

**PHASE FIELD MODEL WITH A VARIABLE
CHEMICAL POTENTIAL**

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PHASE FIELD MODEL WITH A VARIABLE CHEMICAL POTENTIAL

YOSHIHIRO TONEGAWA

ABSTRACT. We study some asymptotic behavior of phase interfaces with variable chemical potential under the uniform energy bound. The problem is motivated by the Cahn-Hilliard equation, where one has a control of the total energy and chemical potential. We show that the limit interface is an integral varifold with generalized L^p mean curvature. The convergence of interfaces as $\varepsilon \rightarrow 0$ is in the Hausdorff distance sense.

1. INTRODUCTION

In this paper, we study some asymptotic behavior of phase interfaces in the van der Waals-Cahn-Hilliard theory of phase transitions. The equation we consider is

$$(1.1) \quad -\varepsilon \Delta u + \frac{W'(u)}{\varepsilon} = f,$$

where $u : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$, $n \geq 2$, is the normalized density distribution of a two-phase fluid and W is a double well potential with strict local minima at ± 1 . The function f may be considered as a chemical potential field in the two-phase fluid model ([23]) and $\varepsilon \approx 0$ is a parameter which gives the order of interface thickness.

The associated energy,

$$(1.2) \quad E_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon},$$

and the behavior of minimizers of this energy with a volume constraint, were initially studied in [30, 39] within the framework of Γ -convergence ([16]) and subsequently generalized by many authors ([6, 20, 26, 31, 35, 40]). In this case, the minimizers satisfy the equation (1.1) with some constant $f \equiv c$, and the functional $E_\varepsilon(\cdot)$ Γ -converges to the area functional. The limit interface as $\varepsilon \rightarrow 0$ is area-minimizing with a given volume constraint. It was also proved ([27]) that the constant mean

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curvature of the limit interface is determined by the chemical potential and $\sigma = \int_{-1}^1 \sqrt{W(s)}/2 ds$.

In [24], we studied the behavior of general critical points of the functional (1.2) with a volume constraint and showed that the interface is close in the Hausdorff distance sense to a locally constant mean curvature hypersurface when $\varepsilon \approx 0$. This corresponds to studying (1.1) again with constant $f \equiv c$, with suitable bounds on the energy and c . Note that due to the non-convexity of the functional, there can be solutions which are only locally energy minimizing or even unstable. In [24], we also proved that the interfaces of locally energy minimizing solutions as $\varepsilon \rightarrow 0$ converge to a locally area-minimizing hypersurface.

One motivation to consider (1.1) with variable f , aside from being a natural generalization, comes from the Cahn-Hilliard equation ([12]), which models various phase separation phenomena in a melted alloy with two stable phases. It is

$$(1.3) \quad \begin{cases} u_t = \Delta f & \text{on } \Omega \times (0, \infty), \\ f = -\varepsilon \Delta u + \frac{W'(u)}{\varepsilon}, \\ \frac{\partial u}{\partial n} = \frac{\partial f}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \Omega. \end{cases}$$

The Neumann boundary condition reflects the insulation from the outside. It is a 4-th order gradient flow of $E_\varepsilon(\cdot)$ with volume conservation:

$$\frac{d}{dt} \int_{\Omega} u = 0, \quad \frac{d}{dt} E_\varepsilon(u) = - \int_{\Omega} |\nabla f|^2 \leq 0.$$

The time scale here is different from the usual setting, where t may be replaced by $t\varepsilon$. For more physical background, derivation of the equation and the related equation such as the Allen-Cahn equation see for example, [2, 4, 5, 8, 10, 11, 12, 13, 15, 17, 19, 25] and the references therein. Even though it is far from a complete picture, we mention the most relevant references to the present article. With the time-scale under consideration, it is known that the limit problem is the so-called Mullins-Sekerka problem ([32]). This was formally derived by Pego [34], and was given a rigorous justification using asymptotic expansions and spectral analysis by Alikakos, Bates and Chen [3]. In case of radial symmetry and Dirichlet conditions, Stoth [41] proved a global convergence to the limit problem in dimension three. For general domains and solutions, Chen [14] showed that the solutions converge to a weak solution of the limit problem using the notion of varifolds. Here, we take the similar approach to that of Chen in this paper, using varifold as our working device.

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Given a sequence of solutions $\{u^i\}_{i=1}^\infty$ to

$$-\varepsilon_i \Delta u^i + \frac{W'(u^i)}{\varepsilon_i} = f^i,$$

$\varepsilon_i \rightarrow 0$, with uniform bounds $\sup_i E_{\varepsilon_i}(u^i) \leq E_0$ and $\sup_i \|f^i\|_{W^{1,n}} < \infty$, we show that there exists a subsequence whose interfaces converge in the Hausdorff distance sense to a hypersurface with the mean curvature determined by $f^\infty = \lim f^i$, σ and the interface multiplicities. The mean curvature belongs to L^p for any $p < \infty$ with respect to the $(n-1)$ -dimensional hypersurface measure of the limit interface. We prove a monotonicity-type formula for the properly scaled energy, which extends the case discussed in [24]. Once this is established, the rectifiability and integrality of the limit varifold follow more or less from the argument in [24]. The proof is technically involved, and we need detailed estimates on the positive part of the so-called discrepancy function $\frac{\varepsilon}{2} |\nabla u|^2 - \frac{W}{\varepsilon}$ using the Aleksandrov-Bakelman-Pucci (ABP) estimates. The results are applied to the Cahn-Hilliard equation with $n = 2$, showing that there exists a subsequence for a.e. time with the convergence properties discussed in Theorem 1 when $\varepsilon \rightarrow 0$.

We note that there is an interesting class of unstable solutions with multiple peaks ([9, 42, 7, 43] and the references therein). They describe nuclei of phase separation phenomena which are experimentally observed. Their singular limit cannot be captured by our method, since the chemical potential or energy has to blow-up in our scale.

The organization of the paper is as follows. In Section 2, we state our assumptions, terminologies and main results. In Section 3, we derive our main monotonicity formula for the scaled energy. In Section 4, we prove that the limit interface measure is supported on rectifiable sets and that the measure has a.e. integral densities. In the language of varifold, we show that the limit interface measure is an integral varifold (after dividing by σ) with the generalized mean curvature in L^p for any $p < \infty$. In Section 5, we discuss some additional comments and some implications to the Cahn-Hilliard equation.

I would like to thank X.-F. Chen and Y. Giga for stimulating conversations in the course of the present research project.

2. ASSUMPTIONS AND MAIN RESULTS

2.1. Hypotheses. We consider the problem with the following assumptions.

Assumptions.

- A:** The function $W : \mathbb{R} \rightarrow [0, \infty)$ is C^3 and $W(\pm 1) = 0$. For some $\gamma \in (-1, 1)$, $W' < 0$ on $(\gamma, 1)$ and $W' > 0$ on $(-1, \gamma)$. For some $\alpha \in (0, 1)$ and $\kappa > 0$, $W''(x) \geq \kappa$ for all $|x| \geq \alpha$.
- B:** $U \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary ∂U . Sequences of $C^3(U)$ functions $\{u^i\}_{i=1}^\infty$ and $C^1(U)$ functions $\{f^i\}_{i=1}^\infty$ satisfy

$$(2.1) \quad \varepsilon_i \Delta u^i = \varepsilon_i^{-1} W'(u^i) - f^i$$

on U . Here, $\lim_{i \rightarrow \infty} \varepsilon_i = 0$, and we assume there exist

c_0, λ_0 and E_0 such that $\sup_U |u^i| \leq c_0$, $\|f^i\|_{W^{1,n}(U)} = (\int_U |f^i|^n + |\nabla f^i|^n)^{1/n} \leq \lambda_0$ and

$$\int_U \frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \leq E_0$$

for all i .

Assumption **A** requires that W has a standard W-shape with non-degenerate minima at ± 1 , and local maximum at γ . The assumptions on u^i and f^i are either motivated by the applications to the Cahn-Hilliard equation (see Section 5.2) or regarded simply as a starting point of the problem. The regularity of u is then standard ([21]). The requirement of $W^{1,n}$ norm bound on f^i comes mainly from a technical limit of our approach using the ABP estimates. It is not clear this can be replaced by a weaker norm bound such as $W^{1,p}$ for some $p > n/2$, which we believe is ultimately the weakest bound for f^i to conclude the similar convergence results (see Section 5.1 for more discussion).

We next discuss a few immediate consequences of the assumptions. Let

$$\Phi(s) = \int_0^s \sqrt{W(s)/2} ds,$$

and define new functions

$$w^i = \Phi \circ u^i$$

for each i .

Since $|\nabla w^i| = \sqrt{W(u^i)/2} |\nabla u^i|$, it follows by the Cauchy-Schwartz inequality that

$$\int_U |\nabla w^i| \leq \frac{1}{2} \int_U \frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \leq \frac{E_0}{2}.$$

We also have $\Phi(-c_0) \leq w^i \leq \Phi(c_0)$. By the compactness theorem for bounded variation functions, there exists a subsequence also denoted

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by $\{w^i\}$ and an a.e. pointwise limit w^∞ , such that

$$\lim_{i \rightarrow \infty} \int_U |w^i - w^\infty| = 0 \quad \text{and} \quad \int_U |Dw^\infty| \leq \liminf_{i \rightarrow \infty} \int_U |\nabla w^i|.$$

Here, $|Dw^\infty|$ is the total variation of the vector-valued Radon measure Dw^∞ .

Let Φ^{-1} be the inverse of Φ and define

$$u^\infty = \Phi^{-1}(w^\infty).$$

Then $u^i \rightarrow u^\infty$ a.e., and by the Lebesgue dominated convergence theorem

$$\int_U |u^i - u^\infty| \rightarrow 0.$$

Also by Fatou's Lemma and the energy bound, we have

$$\int_U W(u^\infty) = \int_U \lim_{i \rightarrow \infty} W(u^i) \leq \liminf_{i \rightarrow \infty} \int_U W(u^i) = 0.$$

This shows that $u^\infty = \pm 1$ a.e. on U , and the sets $\{u^\infty = \pm 1\}$ have finite perimeter in U , since

$$\|\partial\{u^\infty = 1\}\|(U) = \frac{1}{2} \int_U |Du^\infty| = \frac{1}{\sigma} \int_U |Dw^\infty| \leq \frac{E_0}{2\sigma},$$

where we define

$$\sigma = \int_{-1}^1 \sqrt{W(s)/2} ds,$$

and where $\|\partial A\|$ denotes the perimeter of A in the measure-theoretic sense (see [18]).

2.2. The associated varifolds. In this section we associate to each solution of (2.1) a varifold in a natural way. We refer to [1, 38] for a comprehensive treatment of varifolds.

Let $G(n, n-1)$ denote the Grassman manifold of unoriented $(n-1)$ -dimensional planes in \mathbb{R}^n . We say V is an $(n-1)$ -dimensional *varifold* in $U \subset \mathbb{R}^n$ if V is a Radon measure on $G_{n-1}(U) = U \times G(n, n-1)$. Let $V_{n-1}(U)$ denote the set of all $(n-1)$ -dimensional varifolds in U . *Convergence in the varifold sense* means convergence in the usual sense of measures. For $V \in V_{n-1}(U)$, we let the *weight* $\|V\|$ be the Radon measure in U defined by

$$\|V\|(A) = V(\{(x, S) \mid x \in A, S \in G(n, n-1)\})$$

for each Borel set $A \subset U$. If M is a $(n-1)$ -rectifiable subset of U we define $v(M) \in V_{n-1}(U)$ by

$$v(M)(E) = \mathcal{H}^{n-1}(\{x \in U \mid (x, \text{Tan}^{n-1}(\mathcal{H}^{n-1}|_M, x)) \in E\})$$

for each Borel set $E \in G_{n-1}(U)$, where $\text{Tan}^{n-1}(\mathcal{H}^{n-1}|_M, x)$ is the approximate tangent plane to M at x and so exists for \mathcal{H}^{n-1} a.e. $x \in M$.

We associate to each function w^i a varifold V^i defined naturally as follows ([14, 24, 25]). By Sard's theorem, $\{w^i = t\} \subset U$ is a C^3 hypersurface for L^1 almost all t . Define $V^i \in V_{n-1}(U)$ by

$$V^i(A) = \int_{-\infty}^{\infty} v(\{w^i = t\})(A) dt$$

for each Borel set $A \subset G_{n-1}(U)$. By the coarea formula ([18]), we have

$$\|V^i\|(A) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{w^i = t\} \cap A) dt = \int_A |\nabla w^i|$$

for each Borel set $A \subset U$. One may interpret the varifold V^i as a weighted averaging of the level sets of w^i , which is concentrated around the transition region. The first variation of V^i is given by (see [33, Section 2.1])

$$(2.2) \quad \delta V^i(g) = \int_U \left(\text{div} g - \sum_{j,k=1}^n \frac{w_{x_j}^i}{|\nabla w^i|} \frac{w_{x_k}^i}{|\nabla w^i|} g_{x_k}^j \right) |\nabla w^i|$$

for each $g \in C_c^1(U; \mathbb{R}^n)$.

2.3. Main results. With the above terminology and assumptions **A** and **B**, we show the following.

Theorem 1. *Let V^i be the varifold associated with u^i (via w^i) as in Sections 2.1 and 2.2. On passing to a subsequence we can assume*

$f^i \rightarrow f^\infty$ weakly in $W^{1,n}$, $u^i \rightarrow u^\infty$ a.e., $V^i \rightarrow V$ in the varifold sense.

Moreover,

(1) *For each $\phi \in C_c(U)$,*

$$\|V\|(\phi) = \lim_{i \rightarrow \infty} \int \phi \frac{\varepsilon_i |\nabla u^i|^2}{2} = \lim_{i \rightarrow \infty} \int \phi \frac{W(u^i)}{\varepsilon_i} = \lim_{i \rightarrow \infty} \int \phi |\nabla w^i|.$$

(2) *$\text{supp} \|\partial\{u^\infty = 1\}\| \subset \text{supp} \|V\|$, and $\{u^i\}$ converges locally uniformly to ± 1 on $U \setminus \text{supp} \|V\|$.*

(3) *For each $\tilde{U} \subset\subset U$ and $0 < b < 1$, $\{|u^i| \leq 1 - b\} \cap \tilde{U}$ converges to $\tilde{U} \cap \text{supp} \|V\|$ in the Hausdorff distance sense.*

(4) *$\sigma^{-1}V$ is an integral varifold. Moreover, the density $\theta(x) = \sigma N(x)$ of V satisfies*

$$N(x) = \begin{cases} \text{odd} & \mathcal{H}^{n-1} \text{ a.e. } x \in M^\infty, \\ \text{even} & \mathcal{H}^{n-1} \text{ a.e. } x \in \text{supp} \|V\| \setminus M^\infty, \end{cases}$$

where M^∞ is the reduced boundary of $\{u^\infty = 1\}$.

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(5) The generalized mean curvature H of V is given by

$$H(x) = \begin{cases} \frac{f^\infty(x)}{\theta(x)} \nu^\infty(x) & \mathcal{H}^{n-1} \text{ a.e. } x \in M^\infty, \\ 0 & \mathcal{H}^{n-1} \text{ a.e. } x \in \text{supp}\|V\| \setminus M^\infty, \end{cases}$$

where ν^∞ is the inward normal for M^∞ .

(6) The generalized mean curvature H belongs to L^p_{loc} for any $p < \infty$ with respect to $\|V\|$.

Comments which follows Theorem 1 in [24] goes with minor changes here as well. Part (1) shows that in the limit the energy is equally divided between the two terms of the energy functional (1.2), called the *equipartition of energy*. In fact, our result shows the following:

For any $\tilde{U} \subset\subset U$ and for the full sequence (not only a subsequence),

$$\lim_{i \rightarrow \infty} \int_{\tilde{U}} \left| \frac{\varepsilon_i}{2} |\nabla u^i|^2 - \frac{W(u^i)}{\varepsilon_i} \right| = 0.$$

This is even without $\frac{\varepsilon_i}{2} |\nabla u^i|^2 dx$ converging to some measure. Part (4) suggests that folding of the interface as $\varepsilon \rightarrow 0$ occurs locally as an integer multiple of 1-D traveling wave solutions ([29]), almost everywhere in the measure-theoretic sense.

Without loss of generality, we may assume that $M^\infty \subset \text{supp}\|\partial\{u^\infty = 1\}\|$. We were not able to prove or disprove that $\mathcal{H}^{n-1}(\text{supp}\|\partial\{u^\infty = 1\}\| \setminus M^\infty) = 0$ in general. This is due to the lack of a uniform lower density estimate for the measure $\|\partial\{u^\infty = 1\}\|$ (as opposed to $\|V\|$) at \mathcal{H}^{n-1} a.e. x in the closure of M^∞ . On the other hand, if $N(x)$ is odd \mathcal{H}^{n-1} a.e. for $x \in \text{supp}\|V\|$, the result (4) shows that $\mathcal{H}^{n-1}(\text{supp}\|V\| \setminus M^\infty) = 0$ and $\text{supp}\|V\| = \text{supp}\|\partial\{u^\infty = 1\}\|$. If $N(x) = 1$ a.e., then this corresponds to “no energy loss” situation, since

$$\int |Dw^\infty| \phi = \sigma \|\partial\{u^\infty = 1\}\|(\phi) = \|V\|(\phi) = \lim_{i \rightarrow \infty} \int |\nabla w^i| \phi$$

for all $\phi \in C_c(U)$. In case f^i are all constants and with no energy loss, the relation between the curvature of the limit interface and the chemical potential is established by Luckhaus and Modica in [27], and we prove here the direct generalization.

It is well-known that the support of a rectifiable varifold with L^p mean curvature, $p > n - 1$, is locally a $C^{1,\alpha}$ graph on a relatively open dense subset \mathcal{O} ([1]) for $\alpha = 1 - \frac{n-1}{p}$. The density function on \mathcal{O} is locally constant and integer valued. If we additionally assume a better bound on f^i , for example, $C^{k,\alpha}$, $0 < \alpha < 1$, $k \geq 0$, standard elliptic estimates [21] show that the support there is $C^{k+2,\alpha}$. On the other hand, we do not know if $\mathcal{H}^{n-1}(\text{supp}\|V\| \setminus \mathcal{O}) = 0$ in general. The

density function θ is defined everywhere on the support of $\|V\|$ and is upper-semicontinuous.

If $N = 1$, \mathcal{H}^{n-1} a.e. on $\text{supp}\|V\|$, then the support is locally a $C^{1,\alpha}$ hypersurface of mean curvature given by f^∞/σ , except for a closed set of \mathcal{H}^{n-1} measure zero.

3. MONOTONICITY FORMULA

In this section, in addition to Assumption **A**, we assume that the functions $u, f : U \rightarrow \mathbb{R}$ satisfy Assumption **B** with u^i, f^i and ε_i there replaced by u, f and ε respectively. We assume that \tilde{U} is open and $\tilde{U} \subset\subset U$.

The main result here is the energy monotonicity formula for

$$E(r, x) = \frac{1}{r^{n-1}} \int_{B_r(x)} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \right),$$

Proposition 3.6. We first derive the identity (3.1) which gives the expression of the radial derivative of $E(r, x)$. The main difficulty to prove the positivity of $\frac{d}{dr}E(r, x)$ comes from the positive part of the discrepancy function $\xi_\varepsilon = \frac{\varepsilon |\nabla u|^2}{2} - \frac{W(u)}{\varepsilon}$. When f is a constant function, we proved in [24] that $\xi_\varepsilon \leq c$ independent of ε . There we utilized a differential inequality satisfied by $\xi_\varepsilon - G(u)$, where G is a suitable modification function, and the maximum principle type argument. In this paper, since we only have control of ∇f in L^n -norm, we use the Aleksandorov-Bakelman-Pucci type estimate instead. Even though we could not prove the uniform supremum bound, we show $\xi_\varepsilon \leq c\varepsilon^{-2/5}$ for all sufficiently small ε (Proposition 3.2). This estimate gives us an energy lower bound around the interface for the length scale of order $\varepsilon^{2/5}$. Then we use this lower bound and a covering lemma to obtain the estimate on the integral of ξ_ε (Lemma 3.5) satisfied for r larger than the length scale of order $\varepsilon^{2/5}$. Combined with this estimate, we obtain the monotonicity formula (3.10).

We denote $W - \varepsilon u f$ by \tilde{W} .

Lemma 3.1. For $B_r(x) \subset\subset U$, we have

$$(3.1) \quad \frac{d}{dr} \left\{ \frac{1}{r^{n-1}} \int_{B_r(x)} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{\tilde{W}}{\varepsilon} \right) \right\} = \frac{1}{r^n} \int_{B_r(x)} \left(\frac{\tilde{W}}{\varepsilon} - \frac{\varepsilon |\nabla u|^2}{2} \right) \\ + \frac{\varepsilon}{r^{n+1}} \int_{\partial B_r(x)} ((y-x) \cdot \nabla u)^2 - \frac{1}{r^n} \int_{B_r(x)} ((y-x) \cdot \nabla f) u.$$

Proof. Multiply both sides of (1.2) by $\nabla u \cdot g$, where $g = (g^1, \dots, g^n) \in C_c^1(U; \mathbb{R}^n)$. Then, after two integrations by parts, we obtain

$$(3.2) \quad \int_U \left(\left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{\widetilde{W}(u)}{\varepsilon} \right) \operatorname{div} g - \varepsilon \sum_{i,j} u_{y_i} u_{y_j} g_{y_i}^j - (\nabla f \cdot g) u \right) = 0.$$

We let $x = 0$ by a suitable translation and let $g^j(y) = y_j \rho(|y|)$, where $\rho(|y|)$ is a smooth approximation to the characteristic function χ_{B_r} . Writing $r = |y|$, (3.2) becomes

$$\int_U \left(\left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{\widetilde{W}(u)}{\varepsilon} \right) (r \rho' + n \rho) - \varepsilon \frac{\rho'}{r} (y \cdot \nabla u)^2 - \varepsilon |\nabla u|^2 \rho - (\nabla f \cdot g) \rho u \right) = 0.$$

By letting $\rho \rightarrow \chi_{B_r}$ and rearranging terms, we obtain

$$\begin{aligned} & - (n-1) \int_{B_r} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{\widetilde{W}(u)}{\varepsilon} \right) + r \int_{\partial B_r} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{\widetilde{W}(u)}{\varepsilon} \right) \\ & = \int_{B_r} \left(\frac{\widetilde{W}(u)}{\varepsilon} - \frac{\varepsilon |\nabla u|^2}{2} \right) + \frac{\varepsilon}{r} \int_{\partial B_r} (y \cdot \nabla u)^2 - \int_{B_r} (y \cdot \nabla f) u. \end{aligned}$$

By dividing the above expression by r^n , we obtain (3.1). \square

For the moment we assume the following technical result which we prove later.

Proposition 3.2. *There exist constants c_1 and ε_1 which depend only on c_0 , λ_0 , W , n and $\operatorname{dist}(\widetilde{U}, \partial U)$ such that, if $\varepsilon < \varepsilon_1$,*

$$(3.3) \quad \sup_{\widetilde{U}} \left(\frac{\varepsilon |\nabla u|^2}{2} - \frac{W}{\varepsilon} \right) \leq c_1 \varepsilon^{-2/5}.$$

Lemma 3.3. *For $B_r(x) \subset \widetilde{U}$, there exists c_2 which depends only on c_0 , λ_0 , n , W and $\operatorname{dist}(\widetilde{U}, \partial U)$ such that, for any $s < r$ and $\varepsilon < \varepsilon_1$,*

$$(3.4) \quad \begin{aligned} E(r, x) - E(s, x) & \geq -c_2 (r + r^{1/2} + r \varepsilon^{-2/5}) \\ & + \int_s^r \frac{d\tau}{\tau^n} \int_{B_\tau(x)} \left(\frac{W}{\varepsilon} - \frac{\varepsilon |\nabla u|^2}{2} \right)^+. \end{aligned}$$

Proof. By (3.3), the Hölder and Sobolev inequalities, we obtain

$$\begin{aligned} \frac{1}{r^n} \int_{B_r(x)} \left(\frac{\varepsilon |\nabla u|^2}{2} - \frac{\widetilde{W}}{\varepsilon} \right)^+ & \leq \frac{1}{r^n} \int_{B_r(x)} \left\{ \left(\frac{\varepsilon |\nabla u|^2}{2} - \frac{W}{\varepsilon} \right)^+ + |uf| \right\} \\ & \leq \omega_n (c_1 \varepsilon^{-2/5} + c_0 \|f\|_{L^{2n} r^{-1/2}}) \leq \omega_n (c_1 \varepsilon^{-2/5} + c_0 c(n) \lambda_0 r^{-1/2}). \end{aligned}$$

After integrating over $[s, r]$, this gives the bound on the negative part of the first term in (3.1). Also, for the third term in (3.1),

$$\begin{aligned} \left| \frac{1}{r^n} \int_{B_r(x)} ((y-x) \cdot \nabla f) u \right| &\leq c_0 \frac{1}{r^{n-1}} \int_{B_r(x)} |\nabla f| \\ &\leq c_0 c(n) \|\nabla f\|_{L^n} \leq c_0 c(n) \lambda_0. \end{aligned}$$

The difference resulting from W and \widetilde{W} may be estimated by

$$\frac{1}{r^{n-1}} \int_{B_r(x)} |u f| \leq c_0 c(n) \|f\|_{W^{1,n}} r^{1/2} \leq c_0 c(n) \lambda_0 r^{1/2}.$$

With an appropriate choice of c_2 , we obtain (3.4). \square

Proposition 3.4. *There exist constants $0 < c_3, c_4 < 1$ which depend only on c_0, λ_0, n, W and $\text{dist}(\widetilde{U}, \partial U)$ such that, if $B_{\varepsilon^{2/5}}(x) \subset \widetilde{U}$, $|u(x)| \leq \alpha$ and $\varepsilon < \varepsilon_1$, then*

$$(3.5) \quad E(r, x) \geq c_4 \quad \text{for } \varepsilon \leq r \leq c_3 \varepsilon^{2/5}.$$

Proof. Translate x to the origin. By scaling $\tilde{x} = x/\varepsilon$ and $\tilde{u}(\tilde{x}) = u(\varepsilon\tilde{x})$, the energy scales as

$$E(\varepsilon, 0) = \frac{1}{\varepsilon^{n-1}} \int_{B_\varepsilon} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W}{\varepsilon} \right) dx = \int_{B_1} \left(\frac{|\nabla \tilde{u}|^2}{2} + W \right) d\tilde{x}.$$

In this scale, there exists a constant c_5 which depends only on c_0, λ_0, n, W and $\text{dist}(\widetilde{U}, \partial U)$ such that $|\nabla \tilde{u}| \leq c_5$ due to the standard elliptic L^p estimate ([21]). Since $|\tilde{u}(0)| \leq \alpha$, there exists some $c_6 = c(c_5, W) > 0$ such that $W(\tilde{u}) \geq c_6$ on B_{c_6} . Hence, $E(\varepsilon, 0) \geq \omega_n c_6^{n+1}$. Using (3.4) and setting $s = \varepsilon$, if we restrict r to be less than $c_3 \varepsilon^{2/5}$ for sufficiently small $c_3 = c(c_2, c_6, n)$ and setting $c_4 = \omega_n c_6^{n+1}/2$, we obtain the desired inequality (3.5). \square

In the following, we use the energy lower bound (3.5) around the interface. The ball $B_r(x)$ we consider has a radius of order $\varepsilon^{2/5}$ at least, which is larger than $\varepsilon^{4/5}$, where we have the energy lower bound. We cover the interface by a collection of balls with the radius $\varepsilon^{4/5}$.

Proposition 3.5. *There exist constants $c_7, c_8, \varepsilon_2 \leq \varepsilon_1$ which depend only on c_0, λ_0, n, W and $\text{dist}(\widetilde{U}, \partial U)$ such that, if $B_r(x) \subset \widetilde{U}$, $r \geq c_3 \varepsilon^{2/5}$ and $\varepsilon < \varepsilon_2$,*

$$(3.6) \quad \frac{1}{r^n} \int_{B_r(x)} \left(\frac{\varepsilon |\nabla u|^2}{2} - \frac{W}{\varepsilon} \right)^+ \leq c_7 E(r, x) + c_8.$$

Proof. We translate x to the origin. We estimate the integral on

$$\begin{aligned}\mathcal{A} &= \{x \in B_r \setminus B_{r-\varepsilon^{4/5}}\}, \\ \mathcal{B} &= \{x \in B_{r-\varepsilon^{4/5}} \mid \text{dist}(\{|u| \leq \alpha\}, x) < \varepsilon^{4/5}\}, \\ \mathcal{C} &= \{x \in B_{r-\varepsilon^{4/5}} \mid \text{dist}(\{|u| \leq \alpha\}, x) \geq \varepsilon^{4/5}\}.\end{aligned}$$

The set \mathcal{A} is the $\varepsilon^{4/5}$ -shell of the ball B_r , \mathcal{B} is the $\varepsilon^{4/5}$ -neighborhood of the interface and \mathcal{C} is the complement of the two.

(1) Estimate on \mathcal{A} . By (3.3),

$$(3.7) \quad \frac{1}{r^n} \int_{\mathcal{A}} \left(\frac{\varepsilon |\nabla u|^2}{2} - \frac{W}{\varepsilon} \right)^+ \leq \frac{n\omega_n \varepsilon^{4/5}}{r} c_1 \varepsilon^{-2/5} \leq \frac{n\omega_n c_1}{c_3}$$

since $r \geq c_3 \varepsilon^{2/5}$.

(2) Estimate on \mathcal{B} . We first estimate $\mathcal{H}^n(\mathcal{B})$. We apply the Vitali covering lemma ([38, Theorem 3.3]) to the family of balls $\{B_{\varepsilon^{4/5}}(x)\}_{x \in \{|u| \leq \alpha\} \cap \mathcal{B}}$ (which covers \mathcal{B}), so that $\{B_{\varepsilon^{4/5}}(x_i)\}_{i=1}^N$ is a pairwise disjoint subset of the family and so that $\mathcal{B} \subset \cup_{i=1}^N B_{5\varepsilon^{4/5}}(x_i)$. Then we have $\mathcal{H}^n(\mathcal{B}) \leq \omega_n (5\varepsilon^{4/5})^n N$. On the other hand, by (3.5) and since $\varepsilon < \varepsilon^{4/5} < c_3 \varepsilon^{2/5}$ for all sufficiently small ε ,

$$c_4 (\varepsilon^{4/5})^{n-1} \leq \int_{B_{\varepsilon^{4/5}}(x_i)} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W}{\varepsilon} \right)$$

holds for each $i = 1, \dots, N$. Since they are pairwise disjoint, by summing over i , we have

$$N c_4 (\varepsilon^{4/5})^{n-1} \leq \int_{B_r} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W}{\varepsilon} \right).$$

These two estimates show that

$$\mathcal{H}^n(\mathcal{B}) \leq \frac{\omega_n 5^n \varepsilon^{4/5}}{c_4} \int_{B_r} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W}{\varepsilon} \right) = \frac{\omega_n 5^n \varepsilon^{4/5} r^{n-1} E(r, 0)}{c_4}.$$

Finally, with (3.3),

$$(3.8) \quad \frac{1}{r^n} \int_{\mathcal{B}} \left(\frac{\varepsilon |\nabla u|^2}{2} - \frac{W}{\varepsilon} \right)^+ \leq \frac{c_1 \varepsilon^{-2/5} \mathcal{H}^n(\mathcal{B})}{r^n} \leq \frac{\omega_n 5^n c_1 E(r, 0)}{c_3 c_4}$$

since $r \geq c_3 \varepsilon^{2/5}$.

(3) Estimate on \mathcal{C} . Define a Lipschitz function ϕ as follows:

$$\phi(x) = \min \{1, \varepsilon^{-4/5} \text{dist}(\{|x| \geq r\} \cup \{|u| \leq \alpha\}, x)\}.$$

ϕ is 0 on the set $\{|u| \leq \alpha\} \cup \{|x| \geq r\}$, 1 on \mathcal{C} , and $|\nabla \phi| \leq \varepsilon^{-4/5}$.

Using this ϕ , we estimate $\frac{\varepsilon |\nabla u|^2}{2}$ (which is larger than $\left(\frac{\varepsilon |\nabla u|^2}{2} - \frac{W}{\varepsilon}\right)^+$)

on \mathcal{C} . Differentiate the equation (1.2) with respect to the k -th variable, multiply it by $u_{y_k} \phi^2$ and sum over k . Then,

$$\int \sum_{k=1}^n \varepsilon u_{y_k} \Delta u_{y_k} \phi^2 = \int \frac{W''}{\varepsilon} |\nabla u|^2 \phi^2 - \nabla f \cdot \nabla u \phi^2.$$

We integrate by parts, and since $W'' \geq \kappa$ on $\{|u| \geq \alpha\}$,

$$\int \frac{\kappa |\nabla u|^2 \phi^2}{\varepsilon} + \varepsilon |\nabla^2 u|^2 \phi^2 \leq \int |\nabla f| |\nabla u| \phi^2 + 2\varepsilon \sum_{k,l=1}^n |\phi u_{y_k} \phi_{y_l} u_{y_k y_l}|.$$

By the Cauchy-Schwartz inequality, we then obtain

$$(LHS) \leq \int \frac{\varepsilon}{2\kappa} |\nabla f|^2 \phi^2 + \frac{\kappa}{2\varepsilon} |\nabla u|^2 \phi^2 + \frac{\varepsilon}{2} \phi^2 |\nabla^2 u|^2 + 2\varepsilon |\nabla \phi|^2 |\nabla u|^2.$$

Relegating two terms to the (LHS), we have

$$\frac{\kappa}{2\varepsilon} \int |\nabla u|^2 \phi^2 \leq \int \frac{\varepsilon}{2\kappa} |\nabla f|^2 \phi^2 + 2\varepsilon |\nabla \phi|^2 |\nabla u|^2.$$

Since $|\nabla \phi| \leq \varepsilon^{-4/5}$,

$$\int_{\mathcal{C}} \varepsilon |\nabla u|^2 \leq \int_{B_r} \frac{\varepsilon^3 |\nabla f|^2}{\kappa^2} + \frac{4\varepsilon^{7/5} |\nabla u|^2}{\kappa}.$$

By the Hölder inequality,

$$(3.9) \quad \frac{1}{r^n} \int_{\mathcal{C}} \frac{\varepsilon |\nabla u|^2}{2} \leq \frac{\varepsilon^3 (\omega_n r^n)^{\frac{n-2}{n}} \|\nabla f\|_{L^2}^2}{2\kappa^2 r^n} + \frac{2\varepsilon^{2/5} E(r, 0)}{\kappa r} \\ \leq \frac{\lambda_0^2 \omega_n^{\frac{n-2}{n}} \varepsilon^{11/5}}{2\kappa^2 c_3^2} + \frac{2E(r, 0)}{\kappa c_3}$$

by $r \geq c_3 \varepsilon^{2/5}$. With appropriate choices of the constants, we obtain (3.6) by (3.7-9). \square

Proposition 3.6. *There exist constants c_9, r_0 depending only on c_0, λ_0, n, W and $\text{dist}(\tilde{U}, \partial U)$, with the following property: For any $B_r(x) \subset \tilde{U}$, $\varepsilon < \varepsilon_2$ and $c_3 \varepsilon^{2/5} \leq s < r \leq r_0$,*

(3.10)

$$e^{c_7 r} E(r, x) - e^{c_7 s} E(s, x) \geq \int_s^r \frac{e^{c_7 \tau}}{\tau^n} \int_{B_\tau} \left(\frac{W}{\varepsilon} - \frac{\varepsilon |\nabla u|^2}{2} \right)^+ d\tau - c_9 r^{1/2}.$$

Proof. By (3.1) and (3.6), we obtain

$$\frac{d}{dr} E(r, x) + \frac{d}{dr} \left(\frac{1}{r^n} \int_{B_r(x)} u f \right) \geq \frac{1}{r^n} \int_{B_r(x)} \left(\frac{W}{\varepsilon} - \frac{\varepsilon |\nabla u|^2}{2} \right)^+$$

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$$-c_7 E(r, x) - c_8 - \frac{1}{r^n} \int_{B_r(x)} (|uf| + r|\nabla f||u|).$$

The last term is bounded from below by $-c_{10}(n, c_0, \lambda_0)r^{-1/2}$. By multiplying $e^{c_7 r}$ to the both sides and integrating over $[s, r]$,

$$\begin{aligned} e^{c_7 r} E(r, x) - e^{c_7 s} E(s, x) &\geq \int_s^r e^{c_7 \tau} \frac{d}{d\tau} \left(\frac{1}{\tau^{n-1}} \int_{B_\tau(x)} uf \right) d\tau \\ &+ \int_s^r \frac{e^{c_7 \tau}}{\tau^n} \int_{B_\tau} \left(\frac{W}{\varepsilon} - \frac{\varepsilon|\nabla u|^2}{2} \right)^+ d\tau - 2c_{10}r^{1/2}e^{c_7 r} - c_8 c_7^{-1}(e^{c_7 r} - e^{c_7 s}). \end{aligned}$$

The first term in the (RHS) is estimated from below by $-c(r^{1/2} + r^{3/2})$, thus with a suitable restriction on r , we obtain (3.10). \square

For the rest of the section, we show Proposition 3.2. First, we need the following lemma showing that $|u|$ stays within $1 + \varepsilon^\eta$ for $\eta < 1$. In case f is bounded in L^∞ -norm, we can use the pointwise maximum principle as in [24, Proposition 3.2]. Here we need to use integral estimates.

Proposition 3.7. *There exist constants ε_3 which depend only on $\lambda_0, c_0, n, 0 < \eta < 1, \text{dist}(\tilde{U}, \partial U)$ and W such that*

$$(3.11) \quad \sup_{\tilde{U}} |u| \leq 1 + \varepsilon^\eta$$

whenever $\varepsilon \leq \varepsilon_3$.

Proof. Suppose $B_1 \subset \tilde{U}$. For any $p \geq 1$, multiply both sides of the equation (1.2) by $[(u-1)^+]^p \phi^2$, where $\phi \in C_c^\infty(B_1)$, $\phi \geq 0$, and integrate by parts. Then we obtain

$$\begin{aligned} -\varepsilon \int p[(u-1)^+]^{p-1} |\nabla u|^2 \phi^2 + [(u-1)^+]^p 2\phi \nabla \phi \cdot \nabla u \\ = \int \frac{W'}{\varepsilon} [(u-1)^+]^p \phi^2 - \int f [(u-1)^+]^p \phi^2. \end{aligned}$$

For $u \geq 1$, $W'(u) \geq \kappa(u-1)$ by the Assumption **B**, hence

$$\begin{aligned} &\frac{\kappa}{\varepsilon} \int [(u-1)^+]^{p+1} \phi^2 + \int \varepsilon p [(u-1)^+]^{p-1} |\nabla u|^2 \phi^2 \\ &\leq 2\varepsilon \int [(u-1)^+]^p \phi |\nabla \phi| |\nabla u| + \int |f| [(u-1)^+]^p \phi^2 \\ &\leq \frac{p\varepsilon}{2} \int [(u-1)^+]^{p-1} |\nabla u|^2 \phi^2 + \frac{2\varepsilon}{p} \int [(u-1)^+]^{p+1} |\nabla \phi|^2 \\ &\quad + \frac{\kappa}{2\varepsilon} \int [(u-1)^+]^{p+1} \phi^2 + \frac{\varepsilon^p c(p)}{\kappa^p} \int |f|^{p+1} \phi^2, \end{aligned}$$

which shows

$$\frac{\kappa}{2\varepsilon} \int [(u-1)^+]^{p+1} \phi^2 \leq \frac{2\varepsilon}{p} \int [(u-1)^+]^{p+1} |\nabla \phi|^2 + \frac{\varepsilon^p c(p)}{\kappa^p} \int |f|^{p+1} \phi^2.$$

Since $\|f\|_{L^{p+1}} \leq c(n, p) \|f\|_{W^{1, n}} \leq c(n, p) \lambda_0$ by the Sobolev inequality for any $p < \infty$, and since $|u|$ is bounded by c_0 , we obtain

$$\int_{B_{1-s}} [(u-1)^+]^{p+1} \leq c_{12}(s, p, c_0, \lambda_0, \kappa) \varepsilon^{p+1}$$

by iterating the above estimate. To derive a contradiction, assume that $u(x_0) - 1 \geq \varepsilon^\eta$ for some $x_0 \in B_{1-s}$. By the gradient estimate $|\nabla u| \leq c_5 \varepsilon^{-1}$ (as in the proof of (3.5)), thus for $y \in B_{\varepsilon^{1+\eta}/2c_5}(x_0)$,

$$u(y) - 1 \geq u(x_0) - 1 - \sup |\nabla u| \cdot \varepsilon^{1+\eta}/2c_5 \geq \varepsilon^\eta/2.$$

Then

$$c_{12} \varepsilon^{p+1} \geq \int_{B_{\varepsilon^{1+\eta}/2c_5}(x_0)} [(u-1)^+]^{p+1} \geq \frac{\omega_n \varepsilon^{\eta(p+1)}}{2^{p+1}} (\varepsilon^{1+\eta}/2c_5)^n.$$

The right-hand side is of order $\varepsilon^{p\eta + \eta(n+1) + n}$. Since $\eta < 1$, for sufficiently large p ($p > \frac{\eta(n+1) + n - 1}{1 - \eta}$) and small ε , the inequality cannot hold. This would be a contradiction. Hence we obtain (3.11). $u \geq -1 - \varepsilon^\eta$ is proved similarly. \square

It is convenient to work in the ε -scale, namely, we consider $B_{3d} \subset \tilde{U}$, and consider the rescaled problem

$$-\Delta u + W'(u) = \varepsilon f \quad \text{on } B_{3d/\varepsilon}.$$

Without loss of generality, we choose a suitable unit so that $d = 1$. Also we denote the rescaled discrepancy function by

$$\xi = \frac{|\nabla u|^2}{2} - W(u).$$

With this scale, we need to prove

$$(3.12) \quad \sup_{B_{\varepsilon^{-1}}} \xi \leq c_1 \varepsilon^{3/5}.$$

In the following we use

$$\xi_G(x) = \frac{|\nabla u|^2}{2} - W(u) - G(u),$$

where $G : \mathbb{R} \rightarrow \mathbb{R}$ will be fixed shortly. We first obtain a differential inequality for ξ_G , cf. [14, 29, 24]. For the reader's convenience, we supply the proof.

Lemma 3.8. On $|\nabla u| > 0$,

$$(3.13) \quad \Delta \xi_G - \frac{2(W' + G')\nabla u}{|\nabla u|^2} \cdot \nabla \xi_G + 2G''\xi_G \geq (G')^2 \\ + G'W' - 2G''(W + G) + \varepsilon f(W' + G') - \varepsilon \nabla f \cdot \nabla u.$$

Proof. Compute

$$\Delta \xi_G = |\nabla^2 u|^2 + \sum u_{x_i}(\Delta u)_{x_i} - W''|\nabla u|^2 - W'\Delta u - G''|\nabla u|^2 - G'\Delta u \\ = |\nabla^2 u|^2 - (W' + G')\Delta u - G''|\nabla u|^2 - \varepsilon \nabla u \cdot \nabla f \\ \text{(by } \Delta u = W' - \varepsilon f) \\ = |\nabla^2 u|^2 - (W' + G')(W' - \varepsilon f) - 2G''(G + W + \xi_G) - \varepsilon \nabla u \cdot \nabla f,$$

where the last line is by substituting $|\nabla u|^2 = 2(G + W + \xi_G)$. On the other hand,

$$|\nabla u|^2 |\nabla^2 u|^2 \geq \sum_j \left(\sum_i u_{x_i} u_{x_i x_j} \right)^2 \\ = \sum_j \left((\xi_G)_{x_j} + (W' + G')u_{x_j} \right)^2 \\ \geq 2(W' + G')\nabla u \cdot \nabla \xi_G + (W' + G')^2 |\nabla u|^2.$$

We may then conclude (3.13). \square

Lemma 3.9. Suppose $0 < \eta, \beta < 1$, $0 \leq \iota \leq 1$ satisfy $2\beta + \iota \geq 2\eta$, $0 < s < 1$, $0 < c_{11} < 1$. Then, there exist $0 < \varepsilon_4 < \varepsilon_3$, $0 < c_{12}, c_{13} < \infty$ depending only on $\eta, \beta, \iota, s, c_0, \lambda_0, c_{11}, \text{dist}(\tilde{U}, \partial U)$, n and W with the following properties:

Suppose

(A) $u \in C^3(B_{\varepsilon^{-\beta}})$, $f \in C^1(B_{\varepsilon^{-\beta}})$ and $\varepsilon \leq \varepsilon_4$ satisfy

$$-\Delta u + W'(u) = \varepsilon f$$

on $B_{\varepsilon^{-\beta}}$ and

(B) $\sup_{B_{\varepsilon^{-\beta}}} |u| \leq 1 + \varepsilon^\eta$, $\sup_{B_{\varepsilon^{-\beta}}} \xi \leq c_{11}\varepsilon^\iota$.

Then,

(3.14)

$$\sup_{B_{(1-s)\varepsilon^{-\beta}}} \xi \leq c_{12} \left\{ \varepsilon^{1-\beta} (\|\nabla f\|_{L^n(B_{\varepsilon^{-\beta}})} + \|f\|_{L^n(B_{\varepsilon^{-\beta}} \cap \{|f| \geq c_{13}\varepsilon^{\eta-1})}) \right\} + \varepsilon^\eta.$$

Proof. We choose $\tilde{\phi} \in C^\infty(B_1)$ such that

$$\tilde{\phi}(x) = \begin{cases} 1 & \text{on } B_1 \setminus B_{1-\frac{s}{2}}, \\ 0 & \text{on } B_{1-s}, \end{cases}$$

$0 \leq \tilde{\phi} \leq 1$, $|\nabla\tilde{\phi}|$, $|\nabla^2\tilde{\phi}|$, $|\nabla\tilde{\phi}|/\sqrt{\tilde{\phi}} \leq c_s$, where we denote generic positive constants depending only on s and n as c_s . For simplicity, we use the same notation c_s for such constants. Then define

$$\phi(x) = c_{11}\varepsilon^t\tilde{\phi}(\varepsilon^\beta x)$$

for $x \in B_{\varepsilon^{-\beta}}$. With this definition and by the properties of $\tilde{\phi}$, we have

(3.15)

$$|\nabla\phi| \leq c_{11}c_s\varepsilon^{\beta+t}, \quad |\nabla^2\phi| \leq c_{11}c_s\varepsilon^{2\beta+t}, \quad |\nabla\phi|/\sqrt{\phi} \leq c_{11}^{1/2}c_s\varepsilon^{\beta+\frac{t}{2}}.$$

Let $G(u) = c_{14}\varepsilon^\eta(1 - (u - \gamma)^2/8)$, where $c_{14} \geq 1$ will be determined later, and define

$$\tilde{\xi} = \xi - G - \phi \quad (= \xi_G - \phi).$$

By restricting ε depending only on γ and by Assumption (B), we have $G > 0$. We later use $G'W' \geq 0$ on $u \in [-1, 1]$ as well as $G'' < 0$ for any u . On $B_{\varepsilon^{-\beta}} \setminus B_{(1-\frac{s}{2})\varepsilon^{-\beta}}$, $\phi = c_{11}\varepsilon^t$, hence by (B), $\tilde{\xi} \leq 0$ there. Also, $\tilde{\xi} = \xi - G$ on $B_{(1-s)\varepsilon^{-\beta}}$ since $\phi = 0$ there. In particular, if $\tilde{\xi} \leq 0$ on $B_{\varepsilon^{-\beta}}$, then $\xi \leq G \leq c_{14}\varepsilon^\eta$ on $B_{(1-s)\varepsilon^{-\beta}}$. In the following, we denote

$$M = \sup_{B_{\varepsilon^{-\beta}}} \tilde{\xi}^+$$

and assume that $M > 0$ (or else, we are done). We next apply the Aleksandorov-Bakelman-Pucci maximum principle to $\tilde{\xi}^+$ on $B_{\varepsilon^{-\beta}}$. Note that $\tilde{\xi}^+ = 0$ on $\partial B_{\varepsilon^{-\beta}}$. Let Γ^+ be the upper contact set of the graph of $\tilde{\xi}^+$. Then, with the diameter of the domain being $2\varepsilon^{-\beta}$, we have (see [21, Lemma 9.3])

$$\frac{M}{2\varepsilon^{-\beta}} \leq \frac{1}{n\omega_n^{1/n}} \left(\int_{\Gamma^+} (-\Delta\tilde{\xi})^n \right)^{1/n}.$$

Note that $\tilde{\xi}^+ = \tilde{\xi}$ and $\Delta\tilde{\xi} \leq 0$ on Γ^+ , and the slope of the support hyperplane is equal to the gradient of $\tilde{\xi}$. Since such hyperplanes have slope less than $M\varepsilon^\beta/s$, we have

$$(3.16) \quad |\nabla\tilde{\xi}| \leq M\varepsilon^\beta/s \quad \text{on } \Gamma^+.$$

Also, since $\tilde{\xi}^+ = 0$ on $B_{\varepsilon^{-\beta}} \setminus B_{(1-\frac{s}{2})\varepsilon^{-\beta}}$, the height of the contact points of the graph of $\tilde{\xi}^+$ and the supporting hyperplanes have a lower bound:

$$(3.17) \quad \tilde{\xi} = \tilde{\xi}^+ \geq \frac{sM}{2} \quad \text{on } \Gamma^+.$$

In the following, we carefully estimate $-\Delta\tilde{\xi}$ from above. In doing so, we need to consider three cases, and define three subsets of $\Gamma^+ \cap B_{\varepsilon^{-\beta}}$

by

$$\begin{aligned}\mathcal{A} &= \{x \mid |u(x)| \geq 1\}, \\ \mathcal{B} &= \{x \mid (\gamma + 1)/2 \leq |u(x)| \leq 1\}, \\ \mathcal{C} &= \{x \mid |u(x)| \leq (\gamma + 1)/2\}.\end{aligned}$$

By (3.13), we have

$$\begin{aligned}-\Delta \tilde{\xi} &= -\Delta \xi_G + \Delta \phi \\ &\leq \left\{ -\frac{2(W' + G')\nabla u}{|\nabla u|^2} \cdot \nabla \xi_G \right\} + \{2G''\xi_G - (G')^2\} + \{-G'W' + 2G''(W + G)\} \\ &\quad + \{-\varepsilon f(W' + G') + \Delta \phi\} + \{\varepsilon \nabla u \cdot \nabla f\} \equiv \{I\} + \{II\} + \{III\} + \{IV\} + \{V\}.\end{aligned}$$

(I) Since $\xi_G = \tilde{\xi} + \phi$ and using $|\nabla u| = \sqrt{2}\sqrt{\tilde{\xi} + W + G + \phi} \geq \left(\sqrt{\tilde{\xi}} + \sqrt{\phi}\right)$,

$$\begin{aligned}I &\leq \frac{2}{|\nabla u|} (|W'| + c_{14}\varepsilon^\eta) |\nabla \tilde{\xi} + \nabla \phi| \\ &\leq 2(|W'| + c_{14}\varepsilon^\eta) \left(\frac{|\nabla \tilde{\xi}|}{\sqrt{\tilde{\xi}}} + \frac{|\nabla \phi|}{\sqrt{\phi}} \right) \\ &\leq 2(|W'| + c_{14}\varepsilon^\eta) \left(\sqrt{\frac{2M}{s^3}} \varepsilon^\beta + c_{11}^{1/2} c_s \varepsilon^{\beta + \frac{1}{2}} \right) \leq (|W'| + c_{14}\varepsilon^\eta) c_s c_{11}^{1/2} \varepsilon^{\beta + \frac{1}{2}}.\end{aligned}$$

The last two lines are by (3.15-17) as well as $M \leq c_{11}\varepsilon^t$. By separating into three cases, we find

$$I \leq c_s c_{11}^{1/2} \varepsilon^{\beta + \frac{1}{2}} \times \begin{cases} (\sup |W''| \varepsilon^\eta + c_{14}\varepsilon^\eta) & \text{on } \mathcal{A} \\ (|W'| + c_{14}\varepsilon^\eta) & \text{on } \mathcal{B} \\ (\sup |W'| + c_{14}\varepsilon^\eta) & \text{on } \mathcal{C}. \end{cases}$$

(II) Since $\xi_G = \tilde{\xi} + \phi \geq 0$ on Γ^+ and $G'' < 0$, we have

$$II \leq 0.$$

(III) By the choice of G , both terms are non-positive on \mathcal{B} and \mathcal{C} . Moreover, since $G'' = -\frac{c_{14}}{4}\varepsilon^\eta$ and $G \geq c_{14}\varepsilon^\eta/2$ for $|u| \leq 1 + \varepsilon^\eta$ for suitably small ε ,

$$III \leq -\frac{(c_{14}\varepsilon^\eta)^2}{8} \begin{cases} +c_{14}\varepsilon^{2\eta} \sup |W''| & \text{on } \mathcal{A}, \\ -c_{14}\varepsilon^\eta \frac{(1-\gamma)}{8} |W'| & \text{on } \mathcal{B}, \\ -\frac{c_{14}}{4}\varepsilon^\eta \min_{|u| \leq \frac{\gamma+1}{2}} W(u) & \text{on } \mathcal{C}. \end{cases}$$

The first term comes from $G''G$. Note that $\min_{|u| \leq \frac{\gamma+1}{2}} W(u) > 0$ is a strictly positive constant.

(IV) By (3.11) and (3.15),

$$\begin{aligned}
 IV &\leq |f|\varepsilon(|W'| + c_{14}\varepsilon^\eta) + c_{11}c_s\varepsilon^{2\beta+\iota} \\
 &\leq c_{11}c_s\varepsilon^{2\beta+\iota} + \begin{cases} (\varepsilon^{1+\eta} \sup |W''| + \varepsilon^{1+\eta}c_{14})|f| & \text{on } \mathcal{A}, \\ (\varepsilon|W'| + c_{14}\varepsilon^{1+\eta})|f| & \text{on } \mathcal{B}, \\ (\varepsilon \sup |W'| + c_{14}\varepsilon^{1+\eta})|f| & \text{on } \mathcal{C}. \end{cases}
 \end{aligned}$$

Next, we sum the four terms on each set and evaluate them from above. Note that the terms in (III) are “good terms”, giving the necessary negative contributions.

(I + ⋯ + IV on \mathcal{A})

$$\begin{aligned}
 I + \cdots + IV &\leq c_{14}^2\varepsilon^{2\eta} \left(\frac{c_s c_{11}^{1/2}}{c_{14}^2} \sup |W''| \varepsilon^{\beta+\frac{\iota}{2}-\eta} + \frac{c_s c_{11}^{1/2}}{c_{14}} \varepsilon^{\beta+\frac{\iota}{2}-\eta} - \frac{1}{8} \right. \\
 &\quad \left. + \frac{\sup |W''|}{c_{14}} + \frac{c_{11}c_s}{c_{14}^2} \varepsilon^{2\beta+\iota-2\eta} \right) + (\varepsilon^{1+\eta} \sup |W''| + \varepsilon^{1+\eta}c_{14})|f|.
 \end{aligned}$$

Since $2\eta \leq 2\beta + \iota$, we may restrict c_{14} large depending only on c_{11} , c_s , W so that

$$\begin{aligned}
 (3.18) \quad I + \cdots + IV &\leq -\frac{c_{14}^2}{16} \varepsilon^{2\eta} + (\varepsilon^{1+\eta} \sup |W''| + \varepsilon^{1+\eta}c_{14})|f| \\
 &\leq c_{15}\varepsilon^{1+\eta}|f| \cdot \chi_{\{|f| \geq c_{16}\varepsilon^{\eta-1}\}}
 \end{aligned}$$

for suitable choices of c_{15} , $c_{16} = c(c_{11}, c_s, W) > 0$. Here, χ_A denotes the characteristic function of A .

(I + ⋯ + IV on \mathcal{B})

$$\begin{aligned}
 I + \cdots + IV &\leq |W'|c_{14}\varepsilon^\eta \left(\frac{c_s c_{11}^{1/2}}{c_{14}} \varepsilon^{\beta+\frac{\iota}{2}-\eta} - \frac{1-\gamma}{8} \right) \\
 &\quad + (c_{14}\varepsilon^\eta)^2 \left(\frac{c_s c_{11}^{1/2}}{c_{14}} \varepsilon^{\beta+\frac{\iota}{2}-\eta} - \frac{1}{8} + \frac{c_{11}c_s}{c_{14}^2} \varepsilon^{2\beta+\iota-2\eta} \right) \\
 &\quad + (\varepsilon|W'| + c_4\varepsilon^{1+\eta})|f|.
 \end{aligned}$$

Again with $2\eta \leq 2\beta + \iota$ and restricting c_{14} large, we have

$$\begin{aligned}
 (3.19) \quad I + \cdots + IV &\leq -\frac{1-\gamma}{16} c_{14}\varepsilon^\eta |W'| - \frac{c_{14}^2}{16} \varepsilon^{2\eta} + (\varepsilon|W'| + c_{14}\varepsilon^{1+\eta})|f| \\
 &\leq c_{15}\varepsilon|f| \cdot \chi_{\{|f| \geq c_{16}\varepsilon^{\eta-1}\}},
 \end{aligned}$$

where c_{15}, c_{16} are chosen appropriately again.
 ($I + \dots + IV$ on \mathcal{C})

$$I + \dots + IV \leq c_{14}\varepsilon^\eta \left(\frac{c_s c_{11}^{1/2} \sup |W'|}{c_{14}} \varepsilon^{\beta + \frac{1}{2} - \eta} + c_s c_{11}^{1/2} \varepsilon^{\beta + \frac{1}{2}} \right. \\ \left. - \frac{1}{4} \min_{|u| \leq \frac{\gamma+1}{2}} W(u) + \frac{c_{11} c_s}{c_{14}} \varepsilon^{2\beta + \iota - \eta} \right) + (\varepsilon \sup |W'| + c_{14} \varepsilon^{1+\eta}) |f|.$$

Using $0 < \eta \leq \beta + \frac{1}{2}$ and restricting ε small and c_{14} large depending only on η, W, c_{11} and c_s , we have

$$(3.20) \quad I + \dots + IV \leq -\frac{c_{14}}{8} \varepsilon^\eta \min_{|u| \leq \frac{\gamma+1}{2}} W(u) + (\varepsilon \sup |W'| + c_{14} \varepsilon^{1+\eta}) |f| \\ \leq c_{15} \varepsilon |f| \cdot \chi_{\{|f| \geq c_{16} \varepsilon^{\eta-1}\}},$$

for suitable choices of c_{15}, c_{16} .

($I + \dots + V$) Combining (3.18-20) and $|\nabla u| \leq c_5$, we have

$$I + \dots + V \leq c_5 \varepsilon |\nabla f| + c_{15} \varepsilon |f| \cdot \chi_{\{|f| \geq c_{16} \varepsilon^{\eta-1}\}}$$

on Γ^+ . Thus we have

$$\left(\int_{\Gamma^+} (-\Delta \tilde{\xi})^n \right)^{1/n} \leq c_5 \varepsilon \|\nabla f\|_{L^n(\Gamma^+)} + c_{15} \varepsilon \|f\|_{L^n(\Gamma^+ \cap \{|f| \geq c_{16} \varepsilon^{\eta-1}\})}.$$

Since

$$\sup_{B_{\varepsilon^{-\beta}(1-s)}} \tilde{\xi} \leq M \leq \frac{2\varepsilon^{-\beta}}{n\omega_n^{1/n}} \left(\int_{\Gamma^+} (-\Delta \tilde{\xi})^n \right)^{1/n},$$

$\tilde{\xi} = \xi - G$ and $G \leq c_{14} \varepsilon^\eta$, we have the desired estimate by suitably choosing c_{12} and c_{13} . \square

Here we give a proof for (3.12). By the Sobolev inequality (applied to the original scale), for any $p < \infty$, we have

$$\left(\varepsilon^n \int_{B_{(1-s)\varepsilon^{-1}}} |f|^p \right)^{1/p} \leq c(p, n, s) \lambda_0.$$

In particular, for any $t \geq 1$,

$$\left(\int_{B_{(1-s)\varepsilon^{-1}} \cap \{|f| \geq t\}} |f|^n \right)^{1/n} \leq t^{1-\frac{p}{n}} \left(\int_{B_{(1-s)\varepsilon^{-1}}} |f|^p \right)^{1/n} \\ \leq t^{1-\frac{p}{n}} \varepsilon^{-\frac{1}{p}} c(p, n, s) \lambda_0.$$

Thus, in if $t = \varepsilon^{\eta-1}$, with $0 < \eta$, with an appropriately large p , the (LHS) of (3.14) is bounded in terms of λ_0 uniformly for all small ε . Now we use (3.14) with $\eta = \beta = 1/2$ and $\iota = 0$ to any ball $B_{\varepsilon^{-1/2}}(x) \subset B_{\varepsilon^{-1}}$. The condition (B) is satisfied for a suitable constant c_{11} , and $2\beta + \iota \geq 2\eta$ is satisfied. Thus we have

$$\sup_{B_{(1-s)\varepsilon^{-1}}} \xi \leq c_{12}(\varepsilon^{1/2} + \varepsilon^{1/2} \|\nabla f\|_{L^n} + \varepsilon^{1/2} \|f\|_{L^n(B_{\varepsilon^{-1}} \cap \{|f| \geq c_{13}\varepsilon^{-1/2}\})}).$$

Next we apply Lemma 3.9 again with $\eta = 3/5$, $\beta = 2/5$, $\iota = 1/2$ and the new c_{11} . With this choice, $2\eta \leq 2\beta + \iota$ is satisfied. Since $1 - \beta = \eta = 3/5$, we obtain (3.12).

Remark. The exponent $3/5$ in (3.12) is simply a convenient choice for us, but the argument works just as well, as long as the exponent is strictly larger than $1/2$. If we assume that $f \in W^{1,p}$ for p sufficiently larger than n (for example $p \geq 2n$), then we can prove a better estimate with the exponent equal 1. As far as we can see, however, we could not obtain such estimate for the case $p = n$.

4. RECTIFIABILITY AND INTEGRALITY OF THE LIMIT VARIFOLD

Proposition 4.1. *There exist constants $0 < D_1 \leq D_2 < \infty$ and $r_0 > 0$ which depend only on λ_0 , c_0 , E_0 , $\text{dist}(\tilde{U}, \partial U)$ and W , such that*

$$D_1 r^{n-1} \leq \mu(B_r(x)) \leq D_2 r^{n-1}$$

for all $0 < r < r_0$, $x \in \text{supp}\mu \cap \tilde{U}$ and $B_r(x) \subset \tilde{U}$.

Proof. The existence of D_2 is immediate from (3.10).

To establish the lower bound, let $x \in \text{supp}\mu \cap \tilde{U}$.

Claim. *On passing to a subsequence there exist $x_i \in \tilde{U}$ such that $u^i(x_i) \in [-\alpha, \alpha]$ and $x_i \rightarrow x$ as $i \rightarrow \infty$.*

Proof of Claim. Suppose the converse. Then there exists some $s > 0$ such that $B_s(x) \subset \tilde{U}$ and $B_s(x) \cap \{|u^i| \leq \alpha\} = \emptyset$ for all sufficiently large i . For each such i either $u^i > \alpha$ on $B_s(x)$ or $u^i < -\alpha$ on $B_s(x)$. If $u^i > \alpha$, by using the argument in Proposition 3.7 one shows that $u^i \in [1 - \varepsilon_i^{3/4}, 1 + \varepsilon_i^{3/4}]$ on $B_{s/2}(x)$ for all sufficiently large i . Similarly, if $u^i < -\alpha$, then $u^i \in [-1 - \varepsilon_i^{3/4}, -1 + \varepsilon_i^{3/4}]$ on $B_{s/2}(x)$. This implies $W(u^i) = O(\varepsilon_i^{3/2})$ and thus $\frac{W(u^i)}{\varepsilon_i} \rightarrow 0$ uniformly on $B_{s/2}(x)$ as $i \rightarrow \infty$.

For $\frac{\varepsilon_i |\nabla u^i|^2}{2}$, by using the argument in Proposition 3.5 (estimate on \mathcal{C}), one shows that

$$\int_{B_{s/2}(x)} \varepsilon_i |\nabla u^i|^2 \leq O(\varepsilon_i^2) \rightarrow 0$$

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as $i \rightarrow \infty$. Hence we may conclude that $\mu(B_{s/2}(x)) = 0$, which is a contradiction to $x \in \text{supp}\mu$. This ends the proof of the claim.

For any $x \in \tilde{U} \cap \text{supp}\mu$, $B_r(x) \subset \tilde{U}$ and $r \leq r_0$, Proposition 3.4 and 3.6 show that

$$\begin{aligned} \frac{1}{r^{n-1}}\mu(B_r(x)) &\geq \lim_{i \rightarrow \infty} \frac{1}{r^{n-1}} \int_{B_{r/2}(x_i)} \frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \\ &\geq \liminf 2^{1-n} e^{c_7 r/2} \{-c_9 r^{1/2} + e^{-c_7 c_3 \varepsilon_i^{2/5}} E(c_3 \varepsilon_i^{2/5}, x_i)\} \\ &\geq 2^{1-n} e^{-c_7 r/2} \{-c_9 r^{1/2} + e^{-c_7 c_3} c_4\}. \end{aligned}$$

Thus, by suitably restricting r , we prove the existence of D_1 . \square

Proposition 4.2. *Either $u^i \rightarrow +1$ or $u^i \rightarrow -1$ uniformly on each compact subset of $U \setminus \text{supp}\|V\|$. In particular, $\text{supp}\|\partial\{u^\infty = 1\}\| \subset \text{supp}\|V\|$. The terms $\frac{\varepsilon_i}{2} |\nabla u^i|^2$ and $\varepsilon_i^{-1} W(u^i)$ converge uniformly to zero on compact subsets of $U \setminus \text{supp}\|V\|$.*

Proof. This follows immediately from the argument for the previous proposition. \square

Let

$$\xi^i = \frac{\varepsilon_i |\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i},$$

and define (passing to a subsequence if necessary) the measure $|\xi|$ on U by

$$|\xi|(\phi) = \lim_{i \rightarrow \infty} \int |\xi^i| \phi$$

for nonnegative $\phi \in C_c(U)$. Thus $|\xi|$ is the measure theoretic limit of the absolute values of the discrepancy functions.

Proposition 4.3. *$|\xi|$ is the zero measure and so $\xi^i \rightarrow 0$ in $L^1_{loc}(U)$. Moreover, both $\frac{\varepsilon_i}{2} |\nabla u^i|^2 - |\nabla w^i|$ and $\frac{1}{\varepsilon_i} W(u^i) - |\nabla w^i|$ also converge to zero in $L^1_{loc}(U)$.*

Proof. First we claim that

$$(4.1) \quad \liminf_{r \rightarrow 0} \frac{1}{r^{n-1}} |\xi|(B_r(x)) = 0$$

for all $x \in \text{supp}|\xi| \cap \tilde{U}$. Otherwise, there would exist $x \in \text{supp}|\xi| \cap \tilde{U}$, $R > 0$ and $b > 0$ such that $R \leq r_0$ and $|\xi|(B_r(x)) \geq br^{n-1}$ for all $0 < r \leq R$. Define

$$r_1 = \min \{b(8D_2c_7 + c_8)^{-1}, R\},$$

$$r_2 = r_1 \min \{ \exp[-4b^{-1}(c_9 r_1^{1/2} + 4D_2 \exp[c_7 r_1])], 1/2 \}.$$

By Proposition 4.1 and the definition of $|\xi|$, we may choose a large enough i such that $c_3 \varepsilon_i^{2/5} \leq r_2$ and

$$\frac{1}{\tau^{n-1}} \int_{B_\tau(x)} \frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \leq 2D_2, \quad \frac{1}{\tau^{n-1}} \int_{B_\tau(x)} |\xi^i| \geq \frac{b}{2}$$

for all $r_2 \leq \tau \leq r_1$. By Proposition 3.5 and the definition of r_1 ,

$$\begin{aligned} & \frac{1}{\tau^{n-1}} \int_{B_\tau(x)} \left(\frac{W(u^i)}{\varepsilon_i} - \frac{\varepsilon_i |\nabla u^i|^2}{2} \right)^+ \\ & \geq \frac{1}{\tau^{n-1}} \int_{B_\tau(x)} |\xi^i| - (c_7 E(u^i, \tau, x) + c_8) \tau \geq \frac{b}{2} - (c_7 2D_2 + c_8) r_1 \geq \frac{b}{4} \end{aligned}$$

for all $r_2 \leq \tau \leq r_1$. By Proposition 3.6,

$$\begin{aligned} 2D_2 & \geq \frac{1}{r_1^{n-1}} \int_{B_{r_1}(x)} \frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \\ & \geq e^{-c_7 r_1} \left(\frac{b}{4} \int_{r_2}^{r_1} \frac{d\tau}{\tau} - c_9 r_1^{1/2} \right) = e^{-c_7 r_1} \left(\frac{b}{4} \ln \left(\frac{r_1}{r_2} \right) - c_9 r_1^{1/2} \right) \geq 4D_2, \end{aligned}$$

which is a contradiction, so we have proved (4.1).

Combined with Proposition 4.1 and $\text{supp}|\xi| \subset \text{supp}\mu$, we have

$$\liminf_{r \rightarrow 0} \frac{|\xi|(B_r(x))}{\mu(B_r(x))} = 0$$

for all $x \in \text{supp}|\xi|$. A standard result in measure theory then shows that $|\xi| = 0$.

It follows that $\xi^i \rightarrow 0$ in $L^1_{\text{loc}}(U)$.

By completing the square and using $2|\nabla w^i| = \sqrt{2W(u^i)} |\nabla u^i|$, we see that

$$\begin{aligned} \left| \frac{\varepsilon_i}{2} |\nabla u^i|^2 + \frac{W(u^i)}{\varepsilon_i} - 2|\nabla w^i| \right| & = \left(\sqrt{\frac{\varepsilon_i}{2}} |\nabla u^i| - \sqrt{\frac{W(u^i)}{\varepsilon_i}} \right)^2 \\ & \leq \left| \frac{\varepsilon_i |\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i} \right| = |\xi^i|. \end{aligned}$$

This implies the remaining claims in the proposition. \square

Proposition 4.4. *The limit varifold V satisfies $\|V\| = \frac{1}{2}\mu$ and is rectifiable. The first variation of V is given by*

$$\delta V(g) = \frac{1}{2} \int_U u^\infty \text{div}(f^\infty g) = - \int_{M^\infty} f^\infty g \cdot \nu^\infty d\mathcal{H}^{n-1}$$

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for any $g \in C_c^1(U; \mathbb{R}^n)$, where $M^\infty \subset \text{supp}\|V\|$ is the reduced boundary of $\{u^\infty = 1\}$ and f^∞ on M^∞ is the trace of $f^\infty \in W^{1,n}(U)$. The generalized mean curvature vector H is given by

$$H(x) = \begin{cases} \frac{f^\infty(x)}{\theta(x)} \nu^\infty(x) & \mathcal{H}^{n-1} \text{ a.e. } x \in M^\infty, \\ 0 & \mathcal{H}^{n-1} \text{ a.e. } x \in \text{supp}\|V\| \setminus M^\infty, \end{cases}$$

where θ is the density function for $\|V\|$. Moreover, $f^\infty \lfloor M^\infty \in L_{loc}^p(U, \mathcal{H}^{n-1})$ for any $1 \leq p < \infty$.

Proof. Since $\|V\| = \lim \|V^i\|$ and $\|V^i\| = |\nabla w^i| d\mathcal{L}^n$, it follows from Proposition 4.3 and the definition of μ that $\frac{1}{2}\mu = \|V\|$. Next we rearrange terms in (3.2) and using $\frac{w_{x_j}^i}{|\nabla u^i|} = \frac{w_{x_j}^i}{|\nabla w^i|}$, we have

$$\begin{aligned} & \int \left(\text{div} g - \sum_{j,k} \frac{w_{x_j}^i}{|\nabla w^i|} \frac{w_{x_k}^i}{|\nabla w^i|} g_{x_k}^j \right) \varepsilon_i |\nabla u^i|^2 \\ &= \int \left\{ \left(\frac{\varepsilon_i |\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i} + u^i f^i \right) \text{div} g + u^i g \cdot \nabla f^i \right\} \end{aligned}$$

for any $g \in C_c^1(U)$. Since $\varepsilon_i |\nabla u^i|^2 - 2|\nabla w^i|$ and ξ^i converge to 0, u^i converges to u^∞ strongly in L^p for any $1 < p < \infty$ and f^i converges weakly in $W^{1,n}$ to f^∞ , we have

$$\delta V(g) = \frac{1}{2} \int u^\infty (f^\infty \text{div} g + g \cdot \nabla f^\infty).$$

Here we used (2.2), and note that V^i converges to V in the sense of varifold, hence $\delta V^i(g) \rightarrow \delta V(g)$. To justify the integration by parts for $f^\infty g$, we use the following theorem due to Meyers and Ziemer ([28], see also [36, Theorem 1.3]).

Theorem ([44, Theorem 5.12.4]) *Let μ be a positive Radon measure on \mathbb{R}^n satisfying*

$$K(\mu) \equiv \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{r^{n-1}} \mu(B_r(x)) < \infty.$$

Then,

$$\left| \int_{\mathbb{R}^n} \phi d\mu \right| \leq c(n) K(\mu) \int_{\mathbb{R}^n} |\nabla \phi| d\mathcal{L}^n$$

for all $\phi \in C_c^1(\mathbb{R}^n)$.

Since $\mathcal{H}^{n-1} \lfloor M^\infty \leq \frac{\omega_{n-1}}{2D_1} \|V\|$ and $\|V\|(B_r(x)) \leq \frac{D_2 r^{n-1}}{2}$, we have

$K(\mathcal{H}^{n-1} \lfloor M^\infty \cap \tilde{U}) \leq D_2 D_1^{-1} \omega_{n-1} / 4$. Thus

$$\left| \int_{M^\infty} \phi d\mathcal{H}^{n-1} \right| \leq c(n) D_2 D_1^{-1} \int_U |\nabla \phi|$$

for all $\phi \in C_c^1(\tilde{U})$. Using this and smoothly approximating f^∞ , we obtain

$$-\int_{M^\infty} f^\infty g \cdot \nu d\mathcal{H}^{n-1} = \frac{1}{2} \int_U \operatorname{div}(f^\infty g) u^\infty$$

for all $g \in C_c^1(\tilde{U}; \mathbb{R}^n)$. Moreover, for any $1 \leq p < \infty$,

$$\begin{aligned} \int_{M^\infty \cap \tilde{U}} |f^\infty|^p d\mathcal{H}^{n-1} &\leq c(n) D_2 D_1^{-1} \int_U p |f^\infty|^{p-1} |\nabla f^\infty| \\ &\leq c(n, p) D_2 D_1^{-1} \|f^\infty\|_{W^{1,n}(U)}^p < \infty, \end{aligned}$$

by the Sobolev inequality. Since $|\delta V(g)| \leq \sup |g| \int_{M^\infty \cap \tilde{U}} |f^\infty| d\mathcal{H}^{n-1}$ for all $g \in C_c^1(\tilde{U}; \mathbb{R}^n)$, $\|\delta V(g)\|$ defines a Radon measure on U . By the lower density estimate bound of $\|V\|$ and Allard's rectifiability theorem [1, 5.5.(1)], we can conclude that V is rectifiable. It also follows that $\|\delta V\|$ is absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner M^\infty$, and hence with respect to $\|V\|$ as well. Thus,

$$\delta V(g) = - \int g \cdot H d\|V\| = - \int_{\operatorname{supp}\|V\|} g \cdot H \theta d\mathcal{H}^{n-1}.$$

The expression for H follows from the second expression for $\delta V(g)$ in the statement of the proposition. \square

Note that this proves that $H \in L_{\operatorname{loc}}^p(U; \mathbb{R}^n)$ with respect to $\|V\|$ as well for any $1 \leq p < \infty$. By the standard theory for varifold with its mean curvature in L^p , $p \geq n-1$, the density function θ is well defined everywhere on $\operatorname{supp}\|V\|$ and upper-semicontinuous on U . This fact also follows directly from Proposition 3.6.

Next, we prove that $\theta(x) = N\sigma$ for some positive integer \mathcal{H}^{n-1} a.e. on $\operatorname{supp}\|V\|$. With modifications, the line of proof is very similar to that of [24, Section 5], so we point out the difference so that the reader may follow the proof. There are two points which must be dealt with: the first point is that we only have $f \in W^{1,n}$ so that f is "almost bounded" but not quite, and the second is that the discrepancy function has an estimate which is not as good as the case in [24], where we had $\xi \leq c\varepsilon$. These points can be resolved by replacing the pointwise estimates in [24] by suitable integral estimates.

The first proposition shows that there is only small energy uniformly in ε in the region $\{u \approx \pm 1\}$.

Proposition 4.5. ([24, Proposition 5.1.]) *Assume \mathbf{B} is true with u , ε , f and B_3 , and suppose $s > 0$. Then there exist positive constants b*

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and ε_5 depending only on λ_0, c_0, E_0, W and s such that

$$\int_{B_1 \cap \{|u| \geq 1-b\}} \frac{W(u)}{\varepsilon} \leq s$$

whenever $\varepsilon \leq \varepsilon_5$.

Proof. We use two lemmas, [24, Lemma 5.2, 5.3]. For the first lemma, one only need to use the ABP estimate instead of the pointwise estimate. Since it is a straightforward modification, we omit the proof. The rest of the proof goes through with minor modifications of constants. \square

The second proposition deals with the “cutting” the varifold horizontally into stacked single-layered interfaces. Define $T : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ by $T(x) = (x_1, \dots, x_{n-1})$. Also define $\nu = (\nu_1, \dots, \nu_n) = \frac{\nabla u}{|\nabla u|}$ whenever $|\nabla u| \neq 0$ and $\nu = 0$ when $|\nabla u| = 0$. We additionally define

$$e_\varepsilon = \frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}, \quad \xi_\varepsilon = \frac{\varepsilon |\nabla u|^2}{2} - \frac{W(u)}{\varepsilon}.$$

In [24, Proposition 5.5], one needs to change hypothesis (2).

Proposition 4.6. *Corresponding to each R, E_0, s and N such that $0 < R < \infty, 0 < E_0 < \infty, 0 < s < 1$ and N is a positive integer, there exists $\eta > 0$ with the following property:*

Assume the following:

- (1) $Y \subset \mathbb{R}^n$ has no more than $N + 1$ elements, $T(y) = 0$ for all $y \in Y$, $a > 0$, $|y - z| > 3a$ for all $y, z \in Y$ and $\text{diameter} Y \leq \eta R$.
- (2) On $\{x \in \mathbb{R}^n \mid \text{dist}(x, Y) < R\}$, u satisfies (1.2) with $\|f\|_{W^{1,n}} \leq \eta$, $|u| \leq 2$ and

$$\int_a^R \frac{dr}{r^n} \int_{B_r(x)} \left(\frac{\varepsilon |\nabla u|^2}{2} - \frac{W(u)}{\varepsilon} \right)^+ \leq \eta R \quad \text{for each } x \in Y.$$

- (3) For each $y \in Y$ and $a \leq r \leq R$,

$$\int_{B_r(y)} |\xi_\varepsilon| + (1 - (\nu_n)^2) \varepsilon |\nabla u|^2 dy \leq \eta r^{n-1},$$

$$\int_{B_r(y)} \varepsilon |\nabla u|^2 \leq E_0 r^{n-1}.$$

Then we have

$$\sum_{y \in Y} \frac{1}{a^{n-1}} \int_{B_a(y)} e_\varepsilon \leq s + \frac{1+s}{R^{n-1}} \int_{\{x \mid \text{dist}(Y, x) < R\}} e_\varepsilon.$$

For the proof, one also modify the hypothesis (4) in [24, Lemma 5.4] just like above hypothesis (2). With this, the proof of Lemma 5.4 goes through with minor changes, and hence above proposition follows.

The third proposition deals with the ε -scale. Here, one does need a pointwise estimate on ξ_ε , but $\xi_\varepsilon \leq c_1 \varepsilon^{-2/5}$ is sufficient.

Proposition 4.7. *Given $0 < s < 1$ and $0 < b < 1$, there exist $0 < \eta < 1$ and $1 < L < \infty$ depending also on W with the following property: Assume $0 < \varepsilon < 1$ and u satisfies (1.2) and $\xi_\varepsilon \leq \eta \varepsilon^{-1}$ on $B_{4\varepsilon L}(0)$ with $\|f\|_{W^{1,n}} \leq \eta$, $|u(0)| \leq 1 - b$, and*

$$\int_{B_{4\varepsilon L}(0)} (|\xi_\varepsilon| + (1 - (\nu_n)^2)\varepsilon|\nabla u|^2) \leq \eta(4\varepsilon L)^{n-1}.$$

Then, we have $T^{-1}(0) \cap \{x \in B_{3L\varepsilon}(0) \mid u(x) = u(0)\} = \{0\}$ and

$$\left| \frac{1}{\omega_{n-1}(L\varepsilon)^{n-1}} \int_{B_{L\varepsilon}(0)} e_\varepsilon - 2\sigma \right| \leq s.$$

We only point out that the place $\xi_\varepsilon \leq \eta$ was used in the proof of [24, Propostion 5.6] is where we wanted to conclude $|u| \leq 1 - \tilde{b}$ on $B_{4L}(0)$ by suitably restricting η . But this can be done by having only $\xi_\varepsilon \leq \eta \varepsilon^{-1}$ in fact. Also, instead of C^2 estimate for u (or z), we only need $C^{1,\beta}$ estimate for some $\beta > 0$, which is available even if $f \in W^{1,n}$.

To end the proof of Theorem 1, we point out that the main difference from that of [24] for the proof is the condition (2) of Proposition 4.6. For this, note that the constants c_1 , c_7 and c_8 in Proposition 3.2 and 3.5 of this paper scale like r , hence they are small constants in the blowup argument. It is not hard, then, to verify above Prop. 4.6. (2) using Prop. 3.2 and 3.5 for a rescaled sequence of solutions. The argument in [24] then shows that the density of the limit varifold $\|V\|$ is $N\sigma$, where N is an integer, \mathcal{H}^{n-1} a.e. on the support of $\|V\|$.

5. CONCLUDING REMARKS AND APPLICATIONS

5.1. On the Sobolev norm of chemical potential. In this paper we consider the situation where we control the $W^{1,n}$ norm of chemical potential f as $\varepsilon \rightarrow 0$. We encounter a serious difficulty to relax the control when we estimate the supremum bound on $\frac{\varepsilon}{2}|\nabla u|^2 - \frac{W}{\varepsilon}$. As noted in the beginning, it is conjectured that a control of $W^{1,p}$ for some $p > n/2$ should be sufficient to obtain our result, with the limit mean curvature belonging to L^q space for a smaller q . This follows from the following heuristic argument: if the interface M is C^1 hypersurface and if H belongs to $W^{1,p}(U)$ for some $p > n/2$, the trace of H on M belongs to $L^q(M)$ for $q > n - 1$. If the mean curvature of M is given by H , the

well-known result on the regularity of integral varifold [1] shows that the monotonicity formula for the scaled energy ($n-1$ -dimensional area) holds for M . Since f roughly corresponds to the mean curvature field of the interface, one expects that f being in $W^{1,p}(U)$ for some $p > n/2$ may be sufficient to obtain a monotonicity type formula, which may also prove all the subsequent rectifiability and integrality of the limit varifold. Quite relevant to this point are recent articles by Schätzle [36, 37], where he studied convergence of integral varifolds with their mean curvature given by Sobolev functions in $W^{1,p}(U)$, $p > n/2$. His results also suggest strongly that the multiplicity of the limit varifold in this paper is \mathcal{H}^{n-1} a.e. σ , namely, no folding, on $M^\infty \cap \{f^\infty \neq 0\}$. Geometrically, if there is an odd number folding on $M^\infty \cap \{f^\infty \neq 0\}$, it implies that at least one of the interfaces has to bend in a wrong direction as they converge, which seems unlikely to happen. We would like to resolve these points in the future.

5.2. Implications to the Cahn-Hilliard equation. Consider a sequence of smooth initial data u_0^i , $i = 1, \dots$, and ε_i with $\limsup E_{\varepsilon_i}(u_0^i) < \infty$, $\varepsilon_i \rightarrow 0$. Suppose also that there exists a constant $m_0 \in (-|U|, |U|)$ such that $\int_U u_0^i = m_0$ for all $i = \dots$. Let u^i be the solution to the Cahn-Hilliard equation (1.3) with the initial data. As noted, f^i corresponding to u^i satisfies

$$E_{\varepsilon_i}(u_0^i) = E_{\varepsilon_i}(u^i(\cdot, t)) + \int_0^t \int_U |\nabla f^i|^2$$

for all $t \geq 0$. Moreover, Chen proved that ([14, Lemma 3.4])

$$\|f^i(\cdot, t)\|_{L^2(U)} \leq C(E_{\varepsilon_i}(u^i) + \|\nabla f^i(\cdot, t)\|_{L^2(U)})$$

holds for all t and i for all ε_i small. Thus,

$$\int_0^t \|f^i(\cdot, t)\|_{H^1(U)}^2 \leq C$$

for all large i . For L^1 a.e. t , we have $\liminf \|f^i(\cdot, t)\|_{H^1(U)} < \infty$ by Fatou's Lemma. We now restrict our attention to $n = 2$ and such t . With a suitable growth condition on W (such as $W \geq |u|^k$, $k > 2$, for all large u) and the Neumann boundary condition, one may show that there exists a constant $c = c(U, \|f^i\|_{H^1(U)})$ such that $\sup |u^i| \leq c$. Thus, for $n = 2$, assumptions **A** and **B** are satisfied for a subsequence on this time slice. Unfortunately, even though we may conclude that there exists a subsequence for a.e. time slice for which we may apply our result, the choice of the subsequence may differ for each t . Note that the bound on $\|f^i(\cdot, t)\|_{H^1(U)}$ implies that the time derivative of the total energy stays finite, not allowing a violent jump of mass there.

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