

# Blow-up of the viscous heat-conducting compressible flow

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## Abstract

We show the blow-up of smooth solution of viscous heat-conducting flow when the initial density is compactly supported. This is an extension of Z. Xin's result[4] to the case of positive heat conduction coefficient but we do not need any information for the lower bound of the entropy. We control the lower bound of second moment by total energy.

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# 1 Introduction

In this paper, we consider the following equations for a compressible fluid in  $\mathbb{R}^n \times \mathbb{R}_+$  ( $n \geq 1$ ):

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (1)$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div}(\mathcal{T}), \quad (2)$$

$$\partial_t(\rho E) + \operatorname{div}(\rho u E + up) = \operatorname{div}(u\mathcal{T}) + \kappa \Delta \theta. \quad (3)$$

Here  $\rho = \rho(x, t)$ ,  $u = (u_1, \dots, u_n)$ ,  $\theta$ ,  $p$  and  $E$  denote the density, velocity, absolute temperature, pressure and total energy, respectively. The total energy  $E$  can be written by  $E = \frac{1}{2}|u|^2 + e$ , where  $e$  is the internal energy.  $\mathcal{T}$  is the stress tensor given by

$$\mathcal{T} = \mu(\nabla u + (\nabla u)^t) + \lambda(\operatorname{div} u)I,$$

where  $I$  is the identity matrix, and  $\mu$  and  $\lambda$  are the coefficient of viscosity and the second coefficient of viscosity, respectively. We also denote by  $\kappa \geq 0$  the coefficient of heat conduction. From the physical point of view, we assume

$$\mu \geq 0, \quad \lambda + \frac{2}{n}\mu \geq 0.$$

If  $\mu = \lambda = \kappa = 0$ , then we call the equations as compressible Euler equations for gas. On the other hand, if  $\mu > 0$  and  $\lambda + \frac{2}{n}\mu \geq 0$ , then we call the equations as compressible Navier-Stokes equations. In particular, we call the equations as heat-conducting compressible Navier-Stokes equations if  $\mu > 0$ ,  $\lambda + \frac{2}{n}\mu \geq 0$  and  $\kappa > 0$ . A polytropic gas is a gas satisfying the following state of equations:

$$p = R\rho\theta, \quad e = c_\nu\theta \quad \text{and} \quad p = A \exp(S/c_\nu) \rho^\gamma, \quad (4)$$

where  $R > 0$  is the gas constant,  $A$  a positive constant of absolute value,  $\gamma > 1$  the ratio of specific heats,  $c_\nu = \frac{R}{\gamma-1}$  the specific heat at constant volume and  $S$  the entropy.

The blow-up of smooth solutions of compressible Euler equations has been studied by several mathematicians. In 1985[3], T. C. Sideris showed that the life span  $T$  of the  $C^1$  solution of the compressible Euler equations is

finite when the initial data have compact support and the initial flow velocity is sufficiently large (super-sonic) in some region. In 1986[2], the blow-up of smooth solutions of compressible Euler equations, without external force and heat source, was shown on  $\mathbb{R}^3$  by T. Makino, S. Ukai and S. Kawashima, in case that the initial density and velocity have compact supports. In 1998[4], Z. Xin showed, in a different way from [2] and [3], the same blow-up result for the compressible Euler equations, when the initial density and initial velocity have compact supports. In the paper, he also showed the similar results for the compressible Navier-Stokes equations for polytropic gas with zero heat conduction (that is,  $\kappa = 0$ ) and without external forces and heat sources, when the initial density has compact support. His theorem was derived independently of the size of data, but his point of view cannot be applied for  $\kappa > 0$ , since in his argument the estimation for the lower bound of entropy is strongly necessary, which seems hard to be obtained for the case  $\kappa > 0$ .

As for the positive result, one may refer to [1]. In the paper [1], the authors showed the local existence of strong solutions of the compressible Navier-Stokes equations with  $\kappa \geq 0$  and nonnegative density.

In this paper, we extend the Xin's blow-up result to the heat-conducting compressible Navier-Stokes equations, that is, for the case  $\kappa > 0$ .

Before stating our main theorem, we introduce some notations. We denote by  $B_R = B_R(0)$  the ball in  $\mathbb{R}^n$  of radius  $R$  centered at the origin. We will use several physical quantities:

$$\begin{aligned} m_0 &= \int_{\mathbb{R}^n} \rho_0 dx \quad (\text{initial total mass}), \\ m_1 &= \int_{\mathbb{R}^n} \rho_0(x) |x|^2 dx \quad (\text{initial second moment}), \\ m_2 &= \int_{\mathbb{R}^n} \rho_0 u_0(x) \cdot x dx, \\ m_3 &= \int_{\mathbb{R}^n} \rho_0 E_0 dx \quad (\text{initial total energy}). \end{aligned}$$

We always assume that  $m_0 > 0$  and  $m_3 > 0$ . We denote  $H^k = H^k(\mathbb{R}^n)$  by  $L^2$ -Sobolev space of order  $k$  and  $C^1([0, T]; H^k)$  by the space-time function space of functions whose  $C^1$ -time derivative exists in space  $H^k$ -norm sense.

For the proof of blow-up, we have only to prove the following theorem.

**Theorem 1.1.** *We assume  $\mu > 0$ ,  $\lambda + \frac{2}{n}\mu > 0$  and  $\kappa \geq 0$ . Let  $\gamma > 1$  and  $T > 0$ . Suppose that  $(\rho, u, E) \in C^1([0, T]; H^k)$ ,  $k > 2 + [\frac{n}{2}]$  is a solution to the cauchy problem (1), (2) and (3) with initial data  $(\rho_0, u_0, E_0)$ . Suppose that the initial density  $\rho_0$  is compactly supported in a ball  $B_{R_0}$ . Then we have*

$$R_0^2 \geq \frac{m_1}{m_0} + 2\frac{m_2}{m_0}T + \min(2, n(\gamma - 1))\frac{m_3}{m_0}T^2.$$

**Remark 1.2.** *Let  $T^*$  be the life span of the solution  $(\rho, u, E)$ . Then since  $m_0$  and  $m_3$  are strictly positive, the theorem implies that  $T^*$  should be finite for  $\gamma > 1$ . The theorem also shows the relationship between the size of support and the life span. For example, the range of life span can be extended as the initial support of density become larger. Especially, we can expect the global existence of smooth solution of compressible Navier-Stokes equations in case that the initial density is positive but has decay at infinity.*

Our proof is based on some elementary argument like integration by parts and energy estimate. The key idea is to control the lower bound of the second moment of solution by the evolution of total energy via a quantity  $\int \rho u(x, t) \cdot x \, dx$ . The control of second moment by total energy enables us not to rely on the lower bound of entropy. The argument can easily give another proof for the compressible Euler equations with compactly supported initial data. We leave the details of proof to the readers.

## 2 Preliminaries

Since we consider the case of compactly supported initial density, we can assume that there is a positive constant  $R_0$  so that  $\text{supp}\rho_0 \subset B_{R_0}$ . We let  $(\rho, u, E) \in C^1([0, T]; H^k(\mathbb{R}^n))$ ,  $k > 2 + [\frac{n}{2}]$ , be a solution to the Cauchy problem (1), (2) and (3), with initial data  $(\rho_0, u_0, E_0) \in H^k(\mathbb{R}^n)$ . We denote by  $X(\alpha, t)$  the particle trajectory starting at  $\alpha$  when  $t = 0$ , that is,

$$\frac{d}{dt}X(\alpha, t) = u(X(\alpha, t), t) \quad \text{and} \quad X(\alpha, 0) = \alpha.$$

We set

$$\Omega(0) = \text{supp}\rho_0$$

and

$$\Omega(t) = \{x = X(\alpha, t) | \alpha \in \Omega(0)\}.$$

From the transport equation (1), one can easily show that

$$\text{supp}\rho(x, t) = \Omega(t)$$

and hence from the equation of state (4) that

$$p(x, t) = \theta(x, t) = 0 \quad \text{if} \quad x \in \Omega(t)^c.$$

Therefore, from the equation (2) and (3), we observe that

$$\text{div}(\mathcal{T})(x, t) = 0 \quad \text{and} \quad \text{div}(u\mathcal{T})(x, t) = 0 \quad \text{if} \quad x \in \Omega(t)^c.$$

The following lemma was shown by Z. Xin in [4].

**Lemma 2.1.** *We assume  $\mu > 0$ ,  $\lambda + \frac{2}{n}\mu > 0$  and  $\kappa \geq 0$ . Suppose that  $(\rho, u, E) \in C^1([0, T]; H^k(\mathbb{R}^n))$ ,  $k > 2 + [\frac{n}{2}]$ , is the solution of (1), (2) and (3). Then*

$$u(x, t) \equiv 0 \quad \text{in} \quad x \in \Omega(t)^c.$$

Moreover,  $\Omega(t) = \Omega(0)$  for all  $0 < t < T$ .

*Proof.* We observe that

$$\begin{aligned} \text{div}(u\mathcal{T}) - u\text{div}(\mathcal{T}) &= 2\mu \sum_{i=1}^n (\partial_i u_i)^2 + \lambda(\text{div}u)^2 \\ &\quad + \mu \sum_{i \neq j}^n (\partial_i u_j)^2 + 2\mu \sum_{i > j} (\partial_i u_j)(\partial_j u_i). \end{aligned}$$

Assume  $\lambda \leq 0$ . Then

$$\begin{aligned} \text{div}(u\mathcal{T}) - u\text{div}(\mathcal{T}) &\geq (2\mu + n\lambda) \sum_{i=1}^n (\partial_i u_i)^2 + \mu \sum_{i \neq j}^n (\partial_i u_j)^2 \\ &\quad + 2\mu \sum_{i > j} (\partial_i u_j)(\partial_j u_i) \\ &= (2\mu + n\lambda) \sum_{i=1}^n (\partial_i u_i)^2 + \mu \sum_{i > j}^n (\partial_i u_j + \partial_j u_i)^2. \end{aligned}$$

Assume  $\lambda > 0$ . Then

$$\begin{aligned}
\operatorname{div}(u\mathcal{T}) - u\operatorname{div}(\mathcal{T}) &\geq 2\mu \sum_{i=1}^n (\partial_i u_i)^2 + \mu \sum_{i \neq j}^n (\partial_i u_j)^2 \\
&\quad + 2\mu \sum_{i>j} (\partial_i u_j)(\partial_j u_i) \\
&= 2\mu \sum_{i=1}^n (\partial_i u_i)^2 + \mu \sum_{i>j}^n (\partial_i u_j + \partial_j u_i)^2.
\end{aligned}$$

Therefore, both of the cases imply that

$$\begin{aligned}
\partial_i u_i(x, t) &= 0 \\
\partial_i u_j(x, t) + \partial_j u_i(x, t) &= 0,
\end{aligned}$$

for all  $i, j = 1, \dots, n$  and  $x \in \Omega(t)^c$ .

This again implies  $u(x, t) \equiv 0$  on  $\Omega(t)^c$ . That is,  $u(X(\alpha, t), t) = 0$  if  $\alpha \notin \Omega(0)$ .

Thus we observe that

$$\begin{aligned}
X(\alpha, t) &= \alpha + \int_0^t u(X(\alpha, s), s) ds \\
&= \alpha \quad (\text{if } \alpha \in \Omega(0)^c).
\end{aligned}$$

This implies that

$$\Omega(t) = \Omega(0) \quad \text{for } 0 \leq t \leq T.$$

□

### 3 Proof of Theorem 1.1

Throughout this section we assume that  $\Omega(t) = \operatorname{supp}\rho(\cdot, t)$  is contained in a ball  $B_{R(t)}$ .

Multiplying  $|x|^2$  to (1) and integrating it over  $\mathbb{R}^n$ , we get the identity

$$\frac{d}{dt} \int_{\mathbb{R}^n} \rho |x|^2 dx = 2 \int_{\mathbb{R}^n} \rho u \cdot x dx. \quad (5)$$

If we take inner product by  $x$  to (2) and integrate it over  $\mathbb{R}^n$ , then we also get the identity

$$\frac{d}{dt} \int_{\mathbb{R}^n} \rho u \cdot x dx = \int_{\mathbb{R}^n} \rho |u|^2 dx + n \int_{\mathbb{R}^n} p dx. \quad (6)$$

Integrating (3) over  $\mathbb{R}^n$ , we finally get the identity

$$\frac{d}{dt} \int_{\mathbb{R}^n} \rho E \, dx = 0. \quad (7)$$

The integration by parts applied for deriving the above identities can be justified by Lemma 2.1.

Integrating (5), (6) and (7) over  $[0, t]$ , respectively, we obtain the following identities:

$$\begin{aligned} \int_{\mathbb{R}^n} \rho(x, t) |x|^2 \, dx &= \int_{\mathbb{R}^n} \rho_0(x) |x|^2 \, dx \\ &\quad + 2 \int_0^t \int_{\mathbb{R}^n} \rho u(x, s) \cdot x \, dx ds, \end{aligned} \quad (8)$$

$$\begin{aligned} \int_{\mathbb{R}^n} \rho u(x, s) \cdot x \, dx &= \int_{\mathbb{R}^n} \rho_0 u_0(x) \cdot x \, dx + \int_0^s \int_{\mathbb{R}^n} \rho |u|^2(x, \tau) \, dx d\tau \\ &\quad + n \int_0^s \int_{\mathbb{R}^n} p(x, \tau) \, dx d\tau, \end{aligned} \quad (9)$$

$$\int_{\mathbb{R}^n} \rho E(x, s) \, dx = \int_{\mathbb{R}^n} \rho_0 E_0(x) \, dx. \quad (10)$$

Using the definition of  $E$ , we have from (9) and (10)

$$\begin{aligned} \int_{\mathbb{R}^n} \rho u(x, s) \cdot x \, dx &= \int_{\mathbb{R}^n} \rho_0 u_0(x) \cdot x \, dx + 2 \int_0^s \int_{\mathbb{R}^n} \rho_0 E_0(x) \, dx d\tau \\ &\quad + \left( n - \frac{2}{\gamma - 1} \right) \int_0^s \int_{\mathbb{R}^n} p(x, \tau) \, dx d\tau. \end{aligned} \quad (11)$$

Now we first assume  $(n - \frac{2}{\gamma - 1}) \geq 0$ . Then by (11) we obtain

$$\int_{\mathbb{R}^n} \rho u(x, s) \cdot x \, dx \geq m_2 + 2m_3 s. \quad (12)$$

Substituting (12) into (8), we get

$$\int_{\mathbb{R}^n} \rho(x, t) |x|^2 \, dx \geq m_1 + 2m_2 t + 2m_3 t^2. \quad (13)$$

Secondly, we consider the case  $\gamma \in (1, 1 + \frac{2}{n})$ . By the equation of state  $p = (\gamma - 1)\rho e$  and the identity (9), we have

$$\int_{\mathbb{R}^n} \rho u(x, s) \cdot x \, dx = m_2 + 2m_3 s - (2 - n(\gamma - 1)) \int_0^s \int \rho e \, dx d\tau. \quad (14)$$

It follows from (10) and the definition of  $E$  that  $\int \rho e \, dx \leq m_3$ . Substituting this into (14), we have

$$\int \rho u(x, s) \cdot x \, dx \geq m_2 + n(\gamma - 1)m_3 s. \quad (15)$$

and hence from (8) and (15), we have

$$\int_{\mathbb{R}^n} \rho(x, t) |x|^2 \, dx \geq m_1 + 2m_2 t + n(\gamma - 1)m_3 t^2. \quad (16)$$

On the other hand, since  $\Omega(t) \subset B_{R(t)}$ , we can estimate the upper bound of the second moment as follows:

$$\begin{aligned} \int_{\mathbb{R}^n} \rho(x, t) |x|^2 \, dx &= \int_{\Omega(t)} \rho(x, t) |x|^2 \, dx = \int_{|x| \leq R(t)} \rho(x, t) |x|^2 \, dx \\ &\leq (R(t))^2 \int_{\mathbb{R}^n} \rho(x, t) \, dx \\ &= (R(t))^2 \int_{\mathbb{R}^n} \rho_0(x) \, dx. \end{aligned} \quad (17)$$

Thus from (13) and (17), we conclude that

$$m_0 R(t)^2 \geq m_1 + 2m_2 t + 2m_3 t^2$$

for  $\gamma \geq 1 + \frac{2}{n}$ , and from (16) and (17) that

$$m_0 R(t)^2 \geq m_1 + 2m_2 t + n(\gamma - 1)m_3 t^2$$

for  $1 < \gamma < 1 + \frac{2}{n}$ .

Since the solution is smooth in the time interval  $[0, T]$ , from lemma 2.1 we note that  $\Omega(t) = \Omega(0)$  for  $t \in [0, T]$  and hence  $R(t)$  can be chosen to be  $R_0$ . Therefore we have that for  $\gamma \geq 1 + \frac{2}{n}$

$$m_0 R_0^2 \geq m_1 + 2m_2 T + 2m_3 T^2$$

and for  $1 < \gamma < 1 + \frac{2}{n}$

$$m_0 R(0)^2 \geq m_1 + 2m_2 T + n(\gamma - 1)m_3 T^2.$$

This completes the proof of theorem.



## References

- [1] Cho, Y.; Kim, H. Existence results for viscous polytropic fluids with vacuum. in preprint.
- [2] Makino, T.; Ukai, S.; Kawashima, S. Sur la Solution à Support Compact de l'Equation d'Euler Compressible. Japan J. Appl. Math. **1986**, 3, 249-257.
- [3] Sideris, T. C. Formation of singularity in three dimensional compressible fluids. Comm. Math. Phys. **1985**, 101, 475-487.
- [4] Xin, Z. Blow up of smooth solutions to the compressible Navier-Stokes equations with compact density. Comm. Pure Appl. Math. **1998**, 51, 229-240.