\check{\mathfrak{b}}$*: simple*)

$\mathcal{D}(\mathfrak{l})$	Δ_1	$\mathfrak{b}_{-1}$	$Y_{\ell+n+1}$	Σ_1
$A_\ell \times A_n$	$\{\alpha_i\}$	$\varpi_\ell + \pi_1$	$A_{\ell+n+1}$	$\{\gamma_i, \gamma_{\ell+1}\}$
A_ℓ	$\{\alpha_i\}$	ϖ_ℓ	$A_{\ell+1}$	$\{\gamma_i, \gamma_{\ell+1}\}$
A_ℓ	$\{\alpha_i\}$	$2\varpi_l$	$C_{\ell+1}$	$\{\gamma_i, \gamma_{\ell+1}\}$
$A_\ell\ (\ell \geqq 3)$	$\{\alpha_i\}$	$\varpi_{\ell-1}$	$D_{\ell+1}$	$\{\gamma_i, \gamma_{\ell+1}\}$
$B_\ell\ (\ell \geqq 2)$	$\{\alpha_1\}$	ϖ_1	$B_{\ell+1}$	$\{\gamma_1, \gamma_2\}$
$D_\ell\ (\ell \geqq 4)$	$\{\alpha_\ell\}$	ϖ_1	$D_{\ell+1}$	$\{\gamma_1, \gamma_{\ell+1}\}$
$D_\ell\ (\ell \geqq 4)$	$\{\alpha_1\}$	ϖ_1	$D_{\ell+1}$	$\{\gamma_1, \gamma_2\}$
D_5	$\{\alpha_5\}$	ϖ_5	E_6	$\{\gamma_1, \gamma_3\}$
D_5	$\{\alpha_4\}$	ϖ_5	E_6	$\{\gamma_1, \gamma_2\}$
D_5	$\{\alpha_1\}$	ϖ_5	E_6	$\{\gamma_1, \gamma_6\}$
E_6	$\{\alpha_1\}$	ϖ_6	E_7	$\{\gamma_1, \gamma_7\}$
E_6	$\{\alpha_6\}$	ϖ_6	E_7	$\{\gamma_6, \gamma_7\}$

In this case $(Y_{\ell+n+1}, \Sigma_1)$ is the prolongation of $\mathfrak{m}$ except for $(A_{\ell+n+1}, \{\gamma_1, \gamma_{\ell+1}\})$ and $(C_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})$. Moreover the latter two are the prolongations of $(\mathfrak{m}, \mathfrak{g}_0)$.

(b) $\dim \check{\mathfrak{b}} = \infty$

$\mathcal{D}(\mathfrak{l})$	Δ_1	$\mathfrak{b}_{-1}$	$\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$
A_ℓ	$\{\alpha_i\}$	ϖ_ℓ	$(A_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})$
C_ℓ	$\{\alpha_\ell\}$	ϖ_1	$\mathfrak{g}$

In $(C_\ell, \{\alpha_\ell\})$-case, $\mu = 2$

$$S_{-2} = V^*, \quad S_{-1} = V, \quad \mathfrak{L}_1 = S^2(V^*),$$

$$\mathfrak{l}_0 = V\otimes V^*\oplus \mathbb{C}, \qquad \mathfrak{l}_1 = S^2(V)$$

(c) $\mathfrak{g}$ *is a pseudo-projective GLA, i.e.,* $\mathcal{D}(\mathfrak{l}) = (A_\ell \times A_n, \{\alpha_1\})$, $\Xi = k\varpi_\ell + \pi_1$, $(k \geqq 2, n \geqq 1)$, *or* $\mathcal{D}(\mathfrak{l}) = (A_\ell, \{\alpha_1\})$, $\chi = k\varpi_\ell$, $(k \geqq 3, n = 0)$

$$S_{-\mu} = W, \quad S_p = W\otimes S^{\mu+p}(V^*)\ (-\mu < p < 0),$$

$$\mathfrak{l}_{-1} = V, \quad \mathfrak{l}_0 = \mathfrak{gl}\,(V)\oplus \mathfrak{gl}\,(W), \quad \mathfrak{l}_1 = V^*,$$

where $\mu = k+1$, $\dim V = \ell$ *and* $\dim W = n+1$.

In this case $\mathfrak{g}$ *is the prolongation of* $(\mathfrak{m}, \mathfrak{g}_0)$.

6. Second cohomology of pseudo-product graded Lie algebras

Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ satisfying the condition $H^1(\mathfrak{G})_{0,0} = 0$. We use the same notation as in §4. We define elements $\{\alpha_i^\vee\}$ and $\{\varpi_i^\vee\}$ of $\hat{\mathfrak{l}}\cap\mathfrak{h}$ by $\langle\alpha_j, \alpha_i^\vee\rangle = a_{ij}$ and $\langle\varpi_i^\vee, \alpha_j\rangle = \delta_{ij}$, where (a_{ij}) is the Cartan matrix of $\hat{\mathfrak{l}}$. Also let $\{r_1, \ldots, r_\ell\}$ be the set of fundamental reflections of $\hat{\mathfrak{l}}$. Also we assume $\chi = \sum_{i=1}^{\ell} m_i\varpi_i$ $(m_i \in \mathbb{Z})$.

6.1. **Computation of** $H^2(\mathfrak{G})_{r,-1}$. By Theorem 4.1,

$$H^2(\mathfrak{G})_{r,-1} \cong H^2(\mathfrak{l}_-, \mathfrak{b}_{-1})_r.$$

Note that $H^2(\mathfrak{G})_{r,-1} = 0$ for all r provided that $(X_\ell, \Delta_1) = (A_1, \{\alpha_1\})$. Since Δ_1 is the form $\{\alpha_i\}$, an element of W_1^2 becomes the form $\{r_ir_k\}$, where $k\neq i$ and $\langle\alpha_k, \alpha_i^\vee\rangle \neq 0$. Then

$$\langle\xi_{r_ir_k}^{-1}(\chi), E\rangle = \langle w_0(\chi)-\chi, \varpi_i^\vee\rangle - \langle w_0(\chi), \alpha_i^\vee - \langle\alpha_k, \alpha_i^\vee\rangle\alpha_k^\vee\rangle - \langle\alpha_k, \alpha_i^\vee\rangle$$

We compute $\langle\xi_{r_ir_k}^{-1}(\chi), E\rangle$ by a case by case analysis. For convenience, we put $\xi_{r_ir_k} = \xi_{r_ir_k}^{-1}(\chi)$.

Case 1: Take $(X_\ell, \Delta_1) = (A_\ell, \{\alpha_1\})$ $(\ell \geqq 2)$. Then $W_1^2 = \{r_1r_2\}$ and

$$\langle\xi_{r_1r_2}, E\rangle = -\sum_{j=1}^{\ell-2} m_j + 1.$$

Case 2: Take $(X_\ell, \Delta_1) = (A_\ell, \{\alpha_i\})$ $(1 < i \leqq \left[\frac{\ell+1}{2}\right])$. Then $W_1^2 = \{r_ir_{i-1}, r_ir_{i+1}\}$ and

$$\begin{aligned}\langle\xi_{r_ir_{i-1}}, E\rangle = &-\sum_{j=1}^{i-1} jm_j - \sum_{j=i}^{\ell-i} im_j - (i-1)m_{\ell-i+1}\\ &-(i-2)m_{\ell-i+2} - \sum_{j=\ell-i+3}^{\ell}(\ell-j+1)m_j + 1,\end{aligned}$$

$$\begin{aligned}\langle\xi_{r_ir_{i+1}}, E\rangle = &-\sum_{j=1}^{i-1} jm_j - \sum_{j=i}^{\ell-j-1} im_j - (i-1)m_{\ell-i}\\ &-(i-1)m_{\ell-i+1} - \sum_{j=\ell-i+2}^{\ell}(\ell-j+1)m_j + 1.\end{aligned}$$

Case 3: Take $(X_\ell, \Delta_1) = (B_\ell, \{\alpha_1\})$ $(\ell \geqq 3)$. Then $W_1^2 = \{r_1r_2\}$ and

$$\langle\xi_{r_1r_2}, E\rangle = -m_1 - m_2 - 2\sum_{j=3}^{\ell-1} m_j - m_\ell + 1.$$

Case 4: Take $(X_\ell, \Delta_1) = (C_\ell, \{\alpha_\ell\})$ $(\ell \geqq 2)$. Then $W_1^2 = \{r_\ell r_{\ell-1}\}$ and

$$\langle \xi_{r_\ell r_{\ell-1}}, E\rangle = -\sum_{j=1}^{\ell-2} jm_j - (\ell-2)m_{\ell-1} - (\ell-1)m_\ell + 1.$$

Case 5: Take $(X_\ell, \Delta_1) = (D_\ell, \{\alpha_1\})$ $(\ell \geqq 4)$. Then $W_1^2 = \{r_1 r_2\}$ and

$$\langle \xi_{r_1 r_2}, E\rangle = -m_1 - m_2 - 2\sum_{j=3}^{\ell-2} m_j - m_{\ell-1} - m_\ell + 1.$$

Case 6: Take $(X_\ell, \Delta_1) = (D_\ell, \{\alpha_{\ell-1}\})$ $(\ell \geqq 5)$. Then $W_1^2 = \{r_{\ell-1} r_{\ell-2}\}$ and

$$\langle \xi_{r_{\ell-1} r_{\ell-2}}, E\rangle = -\sum_{j=1}^{\ell-3} jm_j - (\ell-3)m_{\ell-2} - \frac{1}{2}(\ell-2-\delta_0)m_{\ell-1}$$
$$-\frac{1}{2}(\ell-2+\delta_0)m_\ell + 1,$$

where $\delta_0 = 0$ if ℓ is an even number and $\delta_0 = 1$ if ℓ is an odd number.

Case 7: Take $(X_\ell, \Delta_1) = (E_6, \{\alpha_1\})$. Then $W_1^2 = \{r_1 r_3\}$ and

$$\langle \xi_{r_1 r_3}, E\rangle = -m_1 - 2m_2 - 2m_3 - 4m_3 - 3m_5 - 2m_6 + 1.$$

Case 8: Take $(X_\ell, \Delta_1) = (E_7, \{\alpha_7\})$. Then $W^2(\alpha_7) = \{r_7 r_6\}$ and

$$\langle \xi_{r_7 r_6}, E\rangle = -2m_1 - 3m_2 - 4m_3 - 6m_4 - 5m_5 - 3m_6 - 2m_7 + 1.$$

Hence we obtain the following proposition

Proposition 6.1. (1) $H^2(\mathfrak{G})_{r,-1} = 0$ *for all* $r \geqq 2$.
(2) $H^2(\mathfrak{G})_{1,-1} \neq 0$ *if and only if the sequence* $(X_\ell, \Delta_1, \lambda)$ *is one of the following*

$$(A_\ell, \{\alpha_1\}, j\varpi_{\ell-1} + k\varpi_\ell) \quad (\ell \geqq 2, j, k \geqq 0, j+k \geqq 1),$$
$$(A_\ell, \{\alpha_2\}, k\varpi_\ell) \quad (\ell \geqq 3, k \geqq 1), \quad (C_2, \{\alpha_2\}, k\varpi_1) \quad (k \geqq 1)$$

6.2. **Computation of $H^2(\mathfrak{G})_{r,0}$.** By Theorem 4.1,

$$H^2(\mathfrak{G})_{r,0} \cong H^1(\mathfrak{l}_-, H^1(\mathfrak{b}_-, \mathfrak{g})_0)_r.$$

and

$$\mathrm{ch}_{\mathfrak{l}}(H^1(\mathfrak{b}_-, \mathfrak{g})_0) = \mathrm{ch}_{\mathfrak{l}}(\mathfrak{b}_{-1} \otimes \mathfrak{b}_{-1}^*) - \mathrm{ch}_{\mathfrak{l}}(\mathfrak{b}_0)$$
$$= \mathrm{ch}_{\mathfrak{l}}(\mathfrak{sl}(T))\,\mathrm{ch}_{\mathfrak{l}}(\mathfrak{gl}(U)) - \mathrm{ch}_{\mathfrak{l}}(\hat{\mathfrak{l}}).$$

Note that $H^1(\mathfrak{b}_-, \mathfrak{g})_0 = 0$ if and only if $\mathcal{D}(\mathfrak{l}) = \hat{\mathfrak{l}}$, $X_\ell = A_\ell$ and $\chi = \varpi_1$ or ϖ_ℓ. Hence we may assume that $H^1(\mathfrak{b}_-, \mathfrak{g})_0 \neq 0$. Let Λ be the highest weight of an irreducible component of $H^1(\mathfrak{b}_-, \mathfrak{g})_0$; then $\langle \Lambda, \alpha_j^\vee\rangle > 0$ for some j. Assume that $\Delta_1 = \{\alpha_i\}$. Then $W_1^1 = \{r_i\}$ and

$$\langle \xi_{r_i}^0(\Lambda), E\rangle = \langle w_0(\Lambda), \varpi_i^\vee - \alpha_i^\vee\rangle + 1.$$

By the table of [Bou68], $\langle \varpi_j, \varpi_i^\vee - \alpha_i^\vee\rangle \geq 0$ for all j except for the case when $(X_\ell, \Delta_1) = (A_\ell, \{\alpha_1\})$ or $(A_\ell, \{\alpha_\ell\})$. Also $\langle \varpi_j, \varpi_i^\vee - \alpha_i^\vee\rangle > 0$ for all j if (X_ℓ, Δ_1) is one of $(A_\ell, \{\alpha_i\})$ $(\ell \geqq 4, 1 < i \leqq \left[\frac{\ell+1}{2}\right])$, $(C_\ell, \{\alpha_\ell\})$ $(\ell \geqq 3)$, $(D_\ell, \{\alpha_{\ell-1}\})$ $(\ell \geqq 5)$, $(E_6, \{\alpha_1\})$, $(E_7, \{\alpha_7\})$.

Now we consider the case when $(X_\ell, \Delta_1) = (A_\ell, \{\alpha_1\})$.
Now we consider the case when $(X_\ell, \Delta_1) = (A_\ell, \{\alpha_1\})$.

Since T is an irreducible $\mathfrak{sl}(\ell+1, \mathbb{C})$-module with highest weight $\lambda = \sum_{i=1}^{\ell} m_i \varpi_i$, T^* is an irreducible $\mathfrak{sl}(\ell+1, \mathbb{C})$-module with highest weight $\lambda = \sum_{i=1}^{\ell} m_{\ell-i+1}\varpi_i$. Let $\hat{T}$ be a Young tableau corresponding to T, that is, a collection of boxes, arranged in left-justified rows, with $\sum_{i=1}^{k} m_i$ in the k-the row. Let $\hat{T}^*$ be a Young tableau corresponding to T^*. We add $s_1^{(1)}$ boxes of the top row in $\hat{T}^*$ to the top row in $\hat{T}$; then we add $s_2^{(1)}$ boxes of the top row in $\hat{T}^*$ to the second row in $\hat{T}$, etc. Next we add $s_1^{(2)}$ boxes of the top row in $\hat{T}^*$ to the top row in $\hat{T}$; then we add $s_2^{(2)}$ boxes of the top row in $\hat{T}^*$ to the second row in $\hat{T}$, etc. Repeating this procedure, we get a Young tableau corresponding to an irreducible $\mathfrak{sl}(\ell+1,\mathbb{C})$-submodule of $T \otimes T^*$ with highest weight

$$\Lambda = \sum_{p=1}^{\ell}(m_p + \sum_{i=1}^{p+1}(s_p^{(i)} - s_{p+1}^{(i)}))\varpi_p.$$

Also, in this procedure, we must impose the rule in [Fis81]. Hence it is necessary to satisfy the following condition:

$$\sum_{i=1}^{\ell-p+1} m_i = \sum_{i=1}^{\ell+1} s_i^{(p)}, s_k^{(p)} = 0 \text{ for } k < p,$$

$$\sum_{i=1}^{j} s_p^{(i)} \leqq m_{p-1} + \sum_{i=1}^{j-1} s_{p-1}^{(i)} \ (1 \leqq j \leqq p, 2 \leqq p \leqq \ell+1)$$

$$\sum_{i=1}^{\ell} s_i^{(j)} \leqq \sum_{i=1}^{\ell-1} s_i^{(j-1)} \ (1 \leqq j \leqq \ell).$$

From these inequalities, we have

$$\sum_{i=1}^{\ell} s_\ell^{(i)} \leqq \min\{\sum_{i=1}^{\ell} m_i, 2m_1 + \sum_{i=2}^{\ell-1} m_i\}.$$

Thus

$$\langle \xi_{r_1}^0(\Lambda), E\rangle = -\sum_{j=1}^{\ell-1} m_j + \sum_{i=1}^{\ell} s_l^{(i)} + 1 \leqq \min\{m_1, m_\ell\} + 1.$$

Hence we obtain the following proposition.

Proposition 6.2. (1) $H^2(\mathfrak{G})_{r,0} = 0$ *for* $r \geqq 2$ *except for the case when* $(X_\ell, \Delta_1) = (A_\ell, \{\alpha_1\})$.

(2) $H^2(\mathfrak{G})_{1,0} = 0$ *except when* (X_ℓ, Δ_1) *is one of* $(A_\ell, \{\alpha_1\})$ $(\ell \geqq 2)$, $(A_3, \{\alpha_2\})$, $(B_\ell, \{\alpha_1\})$ $(\ell \geqq 3)$ *or* $(D_\ell, \{\alpha_1\})$ $(\ell \geqq 4)$.

(3) *If* $(X_\ell, \Delta_1) = (A_\ell, \{\alpha_1\})$, *then we see that* $H^2(\mathfrak{G})_{r,0} = 0$ *for* $r \geqq \min\{m_1, m_\ell\} + 2$.

Remark 6.1. (1) The contents of (1) and (2) in Proposition 4.2 were first observed by Y.Seashi (see Theorem 2 [SYY97]), which essentialy constitutes the proof of Theorem A in §3.1.

(2) Let $(X_\ell, \Delta_1) = (A_3, \{\alpha_2\})$. We set

$$\Gamma(\lambda) = \left\{(a,b,c) \in (\mathbb{Z}_{\geq 0})^3 \mid \begin{array}{c} b \leqq m_2, m_1 - m_3 \leqq c \leqq m_1 \\ a \neq 0, a+b+c = m_1 + m_2 \end{array}\right\}.$$

Let M be the sum of irreducible components of the $\mathcal{D}(\mathfrak{l})$-module $\mathfrak{sl}(T)$ with highest weight Λ such that $\langle \xi_{r_2}^0(\Lambda), E\rangle = 1$. By using the Young tableau method, we get $\mathrm{ch}_{\mathfrak{l}}(M) = \sum_a n_a \mathrm{ch}_{\mathfrak{l}}(L(2a\varpi_2))$, where $n_a = \sharp\{ (b,c) \mid (a,b,c) \in \Gamma(\lambda)\}$. Hence $H^2(\mathfrak{G})_{1,0} = 0$ if and only if $\Gamma(\lambda) = \emptyset$.

6.3. **Computation of** $H^2(\mathfrak{G})_{r,s}$ $(s = 1,2)$**.** By Theorem 4.1,

$$H^2(\mathfrak{G})_{r,1} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_1)_r \oplus H^1(\mathfrak{l}_-, H^1(\mathfrak{b}_-, \mathfrak{g})_1)_r,$$

$$H^2(\mathfrak{G})_{r,2} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_2)_r.$$

6.3.1. *The case* $H^1(\mathfrak{G})_{1,1} \neq 0$. In this case, $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a parabolic graded subalgebra of a finite dimensional SGLA $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ such that $\mathfrak{g}_p = \mathfrak{s}_p$ for all $p \leqq 0$.

Theorem 6.1. *Under the above assumption, we have*

(1) $H^2(\mathfrak{G})_{r,s} \cong H^2(\mathfrak{m}, \mathfrak{s})_{r,s}$ $(s = -1, 0)$, *where* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$.
(2) $H^2(\mathfrak{G})_{r,1} \cong H^2(\mathfrak{m}, \mathfrak{s})_{r,1} \oplus H^1(\mathfrak{l}_-, \mathfrak{b}^*_{-1})_r$.
(3) $H^2(\mathfrak{G})_{r,2} \cong H^2(\mathfrak{m}, \mathfrak{s})_{r,2} \oplus H^0(\mathfrak{l}_-, \mathfrak{b}^*_{-1} \otimes \mathfrak{b}^*_{-1})_r$.

Proof. There exits a gradation $(\mathfrak{c}_q)_{q\in\mathbb{Z}}$ of $\mathfrak{s}$ such that $\mathfrak{b}_q = \mathfrak{c}_q$ for all $q \leqq 0$. We remark that

$$H^2(\mathfrak{m}, \mathfrak{s})_{r,s} \cong \bigoplus_{i=0}^{2} H^{2-i}(\mathfrak{l}_-, H^i(\mathfrak{b}_-, \mathfrak{s})_s)_{r,s}.$$

The proof is similar to that of Theorem 4.1.

The statement of (1) is easy to check. We consider the following sequence

$$0 \longrightarrow \mathfrak{c}_1 \xrightarrow{\tilde{\delta}_0} \mathrm{Hom}(\mathfrak{b}_{-1}, \mathfrak{b}_0) \xrightarrow{\delta_1} \mathrm{Hom}(\textstyle\bigwedge^2 \mathfrak{b}_{-1}, \mathfrak{b}_{-1}) \longrightarrow 0,$$

where $\tilde{\delta}_0$ is the coboundary operator $\tilde{\delta}_0 : C^0(\mathfrak{c}_-, \mathfrak{s})_1 \to C^1(\mathfrak{c}_-, \mathfrak{s})_1$. Since $\tilde{\delta}_0$ is injective, we see that $H^1(\mathfrak{b}_-, \mathfrak{g})_1 \cong \mathfrak{c}_1 \oplus H^1(\mathfrak{b}_-, \mathfrak{s})_1$. Also, since $\mathfrak{c}_1$ is isomorphic to $\mathfrak{b}^*_{-1}$ as a $\mathfrak{b}_0$-module,

$$H^2(\mathfrak{G})_{r,1} \cong H^2(\mathfrak{m}, \mathfrak{s})_{r,1} \oplus H^1(\mathfrak{l}_-, \mathfrak{b}^*_{-1})_r,$$

which proves (2). Next consider the following sequence

$$0 \longrightarrow \mathrm{Hom}(\mathfrak{b}_{-1}, \mathfrak{c}_1) \xrightarrow{\tilde{\delta}_1} \mathrm{Hom}(\textstyle\bigwedge^2 \mathfrak{b}_{-1}, \mathfrak{b}_0) \xrightarrow{\delta_2} \mathrm{Hom}(\textstyle\bigwedge^3 \mathfrak{b}_-, \mathfrak{b}_{-1}) \longrightarrow 0,$$

TABLE 1

$\mathcal{D}(\mathfrak{l})$	Δ_1	Ξ	$\mathfrak{b}^*_{-1}$	$\otimes^2\mathfrak{b}^*_{-1}$
$A_\ell \times A_n$	$\{\alpha_i\}$	$\varpi_\ell + \pi_1$	$\varpi_1 + \pi_n$	$2\varpi_1, \varpi_2 \pmod{\pi_n, \pi_{n-1}}$
A_ℓ	$\{\alpha_i\}$	$2\varpi_\ell$	$2\varpi_1$	$4\varpi_1, 2\varpi_2, 2\varpi_1 + \varpi_2$
A_ℓ $(\ell \geqq 3)$	$\{\alpha_i\}$	$\varpi_{\ell-1}$	ϖ_2	$2\varpi_2, \varpi_4, \varpi_1 + \varpi_3$
B_2	$\{\alpha_1\}$	ϖ_1	ϖ_1	$0, 2\varpi_2, 2\varpi_1$
B_ℓ $(\ell \geqq 3)$	$\{\alpha_1\}$	ϖ_1	ϖ_1	$0, \varpi_2, 2\varpi_1$
D_ℓ $(\ell \geqq 4)$	$\{\alpha_\ell\}$	ϖ_1	ϖ_1	$0, \varpi_2, 2\varpi_1$
D_ℓ $(\ell \geqq 4)$	$\{\alpha_1\}$	ϖ_1	ϖ_1	$0, \varpi_2, 2\varpi_1$
D_5	$\{\alpha_5\}$	ϖ_5	ϖ_4	$2\varpi_4, \varpi_3, \varpi_1$
D_5	$\{\alpha_4\}$	ϖ_5	ϖ_4	$2\varpi_4, \varpi_3, \varpi_1$
D_5	$\{\alpha_1\}$	ϖ_5	ϖ_4	$2\varpi_4, \varpi_3, \varpi_1$
E_6	$\{\alpha_1\}$	ϖ_6	ϖ_1	$\varpi_3, 2\varpi_1, \varpi_6$
E_6	$\{\alpha_6\}$	ϖ_6	ϖ_1	$\varpi_3, 2\varpi_1, \varpi_6$

where $\tilde{\delta}_1$ is the coboundary operator $\tilde{\delta}_1 : C^1(\mathfrak{c}_-, \mathfrak{s})_2 \to C^2(\mathfrak{c}_-, \mathfrak{s})_2$. Since $\tilde{\delta}_1$ is injective, we see that $\operatorname{Ker} \delta_2$ is isomorphic to $H^2(\mathfrak{b}_-, \mathfrak{g})_s \oplus \operatorname{Hom}(\mathfrak{b}_{-1}, \mathfrak{c}_1)$. Since $\mathfrak{c}_1$ is isomorphic to $\mathfrak{b}_{-1}^*$, we get

$$H^2(\mathfrak{G})_{r,2} \cong H^2(\mathfrak{m}, \mathfrak{s})_{r,2} \oplus H^0(\mathfrak{l}_-, \mathfrak{b}_{-1}^* \otimes \mathfrak{b}_{-1}^*)_r,$$

which proves (3). □

By the above theorem, we need to decompose the $\mathfrak{b}_0$-module $\otimes^2 \mathfrak{b}_{-1}^*$ into irreducible $\mathfrak{b}_0$-modules. By the table in (1) of Theorem 5.2 and the table of [OV90], we get the Table 1 of the irreducible decomposition of the $\mathfrak{b}_0$-module $\otimes^2 \mathfrak{b}_{-1}^*$.

By Table 1 and Kostant theorem, we get the following theorem.

Theorem 6.2. *Let $\mathfrak{G}$ be a pseudo-product GLA of type $(\mathfrak{l}, S)$ such that: (i) $\hat{\mathfrak{l}}$ is an SGLA of type (X_ℓ, Δ_1); (ii) S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight Ξ; (iii) $H^1(\mathfrak{G})_{1,1} \neq 0$. Then the following are the triplet $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ and the set of r such that $H^2(\mathfrak{G})_r \neq 0$ $(r \geqq 1)$.*

(1) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_1, \{\alpha_1\}, \varpi_1)$, $r = 2, 3, 4$.
(2) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_1 \times A_1, \{\alpha_1\}, \varpi_1 + \pi_1)$, $r = 1, 2, 3$.
(3) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_1 \times A_n, \{\alpha_1\}, \varpi_1 + \pi_1)$ $(n \geqq 2)$, $r = 2, 3$.
(4) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell \times A_n, \{\alpha_1\}, \varpi_\ell + \pi_1)$ $(\ell \geqq 2, n \geqq 0)$, $r = 1, 2$.
(5) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell \times A_n, \{\alpha_2\}, \varpi_\ell + \pi_1)$ $(\ell \geqq 3, n \geqq 0)$, $r = 1, 2$.
(6) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell \times A_n, \{\alpha_i\}, \varpi_\ell + \pi_1)$ $(3 \leqq i \leqq \ell - 1, 0 \leqq n \leqq 1)$, $r = 1, 2$.
(7) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell \times A_n, \{\alpha_i\}, \varpi_\ell + \pi_1)$ $(3 \leqq i \leqq \ell - 1, 2 \leqq n)$, $r = 2$.
(8) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_2, \{\alpha_2\}, \varpi_1)$, $r = 1, 2, 3$.
(9) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_\ell\}, \varpi_1)$ $(\ell \geqq 3)$, $r = 2, 3$.
(10) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell \times A_n, \{\alpha_\ell\}, \varpi_\ell + \pi_1)$ $(\ell \geqq 2, n \geqq 1)$, $r = 1, 2, 3$.
(11) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_1, \{\alpha_1\}, 2\varpi_1)$, $r = 2, 3, 4$.
(12) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_1\}, 2\varpi_\ell)$ $(\ell \geqq 2)$, $r = 1, 2$.
(13) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_2, \{\alpha_2\}, 2\varpi_2)$, $r = 1, 2, 3, 4$.
(14) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_2\}, 2\varpi_\ell)$ $(\ell \geqq 3)$, $r = 1, 2$.
(15) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_\ell\}, 2\varpi_\ell)$ $(\ell \geqq 3)$, $r = 1, 2, 3, 4$.
(16) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_i\}, 2\varpi_\ell)$ $(3 \leqq i \leqq \ell - 1)$, $r = 2$.
(17) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_3, \{\alpha_1\}, \varpi_2)$, $r = 1, 2, 3$.
(18) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_3, \{\alpha_2\}, \varpi_2)$, $r = 1, 2, 3, 4$.
(19) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_1\}, \varpi_{\ell-1})$ $(\ell \geqq 4)$, $r = 1, 2$.
(20) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_i\}, \varpi_{\ell-1})$ $(2 \leqq i \leqq \ell - 3)$, $r = 2$.
(21) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_{\ell-2}\}, \varpi_{\ell-1})$ $(\ell \geqq 4)$, $r = 2, 3$.
(22) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_{\ell-1}\}, \varpi_{\ell-1})$ $(\ell \geqq 4)$, $r = 2, 3, 4$.
(23) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_\ell\}, \varpi_{\ell-1})$ $(\ell \geqq 4)$, $r = 2, 3$.
(24) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (B_\ell, \{\alpha_1\}, \varpi_1)$ $(\ell \geqq 2)$, $r = 1, 2, 3, 4$
(25) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (D_\ell, \{\alpha_\ell\}, \varpi_1)$ $(\ell \geqq 4)$, $r = 1, 2, 3$.
(26) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (D_\ell, \{\alpha_1\}, \varpi_1)$ $(l \geqq 4)$, $r = 1, 2, 3, 4$.
(27) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (D_5, \{\alpha_5\}, \varpi_5)$, $r = 2, 3, 4$.
(28) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (D_5, \{\alpha_4\}, \varpi_5)$, $r = 2, 3$.
(29) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (D_5, \{\alpha_1\}, \varpi_5)$, $r = 2$.
(30) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (E_6, \{\alpha_1\}, \varpi_6)$, $r = 2$.
(31) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (E_6, \{\alpha_6\}, \varpi_6)$, $r = 2, 3, 4$.

In this case, by Theorem5.2, $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ is the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{g}_0)$. Hence we include the following theorem (Proposition 6.2 [Yam99]) for $H^2(\mathfrak{m}, \mathfrak{s})$.

Theorem 6.3. *Let $\mathfrak{G}$ be a pseudo-product GLA of type $(\mathfrak{l}, S)$ such that: (i) $\hat{\mathfrak{l}}$ is an SGLA of type (X_ℓ, Δ_1); (ii) S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight Ξ; (iii) $H^1(\mathfrak{G})_{1,1} \neq 0$. Then the following are the triplet $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ and the set of r such that $H^2(\mathfrak{m}, \mathfrak{s})_r \neq 0$ $(r \geqq 1)$.*

(1) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_1, \{\alpha_1\}, \varpi_1)$, $r = 4$.
(2) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_1 \times A_1, \{\alpha_1\}, \varpi_1 + \pi_1)$, $r = 1, 2, 3$.
(3) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_1 \times A_n, \{\alpha_1\}, \varpi_1 + \pi_1)$ $(n \geqq 2)$, $r = 2, 3$.
(4) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell \times A_n, \{\alpha_1\}, \varpi_\ell + \pi_1)$ $(\ell \geqq 2, n \geqq 0)$, $r = 1, 2$.
(5) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell \times A_n, \{\alpha_2\}, \varpi_\ell + \pi_1)$ $(\ell \geqq 3, n \geqq 0)$, $r = 1$.
(6) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell \times A_n, \{\alpha_i\}, \varpi_\ell + \pi_1)$ $(3 \leqq i \leqq \ell - 1, 0 \leqq n \leqq 1)$, $r = 1$.
(7) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_2, \{\alpha_2\}, \varpi_1)$, $r = 1, 2, 3$.
(8) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_\ell\}, \varpi_1)$ $(\ell \geqq 3)$, $r = 2, 3$.
(9) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell \times A_n, \{\alpha_\ell\}, \varpi_\ell + \pi_1)$ $(\ell \geqq 2, n \geqq 1)$, $r = 1$.
(10) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_1, \{\alpha_1\}, 2\varpi_1)$, $r = 3, 4$.
(11) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_1\}, 2\varpi_\ell)$ $(\ell \geqq 2)$, $r = 1$.
(12) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_2, \{\alpha_2\}, 2\varpi_2)$, $r = 1$.
(13) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_2\}, 2\varpi_\ell)$ $(\ell \geqq 3)$, $r = 1$.
(14) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_\ell\}, 2\varpi_\ell)$ $(\ell \geqq 3)$, $r = 1$.
(15) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_3, \{\alpha_1\}, \varpi_2)$, $r = 1$.
(16) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_3, \{\alpha_2\}, \varpi_2)$, $r = 1, 2$.
(17) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_\ell, \{\alpha_1\}, \varpi_{\ell-1})$ $(\ell \geqq 4)$, $r = 1$.
(18) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (B_\ell, \{\alpha_1\}, \varpi_1)$ $(\ell \geqq 2)$, $r = 1, 2$
(19) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (D_\ell, \{\alpha_\ell\}, \varpi_1)$ $(\ell \geqq 4)$, $r = 1$.
(20) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (D_\ell, \{\alpha_1\}, \varpi_1)$ $(l \geqq 4)$, $r = 1, 2$.

6.3.2. *The case* $\dim \check{\mathfrak{b}} = \infty$.

By the theorem of Kobayashi and Nagano [KN65], $\check{\mathfrak{b}} = \bigoplus_{p \in \mathbb{Z}} \check{\mathfrak{b}}_p$ is isomorphic to one of $W(n; \mathbf{1})$, $CH(\ell; \mathbf{1}; 2)$ $(\ell \geqq 2)$. On the other hand, if $\check{\mathfrak{b}} = \bigoplus_{p \in \mathbb{Z}} \check{\mathfrak{b}}_p$ is isomorphic to $W(n; \mathbf{1})$, then $H^1(\mathfrak{G})_{1,1} \neq 0$ and the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_\ell, \{\alpha_i\}, \varpi_1)$. In this case, our problem is reduced to Theorem 6.2. Hence we may consider only the case where $\check{\mathfrak{b}} = \bigoplus_{p \in \mathbb{Z}} \check{\mathfrak{b}}_p \cong CH(\ell; \mathbf{1}; 2)$ $(\ell \geqq 2)$. Thus we assume that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(C_\ell, \{\alpha_\ell\}, \varpi_1)$. Since $\mathfrak{csp}\,(\mathfrak{b}_{-1})$ is an involutive subalgebra of $\mathfrak{gl}\,(\mathfrak{b}_{-1})$ (cf.[KN66]), we have

$$H^i(\mathfrak{b}_-, \check{\mathfrak{b}})_s = 0 \quad \text{for } i \leqq s. \tag{6.1}$$

By Theorem 4.1,

$$H^2(\mathfrak{G})_{r,1} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_1)_r \oplus H^1(\mathfrak{l}_-, H^1(\mathfrak{b}_-, \mathfrak{g})_1)_r.$$

Clearly, $\mathrm{ch}_{\mathfrak{l}}(H^1(\mathfrak{b}_-, \mathfrak{g})_1) = \mathrm{ch}_{\mathfrak{l}}(\check{\mathfrak{b}}_1)$. By (6.1), we obtain

$$\mathrm{ch}_{\mathfrak{l}}(H^2(\mathfrak{b}_-, \mathfrak{g})_1) = \mathrm{ch}_{\mathfrak{l}}(\bigwedge^2 \mathfrak{b}_{-1}^* \otimes \mathfrak{b}_{-1}) - \mathrm{ch}_{\mathfrak{l}}(\mathfrak{b}_{-1}^* \otimes \mathfrak{b}_0) + \mathrm{ch}_{\mathfrak{l}}(\check{\mathfrak{b}}_1).$$

Using the tables of [OV90] and [Kac68], we have the following decomposition into irreducible $\mathfrak{l}$-modules.

$$\check{\mathfrak{b}}_1 \cong L(3\varpi_1)$$

$$\mathrm{Hom}(\bigwedge^2 \mathfrak{b}_{-1}, \mathfrak{b}_{-1}) \cong 2L(\varpi_1)) \oplus L(\varpi_1 + \varpi_2) \oplus L(\varpi_3)),$$
$$\mathrm{Hom}(\mathfrak{b}_{-1}, \mathfrak{b}_0) \cong L(\varpi_1) \oplus L(\varpi_1 + \varpi_2) \oplus L(3\varpi_1).$$

Therefore $\mathrm{ch}_{\mathfrak{l}}(H^2(\mathfrak{b}_-, \mathfrak{g})_1) = \mathrm{ch}_{\mathfrak{l}}(L(\varpi_3))$. Here $\mathrm{ch}_{\mathfrak{l}}(L(\varpi_3)) = 0$ in case $\ell = 2$. Moreover

$$\langle \xi^1_{r_\ell}(3\varpi_1), E \rangle = 1, \qquad \langle \xi^1_1(\varpi_3), E \rangle = 0.$$

By Theorem 4.1,

$$H^2(\mathfrak{G})_{r,2} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_2)_r.$$

By (6.1), we obtain

$$\mathrm{ch}_{\mathfrak{l}}(H^2(\mathfrak{b}_-, \mathfrak{g})_2) = \mathrm{ch}_{\mathfrak{l}}(\check{\mathfrak{b}}_1 \otimes \mathfrak{b}_{-1}^*) - \mathrm{ch}_{\mathfrak{l}}(\check{\mathfrak{b}}_2).$$

By the tables of [OV90] (also see [AG93]) and [Kac68]),

$$\check{\mathfrak{b}}_1 \otimes \mathfrak{b}_{-1}^* \cong L(2\varpi_1) \oplus L(2\varpi_1 + \varpi_2) \oplus L(4\varpi_1)$$
$$\check{\mathfrak{b}}_2 \cong L(4\varpi_1).$$

Hence

$$\mathrm{ch}_{\mathfrak{l}}(H^2(\mathfrak{b}_-, \mathfrak{g})_2) = \mathrm{ch}_{\mathfrak{l}}(L(2\varpi_1 + \varpi_2)) + \mathrm{ch}_{\mathfrak{l}}(L(2\varpi_1)).$$

Moreover

$$\langle \xi_1^2(2\varpi_1 + \varpi_2), E\rangle = 1, \qquad \langle \xi_1^2(2\varpi_1), E\rangle = 2.$$

Theorem 6.4. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ such that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(C_\ell, \{\alpha_\ell\}, \varpi_1)$ $(\ell \geqq 2)$. Then we obtain the following list of pairs (r, s) such that $H^2(\mathfrak{G})_{r,s} \neq 0$ $(r \geqq 1, s = 1, 2)$.*

(i) $l = 2$, $(r, s) = (1, 1), (1, 2), (2, 2)$.
(ii) $l \geqq 3$, $(r, s) = (0, 1), (1, 1), (1, 2), (2, 2)$.

6.3.3. *The general case.* Let Λ be the highest weight of the $\mathfrak{l}$-module $H^2(\mathfrak{b}_-, \mathfrak{g})_s$. Then

$$\langle \xi_1^s(\Lambda), E\rangle \leqq s\langle \chi, \varpi_i^\vee\rangle + s. \tag{6.2}$$

Hence we have

Proposition 6.3. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$. Then:*

(1) *For $s = 1, 2$,*

$$H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_s)_r = 0 \quad \textit{for } r \geqq s(\mu - 1) + 1.$$

(2) *If $X_\ell = B_\ell$, C_ℓ or E_7, then*

$$H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_s)_r = 0 \quad \textit{for } r \geqq [s(\mu + 1)/2] + 1.$$

Proof. (1) Since $\mu = \langle \chi - w_0(\chi), \varpi_i^\vee\rangle + 1$ and $\langle w_0(\chi), \varpi_i^\vee\rangle < 0$, we have $s(\langle \chi, \varpi_i^\vee\rangle + 1) = s(\mu + \langle w_0(\chi), \varpi_i^\vee\rangle \leqq s(\mu - 1)$, which proves our assertion.

(2) If $X_\ell = B_\ell, C_\ell$ or E_7, then $w_0(\chi) = -\chi$, so $\mu = 2\langle \chi, \varpi_i^\vee\rangle + 1$. By (6.1), we have $\langle \xi_1^s(\Lambda), E\rangle \leqq s(\mu + 1)/2$, which proves our assertion. □

7. The symbol algebras of the Plücker embedding equations

In this section we will calculate $H^2(\mathfrak{G})$ when the PPGLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$ is associated with the Plücker embedding equations for $M = \mathrm{Gr}(i, \ell + 1)$. Namely let $\mathfrak{G}$ be a pseudo-product GLA of type $(\mathfrak{l}, S)$ such that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_\ell, \{\alpha_i\}, \varpi_{\ell-i+1})$ $(1 \leqq i \leqq \ell - i + 1)$, i.e., $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is a reductive GLA such that

(1) $\mathfrak{l}$ is isomorphic to $\mathfrak{gl}(\ell + 1, \mathbb{C})$.
(2) $\hat{\mathfrak{l}}$ is an SGLA of type $(X_\ell, \Delta_1) = (A_\ell, \{\alpha_i\})$ $(1 \leqq i \leqq \ell)$,

and S is an irreducible $\mathfrak{l}$-module with highest weight $\chi = \varpi_{\ell-i+1}$ (i.e., $S = \bigwedge^{\ell-i+1}(\mathbb{C}^{\ell+1})$). We set $\mathfrak{b}_0 = \mathfrak{l}$, $\mathfrak{b}_{-1} = S$; then $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$. Let $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ be the prolongation of $\mathfrak{b}_{-1} \oplus \mathfrak{b}_0$. Also, we use the notation in §4.

7.1. **The case $i = 1$ or 2 (the case $\check{\mathfrak{b}}_1 \neq 0$).** In this subsection, we assume $\check{\mathfrak{b}}_1 \neq 0$. Then by Theorem 5.1 and Table 1, we see that $H^1(\mathfrak{G})_{1,1} \neq 0$ and the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is either $(A_\ell, \{\alpha_1\}, \varpi_\ell)$ or $(A_\ell, \{\alpha_2\}, \varpi_{\ell-1})$ $(l \geqq 3)$. By Theorem 6.2 (1), (4), (18), (20) and (21), we obtain the following theorem.

Theorem 7.1. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ such that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_\ell, \{\alpha_i\}, \varpi_{\ell-i+1})$. If $\check{\mathfrak{b}}_1 \neq 0$, then the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_\ell, \{\alpha_1\}, \varpi_\ell)$ $(\ell \geqq 1)$ or $(A_\ell, \{\alpha_2\}, \varpi_{\ell-1})$ $(\ell \geqq 3)$ and we have*

(1) *We assume that $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_\ell, \{\alpha_1\}, \varpi_\ell)$ $(\ell \geqq 1)$. Then the set of integers such that $H^2(\mathfrak{G})_r \neq 0$ $(r \geqq 1)$ is the following.*
  (i) $\ell = 1$, $r = 2, 3, 4$.
  (ii) $\ell \geqq 2$, $r = 1, 2$.

(2) *We assume that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_\ell, \{\alpha_2\}, \varpi_{\ell-1})$ $(\ell \geqq 3)$. Then the set of integers r such that $H^2(\mathfrak{G})_r \neq 0$ $(r \geqq 1)$ is the following.*
  (i) $\ell = 3$, $r = 1, 2, 3, 4$.
  (ii) $\ell = 4$, $r = 2, 3$.
  (iii) $\ell \geqq 5$, $r = 2$

In these cases, $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is a parabolic graded subalgebra of a simple GLA $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ such that $\mathfrak{g}_p = \mathfrak{s}_p$ for all $p \leqq 0$. In fact $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ is of type $(A_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})$ and is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ when $i = 1$ and $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ is of type $(D_{\ell+1}, \{\gamma_2, \gamma_{\ell+1}\})$ and is the prolongation of $\mathfrak{m}$ when $i = 2$. Hence we obtain the following theorem for $H^2(\mathfrak{m}, \mathfrak{s})$ (Proposition 6.2 [Yam99]).

Theorem 7.2. *Notations being as above.*

(1) *We assume that $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_\ell, \{\alpha_1\}, \varpi_\ell)$ $(\ell \geqq 1)$. Then the set of integers such that $H^2(\mathfrak{m}, \mathfrak{s})_r \neq 0$ $(r \geqq 1)$ is the following.*
  (i) $\ell = 1$, $r = 4$.
  (ii) $\ell \geqq 2$, $r = 1, 2$.

(2) *We assume that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_\ell, \{\alpha_2\}, \varpi_{\ell-1})$ $(\ell \geqq 3)$. Then the set of integers r such that $H^2(\mathfrak{m}, \mathfrak{s})_r \neq 0$ $(r \geqq 1)$ is the following.*
  (i) $\ell = 3$, $r = 1, 2$.

7.2. **The case $i \geqq 3$ (the case $\check{\mathfrak{b}}_1 = 0$).** Let $\hat{\mathfrak{l}}$ be an SGLA of type $(A_\ell, \{\alpha_i\})$ $(l \geqq 5, 3 \leqq i \leqq \ell - i + 1)$.

7.2.1. *The computation of $H^2(\mathfrak{G})_{r,-1}$.*
We have

$$\langle \xi^{-1}_{r_i r_{i-1}}(\varpi_{\ell-i+1}), E\rangle = \langle \xi^{-1}_{r_i r_{i+1}}(\varpi_{\ell-i+1}), E\rangle = -(i-1) + 1 = -i + 2 \leqq 0$$

7.2.2. *The computation of $H^2(\mathfrak{G})_{r,0}$.*
We have already known that

$$H^2(\mathfrak{G})_{r,0} \cong H^1(\mathfrak{l}_-, H^1(\mathfrak{b}_-, \mathfrak{g})_0)_r.$$

Also $H^1(\mathfrak{b}_-, \mathfrak{g})_0$ is isomorphic to $\mathrm{Hom}(\mathfrak{b}_{-1}, \mathfrak{b}_{-1})/\delta^0(\mathfrak{b}_0)$ as a $\mathfrak{b}_0$-module.

$$\mathrm{Hom}(\mathfrak{b}_{-1}, \mathfrak{b}_{-1}) \cong L(\varpi_{\ell-i+1}) \otimes L(\varpi_i) = \bigoplus_{j \geqq 0} L(\varpi_{\ell-i+j+1} + \varpi_{i-j})$$

$$\mathfrak{b}_0 \cong L(\varpi_1 + \varpi_\ell) \oplus L(0).$$

Hence

$$H^1(\mathfrak{b}_-, \mathfrak{g})_0 \cong \bigoplus_{j=0}^{i-2} L(\varpi_{\ell-i+j+1} + \varpi_{i-j}).$$

We get

$$\begin{aligned}
&\langle \xi_1^0(\varpi_{\ell-i+j+1} + \varpi_{i-j}), E\rangle = \langle r_i(w_0(\varpi_{\ell-i+j+1} + \varpi_{i-j}) - \rho) + \rho, \varpi_i^\vee\rangle \\
&= -\langle \varpi_{i-j} + \varpi_{\ell-i+j+1}, \varpi_i^\vee\rangle + \langle \varpi_{i-j} + \varpi_{\ell-i+j+1}, \alpha_i\rangle \\
&= \frac{-1}{\ell+1}((i-j)(\ell-i+1) + (i-j)i) + \delta_{j,0} + \delta_{\ell-i+j+1,i} + 1 \\
&= j - i + \delta_{j,0} + \delta_{\ell-i+j+1,i} + 1 \leqq 0
\end{aligned}$$

7.2.3. *The computation of $H^2(\mathfrak{G})_{r,1}$.*

We have already known that

$$H^2(\mathfrak{G})_{r,1} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_1)_r.$$

Also $H^2(\mathfrak{b}_-, \mathfrak{g})_1$ is isomorphic to $\mathrm{Hom}(\bigwedge^2 \mathfrak{b}_{-1}, \mathfrak{b}_{-1})/\delta^0(\mathrm{Hom}(\mathfrak{b}_{-1}, \mathfrak{b}_0))$ as a $\mathfrak{b}_0$-module. By the table [OV90] and the Young tableau method, we have

$$\begin{aligned}
\bigwedge^2 \mathfrak{b}_{-1}^* &= \bigoplus_{j=0}^{[(i-1)/2]} L(\varpi_{i+2j+1} + \varpi_{i-2j-1}) \\
\mathfrak{b}_0 \otimes \mathfrak{b}_{-1}^* &\cong (L(\varpi_1 + \varpi_\ell) \otimes L(\varpi_i)) \oplus L(\varpi_i) \\
&= L(\varpi_1 + \varpi_i + \varpi_\ell) \oplus L(\varpi_1 + \varpi_{i-1}) \\
&\oplus L(\varpi_{i+1} + \varpi_\ell) \oplus 2L(\varpi_i) \\
\mathfrak{b}_{-1} \otimes \bigwedge^2 \mathfrak{b}_{-1}^* &\cong \bigoplus_{j=0}^{[(i-1)/2]} \bigoplus_{(a,b,c)\in A(j)} L(\varpi_a + \varpi_{i-2j-1+b} + \varpi_{i+2j+1+c}),
\end{aligned}$$

where $A(j) = \{(a, b, c) \in \mathbb{Z}^3 \mid 0 \leqq a \leqq i-2j-1, 0 \leqq b \leqq 4j+2, 0 \leqq c \leqq \ell-i-2j, a+b+c = \ell-i+1\}$.

Case 1: $i < \ell-i-2j-c \leqq \ell-i+2j+2-b \leqq \ell-a+1$.

$$\begin{aligned}
&\langle \xi_1^1(\varpi_a + \varpi_{i-2j-1+b} + \varpi_{i+2j+c+1}), E\rangle \\
&= -\frac{1}{\ell+1}((\ell+i+1)i - i^2) + 1 = -i+1 \leqq 0
\end{aligned}$$

Case 2: $\ell-i-2j-c < i \leqq \ell-i+2j+2-b \leqq \ell-a+1$.

$$\begin{aligned}
&\langle \xi_1^1(\varpi_a + \varpi_{i-2j-1+b} + \varpi_{i+2j+c+1}), E\rangle \\
&= -(\ell-i-2j-c) + 1 \leqq -\ell+i+2j+1+\ell-i-2j = 1
\end{aligned}$$

Case 3: $\ell-i-2j-c \leqq \ell-i+2j+2-b < i \leqq \ell-a+1$.

$$\begin{aligned}
&\langle \xi_1^1(\varpi_a + \varpi_{i-2j-1+b} + \varpi_{i+2j+c+1}), E\rangle \\
&= -\ell-a+2i \leqq -\ell-a+i+\ell-i+1 = -a+1 \leqq 1
\end{aligned}$$

Case 4: $\ell-i-2j-c \leqq \ell-i+2j+2-b \leqq \ell-a+1 < i$.

$$\begin{aligned}
&\langle \xi_1^1(\varpi_a + \varpi_{i-2j-1+b} + \varpi_{i+2j+c+1}), E\rangle \\
&= -2\ell+3i-1 \leqq -2\ell+i+2(\ell-i+1)-1 \leqq -i+1 \leqq 0
\end{aligned}$$

Proposition 7.1. (1) $H^2(\mathfrak{G})_{r,-1} = 0$ *for all* $r \geqq 1$.

(2) $H^2(\mathfrak{G})_{r,0} = 0$ *for all* $r \geqq 1$.

(3) $H^2(\mathfrak{G})_{r,1} = 0$ *for all* $r \geqq 2$. *Furthermore* $H^2(\mathfrak{G})_{1,1} = 0$ *when* $i = 3$ *and* $\ell \geqq 6$.

Proof. We need to show the case $i = 3$ and $\ell \geqq 6$. In this case, by the above results, we have

$$\begin{aligned} H^2(\mathfrak{b}_-, \mathfrak{g})_1 \cong\ & L(\varpi_2 + \varpi_3 + \varpi_{\ell-1}) \oplus L(2\varpi_2 + \varpi_\ell) \\ & \oplus L(\varpi_1 + \varpi_4 + \varpi_{\ell-1}) \oplus L(\varpi_4 + \varpi_\ell) \\ & \oplus L(\varpi_2 + \varpi_4 + \varpi_{\ell-2})\ (\ell \geqq 6) \oplus L(\varpi_5 + \varpi_{\ell-1})\ (\ell \geqq 7) \\ & \oplus L(\varpi_6 + \varpi_{\ell-2})\ (\ell \geqq 8) \end{aligned}$$

Hence we obtain

$$H^2(\mathfrak{G})_{r,1} = 0 \text{ for all } r \geqq 1$$

□

7.3. **The computation of $H^2(\mathfrak{G})_{r,2}$ when $i = 3$ and $\ell \geqq 5$.** In what follows, we consider the case where $\mathfrak{b}_0 = \mathfrak{gl}(V)$ and $\mathfrak{b}_{-1} = \bigwedge^{\ell-2} V$, where $V = \mathbb{C}^{\ell+1}$ $(\ell \geqq 5)$. We see that

$$H^2(\mathfrak{G})_{r,2} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_2)_r, \quad H^2(\mathfrak{b}_-, \mathfrak{g})_2 \cong \operatorname{Ker}\delta_2,$$

where δ^2 is the coboundary operator $\delta^2 : C^2(\mathfrak{b}_-, \mathfrak{g})_2 \to C^3(\mathfrak{b}_-, \mathfrak{g})_2$.

We investigate a relation between the coboundary operators $\delta^0 : \mathfrak{b}_0 \to \operatorname{Hom}(\mathfrak{b}_{-1}, \mathfrak{b}_{-1})$ and $\delta^2 : \operatorname{Hom}(\bigwedge^2 \mathfrak{b}_{-1}, \mathfrak{b}_0) \to \operatorname{Hom}(\bigwedge^3 \mathfrak{b}_{-1}, \mathfrak{b}_{-1})$. Recall that for $\varphi \in \mathfrak{b}_0$, $\omega \in \operatorname{Hom}(\bigwedge^2 \mathfrak{b}_{-1}, \mathfrak{b}_0)$ and $v_1, v_2, v_3 \in \mathfrak{b}_{-1}$,

$$\delta_0(\varphi)(v_1) = [v_1, \varphi] = -\varphi(v_1)$$

and

$$\begin{aligned} \delta_2(\omega)(v_1, v_2, v_3) &= [v_1, \omega(v_2, v_3)] - [v_2, \omega(v_1, v_3)] + [v_3, \omega(v_1, v_2)] \\ &= -\omega(v_2, v_3)(v_1) + \omega(v_1, v_3)(v_2) - \omega(v_1, v_2)(v_3). \end{aligned}$$

In particular, if $\omega = \varphi \otimes w_1^* \wedge w_2^*$, $\delta^2(\varphi \otimes w_1^* \wedge w_2^*) = \delta^0(\varphi) \wedge w_1^* \wedge w_2^*$.

We use the following notation: Let $\{e_1, \dots, e_{\ell+1}\}$ be the canonical basis of V and let $\{e_1^*, \dots, e_{\ell+1}^*\}$ be the dual basis. Let E_{ij} be the element of $\mathfrak{b}_0$ such that $E_{ij}e_k = \delta_{jk}e_i$. Also we put $e_{i_1 \dots i_{\ell-2}} = e_{i_1} \wedge \dots \wedge e_{i_{\ell-2}}$ and $e^*_{i_1 \quad i_{\ell-2}} = e^*_{i_1} \wedge \dots \wedge e^*_{i_{\ell-2}}$.

Let $\mathfrak{h}$ be the canonical Cartan subalgebra of $\mathfrak{b}_0$ (i.e., $\mathfrak{h} = \sum_{i=1}^{\ell+1} \mathbb{C}E_{ii}$). We define a linear form λ_i of $\mathfrak{h}$ by $\lambda_i(E_{jj}) = \delta_{ij}$. Then

$$\varpi_i = \sum_{j=1}^{i} \lambda_j - \frac{i}{\ell+1} \sum_{j=1}^{\ell+1} \lambda_j$$

and $e^*_{i_1 \dots i_{\ell-2}}$ is a weight vector of $\mathfrak{l}$-module S with weight $-\lambda_{i_1} - \dots - \lambda_{i_{\ell-2}}$. Note that if Λ is a weight of the $\mathcal{D}(\mathfrak{l})$-module $\operatorname{Ker}\delta^2$, then $\Lambda + 2\sigma$ is a weight of the $\mathfrak{l}$-module $\operatorname{Ker}\delta^2$. Since $\operatorname{ad}(I_{\ell+1})|S = (\ell-2)1_S$ and $\operatorname{ad}(J)|S = -1_S$, we have $I_{\ell+1} = -(\ell-2)J$, where $I_{\ell+1} = \sum_{i=1}^{\ell+1} E_{ii}$. Hence $\sigma = -\dfrac{\ell-2}{\ell+1} \sum_{i=1}^{\ell+1} \lambda_i$. Let $\Lambda = \sum_{i=1}^{\ell} m_i \varpi_i$ be a highest weight of the $\mathfrak{l}$-module $\operatorname{Ker}\delta^2$; then

$$\Lambda + 2\sigma = \sum_{i=1}^{\ell} \sum_{j=1}^{i} m_i \lambda_j - \frac{1}{\ell+1} \left\{ \sum_{i=1}^{\ell} m_i i + 2(\ell-2) \right\} \sum_{j=1}^{\ell+1} \lambda_j.$$

The highest weight vectors v_Λ of $\operatorname{Ker}\delta^2$ with highest weight $\Lambda + 2\sigma$ have the following forms:

$$v_\Lambda = \sum a_{i,j,i_1, \dots, i_{\ell-2}, j_1, \dots, j_{\ell-2}} E_{ij} \otimes e^*_{i_1, \dots, i_{\ell-2}} \wedge e^*_{j_1 \dots j_{\ell-2}},$$

where the summation is taken over all $i, j, i_1, \dots, i_{\ell-2}, j_1, \dots, j_{\ell-2}$ such that $\Lambda + 2\sigma = \lambda_i - \lambda_j - \sum_{k=1}^{\ell-2}(\lambda_{i_k} + \lambda_{j_k})$. Since $E_{k,k+1}v_\Lambda = 0$ for $1 \leqq k \leqq \ell$, we get the relations between the coefficients $\{a_{i,j,i_1, \dots, i_{\ell-2}, j_1, \dots, j_{\ell-2}}\}$ of v_Λ.

From these methods, we can calculate $\delta^2(v_\Lambda)$.

7.3.1. *The case $\ell = 5$.* .

We first consider the case $\ell = 5$. By the table of [VG90] and the Young tableau method, we have

$$\begin{aligned}\mathrm{Hom}(\bigwedge^2 \mathfrak{b}_{-1}, \mathfrak{b}_0) &\cong L(\varpi_3 + \varpi_4 + \varpi_5) \oplus L(2\varpi_3)\\ &\oplus 3L(\varpi_2 + \varpi_4) \oplus L(\varpi_1 + \varpi_2 + \varpi_3)\\ &\oplus L(\varpi_2 + 2\varpi_5) \oplus 2L(\varpi_1 + \varpi_5)\\ &\oplus L(\varpi_1 + \varpi_2 + \varpi_4 + \varpi_5) \oplus L(2\varpi_1 + \varpi_4)\\ &\oplus L(0)\end{aligned}$$

By a direct inspection, we see that $\mathrm{Ker}\,\delta_2 \cong L(\varpi_1 + \varpi_5)$ and

$$\langle \xi_1(\varpi_1 + \varpi_5), E\rangle = -\langle \varpi_1 + \varpi_5, \varpi_3^\vee\rangle + 2(\langle \varpi_3, \varpi_3^\vee\rangle + 1) = 3.$$

Also a lowest weight vector of $H^2(\mathfrak{b}_-, \mathfrak{g})_2$ is the following (This vector is also a generator of $H^2(\mathfrak{G})_{3,2}$).

$$\begin{aligned}\omega_{15} = {} & E_{65} \otimes e^*_{641} \wedge e^*_{321} - E_{65} \otimes e^*_{631} \wedge e^*_{421} + E_{65} \otimes e^*_{621} \wedge e^*_{431}\\
& - E_{64} \otimes e^*_{651} \wedge e^*_{321} + E_{64} \otimes e^*_{631} \wedge e^*_{521} - E_{64} \otimes e^*_{621} \wedge e^*_{531}\\
& + E_{63} \otimes e^*_{651} \wedge e^*_{421} - E_{63} \otimes e^*_{641} \wedge e^*_{521} + E_{63} \otimes e^*_{621} \wedge e^*_{541}\\
& - E_{62} \otimes e^*_{651} \wedge e^*_{431} + E_{62} \otimes e^*_{641} \wedge e^*_{531} - E_{62} \otimes e^*_{631} \wedge e^*_{541}\\
& + 2E_{61} \otimes e^*_{654} \wedge e^*_{321} - 2E_{61} \otimes e^*_{653} \wedge e^*_{421} + 2E_{61} \otimes e^*_{652} \wedge e^*_{431}\\
& - E_{61} \otimes e^*_{651} \wedge e^*_{432} + 2E_{61} \otimes e^*_{643} \wedge e^*_{521} - 2E_{61} \otimes e^*_{642} \wedge e^*_{531}\\
& + E_{61} \otimes e^*_{641} \wedge e^*_{532} + 2E_{61} \otimes e^*_{632} \wedge e^*_{541} - E_{61} \otimes e^*_{631} \wedge e^*_{542}\\
& + E_{61} \otimes e^*_{621} \wedge e^*_{543} + E_{55} \otimes e^*_{541} \wedge e^*_{321} - E_{55} \otimes e^*_{531} \wedge e^*_{421}\\
& + E_{55} \otimes e^*_{521} \wedge e^*_{431} + 2E_{54} \otimes e^*_{531} \wedge e^*_{521} - 2E_{53} \otimes e^*_{541} \wedge e^*_{521}\\
& + 2E_{52} \otimes e^*_{541} \wedge e^*_{531} + E_{51} \otimes e^*_{543} \wedge e^*_{521} - E_{51} \otimes e^*_{542} \wedge e^*_{531}\\
& - E_{51} \otimes e^*_{541} \wedge e^*_{532} - 2E_{45} \otimes e^*_{431} \wedge e^*_{421} - 2E_{43} \otimes e^*_{541} \wedge e^*_{421}\\
& + 2E_{42} \otimes e^*_{541} \wedge e^*_{431} + E_{44} \otimes e^*_{541} \wedge e^*_{321} + E_{44} \otimes e^*_{531} \wedge e^*_{421}\\
& - E_{44} \otimes e^*_{521} \wedge e^*_{431} + E_{41} \otimes e^*_{543} \wedge e^*_{421} - E_{41} \otimes e^*_{542} \wedge e^*_{431}\\
& - E_{41} \otimes e^*_{541} \wedge e^*_{432} - 2E_{35} \otimes e^*_{431} \wedge e^*_{321} + 2E_{34} \otimes e^*_{531} \wedge e^*_{321}\\
& + 2E_{32} \otimes e^*_{531} \wedge e^*_{431} - E_{33} \otimes e^*_{541} \wedge e^*_{321} - E_{33} \otimes e^*_{531} \wedge e^*_{421}\\
& - E_{33} \otimes e^*_{521} \wedge e^*_{431} + E_{31} \otimes e^*_{543} \wedge e^*_{321} - E_{31} \otimes e^*_{532} \wedge e^*_{431}\\
& - E_{31} \otimes e^*_{531} \wedge e^*_{432} - 2E_{25} \otimes e^*_{421} \wedge e^*_{321} + 2E_{24} \otimes e^*_{521} \wedge e^*_{321}\\
& - 2E_{23} \otimes e^*_{521} \wedge e^*_{421} - E_{22} \otimes e^*_{541} \wedge e^*_{321} + E_{22} \otimes e^*_{531} \wedge e^*_{421}\\
& + E_{22} \otimes e^*_{521} \wedge e^*_{431} + E_{21} \otimes e^*_{542} \wedge e^*_{321} - E_{21} \otimes e^*_{532} \wedge e^*_{421}\\
& - E_{21} \otimes e^*_{521} \wedge e^*_{432}\end{aligned}$$

We summarize the above results in the following theorem.

Theorem 7.3. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ such that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_5, \{\alpha_3\}, \varpi_3)$. Then we have*

(1) $H^2(\mathfrak{G})_{r,-1} \neq 0$ *if and only if* $r = -1$.

(2) $H^2(\mathfrak{G})_{r,0} \neq 0$ *if and only if* $r = -1, 0$.
(3) $H^2(\mathfrak{G})_{r,1} \neq 0$ *if and only if* $r = -1, 0$.
(4) $H^2(\mathfrak{G})_{r,2} \neq 0$ *if and only if* $r = 3$.
Consequently $H^2(\mathfrak{G})_r \neq 0$ *if and only if* $r = -1, 0, 3$.

7.3.2. *The case* $\ell \geqq 6$. .
In this case, by the table of [VG90] and the Young tableau method, we have

$$\bigwedge^2 \mathfrak{b}_{-1}^* \cong L(\varpi_2 + \varpi_4) \oplus L(\varpi_6)$$

$$\begin{aligned}\mathrm{Hom}(\bigwedge^2 \mathfrak{b}_{-1}, \mathfrak{b}_0) \cong & L(\varpi_3 + \varpi_4 + \varpi_\ell) \oplus L(2\varpi_3) \\ & \oplus L(\varpi_1 + \varpi_2 + \varpi_3) \oplus L(\varpi_7 + \varpi_\ell) \\ & \oplus 3L(\varpi_2 + \varpi_4) \oplus L(\varpi_1 + \varpi_6 + \varpi_\ell) \\ & \oplus L(\varpi_2 + \varpi_5 + \varpi_\ell) \oplus 2L(\varpi_1 + \varpi_5) \\ & \oplus L(\varpi_1 + \varpi_2 + \varpi_4 + \varpi_\ell) \oplus L(2\varpi_1 + \varpi_4) \\ & \oplus 2L(\varpi_6)\end{aligned}$$

By a direct inspection, we see that $\mathrm{Ker}\,\delta^2 = 0$. Hence we obtain the following theorem.

Theorem 7.4. *Let* $\mathfrak{G}$ *be a PPGLA of type* $(\mathfrak{l}, S)$ *such that the triple* $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ *is* $(A_\ell, \{\alpha_3\}, \varpi_{\ell-2})$ $(\ell \geqq 6)$. *Then* $H^2(\mathfrak{G})_{r,2} = 0$ *for all* r.

Thus, by Proposition 7.1 and Theorem 7.4, we obtain the vanishing $H^2(\mathfrak{G})_r = 0$ $(r \geqq 1)$ for the second cohomology when $\ell \geqq 6$.

Summarizing the discussion above, we have the following rigidity theorem for the Plücker embedding equations for $M = \mathrm{Gr}(k, \ell + 1)$, when $k = 2$ $(\ell \geqq 4)$ and $k = 3$ $(\ell \geqq 6)$: Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the pseudo-product GLA of type $(\mathfrak{l}, S)$ such that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_\ell, \{\alpha_2\}, \varpi_{\ell-1})$ or $(A_\ell, \{\alpha_3\}, \varpi_{\ell-2})$. In case $k = 2$, the prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ becomes a simple GLA $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ of type $(D_{\ell+1}, \{\gamma_2, \gamma_{\ell+1}\})$ and we have $H^2(\mathfrak{m}, \mathfrak{s})_r = 0$ for $r \geqq 1$ when $\ell \geqq 4$. In case $k = 3$, $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$ and we have $H^2(\mathfrak{G})_r = 0$ for $r \geqq 1$ when $\ell \geqq 6$. Thus in both cases, the pseudo-product structure reduces to that of regular differential system of type $\mathfrak{m}$. Here $\mathfrak{m}$ is a subalgebra of $\mathfrak{C}^k(V, W)$ as in Lemma 2.1. A submanifold R of J^k is called a system of differential equation of type $\mathfrak{m}$, when (R, D) is a regular differential system of type $\mathfrak{m}$, where D is the restriction to R of the canonical differential system C^k on J^k.

Then, utilizing the Tanaka-Morimoto theory of normal Cartan connections [Tan79] [Mor93], we obtain the following rigidity theorem for the Plücker embedding equations.

Theorem 7.5. *Let* $\mathfrak{G}$ *be a pseudo-product GLA of type* $(\mathfrak{l}, S)$ *such that the triple* $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ *is* $(A_\ell, \{\alpha_2\}, \varpi_{\ell-1})$ $(\ell \geqq 4)$ *or* $(A_\ell, \{\alpha_3\}, \varpi_{\ell-2})$ $(\ell \geqq 6)$. *Then every system* R *of differential equation of type* $\mathfrak{m}$ *is locally isomorphic with the model system* R_S *of type* $(\mathfrak{l}, S)$.

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Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan

Department of Mathematics, Hokkaido University of Education, Asahikawa campus, Asahikawa 070-8261, Japan