Classification of the R-operator

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Classification of the R-operator

—Dedicated to Professor Yoshiyuki Shimizu

on the occasion of his sixtieth birthday—

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Abstract

We classify the R-operator, which is a solution of the quantum Yang-Baxter equation on a function space.

1 Introduction

In the studies of the integrable models, it is one of the most important problems to find the solutions of the Yang-Baxter equation. Almost all solutions of the Yang-Baxter equation have been constructed in the set of matrices, which is called R-matrices. Ueno and the author have introduced the infinite-dimensional R-matrix [1, 2, 3] with spectral parameter, and, by means of the Fourier transformation of this R-matrix, found the solutions of the quantum Yang-Baxter equation on a function space called the R-operators. There are three kinds of the R-operators expressed in terms of the elliptic, trigonometric, and rational functions, respectively.

The elliptic R-operator has been studied most in three kinds. The elliptic R-operator is obtained from the limiting case $n \to \infty$ of Belavin's R-matrix [1]. Felder and Pasquier showed that Belavin's R-matrix can be obtained through restricting the domain of a modified version of the elliptic R-operator to a suitable finite-dimensional subspace [4]. The author constructed the incoming and outgoing intertwining vectors and the factorized L-operators, and proved the vertex-IRF correspondence [5]. By means of

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the results of Felder and Pasquier, he reproduced the vertex-IRF correspondence and the factorized L-operators for Belavin's R-matrix from those of the elliptic R-operator.

Hikami and Komori constructed the commuting difference operators [6], Ruijsenaars operators [7], by virtue of the elliptic R-operator. Hikami found the elliptic K-operator [8], the solutions of the boundary (quantum) Yang-Baxter equation and they, in addition, constructed the commuting difference operators called generalized Ruijsenaars operators [9] by means of the elliptic R-operator and K-operator. Fan, Hou and Shi constructed another elliptic K-operator [10].

It is also one of the most important problems to classify the solutions of the Yang-Baxter equation. Belavin and Drinfel'd studied the solutions of the classical Yang-Baxter equation associated with complex simple Lie algebras and gave an almost complete classification of the nondegenerate r-matrices [11] with spectral parameter, the solutions of the classical Yang-Baxter equation. Later Stolin considered the rational case [12].

The aim of this article is to classify the R-operator with some properties.

First we introduce the R-operator. For $x_1, x_2, \dots, x_n \in \mathbb{C}$ and $r > 0$, let $C(x_1, r) := \{x \in \mathbb{C}\ ;\ |x - x_0| < r\}$ and $C((x_1, x_2, \dots, x_n), r) := C(x_1, r) \times C(x_2, r) \times \cdots \times C(x_n, r)$.

Definition 1.1. Let functions $A(x)$ and $B(u, x)$ be meromorphic on $C(0, r)$ and $C((0,0), r)$, respectively. For a function f meromorphic on $C((0,0), r/2)$, we define the function $(R(u)f)(z_1, z_2)$ meromorphic on $C(0,r) \times C((0,0), r/2) (\ni (u, z_1, z_2))$ as

$$(R(u)f)(z_1, z_2) = A(z_1 - z_2) f(z_1, z_2) - B(u, z_1 - z_2) f(z_2, z_1).$$

We call this operator $R(u)$ the R-operator.

If the functions $A(x)$ and $B(u, x)$ satisfy the equations

$$\begin{aligned} &B(u-v, x+y)(A(x)A(-x) - A(y)A(-y)) \\ =&B(u-v, x+y)(B(u,x)B(u,-x) - B(u,y)B(u,-y)) \qquad (1.1) \\ &(B(u,x)B(-v, x+y) \\ =&B(u-v, x+y)B(u,-y) + B(-v,y)B(u-v,x) \qquad (1.2) \end{aligned}$$

on $C((0,0,0,0), r/2)$, then the R-operator $R(u)$ satisfies the (quantum) Yang-Baxter equation

$$(R_{12}(u)R_{13}(u+v)R_{23}(v)f)(z_1, z_2, z_3) = (R_{23}(v)R_{13}(u+v)R_{12}(u)f)(z_1, z_2, z_3)$$

as the meromorphic functions on $C((0,0,0,0,0),r/2)$, where a function f is meromorphic on $C((0,0,0),r/2)$.

The main purpose of this article is to give the complete classification of the R-operator satisfying the functional equations (1.1) and (1.2).

Theorem 1.1. *The meromorphic solutions $A(x)$ and $B(u,x)$ of the equations (1.1) and (1.2) defined on the polydiscs $C(0,r)$ and $C((0,0),r)$ respectively are one of the following:*

0. *trivial case* $\quad A(x)$ *is arbitrary* ,

$$B(u,x) \equiv 0.$$

1. *generic case*

1 − 1. *elliptic*

$$A(x) = ch(x)\frac{\sigma(x+s;\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)\sigma(s;\tau_1,\tau_2)},$$

$$B(u,x) = c\exp(\rho ux)\frac{\sigma(x+au;\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)\sigma(au;\tau_1,\tau_2)}.$$

$(a,c,\tau_1,\tau_2 \in \mathbb{C}\setminus\{0\}, \mathrm{Im}\tau_2/\tau_1 > 0, s \in \mathbb{C}\setminus(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2),$
$\rho \in \mathbb{C})$

1 − 2. *trigonometric*

$$A(x) = \begin{cases} ch(x)\frac{\sinh\frac{x+s}{\lambda}}{\sinh\frac{x}{\lambda}\sinh\frac{s}{\lambda}}, \\ ch(x)\frac{1}{\sinh\frac{x}{\lambda}}, \end{cases}$$

$$B(u,x) = \begin{cases} c\exp(\rho ux)\frac{\sinh\frac{x+au}{\lambda}}{\sinh\frac{x}{\lambda}\sinh\frac{au}{\lambda}}, \\ c\exp(\rho ux)\frac{\exp(\pm\frac{x}{\lambda})}{\sinh\frac{x}{\lambda}}. \end{cases}$$

$(a,c,\lambda \in \mathbb{C}\setminus\{0\}, s \in \mathbb{C}\setminus\mathbb{Z}\pi\sqrt{-1}\lambda, \rho \in \mathbb{C})$

1 − 3. *rational*

$$A(x) = \begin{cases} ch(x)\frac{x+s}{xs}, \\ ch(x)\frac{1}{x}, \end{cases}$$

$$B(u,x) = \begin{cases} c\exp(\rho ux)\frac{x+au}{axu}, \\ c\exp(\rho ux)\frac{1}{x}. \end{cases}$$

$(a,c,s \in \mathbb{C}\setminus\{0\}, \rho \in \mathbb{C})$

2. *singular case*

$$A(x) = c_1 h(x),$$

$$B(u,x) = c_2\exp(\rho ux)\frac{1}{u}.$$

$(c_1,\rho \in \mathbb{C}, c_2 \in \mathbb{C}\setminus\{0\})$

Here the function $h(x)$ is a meromorphic function defined on the disk $C(0,r)$ satisfying the relation $h(x)h(-x) = 1$, and the function $\sigma(x) = \sigma(x;\tau_1,\tau_2)$

is the Weierstrass sigma function.

$$\sigma(x;\tau_1,\tau_2) = x \prod_{(m_1,m_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \{(1-\frac{x}{m_1\tau_1+m_2\tau_2})e^{\frac{x}{m_1\tau_1+m_2\tau_2}+\frac{1}{2}(\frac{x}{m_1\tau_1+m_2\tau_2})^2}\}.$$

Our strategy to solve the functional equations (1.1) and (1.2) is as follows. We reduce the functional equations (1.1) and (1.2) to the functional equation introduced by Braden and Buchstaber [13].

$$\phi_1(x+y)(\phi_4(x)\phi_5(y)-\phi_4(y)\phi_5(x)) = \phi_2(x)\phi_3(y)-\phi_2(y)\phi_3(x).$$

They proved that the solutions of this functional equation above are characterized by those of the functional equation discussed by Bruschi and Calogero [14, 15].

$$\alpha(x)\alpha'(y)-\alpha'(x)\alpha(y) = (\alpha(x+y)-\alpha(x)\alpha(y))(\eta(x)-\eta(y)).$$

Since Kawazumi and the author [16] gave the complete classification of the meromorphic solutions near the origin of the equation above, we obtain all the meromorphic solutions of (1.1) and (1.2) near the origin.

This article is organized as follows. In Section 2, we review the solutions of the functional equations above, and define the R-operator. After showing that $B \equiv 0$ is a solution of the equations (1.1) and (1.2) with an arbitrary A, we solve the equations (1.1) and (1.2) under the assumption that $B \not\equiv 0$ and $A(x)A(-x)$ is not identically constant in Section 3. There are three kinds of the meromorphic solutions of the equations (1.1) and (1.2) expressed in terms of the elliptic, trigonometric, and rational functions, respectively. We discuss the elliptic case in Section 4, the trigonometric case in Section 5, and the rational case in Section 6, respectively. In Section 7, we solve the equations (1.1) and (1.2) under the assumption that $B \not\equiv 0$ and $A(x)A(-x)$ is identically constant. We can reduce the functional equations (1.1) and (1.2) to the functional equation introduced by Bruschi and Calogero directly. Finally, in Appendix, we give the proof of a proposition in Section 4.

2 Reviews of certain functional equations of addition type and R-operator

In this section, we review the solutions of the functional equations of addition type

$$\alpha(x)\alpha'(y)-\alpha'(x)\alpha(y) = (\alpha(x+y)-\alpha(x)\alpha(y))(\eta(x)-\eta(y)), \qquad (2.1)$$

$$\alpha(x+y) - \alpha(x)\alpha(y) = \varphi(x)\varphi(y)\psi(x+y), \tag{2.2}$$
$$\phi_1(x+y)(\phi_4(x)\phi_5(y) - \phi_4(y)\phi_5(x)) = \phi_2(x)\phi_3(y) - \phi_2(y)\phi_3(x), \tag{2.3}$$

and the properties of the R-operator.

2.1 Solutions of the equations (2.1) and (2.2)

Bruschi and Calogero have investigated the general analytic solution of the equations (2.1) and (2.2) [14, 15]. They have obtained the solution expressed by the elliptic functions in the most general case, and they had some trigonometric and rational solutions by degenerating the periods of the elliptic functions.

Kawazumi and the author classified the meromorphic solutions near the origin of the equations (2.1) and (2.2) [16].

Theorem 2.1 (Kawazumi-Shibukawa [16]).
(1) *Let α and η be holomorphic functions defined on a punctured disk $\{x \in \mathbb{C}; 0 < |x| < r'\}$ for some $r' > 0$. If they satisfy the functional equation* (2.1), *then they are equal to one of the following functions.*

(0 − i) $\alpha(x) = 0$ *or* $e^{\rho x}$ $(\rho \in \mathbb{C})$, η : *arbitrary,*

(0 − ii) $\alpha(x) = Ce^{\rho x}$, η : *constant,*

(I)
$$\alpha(x) = e^{\rho x}\frac{\sigma(\nu;\tau_1,\tau_2)\sigma(x+\mu;\tau_1,\tau_2)}{\sigma(\mu;\tau_1,\tau_2)\sigma(x+\nu;\tau_1,\tau_2)},$$
$$\eta(x) = \zeta(z;\tau_1,\tau_2) - \zeta(z+\nu;\tau_1,\tau_2) + C,$$
$$(\rho,\mu,\nu,C \in \mathbb{C},\ \tau_1,\tau_2 \in \mathbb{C}\setminus\{0\},\ \operatorname{Im}\tau_2/\tau_1 > 0,$$
$$\mu,\nu \notin \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2,\ \mu - \nu \notin \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2)$$

(II)
$$\alpha(x) = e^{\rho x}\frac{a(e^{2x/\lambda}-1)+b}{c(e^{2x/\lambda}-1)+b},$$
$$\eta(x) = \frac{2\lambda^{-1}e^{2x/\lambda}}{e^{2x\lambda}-1} - \frac{2\lambda^{-1}ce^{2x/\lambda}}{c(e^{2x/\lambda}-1)+b} + C,$$
$$(\lambda,\rho,a,b,c,C \in \mathbb{C},\ \lambda \neq 0,\ b(a-c) \neq 0)$$

(III)
$$\alpha(x) = e^{\rho x}\frac{ax+b}{cx+b},\quad \eta(x) = \frac{b}{x(cx+b)} + C.$$
$$(\rho,a,b,c,C \in \mathbb{C},\ b(a-c) \neq 0)$$

All the solutions except for the case (0-i) *extend themselves to meromorphic functions defined on the whole plane* $\mathbb{C}$.

(2) *Let* α, φ *and* ψ *be holomorphic functions defined on a punctured disk* $\{x \in \mathbb{C}; 0 < |x| < r''\}$ *for some* $r'' > 0$. *If they satisfy the functional equation* (2.2), *then they are equal to one of the following functions.*

(0 – i) $\alpha(x) = 0$ *or* $e^{\rho x}$ $(\rho \in \mathbb{C})$,
$\varphi \equiv 0$ *and* ψ : *arbitrary, or* φ : *arbitrary and* $\psi \equiv 0$,

(0 – ii) $\alpha(x) = Ce^{\rho x}, \quad \varphi = C_1 e^{C_2 x}, \ \psi(x) = C(1-C)C_1^{-2}e^{(\rho - C_2)x},$
$(C, \rho, C_1, C_2 \in \mathbb{C}, C \neq 0, 1, C_1 \neq 0)$

(I)
$$\alpha(x) = e^{\rho x}\frac{\sigma(\nu;\tau_1,\tau_2)\sigma(x+\mu;\tau_1,\tau_2)}{\sigma(\mu;\tau_1,\tau_2)\sigma(x+\nu;\tau_1,\tau_2)},$$
$$\varphi(x) = \exp(C_1 x + C_2)\frac{\sigma(x)}{\sigma(x+\nu)},$$
$$\psi(x) = \exp((\rho - C_1)x - 2C_2)\frac{\sigma(\nu)\sigma(\mu-\nu)\sigma(x+\mu+\nu)}{\sigma^2(\mu)\sigma(x+\nu)},$$
$(\rho, \mu, \nu, C_1, C_2 \in \mathbb{C}, \ \tau_1, \tau_2 \in \mathbb{C} \setminus \{0\}, \ \mathrm{Im}\, \tau_2/\tau_1 > 0,$
$\mu, \nu \notin \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2, \ \mu - \nu \notin \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2)$

(II)
$$\alpha(x) = e^{\rho x}\frac{a(e^{2x/\lambda}-1)+b}{c(e^{2x/\lambda}-1)+b}, \quad \varphi(x) = e^{C_1 x + C_2}\frac{e^{2x/\lambda}-1}{c(e^{2x/\lambda}-1)+b},$$
$$\psi(x) = e^{-C_1 x - 2C_2}\frac{(a-c)\{-ac(e^{2x/\lambda}-1)+b^2-b(a+c)\}}{c(e^{2x/\lambda}-1)+b},$$
$(\lambda, \rho, a, b, c, C_1, C_2 \in \mathbb{C}, \ \lambda \neq 0, \ b(a-c) \neq 0)$

(III)
$$\alpha(x) = e^{\rho x}\frac{ax+b}{cx+b}, \quad \varphi(x) = e^{C_1 x + C_2}\frac{x}{cx+b},$$
$$\psi(x) = e^{(\rho - C_1)x - 2C_2}\frac{(c-a)\{acx+b(a+c)\}}{cx+b}.$$
$(\rho, a, b, c, C_1, C_2 \in \mathbb{C}, \ b(a-c) \neq 0)$

All the solutions except for the case (0-i) *extend themselves to meromorphic functions defined on the whole plane* $\mathbb{C}$.

Remark. (1) In Theorem 2.1 (I), we use τ_1, τ_2, μ and ν instead of $\tau_1/\lambda, \tau_2/\lambda, \mu/\lambda$ and ν/λ in the paper [16]. We note that we do not need the parameter λ in the non-degenerate case in the paper [16].
(2) We note that the condition $\mu - \nu \notin \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ in Theorem 2.1 (I) is dropped in the paper [16]. Because $-\mu/\lambda$ is a pole of the function α and $(-\nu/\lambda) + \mathbb{Z}(\tau_1/\lambda) + \mathbb{Z}(\tau_2/\lambda)$ is the set of zeros of α, we conclude $-\mu/\lambda \notin (-\nu/\lambda) + \mathbb{Z}(\tau_1/\lambda) + \mathbb{Z}(\tau_2/\lambda)$, which implies the condition $\mu - \nu \notin \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$.

2.2 Solutions of the equation (2.3)

Braden and Buchstaber have investigated the functional equation (2.3) [13]. They have shown that the solutions of the equation (2.3) are characterized by the solutions of the equation (2.1). We shall briefly review the results obtained by Braden and Buchstaber.

Let $x_0 \in \mathbb{C}$ and $r_0 > 0$. Let ϕ_1 be a holomorphic function on $C(2x_0, 2r_0)$, and let ϕ_2, ϕ_3, ϕ_4 and ϕ_5 be holomorphic functions on $C(x_0, r_0)$ such that

(1) equation (2.3) for all $x, y \in C(x_0, r_0)$,
(2) $\phi_2(x_0)\phi_3'(x_0) - \phi_2'(x_0)\phi_3(x_0) \neq 0$,
(3) $\phi_4(x_0)\phi_5'(x_0) - \phi_4'(x_0)\phi_5(x_0) \neq 0$.

Lemma 2.2. *We define the function $\tilde{\phi}_1$ holomorphic on $C(0, 2r_0)$ and the functions $\tilde{\phi}_2, \dots, \tilde{\phi}_5$ holomorphic on $C(0, r_0)$ as follows.*

$$\tilde{\phi}_1(x) = c\phi_1(x + 2x_0),$$
$$\begin{pmatrix} \tilde{\phi}_{2k}(x) \\ \tilde{\phi}_{2k+1}(x) \end{pmatrix} = \begin{pmatrix} \phi_{2k}'(x_0) & \phi_{2k}(x_0) \\ \phi_{2k+1}'(x_0) & \phi_{2k+1}(x_0) \end{pmatrix}^{-1} \begin{pmatrix} \phi_{2k}(x + x_0) \\ \phi_{2k+1}(x + x_0) \end{pmatrix} \quad (k = 1, 2),$$

where

$$c = \det \begin{pmatrix} \phi_4'(x_0) & \phi_4(x_0) \\ \phi_5'(x_0) & \phi_5(x_0) \end{pmatrix} / \det \begin{pmatrix} \phi_2'(x_0) & \phi_2(x_0) \\ \phi_3'(x_0) & \phi_3(x_0) \end{pmatrix}.$$

Then they satisfy

$$\tilde{\phi}_1(x + y)(\tilde{\phi}_4(x)\tilde{\phi}_5(y) - \tilde{\phi}_4(y)\tilde{\phi}_5(x)) = \tilde{\phi}_2(x)\tilde{\phi}_3(y) - \tilde{\phi}_2(y)\tilde{\phi}_3(x)$$

for all $x, y \in C(0, r_0)$.

By the straightforward computation, we deduce $\tilde{\phi}_{2k}(0) = \tilde{\phi}_{2k+1}'(0) = 0$ and $\tilde{\phi}_{2k}'(0) = \tilde{\phi}_{2k+1}(0) = 1$ for $k = 1, 2$.

Lemma 2.3. *There exist $(0 <) r_2 \leq r_1$, the functions γ_k and ξ_k $(k = 1, 2)$ holomorphic on $C(0, r_2)$ such that $\gamma_k(x) \neq 0$ for all $x \in C(0, r_2)$,*

$$\begin{pmatrix} \tilde{\phi}_{2k}(x) \\ \tilde{\phi}_{2k+1}(x) \end{pmatrix} = \frac{1}{\gamma_k(x)} \begin{pmatrix} \xi_k(x) \\ \xi_k'(x) \end{pmatrix}$$

for all $x \in C(0, r_2)$, $\xi_k(0) = 0$, and $\xi_k'(0) = \gamma_k(0) = 1$.

For $k = 1, 2$, define $\tilde{\xi}_k(x) = \exp(-\lambda_k x)\xi_k(x)$, where $\lambda_k = -\tilde{\phi}''_{2k}(0)/2$. Then the functions $\tilde{\xi}_k(x)$ are holomorphic on $C(0, r_2)$ and satisfy

$$\tilde{\xi}_k(0) = \tilde{\xi}''_k(0) = 0, \quad \tilde{\xi}'_k(0) = 1.$$

Define the functions $\tilde{\xi}_0(x) := e^{(\lambda_1-\lambda_2)x}\tilde{\phi}_1(x)$ for $x \in C(0, 2r_2)$ and $\gamma(x) := e^{2(\lambda_1-\lambda_2)x}\gamma_2(x)/\gamma_1(x)$ for $x \in C(0, r_2)$.

Lemma 2.4. (1) *The function* $\tilde{\xi}_1(x)/\tilde{\xi}_2(x)$ *is holomorphic on* $C(0, r_2)$.
(2) *For all* $x, y \in C(0, r_2)$

$$\tilde{\xi}_0(x+y)(\tilde{\xi}_2(x)\tilde{\xi}'_2(y) - \tilde{\xi}_2(y)\tilde{\xi}'_2(x)) = \gamma(x)\gamma(y)(\tilde{\xi}_1(x)\tilde{\xi}'_1(y) - \tilde{\xi}_1(y)\tilde{\xi}'(x)).$$

The functions $\tilde{\xi}_k$ $(k = 1, 2)$ holomorphic on $C(0, r_2)$ are not identically zero because of $\tilde{\xi}'_k(0) = 1$. Since $\tilde{\xi}_k(0) = 0$, there exists $(0 <) r_3 \leq r_2$ such that $\tilde{\xi}_1(x) \neq 0$ and $\tilde{\xi}_2(x) \neq 0$ for all $x \in C(0, r_3) \setminus \{0\}$. Braden and Buchstaber proved

Theorem 2.5 (Braden-Buchstaber [13]).
(1) $\gamma(x) = (\tilde{\xi}_2(x)/\tilde{\xi}_1(x))^2$ *and* $\tilde{\xi}_0(x) = \tilde{\xi}_2(x)/\tilde{\xi}_1(x)$ *for all* $x \in C(0, r_3)$.
(2) *We define the function* α *holomorphic on* $C(0, r_3)$ *and the function* η *meromorphic on* $C(0, r_3)$ *as*

$$\alpha(x) = \tilde{\xi}_2(x)/\tilde{\xi}_1(x), \quad \eta(x) = \tilde{\xi}'_2(x)/\tilde{\xi}_2(x).$$

Then they satisfy the equation (2.1) *for all* $x, y \in C(0, r_3/2) \setminus \{0\}$.

We can reconstruct the solutions $\phi_1, \ldots, \phi_5$ of the equation (2.3) from the functions α and η in the theorem above.

2.3 R-operator

First we define the R-operator.

Definition 2.1 (Shibukawa-Ueno [1]). For $r > 0$, let functions $A(x)$ and $B(u, x)$ be meromorphic on $C(0, r)$ and $C((0, 0), r)$ respectively. For a function f meromorphic on $C((0, 0), r/2)$, we define the function $(R(u)f)(z_1, z_2)$ meromorphic on $C(0, r) \times C((0, 0), r/2)(\ni (u, z_1, z_2))$ as

$$(R(u)f)(z_1, z_2) = A(z_1 - z_2)f(z_1, z_2) - B(u, z_1 - z_2)f(z_2, z_1). \tag{2.4}$$

We call this operator $R(u)$ the R-operator.

We state a sufficient condition for the R-operator to satisfy the (quantum) Yang-Baxter equation (2.7).

Proposition 2.6 (Hikami-Komori [17]). *If the functions A and B satisfy the equations below*

$$\begin{aligned} &B(u-v,x+y)(A(x)A(-x)-A(y)A(-y)) \\ &\qquad = B(u-v,x+y)(B(u,x)B(u,-x)-B(u,y)B(u,-y)), \qquad (2.5) \\ &B(u,x)B(-v,x+y) \\ &\qquad = B(u-v,x+y)B(u,-y)+B(-v,y)B(u-v,x) \qquad (2.6) \end{aligned}$$

as meromorphic functions on $C((0,0,0,0),r/2)$, then R-operator $R(u)$ satisfies the Yang-Baxter equation

$$\begin{aligned} &(R_{12}(u)R_{13}(u+v)R_{23}(v)f)(z_1,z_2,z_3) \\ =&(R_{23}(v)R_{13}(u+v)R_{12}(u)f)(z_1,z_2,z_3) \qquad (2.7) \end{aligned}$$

as meromorphic functions on $C((0,0,0,0,0),r/2)$, where a function f is meromorphic on $C((0,0,0),r/2)$.

The aim of this article is to give the complete classification of the R-operator (2.4) satisfying the equations (2.5) and (2.6).

3 Trivial and Generic case

3.1 Trivial case

If $B \equiv 0$ on $C((0,0),r)$, the equations (2.5) and (2.6) hold for any A. Thus, for an arbitrary A meromorphic on $C(0,r)$, the R-operator (2.4) satisfies the Yang-Baxter equation (2.7).

3.2 Generic case

In this section, we assume $B \not\equiv 0$ on $C((0,0),r)$. With the aid of Proposition 2.6 and the identity theorem for the meromorphic functions, we have

Lemma 3.1. *The equations* (2.5) *and* (2.6) *on $C((0,0,0,0),r/2)$ are equivalent to the equations* (2.6) *on $C((0,0,0,0),r/2)$ and*

$$A(x)A(-x)-A(y)A(-y)=B(u,x)B(u,-x)-B(u,y)B(u,-y) \qquad (3.1)$$

on $C((0,0,0),r)$.

We shall solve the equations (2.6) on $C((0,0,0,0),r/2)$ and (3.1) on $C((0,0,0),r)$ with the condition $B \not\equiv 0$ on $C((0,0),r)$

We also assume the following.

Assumption 3.1. The meromorphic function $A(x)A(-x)$ is not identically constant on the disk $C(0,r)$.

From the equations (2.6) and (3.1), we get (See Braden-Buchstaber [13, Lemma 5].)

Lemma 3.2. *The meromorphic solutions $A(x)$ and $B(u,x)$ of the equations (2.6) and (3.1) on $C((0,0,0,0),r/2)$ satisfy the equation*

$$\begin{aligned}&B(v,x+y)(A(x)A(-x)-A(y)A(-y))\\=&B(u,-x)B(u+v,x)B(v,y)-B(u,-y)B(u+v,y)B(v,x)\end{aligned}\tag{3.2}$$

as meromorphic functions on $C((0,0,0,0),r/2)$.

In the sequel we solve the equation (3.2) making use of the method in Section 2.2 obtained by Braden and Buchstaber [13].

Theorem 3.3. (1) *Under Assumption* 3.1, *the function $A(x)A(-x)$ meromorphic on the disk $C(0,r)$ is one of the following:*

elliptic: $$A(x)A(-x)=\frac{a_1\wp(x;\tau_1,\tau_2)+a_2}{a_3\wp(x;\tau_1,\tau_2)+a_4},$$

trigonometric: $$A(x)A(-x)=\frac{a_1\sinh^{-2}(\frac{x}{\lambda})+a_2}{a_3\sinh^{-2}(\frac{x}{\lambda})+a_4},$$

rational: $$A(x)A(-x)=\frac{a_1x^{-2}+a_2}{a_3x^{-2}+a_4},$$

where $\wp(x)=\wp(x;\tau_1,\tau_2)$ is the Weierstrass pe function

$$\wp(x;\tau_1,\tau_2)=-\frac{d}{dx}(\frac{\sigma'(x;\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)}),$$

and the constants $\tau_1,\tau_2,\lambda\in\mathbb{C}\backslash\{0\}$ and $a_1,a_2,a_3,a_4\in\mathbb{C}$ satisfy the relations $\mathrm{Im}\ \tau_2/\tau_1>0$ *and* $a_1a_4-a_2a_3\neq 0$.

(2) *Under Assumption* 3.1, *there exists $C(u_1,r_1)\subset C(0,r/4)$ such that the function $B(u,x)$ is one of the following:*

elliptic: $$B(u,x)=e^{\rho(u)x}b(u)\frac{\sigma(x+a(u);\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)}$$

$$\forall (u,x) \in D_1 \cap D^e \cap (C(u_1,r_1) \times C(0,r)),$$

trigonometric: $$B(u,x) = \begin{cases} e^{\rho(u)x} b(u) \dfrac{\sinh(\frac{x+a(u)}{\lambda})}{\sinh(\frac{x}{\lambda})} \\ e^{\rho(u)x} b(u) \dfrac{\exp(\pm\frac{x+a(u)}{\lambda})}{\sinh(\frac{x}{\lambda})} \end{cases}$$

$$\forall (u,x) \in D_1 \cap D^t \cap (C(u_1,r_1) \times C(0,r)),$$

rational: $$B(u,x) = e^{\rho(u)x} \frac{b(u)x + a(u)}{x}$$

$$\forall (u,x) \in D_1 \cap D^r \cap (C(u_1,r_1) \times C(0,r)),$$

where $\rho(u), a(u), b(u) \in \mathbb{C}$ for all $u \in C(u_1,r_1)$. Here $D_1 (\subset C((0,0),r))$ is the domain of the meromorphic function $B(u,x)$ and

$$\begin{aligned} D^e =& C(0,r) \times (C(0,r) \setminus (\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2)), \\ D^t =& C(0,r) \times (C(0,r) \setminus \mathbb{Z}\pi\sqrt{-1}\lambda), \\ D^r =& C(0,r) \times (C(0,r) \setminus \{0\}). \end{aligned}$$

For the proof, we need the lemma below.

Lemma 3.4. *For any $C((u_0',x_0'),r_0') \subset C((0,0),r/2)$, there exist $(u_1,x_1) \in C((u_0',x_0'),r_0')$ and $r_1 (>0)$ such that*
(0) $C((u_1,x_1),r_1) \subset C((u_0',x_0'),r_0')$,
(1) $B(u,x)$ is holomorphic on $C((u_1,x_1),r_1)$, $C(u_1,r_1) \times C(2x_1,2r_1)$ and $C(2u_1,2r_1) \times C(x_1,r_1)$,
(2) $B(u,-x)$ is holomorphic on $C((u_1,x_1),r_1)$ and $C(u_1,r_1) \times C(2x_1,2r_1)$,
(3) $A(x)$ is holomorphic on $C(x_1,r_1)$ and $C(2x_1,2r_1)$,
(4) $A(-x)$ is holomorphic on $C(x_1,r_1)$and $C(2x_1,2r_1)$,
(5) $B(u,x) \neq 0$ for all $(u,x) \in C((u_1,x_1),r_1)$.

Proof. Since the meromorphic functions $B(u,x)$ and $A(x)$ are defined on some dense open subsets of $C((0,0),r)$ and $C(0,r)$ respectively, for any $C((u_0',x_0'),r_0') \subset C((0,0),r/2)$, there exist $(u_1,x_1) \in C((u_0',x_0'),r_0')$ and $r_1(>0)$ such that the conditions (0), (1), (2), (3) and (4) hold.

With the aid of $B(u,x) \not\equiv 0$ on the polydisk $C((0,0),r)$ and the identity theorem for the meromorphic functions, there exists $(u_1',x_1') \in C((u_1,x_1),r_1)$ such that $B(u_1',x_1') \neq 0$. Because the function $B(u,x)$ is continuous on $C((u_1,x_1),r_1)$, there exists $r_1'(>0)$ such that $C((u_1',x_1'),r_1') \subset C((u_1,x_1),r_1)$ $B(u,x) \neq 0$ for all $(u,x) \in C((u_1',x_1'),r_1')$. Then $(u_1,x_1) := (u_1',x_1')$ and $r_1 := r_1'$ satisfy the conditions (0), (1), (2), (3), (4) and a part of (5).

Repeating this procedure, we can take $(u_1,x_1) \in C((u_0',x_0'),r_0')$ and $r_1(>0)$ satisfying the conditions in this lemma. □

By the lemma above for $C((u'_0, x'_0), r'_0) := C((0,0), r/4)$, we take $(u_1, x_1) \in C((0,0), r/4)$ and $r_1 (> 0)$ satisfying the conditions in the lemma above.

Lemma 3.5. (1) $\frac{d}{dx}(A(x)A(-x)) \not\equiv 0$ *on* $C(x_1, r_1)$.
(2) *For all* $u, v \in C(u_1, r_1)$,

$$\begin{vmatrix} B(u,-x)B(u+v,x) & \frac{\partial}{\partial x}(B(u,-x)B(u+v,x)) \\ B(v,x) & \frac{\partial B}{\partial x}(v,x) \end{vmatrix} \not\equiv 0$$

on $C(x_1, r_1)$.

Proof. (1) It is obvious by Assumption 3.1.
(2) The proof is by contradiction. Assume the assertion were false. Then there would exist $u_0, v_0 \in C(u_1, r_1)$ such that

$$\begin{vmatrix} B(u_0,-x)B(u_0+v_0,x) & \frac{d}{dx}(B(u_0,-x)B(u_0+v_0,x)) \\ B(v_0,x) & \frac{\partial B}{\partial x}(v_0,x) \end{vmatrix} \equiv 0$$

on $C(x_1, r_1)$. Then

$$\frac{d}{dx}\left(\frac{B(u_0,-x)B(u_0,+v_0,x)}{B(v_0,x)}\right) \equiv 0 \quad \text{on } C(x_1, r_1).$$

Thus there exists $c \in \mathbb{C}$ such that

$$\frac{B(u_0,-x)B(u_0+v_0,x)}{B(v_0,x)} \equiv c \quad \text{on } C(x_1, r_1).$$

By the equation (3.2), we have $B(v_0, x+y) = 0$ for all $x, y \in C(x_1, r_1)$, that is to say, $B(v_0, x) \equiv 0$ on $C(2x_1, 2r_1)$. Therefore $A(x)A(-x) = A(y)A(-y)$ for any $x, y \in C(2x_1, 2r_1)$ from the equation (3.1), which is a contradiction of Assumption 3.1. □

Let $u_0, v_0 \in C(u_1, r_1)$. Lemma 3.2 says that, for all $x, y \in C(x_1, r_1)$,

$$\begin{aligned} &B(v_0, x+y)(A(x)A(-x) - A(y)A(-y)) \\ =&B(u_0,-x)B(u_0+v_0,x)B(v_0,y) - B(u_0,-y)B(u_0+v_0,y)B(v_0,x). \end{aligned}$$

The lemma above tells us that there exists $x_0 \in C(x_1, r_1)$ such that

$$\frac{d}{dx}(A(x)A(-x))|_{x=x_0} \neq 0,$$

$$\begin{vmatrix} B(u_0,-x_0)B(u_0+v_0,x_0) & \frac{d}{dx}(B(u_0,-x)B(u_0+v_0,x))|_{x=x_0} \\ B(v_0,x_0) & \frac{\partial B}{\partial x}(v_0,x_0) \end{vmatrix} \neq 0,$$

and there consequently exists $r_0 (> 0)$ satisfying the conditions

(1) $C(x_0, r_0) \subset C(x_1, r_1)$,

(2) $\phi_1(x) := B(v_0, x)$ defined on $C(2x_0, 2r_0)$, $\phi_2(x) := B(u_0, -x)B(u_0 + v_0, x)$, $\phi_3(x) := B(v_0, x)$, $\phi_4(x) := A(x)A(-x)$, and $\phi_5(x) \equiv 1$ defined on $C(x_0, r_0)$ satisfy the condition (1), (2) and (3) above Lemma 2.2.

From Theorems 2.1 and 2.5, the function $\alpha(x) = \tilde{\xi}_2(x)/\tilde{\xi}_1(x)$ defined near the origin is one of the following.

$$\begin{aligned}
&(0) && \alpha(x) = Ce^{\rho x},\\
&(\mathrm{I}) && \alpha(x) = e^{\rho x}\frac{\sigma(\mu;\tau_1,\tau_2)\sigma(\lambda x+\nu;\tau_1,\tau_2)}{\sigma(\nu;\tau_1,\tau_2)\sigma(\lambda x+\mu;\tau_1,\tau_2)},\\
&(\mathrm{II}) && \alpha(x) = e^{\rho x}\frac{a(e^{2x/\lambda}-1)+b}{c(e^{2x/\lambda}-1)+b},\\
&(\mathrm{III}) && \alpha(x) = e^{\rho x}\frac{ax+b}{cx+b}.
\end{aligned}$$

Lemma 3.6. $\alpha(x) \neq Ce^{\rho x}$.

Proof. The proof is by contradiction. Assume the assertion were false. With the aid of Lemma 2.2 and Theorem 2.5, $B(v_0, x) = c^{-1}Ce^{(\rho-\lambda_1+\lambda_2)(x-2x_0)}$ near $2x_0$. By virtue of the equation (3.1), $A(x)A(-x) - A(y)A(-y) \equiv 0$ near $2x_0$, which is a contradiction of Assumption 3.1. □

Proof of Theorem 3.3 (1). We first note that Assumption 3.1 implies the condition $a_1a_4 - a_2a_3 \neq 0$.

We derive the functions $\tilde{\xi}_1(x)$ and $\tilde{\xi}_2(x)$ from Theorems 2.1 and 2.5 by means of $\tilde{\xi}_k(0) = \tilde{\xi}''(0) = 0, \tilde{\xi}_k'(0) = 1$ $(k = 1, 2)$.

$$\begin{aligned}
&(\mathrm{I}) && \tilde{\xi}_1(x) = e^{\zeta(\mu)x}\frac{\sigma(\mu)\sigma(x)}{\sigma(x+\mu)}, \quad \tilde{\xi}_2(x) = e^{\zeta(\nu)x}\frac{\sigma(\nu)\sigma(x)}{\sigma(x+\nu)},\\
&(\mathrm{II}) && \tilde{\xi}_1(x) = \exp\left(\frac{2a-1}{\lambda}x\right)\frac{\frac{\lambda}{2}(\exp(\frac{2x}{\lambda})-1)}{a(\exp(\frac{2x}{\lambda})-1)+1},\\
& && \tilde{\xi}_2(x) = \exp\left(\frac{2c-1}{\lambda}x\right)\frac{\frac{\lambda}{2}(\exp(\frac{2x}{\lambda})-1)}{c(\exp(\frac{2x}{\lambda})-1)+1}.\\
&(\mathrm{III}) && \tilde{\xi}_1(x) = e^{ax}\frac{x}{ax+1}, \quad \tilde{\xi}_2(x) = e^{cx}\frac{x}{cx+1},
\end{aligned}$$

where $\zeta(x) = \zeta(x;\tau_1,\tau_2)$ is the Weierstrass zeta function $\zeta(x;\tau_1,\tau_2) = \sigma'(x;\tau_1,\tau_2)/\sigma(x;\tau_1,\tau_2)$. From $\phi_5(x) \equiv 1$, we have $\tilde{\phi}_5(x) \equiv 1$ and $\gamma_2(x) =$

$\xi_2'(x)$, and $\tilde{\phi}_4(x) = \xi_2(x)/\xi_2'(x)$ as a result. By the definition in Lemma 2.2, we are led to

$$
\begin{aligned}
&A(x)A(-x)\\
&=\begin{cases}
\phi_4'(x_0)\phi_4(x_0) + \frac{\phi_4'(x_0)}{\zeta(x-x_0)-\zeta(x-x_0+\nu)+\lambda_2+\zeta(\nu)}, & \text{(I)}\\
\phi_4'(x_0)\phi_4(x_0) & \\
+\frac{\phi_4'(x_0)(\exp(\frac{2(x-x_0)}{\lambda})-1)\{c(\exp(\frac{2(x-x_0)}{\lambda})-1)+1\}}{(\lambda_2+\frac{2c-1}{\lambda})(\exp(\frac{2(x-x_0)}{\lambda})-1)\{c(\exp(\frac{2(x-x_0)}{\lambda})-1)+1\}+\frac{2}{\lambda}\exp(\frac{2(x-x_0)}{\lambda})}, & \text{(II)}\\
\phi_4'(x_0)\phi_4(x_0) + \frac{\phi_4'(x_0)(x--x_0)\{c(x-x_0)+1\}}{(\lambda_2+c)(x-x_0)\{c(x-x_0)+1\}+1}, & \text{(III)}
\end{cases}
\end{aligned}
$$

near x_0. With the aid of the identity theorem for the meromorphic functions, the equation above holds on $C(0,r)$.

Because $A(x)A(-x)$ is an even function on $C(0,r)$, we obtain

$$
\begin{array}{ll}
\text{(I)} & \sigma(\nu - 2x_0) = 0,\\
\text{(II)} & c = \exp(4x_0/\lambda)/(\exp(4x_0/\lambda) - 1),\\
\text{(III)} & cx_0 = 1/2.
\end{array}
$$

By the straightforward computation, we get the desired result. □

Now we prove Theorem 3.3 (2).

Proposition 3.7. *Let $u_0 \in C(u_1, r_1)$. Under Assumption 3.1, for any $v_0 \in C(u_1, r_1)$, there exist $x_0(v_0) \in C(x_1, r_1)$ and $r_2(v_0)(> 0)$ such that the function $B(v_0, x)$ is one of the following: For all $x \in C(2x_0(v_0), r_2(v_0))$*

$$
\begin{array}{ll}
\textit{elliptic:} & B(v_0, x) = e^{\rho(v_0)x} b(v_0) \dfrac{\sigma(x + a(v_0); \tau_1, \tau_2)}{\sigma(x; \tau_1, \tau_2)},\\
\textit{trigonometric:} & B(v_0, x) = \begin{cases} e^{\rho(v_0)x} b(v_0) \frac{\sinh(\frac{x+a(v_0)}{\lambda})}{\sinh(\frac{x}{\lambda})},\\ e^{\rho(v_0)x} b(v_0) \frac{1}{\sinh(\frac{x}{\lambda})},\end{cases}\\
\textit{rational:} & B(v_0, x) = e^{\rho(v_0)x} \dfrac{b(v_0)x + a(v_0)}{x}.
\end{array}
$$

For the sake of brevity we only show the elliptic case.

For any $v_0 \in C(u_1, r_1)$, there exists $C(x_0(v_0), r_0(v_0)) \subset C(x_1, r_1) \setminus (\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2)$ such that $\phi_1(x) := B(v_0, x)$ defined on $C(2x_0(v_0), 2r_0(v_0))$, $\phi_2(x) := B(u_0, -x)B(u_0 + v_0, x)$, $\phi_3(x) := B(v_0, x)$, $\phi_4(x) := A(x)A(-x)$, and $\phi_5(x) \equiv 1$ defined on $C(x_0(v_0), r_0(v_0))$ satisfy the conditions (1), (2) and (3) above Lemma 2.2 by means of Lemma 3.5.

Lemma 3.8. *The function* $\tilde{\xi}_2$ *is expressed as*

$$\tilde{\xi}_2(x) = e^{\zeta(2x_0(v_0);\tau_1,\tau_2)x}\frac{\sigma(2x_0(v_0);\tau_1,\tau_2)\sigma(x;\tau_1,\tau_2)}{\sigma(x+2x_0(v_0);\tau_1,\tau_2)},$$

where τ_1 *and* τ_2 *are in Theorem* 3.3 (1).

The lemma below and the formula (See, for example, [18, pp. 459].) $2\zeta(2x) - 4\zeta(x) = \wp''(x)/\wp'(x)$ imply the lemma above.

Lemma 3.9.

(1) $a_3\wp(x_0(v_0);\tau_1,\tau_2) + a_4 \neq 0.$

(2) $\wp'(x_0(v_0);\tau_1,\tau_2) \neq 0.$

(3)
$$\tilde{\phi}_4(x) = \frac{(a_3\wp(x_0(v_0);\tau_1,\tau_2)+a_4)}{\wp'(x_0(v_0);\tau_1,\tau_2)(a_3\wp(x+x_0(v_0);\tau_1,\tau_2)+a_4)} \times (\wp(x+x_0(v_0);\tau_1,\tau_2) - \wp(x_0(v_0);\tau_1,\tau_2)).$$

(4)
$$\xi_2(x) = \exp((2\zeta(x_0(v_0);\tau_1,\tau_2) + \frac{a_3\wp'(x_0(v_0);\tau_1,\tau_2)}{a_3\wp(x_0(v_0);\tau_1,\tau_2)+a_4})x) \times \frac{\sigma(2x_0(v_0);\tau_1,\tau_2)\sigma(x;\tau_1,\tau_2)}{\sigma(x+2x_0(v_0);\tau_1,\tau_2)}.$$

(5)
$$\lambda_2 = -\frac{\wp''(x_0(v_0);\tau_1,\tau_2)}{2\wp'(x_0(v_0);\tau_1,\tau_2)} + \frac{a_3\wp'(x_0(v_0);\tau_1,\tau_2)}{a_3\wp(x_0(v_0);\tau_1,\tau_2)+a_4}.$$

Proof. We only prove (1) and (2).
(1) By Theorem 3.3 (1), we have $\phi_4(x) = (a_1\wp(x) + a_2)/(a_3\wp(x) + a_4)$. Since the function $\phi_4(x)$ is holomorphic at $x = x_0(v_0)$, $\phi_4(x_0(v_0)) \in \mathbb{C}$. If $a_3\wp(x_0(v_0))+a_4 = 0$, then $a_1\wp(x_0(v_0))+a_2 = 0$. This implies $a_1a_4-a_2a_3 = 0$, which is a contradiction.
(2) By definition, $\phi_4'(x) = (a_1a_4 - a_2a_3)\wp'(x)/(a_3\wp(x) + a_4)^2$. From the condition (4) above Lemma 2.2, $\phi_4'(x_0(v_0)) \neq 0$, which means $\wp'(x_0(v_0)) \neq 0$. □

Proof of Proposition 3.7. Lemma 3.8 tells us that the zeroes of the function $\tilde{\xi}_2$ is $\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$, and, as a consequence, the periods of the Weierstrass sigma function σ in the function α are τ_1 and τ_2 (See [16, Sections 3 and 4].). Thus there exists $(0 <)r_2 < 2r_1$ such that

$$\tilde{\xi}_1(x) = e^{\zeta(\mu;\tau_1,\tau_2)x}\frac{\sigma(\mu;\tau_1,\tau_2)\sigma(x;\tau_1,\tau_2)}{\sigma(x+\mu;\tau_1,\tau_2)}$$

on $C(0, r_2)$. From Theorem 2.5, Lemmas 2.2 and 3.8 and $\phi_1(x) = B(v_0, x)$, we have proved the proposition. □

Proof of Theorem 3.3 (2). For the sake of brevity we only prove the elliptic case.

Because the polydisk $C((0,0),r)$ is Stein with $H^2(C((0,0),r),\mathbb{Z}) = 0$, the sharp form of the Poincaré theorem holds on $C((0,0),r)$ (See, for example, [19, Chapter V, section 2] and [20, Sections I and K].); there exist two functions g and h holomorphic on $C((0,0),r)$ such that h is not identically zero, $B(u,x) = g(u,x)/h(u,x)$, and the functions g and h are coprime locally.

Since the function $B(u,x)$ is holomorphic on $C((u_1,x_1),r_1) \subset C((0,0),r)$, we have $h(v_0,x_1) \neq 0$ for any $v_0 \in C(u_1,r_1)$, which implies $h(v_0,x) \not\equiv 0$ on $C(0,r)$. Thus the function $g(v_0,x)/h(v_0,x)$ is meromorphic on $C(0,r)$. On the other hand, by Proposition 3.7,

$$\frac{g(v_0,x)}{h(v_0,x)} = e^{\rho(v_0)x} b(v_0) \frac{\sigma(x+a(v_0))}{\sigma(x)}$$

on some small disk in $C(0,r)$. Because the right hand side of the equation above is meromorphic on $C(0,r)$, for any $v_0 \in C(u_1,r_1)$,

$$\frac{g(v_0,x)}{h(v_0,x)} = e^{\rho(v_0)x} b(v_0) \frac{\sigma(x+a(v_0))}{\sigma(x)}$$

as the meromorphic functions on $C(0,r)$. This implies

$$B(u,x) = e^{\rho(u)x} b(u) \frac{\sigma(x+a(u))}{\sigma(x)}$$

for any $(u,x) \in D_1 \cap D^e \cap (C(u_1,r_1) \times C(0,r))$, thereby completing the proof. □

4 Elliptic case

In this section, we shall solve the equations (2.6) and (3.1) in the elliptic case of Theorem 3.3; under Assumption 3.1, there exists $C(u_1,r_1) \subset C(0,r/4)$ such that

$$A(x)A(-x) = \frac{a_1\wp(x)+a_2}{a_3\wp(x)+a_4} \quad \text{on } C(0,r), \tag{4.1}$$

$$B(u,x) = e^{\rho(u)x} b(u) \frac{\sigma(x+a(u))}{\sigma(x)}$$
$$\text{for all } (u,x) \in D_1 \cap D^e \cap (C(u_1,r_1) \times C(0,r)). \tag{4.2}$$

Lemma 4.1. *For all $u \in C(u_1, r_1)$, $\sigma(a(u)) \neq 0$ and $b(u) \neq 0$.*

Proof. It is easy to see that $b(u) \neq 0$ for all $u \in C(u_1, r_1)$ because of Lemma 3.4 (5).

The proof of $\sigma(a(u)) \neq 0$ for all $u \in C(u_1, r_1)$ is by contradiction. Assume the assertion were false. Then there would exist $u \in C(u_1, r_1)$ such that $\sigma(a(u)) = 0$, and there consequently exist $m_1, m_2 \in \mathbb{Z}$ such that $a(u) = m_1\tau_1 + m_2\tau_2$. By means of the equation (4.2), there exist functions $\tilde{\rho}$ and $\tilde{b}$ such that

$$B(u, x) = e^{\tilde{\rho}(u)x}\tilde{b}(u)$$

for all $x \in C(x_1, r_1)\backslash(\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2)$. We note that $(u, -x) \in D_1$ if $x \in C(x_1, r_1)$ because of Lemma 3.4 (2). From the equation (1.1), this contradicts Assumption 3.1. $\square$

By virtue of the equations (3.1), (4.1) and (4.2), we conclude the lemma below.

Lemma 4.2. *We have $a_3 = 0$, that is to say,*

$$A(x)A(-x) = \tilde{a}_1\wp(x) + \tilde{a}_2$$

on $C(0, r)$, where $\tilde{a}_1 = a_1/a_4$ and $\tilde{a}_2 = a_2/a_4$.

We note that the relation $a_1a_4 - a_2a_3 \neq 0$ implies $\tilde{a}_1 \neq 0$. The lemma above and the equation (3.1) imply

$$\begin{aligned}&\tilde{a}_1\wp(x) + \tilde{a}_2 - b(u)^2\sigma^2(a(u))(\wp(a(u)) - \wp(x))\\ =&\tilde{a}_1\wp(y) + \tilde{a}_2 - b(u)^2\sigma^2(a(u))(\wp(a(u)) - \wp(y))\end{aligned}$$

for all $(u, x), (u, y) \in D_1 \cap D^e \cap (C(u_1, r_1) \times C(0, r))$ such that $(u, -x), (u, -y) \in D_1 \cap D^e \cap (C(u_1, r_1) \times C(0, r))$, and, consequently, $b(u)^2\sigma^2(a(u)) = -\tilde{a}_1$ for all $u \in C(u_1, r_1)$.

Lemma 4.3. *There exist $c \in \mathbb{C} \setminus \{0\}$ and $C(u_2, r_2) \subset C(u_1, r_1)$ such that $b(u)\sigma(a(u)) = c$ for all $u \in C(u_2, r_2)$, and*

$$B(u, x) = ce^{\rho(u)x}\frac{\sigma(x + a(u))}{\sigma(a(u))\sigma(x)}$$

for all $(u, x) \in D_1 \cap D^e \cap (C(u_2, r_2) \times C(0, r))$ as a result.

For the proof, it suffices to show the lemma below.

Lemma 4.4. *There exists $C((u_2, 0), r_2) \subset C(u_1, r_1) \times C(0, r)$ such that the function $B(u, x)\sigma(x)$ is holomorphic on $C((u_2, 0), r_2)$.*

Proof. Since the sharp form of the Poincaré theorem holds on $C((0, 0), r)$ (See the proof of Theorem 3.3 (2).), there exist two functions g and h holomorphic on $C((0, 0), r)$ such that h is not identically zero, $B(u, x)\sigma(x) = g(u, x)/h(u, x)$, and the functions g and h are coprime locally. By the equation (4.2), $g(u, x) = e^{\rho(u)x}b(u)\sigma(x + a(u))h(u, x)$ for all $(u, x) \in D_1 \cap D^e \cap (C(u_1, r_1) \times C(0, r))$.

We fix $\forall u \in C(u_1, r_1)$. Because the function $e^{\rho(u)x}b(u)\sigma(x + a(u))$ is holomorphic on $C(0, r)$ and $g(u, x) = e^{\rho(u)x}b(u)\sigma(x + a(u))h(u, x)$ for all $x \in C(x_1, r_1) \setminus (\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2)$, we have $g(u, x) = e^{\rho(u)x}b(u)\sigma(x + a(u))h(u, x)$ for all $x \in C(0, r)$. Thus $g(u, 0) = b(u)\sigma(a(u))h(u, 0)$, which tells us that $(u, 0)$ is not a pole of the function $B(u, x)\sigma(x)$ for all $u \in C(u_1', r_1')$. Since the set of points of indeterminacy of the meromorphic function of two variables is isolated, there exists a regular point $(u_2, 0) \in C(u_1', r_1') \times C(0, r)$ of the function $B(u, x)\sigma(x)$. We have thus proved the lemma. □

We note that $c \neq 0$ because the constant c above is one of the square roots of $-\tilde{a}_1$.

Using the equation (4.1), we are led to

Theorem 4.5. *The elliptic solution $A(x)$ defined on $C(0, r)$ is*

$$A(x) = ch(x)\frac{\sigma(x + s)}{\sigma(x)\sigma(s)}, \tag{4.3}$$

where s is a complex constant such that $\wp(s) = -\tilde{a}_2/\tilde{a}_1$, and $h(x)$ is a meromorphic function defined on $C(0, r)$ satisfying the relation $h(x)h(-x) = 1$.

Proposition 4.6. *There exists $C(u_3, r_3) \subset C(u_1, r_1)$ such that*

$$B(u, x) = e^{\rho_1(u)x}\frac{\sigma(x + a_1(u))}{\sigma(a_1(u))\sigma(x)} \tag{4.4}$$

as meromorphic functions on $C(u_3, r_3) \times C(0, r)$, where the functions ρ_1 and a_1 are holomorphic on $C(u_3, r_3)$.

By Lemma 4.3 and Theorem 4.5, we have

$$\wp(a(u)) = c^{-2}(B(u, x)B(u, -x) - A(x)A(-x)) + \wp(s)$$

for all $(u, x) \in C(u_2, r_2) \times (C(x_1, r_1) \setminus (\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2))$. Thus $\wp(a(u))$ is holomorphic on $C(u_2, r_2)$.

First we prove

Lemma 4.7. *There exists $C(u_3', \delta') \subset C(u_2, r_2)$ such that*

$$B(u,x) = e^{\tilde{\rho}(u)x} \frac{\sigma(x + a_1(u))}{\sigma(a_1(u))\sigma(x)} \tag{4.5}$$

for all $(u,x) \in D_1 \cap D^e \cap (C(u_3', \delta') \times C(0,r))$, where the function a_1 is holomorphic on $C(u_3', \delta')$.

Proof. (1) We assume that there would exist $v_1 \in C(u_2, r_2)$ such that $\wp'(a(v_1)) \neq 0$. The function $\wp(x)$ is holomorphic at $x = a(v_1)$ by using Lemma 4.1, and the function $\wp$ has a holomorphic inverse g near $a(v_1)$ as a result (See, for example, [21, pp. 215].). Then there exists $C(v_1, \delta) \subset C(u_2, r_2)$ such that $\wp(a(u))$ is in the domain of the function g for all $u \in C(v_1, \delta)$. Define a function $\tilde{a}$ holomorphic on $C(v_1, \delta)$ as

$$\tilde{a}(u) = g(\wp(a(u))).$$

Hence there exist functions $\varepsilon(u) \in \{0,1\}$ and $m_1(u), m_2(u) \in \mathbb{Z}$ such that

$$a(u) = (-1)^{\varepsilon(u)}\tilde{a}(u) + m_1(u)\tau_1 + m_2(u)\tau_2$$

for all $u \in C(v_1, \delta)$, and consequently

$$B(u,x) = e^{\tilde{\rho}(u)x}(-1)^{\varepsilon(u)} \frac{\sigma(x + (-1)^{\varepsilon(u)}\tilde{a}(u))}{\sigma(\tilde{a}(u))\sigma(x)},$$

for all $(u,x) \in D_1 \cap D^e \cap (C(v_1, \delta) \times C(0,r))$, where $\tilde{\rho}(u) \in \mathbb{C}$.

The proposition below implies the desired result immediately, which we shall prove in Appendix.

Proposition 4.8. *There exist $C(v_1', \delta') \subset C(v_1, \delta)$ and $m_1'(u), m_2'(u) \in \mathbb{Z}$ such that $a(u) = \tilde{a}(u) + m_1'(u)\tau_1 + m_2'(u)\tau_2$ for all $u \in C(v_1', \delta')$ or $a(u) = -\tilde{a}(u) + m_1'(u)\tau_1 + m_2'(u)\tau_2$ for all $u \in C(v_1', \delta')$.*

(2) We assume that $\wp'(a(u)) \equiv 0$ on $C(u_2, r_2)$. Then

$$a(u) = \tau_1/2, (\tau_1 + \tau_2)/2, \tau_2/2 \pmod{\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2},$$

and, consequently, $\wp(a(u)) = \wp(\tau_1/2), \wp((\tau_1 + \tau_2)/2), \wp(\tau_2/2)$. By virtue of the identity theorem for the holomorphic functions, the function $\wp(a(u))$ is constant on $C(u_1, r_1)$. Without loss of generality, we may assume $\wp(a(u)) \equiv \wp(\tau_1/2)$. Hence, for all $u \in C(u_2, r_2)$, there exist $m_1(u), m_2(u) \in \mathbb{Z}$ such that $a(u) = \tau_1/2 + m_1(u)\tau_1 + m_2(u)\tau_2$, and, as a result,

$$B(u,x) = c\exp((\rho(u) + \eta(u))x) \frac{\sigma(x + \frac{\tau_1}{2})}{\sigma(\frac{\tau_1}{2})\sigma(x)}$$

for all $(u,x) \in D_1 \cap D^e \cap (C(u_2,r_2) \times C(0,r))$. Here $\eta(u) = m_1(u)\eta_1 + m_2(u)\eta_2$. Thus we have proved Proposition 4.7. □

Proof of Proposition 4.6. For the proof, we need

Lemma 4.9. *There exists $C((u_3'',0),\delta'') \subset C((u_3',0),\delta')$ such that the function $f(u,x) := e^{\tilde{\rho}(u)}\sigma(x+a_1(u))$ is holomorphic on $C((u_3'',0),\delta'')$.*

Proof. We note that $f(u,x) = c^{-1}B(u,x)\sigma(x)\sigma(a_1(u))$. The proof is similar to that of Lemma 4.4, so we omit it. □

By virtue of $f(u,x) = e^{\tilde{\rho}(u)}\sigma(x+a_1(u))$,

$$\frac{\partial f}{\partial x}(u,0) = \tilde{\rho}(u)\sigma(a_1(u)) + \sigma'(a_1(u)).$$

If there exists $v \in C(u_3'',\delta'')$ such that $\sigma(a_1(v)) = 0$, then $\sigma(a(v)) = \sigma(\pm a_1(v)) = 0$, which contradicts Lemma 4.1. Hence $\sigma(a_1(u)) \neq 0$ for all $u \in C(u_3'',\delta'')$. Since all the functions $(\partial f/\partial x)(u,0)$, $\sigma(a_1(u))$ and $\sigma'(a_1(u))$ are holomorphic on $C(u_3'',\delta'')$, so is the function $\tilde{\rho}$. □

Now we are in the position to determine the functions ρ_1 and a_1 by means of the equation (3.2).

Lemma 4.10. *There exist $C(u_4,r_4) \subset C(u_3,r_3)$ and a function a_4 holomorphic on $C(u_4,r_4)$ such that $\sigma(a_4(u)+a_1(v)) \neq 0$ for all $u,v \in C(u_4,r_4)$ and $\sigma(a_4(u)) \neq 0$ for all $u \in C(u_4,r_4)$.*

Proof. If there exists $C(u_4,r_4) \subset C(u_3,r_3)$ such that $\sigma(a_1(u)+a_1(v)) \neq 0$ for all $u,v \in C(u_4,r_4)$, then $a_4(u) := a_1(u)$.

We assume that, for all $C(u,r) \subset C(u_3,r_3)$, there would exist $u_0, v_0 \in C(u,r)$ such that $\sigma(a_1(u_0)+a_1(v_0)) = 0$. Then $\sigma(2a_1(u)) = 0$ for all $u \in C(u_3,r_3)$, and there exists $d \in \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ such that $a_1(u) = d/2$ for all $u \in C(u_3,r_3)$ as a result. We note that $a_1(u) \notin \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ by means of Lemma 4.1. Define the function a_4 holomorphic on $C(u_3,r_3)$ as

$$a_4(u) = \begin{cases} \frac{\tau_2}{2} & a_1(u) \equiv \frac{\tau_1}{2} \pmod{\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2}, \\ \frac{\tau_1}{2} & a_1(u) \equiv \frac{\tau_1+\tau_2}{2} \pmod{\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2}, \\ \frac{\tau_1+\tau_2}{2} & a_1(u) \equiv \frac{\tau_2}{2} \pmod{\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2}. \end{cases}$$

This a_4 is the desired one. □

We take $C(\tilde{x}_1, \tilde{r}_1) \subset (C(x_1, r_1) \setminus (\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2))$ such that $C(2\tilde{x}_1, 2\tilde{r}_1) \cap (\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2) = \emptyset$. From Theorem 4.5 and the three term identity of σ (See, for example, [22, pp. 377] and [18, pp. 461].),

$$\begin{aligned}&\frac{B(v, x+y)(A(x)A(-x) - A(y)A(-y))}{B(v,x)B(v,y)}\\ =&\frac{c\sigma(a_1(v))}{\sigma(a_4(u))\sigma(a_4(u)+a_1(v))}\\ &\times\left(\frac{\sigma(x+a_4(u)+a_1(v))\sigma(x-a_4(u))}{\sigma(x)\sigma(x+a_1(v))} - \frac{\sigma(y+a_4(u)+a_1(v))\sigma(y-a_4(u))}{\sigma(y)\sigma(y+a_1(v))}\right)\end{aligned}$$

for all $u, v \in C(u_4, r_4)$ and $x, y \in C(\tilde{x}_1, \tilde{r}_1)$. By virtue of the equation (3.2), for all $u, v \in C(u_4, r_4)$, there exists a constant $\gamma(u, v) \in \mathbb{C}$ such that

$$\begin{aligned}&\frac{B(u,-x)B(u+v,x)}{B(v,x)}\\ =&\frac{c\sigma(a_1(v))\sigma(x+a_4(u)+a_1(v))\sigma(x-a_4(u))}{\sigma(x)\sigma(x+a_1(v))\sigma(a_4(u))\sigma(a_4(u)+a_1(v))} + \gamma(u,v)\end{aligned} \tag{4.6}$$

for all $x, y \in C(\tilde{x}_1, \tilde{r}_1)$.

By definition, $C((2u_4, 2x_1), 2r_4) \subset C((0,0), r/2)$. From Lemma 3.4, there exist $(u_1', x_1') \in C((2u_4, 2x_1), 2r_4)$ and $r_1'(> 0)$ such that the conditions in Lemma 3.4 hold. Making use of u_1', x_1' and r_1' instead of u_1, x_1 and r_1 in Section 3.2 and this section, we obtain

Lemma 4.11. *There exists $C(u_3', r_3') \subset C(u_1', r_1')$ such that*

$$B(u,x) = \pm c e^{\rho_2(u)x} \frac{\sigma(x+a_2(u))}{\sigma(x)\sigma(a_2(u))} \tag{4.7}$$

as meromorphic functions on $C(u_3', r_3') \times C(0, r)$ with the functions ρ_2 and a_2 holomorphic on $C(u_3', r_3')$.

Lemma 4.12. *$\rho_1(u) = \rho u + \rho_3$, $\rho_2(u) = \rho u + \rho_4$, $a_1(u) = au + a_3$, and $a_2(u) = au + a_4$, where ρ, ρ_3, ρ_4, a, a_3 and a_4 are complex constants.*

Proof. We note $C(u_3'/2, r_3'/2) \subset C(u_3, r_3)$. By the equations (4.4), (4.6) and (4.7), for $u, v \in C(u_3'/2, r_3'/2)$,

$$\begin{aligned}&\pm e^{(\rho_2(u+v)-\rho_1(u)-\rho_1(v))x}\sigma(x-a_1(u))\sigma(x+a_2(u+v))\sigma(a_1(v))\sigma(a_4(u))\\ &\times\sigma(a_4(u)+a_1(v))\\ =&\sigma(a_1(v))\sigma(x+a_4(u)+a_1(v))\sigma(x-a_4(u))\sigma(a_1(u))\sigma(a_2(u+v))\\ &+\gamma(u,v)\sigma(a_4(u))\sigma(x)\sigma(a_2(u+v))\sigma(x+a_1(v))\sigma(a_4(u)+a_1(v))\end{aligned}$$

for all $x \in C(\tilde{x}_1, \tilde{r}_1)$. The equation above also holds on $\mathbb{C}(\ni x)$ by means of the identity theorem for the holomorphic functions. The both sides of the equation above is quasi-periodic with the periods τ_1 and τ_2, and, consequently,

$$\begin{aligned}&\exp(\eta_i(a_2(u+v) - a_1(u) + 2x + \tau_i) + \tau_i(\rho_2(u+v) - \rho_1(u) - \rho_1(v)))\\ =&\exp(\eta_i(a_1(v) + 2x + \tau_i))\end{aligned}$$

for $i = 1, 2$. This implies that, for all $u, v \in C(u_3'/2, r_3'/2)$, there exist $n_1(u, v), n_2(u, v) \in \mathbb{Z}$ such that

$$\rho_2(u+v) - \rho_1(u) - \rho_1(v) = n_2(u, v)\eta_1 - n_1(u, v)\eta_2, \tag{4.8}$$

$$a_2(u+v) - a_1(u) - a_1(v) = -n_2(u, v)\tau_1 + n_1(u, v)\tau_2. \tag{4.9}$$

Since the functions a_1 and a_2 are holomorphic and the set $\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ is discrete, $n_1(u, v)$ and $n_2(u, v)$ are constant on $C((u_3'/2, u_3'/2), r_3'/2)$, and, as a result, the functions $\rho_2(u+v) - \rho_1(u) - \rho_1(v)$ and $a_2(u+v) - a_1(u) - a_1(v)$ are constant functions on $C((u_3'/2, u_3'/2), r_3'/2)$. In view of this, we get the desired result. □

Theorem 4.13. *The elliptic solution $B(u, x)$ of the equations (2.6) and (3.1) defined on the polydisc $C((0, 0), r)$ is*

$$B(u, x) = c\exp(\rho u x)\frac{\sigma(x + au)}{\sigma(x)\sigma(au)}. \tag{4.10}$$

Here a is a non-zero complex constant and ρ is an arbitrary complex constant.

Proof. From the lemma above, we deduce

$$\pm c e^{(\rho u + \rho_4)x}\frac{\sigma(x + au + a_4)}{\sigma(x)\sigma(au + a_4)} = c e^{(\rho u + \rho_3)x}\frac{\sigma(x + au + a_3)}{\sigma(x)\sigma(au + a_3)}.$$

By virtue of the quasi-periodicity of the equation above combined with the equations (4.8) and (4.9) there exist $n_1, n_2, n_1', n_2' \in \mathbb{Z}$ such that

$$\begin{cases}\rho_3 = (n_2' - n_2)\eta_1 + (n_1 - n_1')\eta_2,\\ a_3 = (n_2 - n_2')\tau_1 + (n_1' - n_1)\tau_2,\end{cases}$$

which implies the desired result.

We note that the constant a is not zero on account of Lemma 4.1, thereby completing the proof of the theorem. □

Conversely we can show the propsition below by using the three term identity of σ.

Proposition 4.14. *The functions* A (4.3) *and* B (4.10) *meromorphic on* $C(0,r)$ *and* $C((0,0),r)$ *respectively satisfy the equations* (2.6) *and* (3.1).

5 Trigonometric case

In this section, we solve the equations (2.6) and (3.1) under the conditions in Theorem 3.3

$$A(x)A(-x)=\frac{a_1\sinh^{-2}(x)+a_2}{a_3\sinh^{-2}(x)+a_4}\quad \text{on } C(0,r),$$

$$B(u,x)=\begin{cases} e^{\rho(u)x}b(u)\dfrac{\sinh(\frac{x+a(u)}{\lambda})}{\sinh(\frac{x}{\lambda})}, \\ e^{\rho(u)x}b(u)\dfrac{\exp(\pm\frac{x+a(u)}{\lambda})}{\sinh(\frac{x}{\lambda})}, \end{cases}$$

$$\forall (u,x)\in D_1\cap D^t\cap (C(u_1,r_1)\times C(0,r)).$$

The proof of the theorem below is the same as that in Section 4, so we omit the proof.

Theorem 5.1. (1) *The trigonometric solution* $A(x)$ *of the equations* (2.6) *and* (3.1) *defined on the polydisc* $C(0,r)$ *is*

$$A(x)=\begin{cases} ch(x)\dfrac{\sinh\frac{x+s}{\lambda}}{\sinh\frac{x}{\lambda}\sinh\frac{s}{\lambda}}, \\ ch(x)\dfrac{1}{\sinh\frac{x}{\lambda}}, \end{cases} \tag{5.1}$$

where $c\in\mathbb{C}\setminus\{0\}$, $s\in\mathbb{C}\setminus\mathbb{Z}\pi\sqrt{-1}\lambda$ *and* $h(x)$ *is a meromorphic function defined on* $C(0,r)$ *satisfying the relation* $h(x)h(-x)=1$.
(2) *There exists* $C(u_3,r_3)\subset C(u_1,r_1)$ *such that the trigonometric solution* $B(u,x)$ *of the equations* (2.6) *and* (3.1) *is expressed as*

$$B(u,x)=\begin{cases} ce^{\rho_1(u)x}\dfrac{\sinh\frac{x+a_1(u)}{\lambda}}{\sinh\frac{a_1(u)}{\lambda}\sinh\frac{x}{\lambda}}, \\ \\ ce^{\rho_1(u)x}\dfrac{1}{\sinh\frac{x}{\lambda}}, \end{cases}\quad \text{on } C(u_3,r_3)\times C(0,r).$$

Here the functions ρ_1 *and* a_1 *are holomorphic on* $C(u_3,r_3)$.
(3) *There exist* $C(u_4,r_4)\subset C(u_3,r_3)$ *and a function* a_4 *holomorphic on*

$C(u_4, r_4)$ *such that*

$$\begin{cases} \sinh \frac{a_4(u)+a_1(v)}{\lambda} \neq 0 & \forall u, v \in C(u_4, r_4), \\ \sinh \frac{a_4(u)}{\lambda} \neq 0 & \forall u \in C(u_4, r_4). \end{cases}$$

(4) *There exists* $C(u_3', r_3') \subset C(2u_4, 2r_4)$ *such that the trigonometric solution* $B(u, x)$ *of the equations* (2.6) *and* (3.1) *is expressed as follows.*

$$B(u,x) = \begin{cases} \pm c e^{\rho_2(u)x} \frac{\sinh \frac{x+a_2(u)}{\lambda}}{\sinh \frac{a_2(u)}{\lambda} \sinh \frac{x}{\lambda}}, \\ \\ \pm c e^{\rho_2(u)x} \frac{1}{\sinh \frac{x}{\lambda}}, \end{cases} \quad \text{on } C(u_3', r_3') \times C(0, r).$$

Here the functions ρ_2 *and* a_2 *are holomorphic on* $C(u_3', r_3')$.

We take $C(\tilde{x}_1, \tilde{r}_1) \subset (C(x_1, r_1) \backslash \mathbb{Z}\pi\sqrt{-1}\lambda)$ as $C(2\tilde{x}_1, 2\tilde{r}_1) \cap \mathbb{Z}\pi\sqrt{-1}\lambda = \emptyset$, and fix $\forall u, v \in C(u_3'/2, r_3'/2)$. Because there exists $\gamma(u, v) \in \mathbb{C}$ such that

$$\frac{B(u,-x)B(u+v,x)}{B(v,x)} = \begin{cases} \frac{c \sinh \frac{a_1(v)}{\lambda} \sinh \frac{x+a_4(u)+a_1(v)}{\lambda} \sinh \frac{x-a_4(u)}{\lambda}}{\sinh \frac{a_4(u)+a_1(v)}{\lambda} \sinh \frac{a_4(u)}{\lambda} \sinh \frac{x}{\lambda} \sinh \frac{x+a_1(v)}{\lambda}} + \gamma(u,v), \\ \frac{-ce^{-\frac{x}{\lambda}}}{\sinh \frac{x}{\lambda}} + \gamma(u,v), \end{cases}$$

for all $x \in C(\tilde{x}_1, \tilde{r}_1)$, we are led to the four cases below.

$$\mp e^{(\rho_2(u+v)-\rho_1(u)-\rho_1(v))x} \frac{\sinh \frac{x+a_2(u+v)}{\lambda} \sinh \frac{-x+a_1(u)}{\lambda} \sinh \frac{a_1(v)}{\lambda}}{\sinh \frac{a_2(u+v)}{\lambda} \sinh \frac{a_1(u)}{\lambda}}$$
$$= \frac{c \sinh \frac{a_1(v)}{\lambda} \sinh \frac{x+a_4(u)+a_1(v)}{\lambda} \sinh \frac{x-a_4(u)}{\lambda}}{\sinh \frac{a_4(u)+a_1(v)}{\lambda} \sinh \frac{a_4(u)}{\lambda}} + \gamma(u,v) \sinh \frac{x}{\lambda} \sinh \frac{x+a_1(v)}{\lambda}, \tag{5.2}$$

$$\mp e^{(\rho_2(u+v)-\rho_1(u)-\rho_1(v))x} \frac{\sinh \frac{-x+a_1(u)}{\lambda} \sinh \frac{a_1(v)}{\lambda}}{\sinh \frac{a_1(u)}{\lambda}}$$
$$= \frac{c \sinh \frac{a_1(v)}{\lambda} \sinh \frac{x+a_4(u)+a_1(v)}{\lambda} \sinh \frac{x-a_4(u)}{\lambda}}{\sinh \frac{a_4(u)+a_1(v)}{\lambda} \sinh \frac{a_4(u)}{\lambda}} + \gamma(u,v) \sinh \frac{x}{\lambda} \sinh \frac{x+a_1(v)}{\lambda}, \tag{5.3}$$

$$\mp c e^{(\rho_2(u+v)-\rho_1(u)-\rho_1(v))x} \frac{\sinh \frac{x+a_2(u+v)}{\lambda}}{\sinh \frac{a_2(u+v)}{\lambda}} = -ce^{-\frac{x}{\lambda}} + \gamma(u,v) \sinh \frac{x}{\lambda}, \tag{5.4}$$

$$\mp c e^{(\rho_2(u+v)-\rho_1(u)-\rho_1(v))x} = -ce^{-\frac{x}{\lambda}} + \gamma(u,v) \sinh \frac{x}{\lambda}, \tag{5.5}$$

for all $x \in C(\tilde{x}_1, \tilde{r}_1)$. We note that the equation above holds on $\mathbb{C}$. By $x = 0$, all the signatures of the equations above are -1.

From the periodicity of the equations above

$$\pm e^{(\rho_2(u+v)-\rho_1(u)-\rho_1(v))\pi\sqrt{-1}\lambda} = 1, \tag{5.6}$$

and, as a result,

Lemma 5.2. *There exist $\rho, \rho_3, \rho_4 \in \mathbb{C}$ such that $\rho_1(u) = \rho u + \rho_3$ for all $u \in C(u_3, r_3)$ and $\rho_2(u) = \rho u + \rho_4$ for all $u \in C(u_3', r_3')$.*

In case of (5.4),

$$B(u,x) = ce^{(\rho u+\rho_3)x}\frac{1}{\sinh\frac{x}{\lambda}}$$

on $C((0,0),r)$ because of the identity theorem. From

$$B(u,x) = ce^{(\rho u+\rho_4)x}\frac{\sinh\frac{x+a_2(u)}{\lambda}}{\sinh\frac{a_2(u)}{\lambda}\sinh\frac{x}{\lambda}}$$

on $C(u_3', r_3') \times C(0,r)$, we deduce a contradiction.

Proposition 5.3. *On $C((0,0),r)$*

$$B(u,x) = \begin{cases} ce^{(\rho u+\rho_3)x}\frac{\sinh\frac{x+au+a_3}{\lambda}}{\sinh\frac{x}{\lambda}\sinh\frac{au+a_3}{\lambda}} & \textit{for } (5.2), \\ ce^{(\rho u+\rho_4)x}\frac{1}{\sinh\frac{x}{\lambda}} & \textit{for } (5.3) \textit{ and } (5.5), \end{cases}$$

where $a, a_3 \in \mathbb{C}$.

Proof. We prove in case of (5.2) only.

There exists $n \in \mathbb{Z}$ such that $\rho_4 - 2\rho_3 = n/\lambda$ from the equation (5.6). One can regard the equation (5.2) as a polynomial of the variable $e^{(x/\lambda)}$. Hence we conclude $n = 0$, and consequently

$$-c\frac{\sinh\frac{x+a_2(u+v)}{\lambda}\sinh\frac{-x+a_1(u)}{\lambda}\sinh\frac{a_1(v)}{\lambda}}{\sinh\frac{a_2(u+v)}{\lambda}\sinh\frac{a_1(u)}{\lambda}}$$
$$=\frac{c\sinh\frac{a_1(v)}{\lambda}\sinh\frac{x+a_4(u)+a_1(v)}{\lambda}\sinh\frac{x-a_4(u)}{\lambda}}{\sinh\frac{a_4(u)+a_1(v)}{\lambda}\sinh\frac{a_4(u)}{\lambda}} + \gamma(u,v)\sinh\frac{x}{\lambda}\sinh\frac{x+a_1(v)}{\lambda}$$

on $\mathbb{C}$. Thus we are led to

$$e^{\frac{2(a_2(u+v)-a_1(u)-a_1(v))}{\lambda}} = 1$$

for all $u, v \in C(u_3'/2, r_3'/2)$, thereby completing the proof. □

From the equation (2.6), we get $\rho_3 = \rho_4 = a_3 = 0$, that is to say,

Theorem 5.4. *The trigonometric solution* $B(u,x)$ *of the equations* (2.6) *and* (3.1) *defined on the polydisc* $C((0,0),r)$ *is*

$$B(u,x) = \begin{cases} ce^{\rho ux}\frac{\sinh\frac{x+au}{\lambda}}{\sinh\frac{x}{\lambda}\sinh\frac{au}{\lambda}}, \\ ce^{\rho ux}\frac{e^{\pm\frac{x}{\lambda}}}{\sinh\frac{x}{\lambda}}. \end{cases} \tag{5.7}$$

Here c *is in Theorem* 5.1, a *is a non-zero complex constant and* ρ *is an arbitrary complex constant.*

Conversely

Proposition 5.5. *The functions* A (5.1) *and* B (5.7) *meromorphic on* $C(0,r)$ *and* $C((0,0),r)$ *respectively satisfy the equations* (2.6) *and* (3.1).

6 Rational case

In this section, we solve the equations (2.6) and (3.1) under the conditions in Theorem 3.3

$$\begin{aligned} &A(x)A(-x) = \frac{a_1x^{-2}+a_2}{a_3x^{-2}+a_4} \quad \text{on } C(0,r), \\ &B(u,x) = e^{\rho(u)x}\frac{b(u)x+a(u)}{x} \\ &\forall(u,x) \in D_1 \cap D^r \cap (C(u_1,r_1)\times C(0,r)). \end{aligned}$$

The proof of the theorem below is the same as that in Section 4, so we omit the proof.

Theorem 6.1. (1) *The rational solution* $A(x)$ *of the equations* (2.6) *and* (3.1) *defined on the polydisc* $C(0,r)$ *is*

$$A(x) = \begin{cases} ch(x)\frac{x+s}{xs}, \\ ch(x)\frac{1}{x}, \end{cases} \tag{6.1}$$

where c *and* s *are non-zero complex constants and* $h(x)$ *is a meromorphic function defined on* $C(0,r)$ *satisfying the relation* $h(x)h(-x) = 1$.
(2) *There exist* $C(u_3,r_3) \subset C(u_1,r_1)$ *and* $C(u_3',r_3') \subset C(2u_3,2r_3)$ *such that*

the rational solution $B(u,x)$ of the equations (2.6) and (3.1) is expressed as follows.

$$B(u,x)=\begin{cases} e^{\rho_1(u)x}\frac{a_1(u)x+c}{x}, & \text{on } C(u_3,r_3)\times C(0,r),\\ e^{\rho_2(u)x}\frac{a_2(u)x\pm c}{x}, & \text{on } C(u_3',r_3')\times C(0,r).\end{cases}$$

Here the functions ρ_1 and a_1 are holomorphic on $C(u_3,r_3)$ and the functions ρ_2 and a_2 are holomorphic on $C(u_3',r_3')$.

We fix $\forall u,v\in C(u_3'/2,r_3'/2)$. Because there exists $\gamma(u,v)\in\mathbb{C}$ such that

$$\frac{B(u,-x)B(u+v,x)}{B(v,x)}=-\frac{c^2}{x(a_1(v)x+c)}+\gamma(u,v)$$

for all $x\in C(x_1,r_1)\setminus\{0\}$, we are led to

$$\begin{aligned}&e^{(\rho_2(u+v)-\rho_1(u)-\rho_1(v))x}\\ =&\frac{c^2}{(-a_1(u)x+c)(a_2(u+v)x\pm c)}-\frac{\gamma(u,v)x(a_1(v)x+c)}{(-a_1(u)x+c)(a_2(u+v)x\pm c)}.\end{aligned}\tag{6.2}$$

for all $x\in C(x_1,r_1)\setminus\{0\}$. We note that the equation above holds on $\mathbb{C}$.

If $\rho_2(u+v)-\rho_1(u)-\rho_1(v)\neq 0$, then the left hand side of the equation above has an essential singularity at infinity. On the other hand the right hand side of the equation above has a pole or a regular point at infinity since the right hand side is a rational function. It is a contradiction. Hence $\rho_2(u+v)-\rho_1(u)-\rho_1(v)=0$ for all $u,v\in C(u_3'/2,r_3'/2)$, and, as a result,

Lemma 6.2. *There exist $\rho,\rho_3\in\mathbb{C}$ such that $\rho_1(u)=\rho u+\rho_3$ for all $u\in C(u_3,r_3)$ and $\rho_2(u)=\rho u+2\rho_3$ for all $u\in C(u_3',r_3')$.*

From the equation (6.2)

$$a_1(u)a_2(u+v)=(a_1(u)-a_2(u+v))a_1(v)$$

for all $u,v\in C(u_3'/2,r_3'/2)$, which implies

Lemma 6.3. *The function $a_1(u)$ is identically zero on $C(u_3,r_3)$, or there exists $a,a_3\in\mathbb{C}$ such that*

$$\begin{cases}\frac{1}{a_1(u)}=\frac{au}{c}+\frac{a_3}{c}, & \forall u\in C(u_3,r_3),\\ \frac{1}{a_2(u)}=\frac{au}{c}+\frac{2a_3}{c}, & \forall u\in C(u_3',r_3').\end{cases}$$

By the straightforward computation

Theorem 6.4. *The rational solution $B(u,x)$ of the equations* (2.6) *and* (3.1) *defined on the polydisc $C((0,0),r)$ is*

$$B(u,x) = \begin{cases} ce^{\rho ux}\frac{x+au}{aux}, \\ ce^{\rho ux}\frac{1}{x}. \end{cases} \tag{6.3}$$

Here c is in Theorem 6.1, *$a \in \mathbb{C}\setminus\{0\}$ and $\rho \in \mathbb{C}$.*

Conversely

Proposition 6.5. *The functions A* (6.1) *and B* (6.3) *meromorphic on $C(0,r)$ and $C((0,0),r)$ respectively satisfy the equations* (2.6) *and* (3.1).

7 Singular case

In this section, we assume that, for some complex constant a,

$$A(x)A(-x) \equiv a$$

on $C(0,r)$. The assumption above and the equation (3.1) imply

$$B(u,x)B(u,-x) = B(u,y)B(u,-y) \tag{7.1}$$

on $C((0,0),r)$. Let $D_1, D_2 \subset C((0,0),r)$ be the domains of the meromorphic function $B(u,x)$ and $B(u,-x)$, respectively. From the equation (7.1), for all $u \in C(0,r)$ such that $(u,x) \in D_1 \cap D_2$, there exists $a(u) \in \mathbb{C}$ such that

$$B(u,x)B(u,-x) = a(u) \quad \forall x \in C(0,r) \text{ s. t. } (u,x) \in D_1 \cap D_2. \tag{7.2}$$

It follows immediately that $a(u)$ is holomorphic at $u = u_0$ if $(u_0, y_0) \in D_1 \cap D_2$.

Lemma 7.1. *If $(u_0,y_0) \in D_1 \cap D_2$, then $(u_0,0)$ is not a pole of the function $B(u,x)$.*

Proof. The proof is by contradiction. Assume the assertion were false. For all $n \in \mathbb{N}$, there would exist $(u'_n, x'_n) \in C((0,0),r)$ such that $(u'_n, x'_n) \in C((u_0,0),1/n) \cap D_1$. Then there exists $r'_n > 0$ such that $C((u'_n,x'_n),r'_n) \subset C((u_0,0),1/n) \cap D_1$. Hence there exists $(u_n,x_n) \in D_2$ such that $(u_n,x_n) \in C((u'_n,x'_n),r'_n)$. Because $(u_n,x_n) \in C((u_0,0),1/n) \cap D_1 \cap D_2$, $\lim_{n\to\infty} u_n = u_0$ and $\lim_{n\to\infty} x_n = 0$. Since $(u_0,0)$ is a pole of $B(u,x)$, $\lim_{n\to\infty} |B(u_n,x_n)| = \lim_{n\to\infty} |B(u_n,-x_n)| = \infty$, and $\lim_{n\to\infty} |B(u_n,x_n)B(u_n,-x_n)| = \infty$ as a consequence.

On the other hand, the lemma above says that $\lim_{n\to\infty} a(u_n) = a(u_0)$, which is a contradiction of the equation (7.2). □

Thus the point $(u_0, 0)$ in the lemma above is a regular point or a point of indeterminacy of $B(u, x)$.

Lemma 7.2. *For any* $(0 <)r' \leq r$, *there exists* $u_0 \in C(0, r')$ *such that* $(u_0, 0)$ *is a regular point of* $B(u, x)$.

Proof. It is enough to consider the case that $(u_0, 0) \in C((0, 0), r')$ in the lemma above is a point of indeterminacy of the function $B(u, x)$.

Because the set of the points of indeterminacy of the meromorphic function with two variables is isolated, there exists $r_0 > 0$ such that $B(u, x)$ has no points of indeterminacy in $C((u_0, 0), r_0) \setminus \{(u_0, 0)\}$ and $C((u_0, 0), r_0) \subset C((0, 0), r')$. That is to say, for any $u_1 \in C(u_0, r_0) \setminus \{u_0\}$, $(u_1, 0)$ is not a point of indeterminacy of $B(u, x)$, and there exists $C((u_1, 0), s) \subset C((u_0, 0), r_0) \setminus \{(u_0, 0)\}$ as a result. The meromorphic function $B(u, x)$ has no points of indeterminacy in $C((u_1, 0), s)$.

Since the set D_1 and D_2 are dense in $C((0, 0), r)$, there exists $(u_3, y_3) \in D_1 \cap D_2 \cap (C(u_1, s) \times C(0, r'))$. By means of the lemma above and $(u_3, y_3) \in D_1 \cap D_2$, $(u_3, 0)$ is not a pole of $B(u, x)$. From $(u_3, 0) \in C((u_1, 0), s)$, $(u_3, 0)$ is not a point of indeterminacy of $B(u, x)$. This point u_3 is the one what we desire. □

Proposition 7.3. *There exist* $r_0 (> 0)$ *and* $u_0 \in C(0, r)$ *such that*

(1)$C((4u_0, 0), 4r_0) \subset C((0, 0), r)$,
(2)$B(u, x)$ *is holomorphic on* $C((u_0, 0), r_0) \cup C((2u_0, 0), 2r_0) \cup C((4u_0, 0), 4r_0)$,
(3)$B(u, x) \neq 0$ *for all* $(u, x) \in C((u_0, 0), r_0) \cup C((2u_0, 0), 2r_0) \cup C((4u_0, 0), 4r_0)$.

It suffices to show the lemma below.

Lemma 7.4. (1) *If there exists* $C((u_0, 0), r_0) \subset C((0, 0), r)$ *such that* $B(u, x)$ *is holomorphic on* $C((u_0, 0), r_0)$ *and* $C((2u_0, 0), 2r_0) \subset C((0, 0), r)$, *then there exists* $C((u_1, 0), r_1) \subset C((u_0, 0), r_0)$ *such that* $B(u, x)$ *is holomorphic on* $C((2u_1, 0), 2r_1)$.
(2) *If there exists* $C((u_0, 0), r_0) \subset C((0, 0), r)$ *such that* $B(u, x)$ *is holomorphic on* $C((u_0, 0), r_0)$, *then there exists* $C((u_1, 0), r_1) \subset C((u_0, 0), r_0)$ *such that* $B(u, x) \neq 0$ *for all* $(u, x) \in C((u_1, 0), r_1)$.

Proof. (1) We take $C((u_2, y_2), r_2) \subset (C(2u_0, 2r_0) \times C(0, r)) \cap D_1 \cap D_2$, and, for all $u \in C(u_2, r_2)$, there exists $y \in C(y_2, r_2)$ such that $(u, y) \in D_1 \cap D_2$ as a result. By Lemma 7.1, $(u, 0)$ is not a pole of $B(u, x)$ for all $u \in C(u_2, r_2)$.

Because the set of the points of indeterminacy of the meromorphic function of two variables is isolated and $u/2 \in C(u_0, r_0)$ for all $u \in C(u_2, r_2)$,

there exists $u_1 \in C(u_0, r_0)$ such that $(2u_1, 0) \in D_1$. Thus there exists $r_1 > 0$ such that $C((u_1, 0), r_1) \subset C((u_0, 0), r_0)$ and $C((2u_1, 0), 2r_1) \subset D_1$, thereby completing the proof.
(2) There exists $u_1 \in C(u_0, r_0)$ such that $B(u_1, 0) \neq 0$ by using $B \not\equiv 0$ and the equation (7.1). Since $B(u, x)$ is continuous at $(u, x) = (u_1, 0)$, we get the desired result. □

Lemma 7.5. *For all $u, v \in C(u_0, r_0/2)$, there exists $\gamma(u, v) \in \mathbb{C}$ such that*

$$\frac{B(u+v, x)}{B(u, x)B(v, x)} = \gamma(u, v) \tag{7.3}$$

for all $x \in C(0, r_0/2)$.

Proof. Let u_0 and r_0 be given in Proposition 7.3. From the equation (2.6) for $u, v \in C(u_0, r_0/2)$ and $x, y \in C(0, r_0/2)$,

$$\begin{aligned} &B(u, x)B(u, y)B(v, x+y) \\ =&B(u+v, x+y)a(u) + B(u, y)B(v, y)B(u+v, x). \end{aligned} \tag{7.4}$$

We interchange x with y in the equation above and subtract the result from the equation above.

$$\frac{B(u+v, x)}{B(u, x)B(v, x)} = \frac{B(u+v, y)}{B(u, y)B(v, y)}$$

for $u, v \in C(u_0, r_0/2)$ and $x, y \in C(0, r_0/2)$. This completes the proof. □

Lemma 7.6. *We fix $\forall u, v \in C(u_0, r_0/2)$ and put*

$$\begin{aligned} \alpha(x) &= \frac{B(u+v, x)}{B(u, x)}, \\ \varphi(x) &= \frac{1}{B(u, x)}, \\ \psi(x) &= a(u)\gamma(u, v)B(u+v, x). \end{aligned}$$

They satisfy the equation (2.2) *for all $x, y \in C(0, r_0/4)$.*

Proof. We note that $\gamma(u, v) \neq 0$ for all $u, v \in C(u_0, r_0/2)$ on account of Proposition 7.3. From the equations (7.3)

$$\frac{B(u+v, x+y)}{B(u, x+y)B(v, x+y)} = \gamma(u, v)$$

and (7.4) for $u, v \in C(u_0, r_0/2)$ and $x, y \in C(0, r_0/4)$,

$$B(u,x)B(u,y)\frac{B(u+v,x+y)}{\gamma(u,v)B(u,x+y)}$$
$$=B(u+v,x+y)a(u)+\frac{B(u+v,y)}{\gamma(u,v)}B(u+v,x).$$

We have thus proved the lemma. □

With the aid of Proposition 7.3, the functions α, φ and ψ are all holomorphic on $C(0, r_0/2)$.

Lemma 7.7. *For all* $x \in C(0, r_0/2)$, $\varphi(x) \neq 0$ *and* $\psi(x) \neq 0$.

Proof. We only show that $a(u) \neq 0$ for all $u \in C(u_0, r_0/2)$. The proof is by contradiction. Assume the assertion were false. Then there would exist $u \in C(u_0, r_0/2)$ such that $a(u) = 0$. Then $B(u,x)B(u,-x) = 0$ for all $x \in C(0,r)$ such that $(u,x) \in D_1 \cap D_2$. Because $\{u\} \times C(0, r_0) \subset C((u_0, 0), r_0)$, $B(u,x)B(u,-x) = 0$ for all $x \in C(0, r_0)$, which is a contradiction of Proposition 7.3. □

The lemma above tells us that the functions α, φ and ψ are the solution of the equation (2.2) with the condition $\varphi(0) \neq 0$ and $\alpha(x+y) - \alpha(x)\alpha(y) \neq 0$ for all $x, y \in C(0, r_0/4)$. By virtue of Theorem 2.1 (2) we conclude

Proposition 7.8.

$$B(u,x) = c_1(u)\exp(\rho_1(u)x)$$

for $u \in C(u_0, r_0/2)$ *and* $x \in C(0, r_0/4)$, *where* c_1 *and* ρ_1 *are holomorphic on* $C(u_0, r_0/2)$. *The function* c_1 *satisfies* $c_1(u) \neq 0$ *for all* $u \in C(u_0, r_0/2)$.

Proof. We prove that the functions c_1 and ρ_1 are holomorphic on $C(u_0, r_0/2)$. The function $B(u,x)$ is holomorphic on $C((u_0,0), r_0)$ because of Proposition 7.3, and the functions $c_1(u) = B(u,0)$ and $(\partial B/\partial x)(u,0) = c_1(u)\rho_1(u)$ are holomorphic on $C(u_0, r_0/2)$ as a consequence. The function c_1 satisfies $c_1(u) \neq 0$ for all $u \in C(u_0, r_0/2)$ because of Proposition 7.3, which implies the desired result. □

Making use of $2u_0$ and $2r_0$ instead of u_0 and r_0 in this section, we obtain

Proposition 7.9.

$$B(u,x) = c_2(u)\exp(\rho_2(u)x)$$

for $u \in C(2u_0, r_0)$ *and* $x \in C(0, r_0/2)$, *where* c_2 *and* ρ_2 *are holomorphic on* $C(2u_0, r_0)$. *The function* C_2 *satisfies* $C_2(u) \neq 0$ *for all* $u \in C(2u_0, r_0)$.

By virtue of the equations (2.6) and (7.3) we deduce the theorem below.

Theorem 7.10. *The singular solutions $A(x)$ and $B(u,x)$ of the equations* (2.6) *and* (3.1) *defined on the polydiscs $C(0,r)$ and $C((0,0),r)$ respectively are*

$$A(x) = c_1 h(x),$$

and

$$B(u,x) = c_2 \exp(\rho u x)\frac{1}{u}.$$

Here c_1 and ρ are arbitrary complex constants, c_2 is non-zero complex constant, and $h(x)$ is a meromorphic function defined on $C(0,r)$ satisfying the relation $h(x)h(-x) = 1$.

Conversely

Proposition 7.11. *The functions A and B in the theorem above meromorphic on $C(0,r)$ and $C((0,0),r)$ respectively satisfy the equations* (2.6) *and* (3.1).

8 Conclusion

We need not use the sharp form of the Poincaré theorem in this article.

We have Theorem 3.3 (1) in exactly the same way. By the definition of the meromorphic function, there exists $(0 <)r' \leq r$ satisfying the conditions below.

(1) The sharp form of the Poincaré theorem for the function B holds on $C((0,0),r')$.

(2) The sharp form of the Poincaré theorem for the functions $B(u,x)\sigma(x)$, $B(u,x)\sinh(x/\lambda)$ and $B(u,x)x$ holds on $C((0,0),r')$, respectively.

We regard r' as r in this article and, consequently, we obtain Theorem 1.1 for $C(0,r')$ and $C((0,0),r')$. From the identity theorem, Theorem 1.1 also holds for $C(0,r)$ and $C((0,0),r)$.

We use the sharp form of the Poincaré theorem for the sake of brevity.

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Appendix: Proof of Proposition 4.8

In this appendix, we shall prove Proposition 4.8.

Proposition A.1 (Proposition 4.8). *There exist $C(v_1', \delta') \subset C(v_1, \delta)$ and $m_1'(u), m_2'(u) \in \mathbb{Z}$ such that $a(u) = \tilde{a}(u) + m_1'(u)\tau_1 + m_2'(u)\tau_2$ for all $u \in C(v_1', \delta')$ or $a(u) = -\tilde{a}(u) + m_1'(u)\tau_1 + m_2'(u)\tau_2$ for all $u \in C(v_1', \delta')$.*

If there exists $C(v_1', \delta') \subset C(v_1, \delta)$ such that $\varepsilon(u) = 0$ for all $u \in C(v_1', \delta')$ or $\varepsilon(u) = 1$ for all $u \in C(v_1', \delta')$, we have nothing to do.

In the sequel, we assume that, for all $C(u, \delta_1) \subset C(v_1, \delta)$, there exist $v, w \in C(u, \delta_1)$ such that $\varepsilon(v) \neq \varepsilon(w)$. Since the sharp form of the Poincaré theorem holds on $C((0,0), r)$ (See the proof of Theorem 3.3 (2).), there exist two functions g and h holomorphic on $C((0,0), r)$ such that h is not identically zero, $B(u,x)\sigma(x) = g(u,x)/h(u,x)$, and the functions g and h are coprime locally. We omit the proof of the lemma below because it is similar to that of Lemma 4.4.

Lemma A.2. *There exists $C(u_3', \delta') \subset C(v_1, \delta)$ such that*
(1) *the function $B(u,x)\sigma(x)$ is holomorphic on $C((u_3', 0), \delta')$,*
(2) *for all $(u, x) \in C((u_3', 0), \delta')$,*

$$ce^{\tilde{\rho}(u)x}(-1)^{\varepsilon(u)}\frac{\sigma(x + (-1)^{\varepsilon(u)}\tilde{a}(u))}{\sigma(\tilde{a}(u))}h(u,x) = g(u,x).$$

By means of the lemma above,

$$e^{\tilde{\rho}(u)x}(-1)^{\varepsilon(u)}\sigma(x + (-1)^{\varepsilon(u)}\tilde{a}(u)) = c^{-1}B(u,x)\sigma(x)\sigma(\tilde{a}(u))$$

for all $(u, x) \in C((u_3', 0), \delta')$. Since the function $\tilde{a}(u)$ is holomorphic on $C(u_3', \delta')$, the function $f(u,x)$

$$\begin{aligned} f(u,x) &:= c^{-1}B(u,x)\sigma(x)\sigma(\tilde{a}(u)) \\ &= e^{\tilde{\rho}(u)x}(-1)^{\varepsilon(u)}\sigma(x + (-1)^{\varepsilon(u)}\tilde{a}(u)) \end{aligned} \tag{A.1}$$

is holomorphic on $C((u_3', 0), \delta')$. The function $(\partial f/\partial x)(u, 0)$ is consequently holomorphic on $C(u_3', \delta')$, and

$$\frac{\partial f}{\partial x}(u, 0) = \tilde{\rho}(u)\sigma(\tilde{a}(u)) + (-1)^{\varepsilon(u)}\sigma'(\tilde{a}(u)) \tag{A.2}$$

for all $u \in C(u_3', \delta')$.

By $B \not\equiv 0$ and Lemma 4.1, we conclude

Lemma A.3. *There exists* $C((u_3'', x_3''), \delta'') \subset C((u_3', 0), \delta')$ *such that*
(1) $C(x_3'', \delta'') \not\ni 0$,
(2) $f(u,x) \neq 0$ *for all* $(u,x) \in C((u_3'', x_3''), \delta'')$,
(3) $\sigma(x - \tilde{a}(u)) \neq 0$ *for all* $(u,x) \in C((u_3'', x_3''), \delta'')$,
(4) $\sigma(x + \tilde{a}(u)) \neq 0$ *for all* $(u,x) \in C((u_3'', x_3''), \delta'')$.

By means of the lemma above, the functions $-f(u,x)/\sigma(x - \tilde{a}(u))$ and $f(u,x)/\sigma(x + \tilde{a}(u))$ are holomorphic on $C((u_3'', x_3''), \delta'')$, and

$$-f(u,x)/\sigma(x - \tilde{a}(u)) \neq 0, \ f(u,x)/\sigma(x + \tilde{a}(u)) \neq 0$$

for all $(u,x) \in C((u_3'', x_3''), \delta'')$.

Lemma A.4. *For all* $u \in C(u_3'', \delta'')$, $2\tilde{a}(u) \in \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$

To prove Lemma A.4, it suffices to show the following.

Lemma A.5. *For all* $(u_3, x_3) \in C((u_3'', x_3''), \delta'')$,

$$-\frac{\sigma(x_3 + \tilde{a}(u_3))}{\sigma(x_3 - \tilde{a}(u_3))} = e^{2x_3\zeta(\tilde{a}(u_3))}.$$

Proof. We fix $\forall(u_3, x_3) \in C((u_3'', x_3''), \delta'')$. Let $\mathrm{Log}^{(1)}(x)$ and $\mathrm{Log}^{(2)}(x)$ be branches of logarithm defined on open connected sets $V_1, V_2 \subset \mathbb{C}$ such that $e^{\tilde{\rho}(u_3)x_3} \in V_1$ and $(-1)^{\varepsilon(u_3)+1} f(u_3, x_3)/\sigma(x_3 + (-1)^{\varepsilon(u_3)+1}\tilde{a}(u_3)) \in V_2$, respectively. Because the function $(-1)^{\varepsilon(u_3)+1} f(u,x)/\sigma(x + (-1)^{\varepsilon(u_3)+1}\tilde{a}(u))$ is continuous at $(u,x) = (u_3, x_3)$, there exist $\tilde{\varepsilon} > 0$ and $\tilde{\delta} > 0$ such that
(1) $C((-1)^{\varepsilon(u_3)+1} f(u_3, x_3)/\sigma(x_3 + (-1)^{\varepsilon(u_3)+1}\tilde{a}(u_3)), \tilde{\varepsilon}) \subset V_2$,
(2) $C((u_3, x_3), \tilde{\delta}) \subset C((u_3'', x_3''), \delta'')$,
and
(3) for all $(u,x) \in C((u_3, x_3), \tilde{\delta})$,

$$\frac{(-1)^{\varepsilon(u_3)+1} f(u,x)}{\sigma(x + (-1)^{\varepsilon(u_3)+1}\tilde{a}(u))} \in C\left(\frac{(-1)^{\varepsilon(u_3)+1} f(u_3, x_3)}{\sigma(x_3 + (-1)^{\varepsilon(u_3)+1}\tilde{a}(u_3))}, \tilde{\varepsilon}\right).$$

Let $N \in \mathbb{N}$ such that $1/N < \tilde{\delta}$. By virtue of the assumption, for all $n \geq N$, there exists $\tilde{u}_n \in C(u_3, 1/n)$ such that $\varepsilon(\tilde{u}_n) \neq \varepsilon(u_3)$. Then we have $\varepsilon(\tilde{u}_n) \equiv \varepsilon(u_3) + 1 \pmod 2$, $\lim_{n\to\infty} \tilde{u}_n = u_3$, and

Lemma A.6. $(-1)^{\varepsilon(u_3)+1} f(\tilde{u}_n, x_3)/\sigma(x_3 + (-1)^{\varepsilon(u_3)+1}\tilde{a}(\tilde{u}_n)) \in V_2$ *for all* $n \geq N$.

From the equation (A.1) and the lemma above, $e^{\tilde{\rho}(\tilde{u}_n)x_3} \in V_2$ for all $n \geq N$, and, consequently,

$$\tilde{\rho}(\tilde{u}_n)x_3 = \mathrm{Log}^{(2)}\left(\frac{(-1)^{\varepsilon(u_3)+1}f(\tilde{u}_n, x_3)}{\sigma(x_3 + (-1)^{\varepsilon(u_3)+1}\tilde{a}(\tilde{u}_n))}\right)$$

for all $n \geq N$. On account of the equation (A.2)

$$\begin{aligned}&\frac{\partial f}{\partial x}(\tilde{u}_n, 0)\\ \to&\frac{1}{x_3}\mathrm{Log}^{(2)}\left(\frac{(-1)^{\varepsilon(u_3)+1}f(u_3, x_3)}{\sigma(x_3 + (-1)^{\varepsilon(u_3)+1}\tilde{a}(u_3))}\right)\sigma(\tilde{a}(u_3)) + (-1)^{\varepsilon(u_3)+1}\sigma'(\tilde{a}(u_3))\end{aligned}$$

as $n \to \infty$. Hence

$$\begin{aligned}&\frac{\partial f}{\partial x}(u_3, 0)\\ =&\frac{1}{x_3}\mathrm{Log}^{(2)}\left(\frac{(-1)^{\varepsilon(u_3)+1}f(u_3, x_3)}{\sigma(x_3 + (-1)^{\varepsilon(u_3)+1}\tilde{a}(u_3))}\right)\sigma(\tilde{a}(u_3)) + (-1)^{\varepsilon(u_3)+1}\sigma'(\tilde{a}(u_3)).\end{aligned}$$

On the other hand, in view of $e^{\tilde{\rho}(u_3)x_3} \in V_1$,

$$\tilde{\rho}(u_3)x_3 = \mathrm{Log}^{(1)}\left(\frac{(-1)^{\varepsilon(u_3)}f(u_3, x_3)}{\sigma(x_3 + (-1)^{\varepsilon(u_3)}\tilde{a}(u_3))}\right),$$

which implies

$$\begin{aligned}&\frac{\partial f}{\partial x}(u_3, 0)\\ =&\frac{1}{x_3}\mathrm{Log}^{(2)}\left(\frac{(-1)^{\varepsilon(u_3)}f(u_3, x_3)}{\sigma(x_3 + (-1)^{\varepsilon(u_3)}\tilde{a}(u_3))}\right)\sigma(\tilde{a}(u_3)) + (-1)^{\varepsilon(u_3)}\sigma'(\tilde{a}(u_3)).\end{aligned}$$

By the straightforward calculation, we obtain the desired result. □

Proof of Proposition 4.8 (*Proposition* A.1). Because the function $2\tilde{a}(u)$ is continuous on $C(u_3'', \delta'')$ and the set $\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ is discrete, there exist $m, n \in \mathbb{Z}$ such that $2\tilde{a}(u) = m\tau_1 + n\tau_2$ for all $u \in C(u_3'', \delta'')$. Hence $a(u) \equiv (-1)^{\varepsilon(u)}\tilde{a}(u) \equiv \tilde{a}(u) \pmod{\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2}$ for all $u \in C(u_3'', \delta'')$. □

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