SELF-DUAL CENTROAFFINE SURFACES OF CODIMENSION TWO WITH CONSTANT AFFINE MEAN CURVATURE

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SELF-DUAL CENTROAFFINE SURFACES OF CODIMENSION TWO WITH CONSTANT AFFINE MEAN CURVATURE

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ABSTRACT. We explicitly determine the minimal, self-dual centroaffine surfaces in $\mathbb{R}^4 \setminus \{0\}$ by giving a representation formula. Moreover, we describe the self-dual centroaffine surfaces with affine mean curvature -1.

1. INTRODUCTION

An immersion f of an n-dimensional manifold M into $\mathbb{R}^{n+2} \setminus \{0\}$ is called a *centroaffine immersion* of codimension two if the position vector $f(x)$ is transversal to $f_* T_x M$ at each point x of M.

Centroaffine immersions of codimension two were studied by Walter, and more generally by Nomizu and Sasaki. One of the most fundamental results given by them is: if f is non-degenerate, then f uniquely determines a pseudo-Riemannian metric on M, called the *affine fundamental form* of f; moreover, the affine fundamental form is invariant under the change $f \mapsto Af$ of centroaffine immersions by an element A of $SL(n+2;\mathbb{R})$.

In [1], the first author considered a certain area-variational problem with respect to the affine fundamental form and studied its extremals, which he called *minimal centroaffine immersions*. Furthermore, he showed that a space of the $SL(4;\mathbb{R})$-congruence classes of minimal ISDC immersions $\mathbb{R}^2 \to \mathbb{R}^4 \setminus \{0\}$ is in one-to-one correspondence with a space of the solutions for a wave equation on $\mathbb{R}^2$. Here, we refer to 'self-dual centroaffine immersions with indefinite affine fundamental form' as 'ISDC immersions'.

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In this paper, our main goal is to give a representation formula for the minimal ISDC surfaces (Thorem 3.1(1)). We also give a formula for the ISDC surfaces with affine mean curvature -1 (Theorem 3.1(2)).

In Section 4, we show another representation formula for the minimal ISDC surfaces (Theorem 4.2). As an application of the second formula, we give a one-parameter family of minimal ISDC surfaces joining two typical examples: a quadric and the Clifford torus (Example 4.3).

We also give the affine mean curvature formula for a non-parametric centroaffine immersion in Section 5. Section 2 is devoted to collect basic definitions and facts on centroaffine immersions.

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2. Preliminaries

In this section, we briefly review geometry of centroaffine immersions of a simply-connected, oriented, 2-dimensional manifold M into $\mathbb{R}^4\setminus\{0\}$. For further detail, we refer the reader to [2].

Let $f : M \to \mathbb{R}^4\setminus\{0\}$ be a centroaffine immersion. We denote by D the standard flat affine connection of $\mathbb{R}^4$, and fix a parallel volume form Det once and for all. A vector field ξ along f is called a *normal vector field* if it satisfies that at each point x of M, the tangent space $T_{f(x)}\mathbb{R}^4$ is decomposed as

$$T_{f(x)}\mathbb{R}^4 = f_*T_xM \oplus \mathbb{R}\,\xi_x \oplus \mathbb{R}\,f(x), \tag{2.1}$$

and that the volume form θ defined by

$$\theta(X,Y) := \mathrm{Det}(f_*X, f_*Y, \xi, f), \quad X, Y \in \Gamma(TM), \tag{2.2}$$

is compatible with the orientation of M.

When we choose a normal vector field ξ, we can determine a torsion-free affine connection ∇, two symmetric $(0,2)$-tensor fields h and T, a $(1,1)$-tensor field S, and two 1-forms τ and P by

$$\begin{aligned} D_X f_*Y &= f_*\nabla_X Y + h(X,Y)\xi + T(X,Y)f, \\ D_X\xi &= -f_*SX + \tau(X)\xi + P(X)f, \end{aligned} \tag{2.3}$$

according to the decomposition (2.1).

It is easily shown that the conformal class of h does not depend on the choice of ξ. When h is non-degenerate (resp. definite, indefinite), f is said to be *non-degenerate* (resp. *definite*, *indefinite*). If f is non-degenerate, there is a normal vector field ξ satisfying that

$$
(2.4)\qquad
\begin{aligned}
&\tau = 0,\\
&\theta = \mathrm{Vol}_h, \quad \text{where } \mathrm{Vol}_h \text{ is the volume form with respect to } h,\\
&\mathrm{tr}_h\{(X,Y) \mapsto T(X,Y) + h(SX,Y)\} = 0.
\end{aligned}
$$

Moreover, such a ξ is unique. We call the ξ the *Blaschke normal vector field.* From now on, we always choose the Blaschke normal vector field as ξ for a given non-degenerate f, and we call the metric h the *affine fundamental form* of f.

For an element A of $SL(4;\mathbb{R})$, both D and Det are invariant under a transformation $v \mapsto Av$ of $\mathbb{R}^4$. Hence, Af is also a non-degenerate centroaffine immersion and its Blaschke normal vector field is $A\xi$; moreover, f and Af induce the same set of the geometric quantities ∇, h, T, S and P. Conversely, if two non-degenerate centroaffine immersions f_1, f_2 induce completely the same quantities, f_1 and f_2 are *congruent*, that is, there exists an element A of $SL(4;\mathbb{R})$ such that $f_2 = Af_1$.

For later use, we recall the equations of Gauss, of Codazzi, and of Ricci for a centroaffine immersion $f : M \to \mathbb{R}^4\setminus\{0\}$: Let ∇, h, T, S and P be the objects determined by (2.3). Then they satisfy

$$
(2.5)\qquad
\begin{aligned}
&R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY - T(Y,Z)X + T(X,Z)Y,\\
&(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z),\\
&(\nabla_X T)(Y,Z) + h(Y,Z)P(X) = (\nabla_Y T)(X,Z) + h(X,Z)P(Y),\\
&(\nabla_X S)Y + P(X)Y = (\nabla_Y S)X + P(Y)X,\\
&h(X,SY) = h(Y,SX),\\
&T(X,SY) - T(Y,SX) = dP(X,Y),
\end{aligned}
$$

where R denotes the curvature tensor field of the induced connection ∇.

Conversely, if a torsion-free affine connection ∇ and tensor fields h, T, S, P are given on M, and if they satisfy the relations in (2.5), then we can construct a non-degenerate centroaffine immersion of M into $\mathbb{R}^4\setminus\{0\}$ with Blaschke normal vector field ξ such that decomposition (2.3) of $D_X f_* Y$ and $D_X\xi$ holds.

Let $\mathbb{R}_4$ denote the dual space of $\mathbb{R}^4$ endowed with the volume form induced by Det. For a given centroaffine immersion $f : M \to \mathbb{R}^4\setminus\{0\}$, we define the *dual map* $f^* : M \to \mathbb{R}_4\setminus\{0\}$ by

$$
(2.6)\qquad f^*(x)(f_*X) = 0, \quad f^*(x)(\xi(x)) = 1, \quad \text{and} \quad f^*(x)(f(x)) = 0,
$$

for each $x \in M$ and $X \in T_xM$. When f is non-degenerate, f^* is also a non-degenerate centroaffine immersion with Blaschke normal vector field ξ^*, which satisfies

(2.7) $$\xi^*(x)(f_*X) = 0, \quad \xi^*(x)(\xi(x)) = 0, \quad \text{and} \quad \xi^*(x)(f(x)) = 1.$$

Moreover, the dual map of f^* is congruent to f.

We note that the differentials of f and of f^* are related by the affine fundamental form h as follows:

(2.8) $$\big((f^*)_*X\big)(f_*Y) = -h(X,Y).$$

For the proof, see [2, Lemma 3.1].

Definition 2.1. A centroaffine immersion $f : M \to \mathbb{R}^4\backslash\{0\}$ is said to be *self-dual* if there exists a volume-preserving linear map $L : \mathbb{R}^4 \to \mathbb{R}_4$ such that $f^* = Lf$.

Definition 2.2. For a non-degenerate centroaffine immersion $f : M \to \mathbb{R}^4\backslash\{0\}$, the *affine mean curvature* H is defined to be $(1/2)\,\mathrm{tr}\, S$. A non-degenerate centroaffine immersion f is said to be *minimal* if the affine mean curvature H vanishes everywhere.

We abbreviate the phrase 'self-dual centroaffine immersion with indefinite affine fundamental form' to 'ISDC immersion'.

Example 2.3 ([1]). The Clifford torus $\phi : \mathbb{R}^2 \to \mathbb{R}^4$ and a quadric $\phi_0 : \mathbb{R}^2 \to \mathbb{R}^4$ defined below are minimal ISDC immersions.

(2.9) $$\phi(u^1,u^2) := \frac{1}{\sqrt{2}}\begin{bmatrix} \cos(u^1+u^2) \\ \sin(u^1+u^2) \\ \cos(-u^1+u^2) \\ \sin(-u^1+u^2) \end{bmatrix},$$

(2.10) $$\phi_0(u^1,u^2) := \frac{1}{\sqrt{2}}\begin{bmatrix} u^1+u^2 \\ -u^1+u^2 \\ \sqrt{2}u^1u^2 \\ \sqrt{2} \end{bmatrix}.$$

They have the same induced connection and affine fundamental form

(2.11) $$\nabla\partial_i = \nabla^0\partial_i = 0 \quad \text{and} \quad h = h^0 = 2du^1du^2.$$

Moreover, the Blaschke normal vector field of ϕ_0 is constant.

Remark 2.4. Nomizu and Sasaki [2] proved that the image of a centroaffine immersion lies in an affine hyperplane which does not contain the origin if T determined by (2.3) vanishes identically. In this case, a minimal centroaffine surface is reduced to an affine minimal surface (or, one may say, an affine maximal surface) in $\mathbb{R}^3$.

3. Representation Formula for Self-Dual Centroaffine Surfaces with Constant Affine Mean Curvature

Throughout Sections 3 and 4, we discuss problems locally, hence we may assume that M is a simply-connected domain of $\mathbb{R}^2$ with coordinates (u^1, u^2). Furthermore, we always identify two centroaffine immersions $M \to \mathbb{R}^4 \setminus \{0\}$ if they are congruent.

The aim of this section is to prove the following theorem:

Theorem 3.1. *Let ϕ_0 be the quadric* (2.10).

(1) *For arbitrary two functions μ, ν in one variable,*

$$f(u^1, u^2) := e^{\mu(u^1)+\nu(u^2)}\phi_0(u^1, u^2)$$

is a minimal ISDC surface. Conversely, any minimal ISDC surface is locally represented in this form.

(2) *For arbitrary two functions μ, ν in one variable,*

$$f(u^1, u^2) := \frac{\exp\{(\mu(u^1) - \nu(u^2))/2\}}{\int^{u^1} \exp \mu(t)dt + \int^{u^2} \exp(-\nu(t))dt}\phi_0(u^1, u^2)$$

is an ISDC surface with affine mean curvature -1. Conversely, any ISDC surface with affine mean curvature -1 is locally represented in this form.

In order to show the theorem, we prove the following two lemmas.

Lemma 3.2. *For any ISDC surface f, there exists a function ω so that $f = e^\omega \phi_0$.*

Lemma 3.3. *The affine mean curvature H of an ISDC surface $f = e^\omega \phi_0$ is give by*

$$H = -e^{-2\omega}\partial_1\partial_2\omega. \tag{3.1}$$

Since equation (3.1) with $H = 0$ is a (linear) wave equation, we easily obtain Theorem 3.1(1). When $H = -1$, (3.1) is Liouville's equation; the solutions are derived from those of the wave equation above through a Bäcklund transformation. We refer the reader to [3] for further information about this subject, and we note only the following fact that establishes Theorem 3.1: a function ω is a solution of $\partial_1\partial_2\omega = e^{2\omega}$ if and only if

$$\omega = \frac{\mu(u^1) - \nu(u^2)}{2} - \log\left(\int^{u^1} e^{\mu(t)}dt + \int^{u^2} e^{-\nu(t)}dt\right).$$

Proof of Lemma 3.2. For any self-dual centroaffine surface f, there exists a non-degenerate quadratic form Q_f such that the quadratic cone C_f determined by Q_f contains the image of f:

$$f(M) \subset C_f = \{x \in \mathbb{R}^4 \,|Q_f(x) = 0\}.$$

This fact was shown in [2, Proposition 3.5], however, we give a proof here since we need more precise information about the cone C_f.

By definition, there exists a linear isomorphism $L_f : \mathbb{R}^4 \to \mathbb{R}_4$ such that $f^* = L_f f$. From $f^*(f) = 0$, we see that the quadratic form Q_f determined by $Q_f(x) = (L_f x)(x)$ satisfies $Q_f(f) = 0$.

The quadratic form Q_f can be explicitly written as follows: let $\{e_1, e_2\}$ be a basis of the tangent space $T_u M$ at $u \in M$. We put

$$\begin{cases} E_1 := f_* e_1, \\ E_2 := f_* e_2, \\ E_3 := 1/\sqrt{2}\,(\xi(u) + f(u)), \\ E_4 := 1/\sqrt{2}\,(\xi(u) - f(u)), \end{cases} \qquad \begin{cases} E_1^* := (f^*)_* e_1, \\ E_2^* := (f^*)_* e_2, \\ E_3^* := 1/\sqrt{2}\,(\xi^*(u) + f^*(u)), \\ E_4^* := 1/\sqrt{2}\,(\xi^*(u) - f^*(u)), \end{cases}$$

where ξ^* is the Blaschke normal vector field for f^*. Since $L_f f = f^*$, we have $L_f E_i = E_i^*$, $i = 1, 2, 3, 4$. Hence, we see that for any vector $x = \sum_{i=1}^4 x^i E_i \in \mathbb{R}^4$,

$$\begin{aligned} Q_f(x) = (L_f x)(x) &= \sum_{i,j=1}^{4} x^i x^j E_i^*(E_j) \\ &= \begin{bmatrix} x^1\, x^2\, x^3\, x^4 \end{bmatrix} \begin{bmatrix} -h(e_1,e_1) & -h(e_1,e_2) & 0 & 0 \\ -h(e_2,e_1) & -h(e_2,e_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}, \end{aligned}$$

where the last equality follows from (2.6), (2.7) and (2.8).

Since both f and ϕ_0 have the indefinite affine fundamental forms, the cones C_f and C_{ϕ_0} are $SL(4;\mathbb{R})$-congruent. This completes the proof. □

Proof of Lemma 3.3. Step 1. Let ξ_0 and ξ be the Blaschke normal vector fields of the quadric ϕ_0 and an ISDC surface $f = e^{\omega}\phi_0$, respectively. We choose a positive function ρ, a function a and a vector field U on M so that

$$\xi = \rho^{-1}(\xi_0 + a\phi_0 + \phi_{0*}U).$$

By definition, we have

$$D_X f_* Y = f_* \nabla_X Y + h(X,Y)\xi + T(X,Y)f,$$

and

$$D_X f_* Y = D_X\big((e^\omega \phi_0)_* Y\big) = D_X\left[(Ye^\omega)\phi_0 + e^\omega \phi_{0*} Y\right].$$

Comparing the tangent components with respect to ϕ_0, the ξ_0-components, and the ϕ_0-components of the right-hand sides of both equations, we get

$$(3.2)\qquad \begin{aligned} &\nabla_X Y + e^{-\omega}\rho^{-1} h(X,Y) U = \nabla^0_X Y + (X\omega) Y + (Y\omega) X,\\ &h(X,Y) = e^\omega \rho h^0(X,Y),\\ &T(X,Y) + (\nabla_X Y)\,\omega + a e^{-\omega}\rho^{-1} h(X,Y)\\ &\qquad\qquad = T^0(X,Y) + X(Y\omega) + (X\omega)\,(Y\omega), \end{aligned}$$

where ∇^0, h^0, T^0, S^0, P^0 are the objects determined by (2.3) for ϕ_0. A similar calculation on $D_X\xi$ derives

$$(3.3)\qquad \begin{aligned} &e^\omega \rho S X = S^0 X - aX - \nabla^0_X U + (X \log\rho) U,\\ &0 = -X\,(\log\rho) + h^0(X,U),\\ &e^\omega \rho P(X) - e^\omega \rho (SX)\omega = P^0(X) + T^0(X,U) + Xa - aX\,(\log\rho). \end{aligned}$$

By the second equation of (3.2) and the second condition of (2.4), we have

$$\begin{aligned} e^\omega \rho \sqrt{|\det h^0(e_i,e_j)|} &= \sqrt{|\det h(e_i,e_j)|}\\ &= \mathrm{Det}(f_* e_1, f_* e_2, \xi, f)\\ &= e^{3\omega}\rho^{-1}\,\mathrm{Det}(\phi_{0*} e_1, \phi_{0*} e_2, \xi_0, \phi_0). \end{aligned}$$

This implies that $\rho = e^\omega$. By the second equation of (3.3), we obtain $U = \mathrm{grad}_{h^0}\,\omega$.

It should be remarked that we have not yet used the fact that ϕ_0 is a quadric. In the next step, we will determine the function a, using the data of ϕ_0.

Step 2. By Example 2.3 and Step 1, we get

$$(3.4)\qquad \begin{aligned} h &= 2e^{2\omega} du^1 du^2,\\ U &= (\partial_2\omega)\,\partial_1 + (\partial_1\omega)\,\partial_2, \end{aligned}$$

where we denote $\partial/\partial u^i$ by ∂_i for $i = 1, 2$. Thus we have

$$(3.5)\qquad \nabla \partial_i = 2\partial_i\omega du^i \otimes \partial_i, \quad i = 1, 2,$$

$$\begin{cases} T(\partial_1,\partial_1) &= \partial_1\partial_1\omega - (\partial_1\omega)^2,\\ T(\partial_1,\partial_2) &= \partial_1\partial_2\omega + \partial_1\omega\partial_2\omega - a,\\ T(\partial_2,\partial_2) &= \partial_2\partial_2\omega - (\partial_2\omega)^2, \end{cases}$$

and

$$\begin{cases} S\partial_1 &= -e^{-2\omega}\left[\{\partial_1\partial_2\omega - \partial_1\omega\partial_2\omega + a\}\,\partial_1 + \{\partial_1\partial_1\omega - (\partial_1\omega)^2\}\,\partial_2\right], \\ S\partial_2 &= -e^{-2\omega}\left[\{\partial_2\partial_2\omega - (\partial_2\omega)^2\}\,\partial_1 + \{\partial_1\partial_2\omega - \partial_1\omega\partial_2\omega + a\}\,\partial_2\right]. \end{cases}$$

Here we used the first equation of (3.2), the last equation of (3.2), and the first equation of (3.3). As a consequence of resulting formulas, the last condition of (2.4) implies that $0 = \operatorname{tr}_h T + \operatorname{tr} S = 4e^{-2\omega}\left[\partial_1\omega\partial_2\omega - a\right]$. Hence, we have

$$a = \partial_1\omega\partial_2\omega.$$

Accordingly, we obtain

$$\begin{aligned} T &= \left\{\partial_1\partial_1\omega - (\partial_1\omega)^2\right\}(du^1)^2 + 2\partial_1\omega\partial_2\omega du^1 du^2 \\ &\quad + \left\{\partial_2\partial_2\omega - (\partial_2\omega)^2\right\}(du^2)^2 \\ S &= -e^{-2\omega}\left[\partial_1\partial_2\omega du^1 \otimes \partial_1 + \left\{\partial_1\partial_1\omega - (\partial_1\omega)^2\right\} du^1 \otimes \partial_2\right. \\ &\quad \left. + \left\{\partial_2\partial_2\omega - (\partial_2\omega)^2\right\} du^2 \otimes \partial_1 + \partial_1\partial_2\omega du^2 \otimes \partial_2\right], \\ P &= 0, \end{aligned} \tag{3.6}$$

where the last equation is deduced from the third equation of (3.3). By (3.6), we have

$$H = \frac{1}{2}\operatorname{tr} S = -e^{-2\omega}\partial_1\partial_2\omega,$$

thereby completing the proof. □

A representation formula for definite centroaffine immersions can be verified in a similar fashion, and we give the result without proof.

Theorem 3.4. *We define an immersion* $\psi : \mathbb{R}^2 \to \mathbb{R}^4$ *by*

$$\psi(u^1, u^2) = \frac{1}{2}\begin{bmatrix} 1 + (u^1)^2 + (u^2)^2 \\ 1 - (u^1)^2 - (u^2)^2 \\ 2u^1 \\ 2u^2 \end{bmatrix}, \quad (u^1, u^2) \in \mathbb{R}^2.$$

Then, we have:

(1) *For a holomorphic function* λ *in the complex variable* $z = u^1 + \sqrt{-1}u^2$,

$$f(u^1, u^2) := e^{-2\operatorname{Im}\lambda(z)}\psi(u^1, u^2)$$

is a definite minimal self-dual centroaffine surface. Conversely, any definite minimal self-dual centroaffine surface is locally represented in this form.

(2) *For a holomorphic function* λ *in* z,

$$f(u^1, u^2) = \frac{\exp\left(\operatorname{Re}\lambda(z)\right)}{2\operatorname{Re}\int^z \exp\lambda(w)dw}\psi(u^1, u^2)$$

is a definite self-dual centroaffine surface with affine mean curvature -1. *Conversely, any definite self-dual centroaffine surface with affine mean curvature* -1 *is locally represented in this form.*

4. Another Representation Formula for Minimal Self-Dual Centroaffine Surfaces

In this section, we give another representation formula for the minimal self-dual centroaffine surfaces of codimension two.

Let $\widetilde{f} : M \to \mathbb{R}^4 \backslash \{0\}$ be a minimal ISDC surface. By Theorem 3.1, there exist two functions μ and ν in one variable such that $\widetilde{f} = \exp\big(\mu(u^1)+\nu(u^2)\big)\phi_0$. We re-parametrize the surface as follows:

$$f(u) = \widetilde{f}(\widetilde{u}(u)), \quad \text{where } \widetilde{u}(u) = \left(\int^{u^1} e^{-2\mu(t)} dt, \int^{u^2} e^{-2\nu(t)} dt \right).$$

Then, it follows from (3.4)–(3.6) that the objects induced by f are:

$$\begin{aligned} &\nabla \partial_i = 0, \\ &h = 2du^1 du^2, \\ &T = a(u^1)(du^1)^2 + b(u^2)(du^2)^2, \\ &S = -a(u^1)du^1 \otimes \partial_2 - b(u^2)du^2 \otimes \partial_1, \\ &P = 0, \end{aligned} \tag{4.1}$$

where a, b are functions in one variable given by $a = e^{-4\mu}\{\mu'' - (\mu')^2\}$ and $b = e^{-4\nu}\{\nu'' - (\nu')^2\}$.

Conversely, if functions a and b in one variable are given, then ∇, h, T, S and P defined by (4.1) satisfy the Gauss, Codazzi and Ricci equations (2.5). We can therefore construct a minimal ISDC surface with these quantities. The following lemma gives this surface in a more explicit form.

Lemma 4.1. *Let* p_{j0}, p'_{j0}, q_{j0} *and* q'_{j0} $(j = 1, 2)$ *be any fixed constants satisfying*

$$\begin{vmatrix} p'_{10}q_{10} & p_{10}q'_{10} & p'_{10}q'_{10} & p_{10}q_{10} \\ p'_{10}q_{20} & p_{10}q'_{20} & p'_{10}q'_{20} & p_{10}q_{20} \\ p'_{20}q_{10} & p_{20}q'_{10} & p'_{20}q'_{10} & p_{20}q_{10} \\ p'_{20}q_{20} & p_{20}q'_{20} & p'_{20}q'_{20} & p_{20}q_{20} \end{vmatrix} = 1. \tag{4.2}$$

For given functions $a, b \in C^\infty(I)$, *let* p_j *and* $q_j \in C^\infty(I)$ $(j = 1, 2)$ *be the solutions of the following ordinary differential equations*

$$p''_j = ap_j, \qquad q''_j = bq_j, \tag{4.3}$$

with initial conditions

$$\begin{cases} p_j(0) = p_{j0}, \\ p'_j(0) = p'_{j0}, \end{cases} \qquad \begin{cases} q_j(0) = q_{j0}, \\ q'_j(0) = q'_{j0}. \end{cases} \tag{4.4}$$

If we set

$$f(u^1, u^2) := \begin{bmatrix} p_1(u^1)q_1(u^2) \\ p_1(u^1)q_2(u^2) \\ p_2(u^1)q_1(u^2) \\ p_2(u^1)q_2(u^2) \end{bmatrix}, \tag{4.5}$$

then f is a minimal ISDC surface with invariants given by (4.1).

Proof. We prove the lemma while we verify that the Blaschke normal vector field ξ is given by

$$\xi = \partial_1 \partial_2 f. \tag{4.6}$$

By (4.5) and (4.3), we have

$$\partial_1 \partial_1 f = af, \quad \partial_2 \partial_2 f = bf. \tag{4.7}$$

Moreover, we have $\partial_j \operatorname{Det}(\partial_1 f, \partial_2 f, \xi, f) = 0$ by a direct calculation. Accordingly, the initial conditions (4.4) with (4.2) imply

$$\operatorname{Det}(\partial_1 f, \partial_2 f, \xi, f) = 1,$$

from which we see that ξ satisfies the condition (2.1).

Equations (4.6) and (4.7) imply that

$$D_{\partial_1} \xi = a \partial_2 f, \quad D_{\partial_2} \xi = b \partial_1 f. \tag{4.8}$$

It follows from (4.7) and (4.8) that the quantities ∇, h, T, S, τ and P are given by (4.1).

We can now easily see that each condition of (2.4) holds. Then ξ is the Blaschke normal vector field of f, and hence f is a minimal ISDC surface. $\square$

Theorem 4.2. *Let p, q be functions in one variable such that*

$$p(0) = 1, \quad p'(0) = 0, \quad q(0) = 1, \quad q'(0) = 0.$$

If we set

$$f(u^1, u^2) = \begin{bmatrix} p(u^1)q(u^2) \\ p(u^1)\widehat{q}(u^2) \\ \widehat{p}(u^1)q(u^2) \\ \widehat{p}(u^1)\widehat{q}(u^2) \end{bmatrix},$$

where

$$\widehat{p}(u^1) := p(u^1) \int_0^{u^1} p^{-2}(t)dt, \quad \widehat{q}(u^2) := q(u^2) \int_0^{u^2} q^{-2}(t)dt,$$

then f is a minimal ISDC surface.

Conversely, any minimal ISDC surface is locally represented as above.

Proof. Setting

$$\begin{aligned} p_{10} = p'_{20} = q_{10} = q'_{20} = 1, \\ p'_{10} = p_{20} = q'_{10} = q_{20} = 0, \end{aligned}$$

we can check that $p_1 := p$, $p_2 := \widehat{p}$, $q_1 := q$ and $q_2 := \widehat{q}$ satisfy conditions (4.2), (4.3) and (4.4) in Lemma 4.1 for $a := p''p^{-1}$ and $b := q''q^{-1}$. □

Example 4.3. We take

$$p(t) = q(t) = \cos(\sqrt{l}t), \quad l > 0,$$

as p and q in Theorem 4.2. Then $\widehat{p}(t) = \widehat{q}(t) = \sqrt{l}^{-1}\sin(\sqrt{l}t)$, and we obtain a one-parameter family $\{f_l;\ l > 0\}$ of minimal ISDC surfaces containing the Clifford torus ϕ and the quadric ϕ_0; indeed, $f_1 = \phi$ and $\lim_{l\to 0} f_l = \phi_0$.

In the case where f is definite, the following theorem holds.

Theorem 4.4. *Let λ be a holomorphic function in $z = u^1 + \sqrt{-1}u^2$ such that*

$$\lambda(0) = 1, \quad \lambda'(0) = 0.$$

If we set

$$\lambda_1(z) = \lambda(z), \quad \lambda_2(z) = \lambda(z) \int_0^z \lambda^{-2}(w)dw,$$

and

$$f(z) = \frac{1}{2} \begin{bmatrix} \overline{\lambda_1(z)}\lambda_1(z) + \overline{\lambda_2(z)}\lambda_2(z) \\ \overline{\lambda_1(z)}\lambda_1(z) - \overline{\lambda_2(z)}\lambda_2(z) \\ \overline{\lambda_2(z)}\lambda_1(z) + \overline{\lambda_1(z)}\lambda_2(z) \\ \sqrt{-1}\left(\overline{\lambda_2(z)}\lambda_1(z) - \overline{\lambda_1(z)}\lambda_2(z)\right) \end{bmatrix},$$

then f is a minimal self-dual centroaffine immersion with affine fundamental form $h = dzd\overline{z}$.

Conversely, any minimal self-dual centroaffine surface with definite affine fundamental form is locally represented as above.

Example 4.5. For a non-negative constant k, we take $\cosh(kz)$ as λ in the theorem above. Then $\lambda_1(z) = \cosh(kz)$ and $\lambda_2(z) = \sinh(kz)/k$, and hence we obtain a family $\{f_k;\ k \in [0, \infty)\}$ of minimal self-dual centroaffine immersions of $\mathbb{C}$ into $\mathbb{R}^4 \setminus \{0\}$. For instance, f_0 coincides with ψ in Theorem 3.4 and

$$f_1(z) = \frac{1}{2}\begin{bmatrix} \cosh(2u^1) \\ \cosh(2\sqrt{-1}u^2) \\ \sinh(2u^1) \\ \sqrt{-1}\sinh(-2\sqrt{-1}u^2) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \cosh(2u^1) \\ \cos(2u^2) \\ \sinh(2u^1) \\ \sin(2u^2) \end{bmatrix}.$$

5. Affine Mean Curvature of Centroaffine Surfaces of Non-parametric Type

In this section, we describe the affine mean curvature of a centroaffine surface of non-parametric type.

Theorem 5.1. *Let $f : M \to \mathbb{R}^4 \setminus \{0\}$ be a centroaffine immersion given as $f(u) := {}^t(u^1, u^2, \varphi(u), \psi(u))$. The affine mean curvature of f is given by*

$$H = -\frac{1}{4}|\det_{\theta^0} h^0|^{1/4}\{|\det_{\theta^0} h^0|^{1/4}\triangle_{h^0}|\det_{\theta^0} h^0|^{-1/4} + \operatorname{tr}_{h^0} T^0 - \operatorname{tr} S^0\} \tag{5.1}$$

with

$$\begin{aligned} \theta^0 &:= (\psi - \sum_{l=1}^{2} u^l\partial_l\psi)du^1 \wedge du^2, \\ h^0_{ij} &:= \frac{(\psi - \sum_{l=1}^{2} u^l\partial_l\psi)\partial_i\partial_j\varphi - (\varphi - \sum_{l=1}^{2} u^l\partial_l\varphi)\partial_i\partial_j\psi}{\psi - \sum_{l=1}^{2} u^l\partial_l\psi}, \\ T^0_{ij} &:= (\psi - \sum_{l=1}^{2} u^l\partial_l\psi)^{-1}\partial_i\partial_j\psi, \\ S^0 &:= 0, \end{aligned} \tag{5.2}$$

where $\triangle_{h^0}$ denotes the Laplacian with respect to h^0, and $\det_{\theta^0} h^0 := \det\big(h^0(e_i, e_j)\big)$ for a basis (e_1, e_2) of T_uM with $\theta^0(e_1, e_2) = 1$.

Proof. Step 1. We put $\xi_0 = {}^t(0,0,1,0)$ and may assume that $M \subset \{u \in \mathbb{R}^2 \,|\, \det \Omega(u) > 0\}$ where

$$\Omega(u) := [f_*\partial_1, f_*\partial_2, \xi_0, f]\,(u) = \begin{bmatrix} 1 & 0 & 0 & u^1 \\ 0 & 1 & 0 & u^2 \\ \partial_1\varphi & \partial_2\varphi & 1 & \varphi \\ \partial_1\psi & \partial_2\psi & 0 & \psi \end{bmatrix}(u) \in GL(4;\mathbb{R}).$$

Let ∇^0, h^0, T^0, S^0, τ^0, P^0 and θ^0 be the geometric quantities defined as in (2.3) and (2.2) with respect to ξ_0. Because ξ_0 is constant, S^0, τ^0 and P^0 vanish identically. We calculate (5.2) as

$$\theta^0\,(\partial_1, \partial_2) = \det \Omega,$$

$$\begin{bmatrix} \Gamma^{0\,1}_{ij} \\ \Gamma^{0\,2}_{ij} \\ h^0_{ij} \\ T^0_{ij} \end{bmatrix} = \Omega^{-1} \begin{bmatrix} 0 \\ 0 \\ \partial_i\partial_j\varphi \\ \partial_i\partial_j\psi \end{bmatrix},$$

where $\nabla^0_{\partial_i}\partial_j = \sum_{k=1}^{2} \Gamma^{0\,k}_{ij}\partial_k$.

Step 2. We choose a positive function ρ, a function a and a vector field U on M so that

$$\xi = \rho^{-1}(\xi_0 + af + f_*U)$$

is the Blaschke normal vector field of f. Then we have

$$\begin{aligned} \rho &= |\det_{\theta^0} h^0|^{-1/4}, \\ U &= \operatorname{grad}_{h^0} \log \rho, \\ a &= \frac{1}{4}(\operatorname{tr}_{h^0} T^0 + \operatorname{tr} S^0 - \rho^{-1}\triangle_{h^0}\rho). \end{aligned} \tag{5.3}$$

To prove it, we remark that (3.2) and (3.3) hold in this case with $\omega = 1$ (see also [2]). We get the first equation of (5.3) from $\theta = \rho^{-1}\theta^0$, the second equation of (3.2) and the third condition of (2.4), and the second equation of (5.3) from the second equation of (3.3). The third equation of (5.3) is obtained as follows. From (3.2) and (3.3), we have

$$T(X,Y) + h(SX,Y) = T^0(X,Y) + h^0(S^0X,Y) - 2ah^0(X,Y) - h^0(\nabla_X U, Y),$$

which implies

$$\begin{aligned} 0 &= \operatorname{tr}_h\{(X,Y) \mapsto T(X,Y) + h(SX,Y)\} \\ &= \rho^{-1}\left(\operatorname{tr}_{h^0} T^0 + \operatorname{tr} S^0 - 4a - \operatorname{div}^{\nabla} U\right). \end{aligned}$$

Noting $\nabla \operatorname{Vol}_h = 0$, we calculate

$$\begin{aligned} a &= \frac{1}{4}\left(\operatorname{tr}_{h^0} T^0 + \operatorname{tr} S^0 - \operatorname{div}^\nabla \operatorname{grad}_{h^0} \log \rho\right) \\ &= \frac{1}{4}\left(\operatorname{tr}_{h^0} T^0 + \operatorname{tr} S^0 - \operatorname{div}^\nabla \operatorname{grad}_h \rho\right) \\ &= \frac{1}{4}\left(\operatorname{tr}_{h^0} T^0 + \operatorname{tr} S^0 - \triangle_h \rho\right). \end{aligned}$$

Since $\dim M = 2$, we obtain the third equation of (5.3).

By (5.3), we obtain that

$$\begin{aligned} \operatorname{tr} S &= -\operatorname{tr}_h T \\ &= -\frac{1}{\rho}\left(\operatorname{tr}_{h^0} T^0 - 2a\right) \\ &= -\frac{1}{2\rho}\left(\operatorname{tr}_{h^0} T^0 - \operatorname{tr} S^0 + \rho^{-1} \triangle_{h^0} \rho\right), \end{aligned}$$

which implies (5.1). □

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