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ABSTRACT. We study hyperbolic invariants of hyperbolic plane curves as applications of the singularity theory of smooth functions

1. INTRODUCTION

One of the main techniques for applying the singularity theory to Euclidean differential geometry is to consider the distance squared function and the height function on a submanifold in Euclidean space ([1,2,8]). This is the Thom's idea for generic differential geometry and Porteous[10] is the first person who realized this idea. Recently, the authors apply this idea to several kinds of geometry ([4,5,6]). In these cases the corresponding functions depend on each geometry. In this paper we apply this idea to hyperbolic differential geometry on plane curves. We adopt the hyperbola H^2_+ in Minkowski 3-space as the model of the hyperbolic plane. Since H^2_+ is a Riemannian manifold, the direct analogous method is to consider the geodesic distance function instead of the distance squared function in Euclidean differential geometry. It is, however, very hard to proceed the calculation. In this paper we consider the Lorenzian height function on a curve in H^2_+. As an application of singularity theory to the Lorenzian height function, we detect the hyperbolic evolute and classify singularities of it (Theorems 2.1, 2.2). By the main result, we understand that the hyperbolic evolute can be defined in the case when the geodesic curvature κ_g of the curve does not equal to ± 1. Moreover, if $\kappa_g^2 > 1$, then the evolute is located in H^2_+, otherwise it is in the pseudo sphere S^2_1.

On the other hand, we can define the notion of *the lightcone Gauss map* of a curve γ in H^2_+. For the study of the lightcone Gauss map, we define the notion of the lightcone height function of the curve and apply the singularity theory again. We use the basic notions and results in Lorentzian geometry (cf., [9]). The basic techniques in this paper depend heavily on those in the book of Bruce and Giblin [3]. In §2 we introduce the notion of lightcone height functions and Lorentzian height functions on curves in H^2_+ and study these properties. The Lorentzian height function is just a direct analogy of the height function in Euclidean 3-space. It is, however, used in the different situation compared with the Euclidean case. We study some hyperbolic invariants in §3. The proof of Theorem 2.2 is given in §4. In §5 we consider the generic properties by using the analogy of the notion of Monge-Taylor maps of curves in [3]. These arguments give the proof of Theorem 2.1. In §6 we give the program by using `Mathematica` that draw pictures of hyperbolic evolutes of curves in the Poincaé's disk.

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2. BASIC NOTIONS AND MAIN RESULTS

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R}\}$ be a 3-dimensional vector space, $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3$. We call $(\mathbb{R}^3, \langle , \rangle)$ a 3-dimensional *pseudo Euclidean space*, or *Minkowski 3-space*. We denote $\mathbb{R}^3_1$ instead of $(\mathbb{R}^3, \langle , \rangle)$.

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^3_1$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. We now define pseudo-spheres in $\mathbb{R}^3_1$ as follows:

$$\begin{aligned} H^2_+ &= \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid -x_1^2 + x_2^2 + x_3^2 = -1, x_1 \geq 1\}, \\ H^2_- &= \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid -x_1^2 + x_2^2 + x_3^2 = -1, x_1 \leq 1\} \\ S^2_1 &= \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid -x_1^2 + x_2^2 + x_3^2 = 1\}. \end{aligned}$$

We call $H^2_\pm$ *a hyperbola* and S^2_1 *a pseudo-sphere.*

Let $\gamma : I \longrightarrow H^2_+ \subset \mathbb{R}^3_1$; $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ be a smooth regular curve in H^2_+ (i.e.,$\dot\gamma(t) \neq 0$ for any $t \in I$), where I is an open interval. We can show that $\langle \dot\gamma(t), \dot\gamma(t) \rangle > 0$ for any $t \in I$. We call such the curve a *spacelike curve*. The *norm* of the vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. The *arc-length* of a spacelike curve γ, measured from $\gamma(t_0)$, $t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} \|\dot\gamma(t)\| \, dt.$$

Then the parameter s is determined such that $\|\gamma'(s)\| = 1$, where $\gamma'(s) = d\gamma/ds(s)$. So we say that a spacelike curve γ is *parameterized by arc-length* if it satisfies that $\|\gamma'(s)\| = 1$. Throughout the reminder in this paper we denote the parameter s of γ as the arc-length parameter. Let us denote $\boldsymbol{t}(s) = \gamma'(s)$, and we call $\boldsymbol{t}(s)$ a *unit tangent vector* of γ at s.

For any $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_1$, the *pseudo vector product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as follows:

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{vmatrix} -\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (-(x_2y_3 - x_3y_2), x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

We now set a unit vector $\boldsymbol{e}(s) = \gamma(s) \wedge \boldsymbol{t}(s)$. Then we have a pseudo-orthonormal frame $\{\gamma(s), \boldsymbol{t}(s), \boldsymbol{e}(s)\}$ along γ. of the curve γ at s. Since $\boldsymbol{t}(s)$ is spacelike, we have $\langle \boldsymbol{b}(s), \boldsymbol{b}(s) \rangle = -\delta(\gamma(s))$ and sign $(\gamma'(s)) = 1$. By the similar arguments as those as in the ordinary Frenet-Serret formula for the Euclidean space curve, the following Frenet-Serret type formula holds:

$$\begin{cases} \gamma'(s) &= \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) &= \gamma(s) + \kappa_g(s)\boldsymbol{e}(s) \\ \boldsymbol{e}'(s) &= -\kappa_g(s)\boldsymbol{t}(s), \end{cases}$$

where $\kappa_g(s)$ is *the geodesic curvature* of the curve γ in H^2_+ which is given by

$$\kappa_g(s) = \det(\gamma(s), \boldsymbol{t}(s), \boldsymbol{t}'(s)).$$

We call the above formula the hyperbolic Frenet-Serret formula of γ.

Since $\langle \gamma(s), \gamma(s) \rangle = -1$, we have $\langle \boldsymbol{e}(s), \boldsymbol{e}(s) \rangle = -\langle \gamma(s), \gamma(s) \rangle \langle \boldsymbol{t}(s), \boldsymbol{t}(s) \rangle = 1$. Therefore, we can show that $\|\gamma(s) \pm \boldsymbol{e}(s)\| = 0$. This means that $\gamma(s) \pm \boldsymbol{e}(s)$ is a lightlike vector for any $s \in I$. Define

$$S^1_+ = \{x \in \mathbb{R}^3_1 | x = (1, x_2, x_3), x_2^2 + x_3^2 = 1\}.$$

We call S^1_+ a *lightlike unit circle.* For any lightlike vector $\boldsymbol{x} = (x_1, x_2, x_3)$, we denote that $\tilde{\boldsymbol{x}} = (1, \frac{x_2}{x_1}, \frac{x_3}{x_1}) \in S^1_+$.

We now define a map $HG^+_\gamma : I \longrightarrow S^1_+$ by $HG^+_\gamma(s) = \widetilde{\gamma(s) + e(s)}$. We call HG^+_γ the *hyperbolic lightcone Gauss map* of γ. Under the assumption that $\kappa_g(s) \neq \pm 1$, we also define a space curve

$$HE_\gamma(s) = \frac{1}{\sqrt{|\kappa_g^2(s) - 1|}}(\kappa_g(s)\gamma(s) + e(s)).$$

We remark that $HE_\gamma(s)$ is located in H^2_+ if and only if $\kappa_g^2(s) > 1$, otherwise it is in S^2_1. We call HE_γ the *hyperbolic evolute* of γ. The geometric meanings of the above subjects will be discussed in §4.

Let $\gamma : S^1 \longrightarrow H^2_+$ be a regular curve. We consider the following conditions on γ

(A 1) The number of points $p = \gamma(t)$ where $\kappa'_g(t) = 0$ is finite.

(A 2) There is no point $p = \gamma(t)$ where $\kappa_g(t) = \kappa'_g(t) = 0$.

(A 3) There is no point $p = \gamma(t)$ where $\kappa'_g(t) = \kappa''_g(t) = 0$.

(A 4) The number of points $p = \gamma(t)$ where $\kappa_g^2(t) = 1$ is finite.

(A 5) There is no point $p = \gamma(t)$ where $\kappa'_g(t)\kappa_g(t) = 0$ and $\kappa_g^2(s) = 1$.

Our main results are formulated as follows:

Theorem 2.1. *Let $Imm(S^1, H^2_+)$ be the space of immersions equipped with Whitney C^∞-topology. Then the set of curves which satisfy* (A 1), (A 2), (A 3), (A 4), (A 5) *is a residual set in $Imm(S^1, H^2_+)$.*

Theorem 2.2. *Let $\gamma : I \longrightarrow H^2_+$ be a regular curve which satisfies the conditions* (A 1), (A 2), (A 3), (A 4), (A 5). *Then*

(1) *The hyperbolic lightcone Gauss map HG^+_γ has a fold point at s_0 if and only if $\kappa_g^2(s_0) = 1$.*

(2) *The germ of hyperbolic evolute at $HE_\gamma(s_0)$ is smooth if and only if $\kappa'_g(s_0) \neq 0$.*

(3) *The germ of hyperbolic evolute at $HE_\gamma(s_0)$ is locally diffeomorphic to the ordinary cusp if and only if $\kappa'_g(s_0) = 0$.*

Here, the ordinary cusp is the plane curve whichi is given by $C = \{(x_1, x_2) | {x_1}^2 = {x_2}^3\}$. We also say that a point $x_0 \in \mathbb{R}^r$ is a fold point of a map germ $f : (\mathbb{R}^r, x_0) \longrightarrow (\mathbb{R}^r, f(x_0))$ if there exist diffeomorphism germs $\phi : (\mathbb{R}^r, x_0) \longrightarrow (\mathbb{R}^r, 0)$ and $\psi : (\mathbb{R}^r, f(x_0)) \longrightarrow (\mathbb{R}^r, 0)$ such that $\psi \circ f \circ \phi^{-1}(x_1, \dots, x_r) = (x_1, \dots, x_{r-1}, x_r^2)$.

In [7] M. Kossowski introduced the notion of $S^1 \times S^1$-valued Gauss maps associated with spacelike curves in Minkowski 3-space. Since curves in H^2_+ are spacelike, so we can define the $S^1 \times S^1$-valued Gauss maps. As a matter of fact, the notion of hyperbolic lightcone Gauss maps in this paper is equal to the S^1_+-component of the $S^1 \times S^1$-valued Gauss maps. It is, however, given by the explicit form HG^+_γ in this special case.

3. Lorentzian invariant functions on curves in H^2_+

In this section we introduce three different families of functions on a regular curve $\gamma : I \to H^2_+$.

Hyperbolic height function

We now define a function $H^T : I \times H^2_+ \longrightarrow \mathbb{R}$ by $H^T(s, u) = \langle \gamma(s), u \rangle$. We call H^T the *hyperbolic timelike height function* on a curve γ. We also define a function $H^S : I \times S^2_1 \longrightarrow \mathbb{R}$ by $H^S(s, u) = \langle \gamma(s), u \rangle$. We call H^S the *hyperbolic spacelike height function* on a curve γ. We denote that $(h^T_u)(s) = H^T(s, u)$ and $(h^S_u)(s) = H^S(s, u)$. We have the following proposition.

Proposition 3.1. Let $\gamma : I \longrightarrow H^2_+$ be a unit speed curve.

(A) For any $(s, u) \in I \times H^2_+$,

(a) $(h_u^T)'(s) = 0$ if and only if $u \in \langle \gamma(s), \boldsymbol{e}(s) \rangle_{\mathbb{R}}$.
(b) $(h_u^T)'(s) = (h_u^T)''(s) = 0$ if and only if $u = \pm \dfrac{1}{\sqrt{\kappa_g^2(s) - 1}} (\kappa_g(s)\gamma(s) + \boldsymbol{e}(s))$ and $\kappa_g^2(s) > 1$.
(c) $(h_u^T)'(s) = (h_u^T)''(s) = (h_u^T)^{(3)}(s) = 0$ if and only if

$$u = \pm \frac{1}{\sqrt{\kappa_g^2(s) - 1}} (\kappa_g(s)\gamma(s) + \boldsymbol{e}(s)),$$

$\kappa_g^2(s) > 1$ and $\kappa_g'(s) = 0$.
(d) $(h_u^T)'(s) = (h_u^T)''(s) = (h_u^T)(3)(s) = (h_u^T)^{(4)}(s) = 0$ if and only if

$$u = \pm \frac{1}{\sqrt{\kappa_g^2(s) - 1}} (\kappa_g(s)\gamma(s) + \boldsymbol{e}(s)),$$

$\kappa_g^2(s) > 1$ and $\kappa_g'(s) = \kappa_g''(s) = 0$.
(B) For any $(s, u) \in I \times S_1^2$,
(a) $(h_u^S)'(s) = 0$ if and only if $u \in \langle \lambda\gamma(s), \boldsymbol{e}(s) \rangle_{\mathbb{R}}$.
(b) $(h_u^S)'(s) = (h_u^S)''(s) = 0$ if and only if $u = \pm \dfrac{1}{\sqrt{1 - \kappa_g^2(s)}} (\kappa_g(s)\gamma(s) + \boldsymbol{e}(s))$ and $\kappa_g^2(s) < 1$.
(c) $(h_u^S)'(s) = (h_u^S)''(s) = (h_u^S)^{(3)}(s) = 0$ if and only if

$$u = \pm \frac{1}{\sqrt{1 - \kappa_g^2(s)}} (\kappa_g(s)\gamma(s) + \boldsymbol{e}(s)),$$

$\kappa_g^2(s) < 1$ and $\kappa_g'(s) = 0$.
(d) $(h_u^S)'(s) = (h_u^S)''(s) = (h_u^S)^{(3)}(s) = (h_u^S)^{(4)}(s) = 0$ if and only if

$$u = \pm \frac{1}{\sqrt{1 - \kappa_g^2(s)}} (\kappa_g(s)\gamma(s) + \boldsymbol{e}(s)),$$

$\kappa_g^2(s) < 1$ and $\kappa_g'(s) = \kappa_g''(s) = 0$.

Proof. By the hyperbolic Frenet-Serret formula, we have the following calculations:
(1) $(h_u^T)'(s) = \langle \boldsymbol{t}(s), u \rangle$.
(2) $(h_u^T)''(s) = \langle \gamma(s) + \kappa_g(s)\boldsymbol{e}(s), u \rangle$.
(3) $(h_u^T)^{(3)}(s) = \langle (1 - \kappa_g^2(s))\gamma(s) + \kappa_g'(s)\boldsymbol{e}(s), u \rangle$.
(4) $(h_u^T)^{(4)}(s) = \langle (1 - \kappa_g^2(s))\gamma(s) + \kappa_g(s)\kappa_g'(s)\boldsymbol{t}(s) + (\kappa_g(s) - \kappa_g^3(s) + \kappa_g''(s))\boldsymbol{e}(s), u \rangle$.

The assertion (a) follows from the above formula (1). By the assertion (a), there exists $\lambda, \mu \in \mathbb{R}$ such that $u = \lambda\gamma(s) + \mu\boldsymbol{e}(s)$. By the formula (2), we have $0 = \langle \gamma(s) + \kappa_g(s)\boldsymbol{e}(s), \lambda\gamma(s) + \mu\boldsymbol{e}(s) \rangle = \lambda\langle \gamma(s), \gamma(s) \rangle + \mu\kappa_g(s)\langle \boldsymbol{e}(s), \boldsymbol{e}(s) \rangle = -\lambda + \mu\kappa_g(s)$. Thus we have $u = \mu(\kappa_g(s)\gamma(s) + \boldsymbol{e}(s))$. Since $\langle u, u \rangle = -1$, we have $\mu = \pm \dfrac{1}{\sqrt{\kappa_g^2(s) - 1}}$.

Other assertions (containing the assertions (B)) are also followed by the similar arguments. □

Lightcone height functions

We define a function $H : I \times S_+^1 \longrightarrow \mathbb{R}$ by $H(s, v) = \langle \gamma(s), v \rangle$. We call H the *Lightcone height function* on a curve γ. In [4] we introduced the notion of Lightcone height function on a spacelike curve in Minkowski 3-space. Since curves on H^2_+ are always spacelike, the definition in here is exactly the same as the definition in [4]. However, we adopt different pseudo orthonormal frame along the curve γ, so that we have the different features as follows:

Proposition 3.2. Let $\gamma : I \longrightarrow H^2_+$ be a regular curve γ. We also denote that $h_v(s) = H(s, v)$. Then

(a) $h'_v(s) = 0$ if and only if $v = \gamma(s) \pm \boldsymbol{e}(s)$.

(b) $h'_v(s) = h''_v(s) = 0$ if and only if $v = \gamma(s) \pm \boldsymbol{e}(s)$ and $\kappa^2_g(s) = 1$.

(c) $h'_v(s) = h''_v(s) = h^{(3)}_v(s) = 0$ if and only if $v = \gamma(s) \pm \boldsymbol{e}(s)$, $\kappa^2_g(s) = 1$ and $\kappa'_g(s) = 0$.

(d) $h'_v(s) = h''_v(s) = h^{(3)}_v(s) = 0$ if and only if $v = \gamma(s) \pm \boldsymbol{e}(s)$, $\kappa^2_g(s) = 1$ and $\kappa'_g(s) = \kappa''_g(s) = 0$.

Proof. If we calculate the derivative of $h_v(s)$ with respect to s-variable, we have the formulae exactly the same form as those in the proof of Proposition 3.1. We only change the variable u to v. So we use the same number (1) to (4) as the corresponding formula.

By the formula (1), there exists $\lambda, \mu \in \mathbb{R}$ such that $v = \lambda\gamma(s) + \mu\boldsymbol{e}(s)$. Since v is lightlike, $\langle\gamma(s), \gamma(s)\rangle = -1$ and $\langle\boldsymbol{e}(s), \boldsymbol{e}(s)\rangle = 1$, we have $-\lambda^2 + \mu^2 = 0$. So we have $v = \lambda(\gamma(s) \pm \boldsymbol{e}(s)) = \gamma(s) \pm \boldsymbol{e}(s)$. Other assertions are followed by the similar arguments as those in the proof of Proposition 3.1. □

4. HYPERBOLIC INVARIANTS OF CURVES

In this section we study the geometric properties of the hyperbolic evolute and the lightcone Gauss map of a curve in H^2_+. For any $r \in \mathbb{R}$ and $u_0 \in H^2_+$ or $u_0 \in S^2_1$, we denote that

$$PS^1(u_0, r) = \{u \in H^2_+ \mid \langle u, u_0\rangle = r\}.$$

We call $PS^1(u_0, r)$ *a pseudo-circle* in H^2_+. We call u_0 *the center* of the pseudo-circle $PS^1(u_0, r)$. Then we have the following proposition.

Proposition 4.1. (1) Let $\gamma : I \longrightarrow H^2_+$ be a unit speed curve with $\kappa^2_g(s) \neq 1$. Then $\kappa'_g(s) \equiv 0$ if and only if $u_0 = \pm\dfrac{1}{\sqrt{|\kappa^2_g(s) - 1|}}(\kappa_g(s)\gamma(s) + \boldsymbol{e}(s))$ are constant vectors. Under this condition, γ is a part of a pseudo-circle in H^2_+ whose center is u_0.

(2) Let $\gamma : I \longrightarrow H^2_+$ be a unit speed curve. Then $\kappa^2_g(s) \equiv 1$ if and only if $\gamma''(s) \in C_0$.
Here, $C_0 = \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle\boldsymbol{x}, \boldsymbol{x}\rangle = 0\}$ is *the lightcone* in $\mathbb{R}^3_1$.

Proof. (1) We denote that $P_\pm(s) = \pm u_0 = \pm\dfrac{1}{\sqrt{|\kappa^2_g(s) - 1|}}(\kappa_g(s)\gamma(s) + \boldsymbol{e}(s))$, then we have

$$P'_\pm(s) = \frac{\mp\kappa'_g(s)}{(|\kappa^2_g(s) - 1|)^{\frac{3}{2}}}\gamma(s) \mp \frac{\kappa_g(s)\kappa'_g(s)}{(|\kappa^2_g(s) - 1|)^{\frac{3}{2}}}\boldsymbol{e}(s).$$

Then $P'_\pm(s) \equiv \boldsymbol{0}$ if and only if $\kappa'_g(s) \equiv 0$.

Under this condition, we put $r = \mp\dfrac{\kappa_g(s)}{\sqrt{|\kappa^2_g(s) - 1}}$ and $u_0 = \pm\dfrac{1}{\sqrt{|\kappa^2_g(s) - 1|}}(\kappa_g(s)\gamma(s) + \boldsymbol{e}(s))$.

Then it is easy to show that $\gamma(s)$ is a part of the pseudo-circle $PS^1(u_0, r)$.

(2) By the Frenet-Serret type formula, we have $\gamma''(s) = \gamma(s) \pm \boldsymbol{e}(s)$ when $\kappa^2_g(g) \equiv 1$. Since $\gamma(s) \pm \boldsymbol{e}(s)$ is always lightlike, the assertion (2) follows. □

Let $\gamma : I \longrightarrow H^2_+$ be a unit speed curve with $\kappa_g^2(s) \neq 1$. For any $s_0 \in I$, we consider the pseudo-circle $PS^1(u_0, r)$, where $u_0 = HE_\gamma(s_0)$ and $r = \pm \dfrac{\kappa_g(s_0)}{\sqrt{|\kappa_g^2(s_0) - 1|}}$. Then we have the following proposition.

Proposition 4.2. Under the above notations, γ and $PS^1(u_0, r)$ have at least 3-point contact at $\gamma(s_0)$.

Proof. Firstly, we assume that $PS^1(u_0, r) \subset H^2_+$. In this case we consider the hyperbolic timelike height function H^T. By definition, we have $PS^1(u_0, r) = (h^T_{u_0})^{-1}(r)$. Proposition 3.1 (A) (b) means that γ and $PS^1(u_0, r)$ have at least 3-point contact at $\gamma(s_0)$. If $PS^1(u_0, r) \subset S^2_1$, we adopt the hyperbolic spacelike height function H^S, and the assertion follows from exactly the same arguments as those of the previous case. □

We call $PS^1(u_0, r)$ in Proposition 4.2 *the osculating pseudo-circle* (*or, the pseudo-circle of geodesic curvature*) ; its center u_0 is called *the center of geodesic curvature.* So the hyperbolic evolute is the locus of the center of geodesic curvature. Moreover, we have the following corollary of Propositions 3.1 and 4.2.

Corollary 4.3. The osculating pseudo-circle and γ have 4-point contact at $\gamma(s_0)$ if and only if $\kappa_g'(s_0) = 0$ and $\kappa_g''(s_0) \neq 0$.

If the curve γ satisfies the conditions (A 1)–(A 5), Theorem 2.2 asserts that the cusp point of the hyperbolic evolute corresponds to the point $\gamma(s_0)$ where the osculating pseudo-circle and γ have 4-point contact.

5. Unfoldings of functions of one-variable

In this section we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [3]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \longrightarrow \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s) = F_{x_0}(s, x_0)$. We say that f has the *A_k-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leqq p \leqq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has the *$A_{\geq k}$-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leqq p \leqq k$. Let F be an unfolding of f and $f(s)$ has A_k-singularity ($k \geqq 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, x_0))(s_0) = \sum_{j=1}^{k-1} \alpha_{ji} s^j$ for $i = 1, \ldots, r$. Then F is called a *(p)versal unfolding* if the $(k-1) \times r$ matrix of coefficients (α_{ji}) has rank $k-1$ ($k - 1 \leqq r$).

We now introduce important sets concerning the unfoldings relative to the above notions. The *singular set* of F is the set $S_F = \{(s, x) | \frac{\partial F}{\partial s}(s, x) = 0\}$. The *bifurcation set* $\mathfrak{B}_F$ of F is the critical value set of the restriction to S_F of the canonical projection $\pi : \mathbb{R} \times \mathbb{R}^r \longrightarrow \mathbb{R}^r$:

$$\mathfrak{B}_F = \{x \in \mathbb{R}^r | \text{ there exists } s \text{ with } \frac{\partial F}{\partial s} = \frac{\partial^2 F}{\partial s^2} = 0 \text{ at } (s, x)\}$$

Then we have the following well-known result (cf., [3]).

Theorem 5.1. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \longrightarrow \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has the A_k singularity at s_0. Suppose that F is an (p)versal unfolding.*
(1) *If $k = 2$, then (s_0, x_0) is the fold point of $\pi | S_F$ and $\mathfrak{B}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*
(2) *If $k = 3$, then $\mathfrak{B}_F$ is diffeomorphic to $C \times \mathbb{R}^{r-2}$.*

As an application of the above theorem, we have the following fundamental proposition in this paper.

Proposition 5.2. Let $\gamma : I \longrightarrow H^2_+$ be a unit speed curve with $\kappa_g(s_0) \neq 0$ and $\kappa_g^2(s_0) \neq 1$.
(1) If $h^T_{u_0}(s)$ has the A_k-singularity $(k = 2, 3)$ at s_0, then H^T is the (p)versal unfolding of $h^T_{u_0}$.
(2) If $h^S_{u_0}(s)$ has the A_k-singularity $(k = 2, 3)$ at s_0, then H^S is the (p)versal unfolding of $h^S_{u_0}$.
(3) If $h_{u_0}(s)$ has the A_2-singularity at s_0, then H is the (p)versal unfolding of h_{u_0}.

Proof. (1) We denote by $\gamma(s) = (X(s), Y(s), Z(s))$, $u = (x_1, x_2, \sqrt{-1 + x_1^2 - x_2^2}) \in H^2_+$. Therefore we have

$$H^T(s, u) = -x_1 X(s) + x_2 Y(s) + \sqrt{-1 + x_1^2 - x_2^2}\, Z(s)$$

and

$$\begin{aligned} j^2\left(\frac{\partial H^T}{\partial x_1}(s, u_0)\right)(s_0) =& \left(-X'(s_0) + \frac{x_1}{\sqrt{-1 + x_1^2 - x_2^2}} Z'(s_0)\right) s \\ &+ \frac{1}{2}\left(-X''(s_0) + \frac{x_1}{\sqrt{-1 + x_1^2 - x_2^2}} Z''(s_0)\right) s^2 \\ j^2\left(\frac{\partial H^T}{\partial x_2}(s, u_0)\right)(s_0) =& \left(Y'(s_0) - \frac{x_2}{\sqrt{-1 + x_1^2 - x_2^2}} Z'(s_0)\right) s \\ &+ \frac{1}{2}\left(Y''(s_0) - \frac{x_2}{\sqrt{-1 + x_1^2 - x_2^2}} Z''(s_0)\right) s^2 \end{aligned}$$

Case (1) If $h^T_{u_0}$ has the A_2-singularity at s_0, then $\dfrac{\partial H^T}{\partial s}(s_0, u_0) = \langle \gamma'(s_0), u_0 \rangle = 0$
Suppose that the rank of the matrix

$$A = \left(-X'(s_0) + \frac{x_1}{\sqrt{-1 + x_1^2 - x_2^2}} Z'(s_0), Y'(s_0) - \frac{x_2}{\sqrt{-1 + x_1^2 - x_2^2}} Z'(s_0)\right)$$

is zero, then we have

$$X'(s_0) = \frac{x_1}{\sqrt{-1 + x_1^2 - x_2^2}} Z'(s_0), \qquad Y'(s_0) = \frac{x_2}{\sqrt{-1 + x_1^2 - x_2^2}} Z'(s_0).$$

Since $\|\gamma'(s_0)\| = 1$, we have $Z'(s_0) \neq 0$, so that we have the contradiction as follows:

$$\begin{aligned} 0 &= \left(\frac{x_1}{\sqrt{-1 + x_1^2 - x_2^2}} Z', \frac{x_2}{\sqrt{-1 + x_1^2 - x_2^2}} Z', Z'\right)\left(x_1, x_2, \sqrt{-1 + x_1^2 - x_2^2}\right) \\ &= -\frac{x_1^2}{\sqrt{-1 + x_1^2 - x_2^2}} Z' + \frac{x_2^2}{\sqrt{-1 + x_1^2 - x_2^2}} Z' + \sqrt{-1 + x_1^2 - x_2^2}\, Z' \\ &= -\frac{Z'}{\sqrt{-1 + x_1^2 - x_2^2}} \\ &\neq 0 \end{aligned}$$

This means that the rank of A is one. By Theorem 5.1, H^T is the (p)versal unfolding of $h^T_{u_0}$.

Case (2) It follows from Proposition 3.1 that $h^T_{u_0}(s)$ has the A_3-singularity at s_0 if and only if

$$u_0 = \pm \frac{1}{\sqrt{\kappa_g^2(s_0) - 1}} (\kappa_g(s_0)\gamma(s_0) + \boldsymbol{e}(s_0)),$$

$\kappa_g^2(s_0) > 1$, $\kappa_g'(s_0) = 0$ and $\kappa_g''(s_0) \neq 0$. For the purpose, we require the 2×2 matrix

$$B = \begin{pmatrix} -X' + \dfrac{x_1}{\sqrt{-1+x_1^2-x_2^2}}\,Z' & Y' - \dfrac{x_2}{\sqrt{-1+x_1^2-x_2^2}}\,Z' \\ -X'' + \dfrac{x_1}{\sqrt{-1+x_1^2-x_2^2}}\,Z' & Y'' - \dfrac{x_2}{\sqrt{-1+x_1^2-x_2^2}}\,Z' \end{pmatrix}$$

to be nonsingular. The determinant of this matrix at s_0 is

$$\begin{aligned}
\det B &= (-(Y'Z''-Z'Y''), Z'X''-X'Z'', X'Y''-Y'X'') \begin{pmatrix} -\dfrac{x_1}{\sqrt{-1+x_1^2-x_2^2}}\,Z' \\ -\dfrac{x_2}{\sqrt{-1+x_1^2-x_2^2}}\,Z' \\ -1 \end{pmatrix} \\
&= \{(X',Y',Z')\wedge(X'',Y'',Z'')\} \begin{pmatrix} -\dfrac{x_1}{\sqrt{-1+x_1^2-x_2^2}}\,Z' \\ -\dfrac{x_2}{\sqrt{-1+x_1^2-x_2^2}}\,Z' \\ -1 \end{pmatrix} \\
&= -\frac{1}{\sqrt{-1+x_1^2-x_2^2}}(\boldsymbol{t}\wedge\boldsymbol{e}) \begin{pmatrix} x_1 \\ x_2 \\ \sqrt{-1+x_1^2-x_2^2} \end{pmatrix} \\
&= \left(-\frac{1}{\sqrt{-1+x_1^2-x_2^2}}\right)\langle\gamma, u_0\rangle \\
&= \left(-\frac{1}{\sqrt{-1+x_1^2-x_2^2}}\right) \pm \frac{1}{\sqrt{\kappa_g^2-1}}\langle\gamma, (k_g\gamma+\boldsymbol{e})\rangle \\
&= \pm\frac{k_g}{\sqrt{-1+x_1^2-x_2^2}}\frac{1}{\sqrt{\kappa_g^2-1}} \neq 0.
\end{aligned}$$

(2) For H^S, the arguments for the proof is exactly the same as those for H^T, so that we omit it.

(3) We denote by $\gamma(s) = (X(s), Y(s), Z(s))$ and $v = (1, \cos\theta, \sin\theta)$. By definition we have $H(s,\theta) = -X(s) + \cos\theta Y(s) + \sin\theta Z(s)$. Therefore, we have $\dfrac{\partial H}{\partial\theta}(s,\theta) = -\sin\theta Y(s) + \cos\theta Z(s)$ and

$$j^1\left(\frac{\partial H}{\partial\theta}(s,\theta_0)\right)(s_0) = (-\sin\theta_0 Y'(s_0) + \cos\theta_0 Z'(s_0))\, s.$$

So we require the matrix

$$C = (-\sin\theta_0 Y(s_0) + \cos\theta_0 Z(s_0), -\sin\theta_0 Y'(s_0) + \cos\theta_0 Z'(s_0))$$

to have rank 1, which it always does since $-\sin\theta_0 Y'(s_0) + \cos\theta_0 Z'(s_0) \neq 0$. In fact, $-Y'(s_0)\sin\theta + Z'(s_0)\cos\theta$ is equal to the first component of $\gamma'\wedge v$. Suppose that $-Y'(s_0)\sin\theta + Z'(s_0)\cos\theta = 0$. Since $\langle\gamma'\wedge v, \gamma'\wedge v\rangle = \langle \boldsymbol{t}(s_0)\wedge\widetilde{(\gamma\pm\boldsymbol{e})}(s_0), \boldsymbol{t}(s_0)\wedge\widetilde{(\gamma\pm\boldsymbol{e})}(s_0)\rangle = 0$, we have

$$\langle (0, Z'-X'\sin\theta, X'\cos\theta - Y'), (0, Z'-X'\sin\theta, X'\cos\theta - Y')\rangle = 0$$

This is equivalent to the condition that

$$Z'^2 - 2X'Z'\sin\theta + X'^2\sin^2\theta + X'^2\cos^2\theta - 2X'Y'\cos\theta + Y'^2 = 0.$$

So we have

$$Z'^2 + Y'^2 + X'^2 - 2X'(Y'\sin\theta + Z'\cos\theta) = 0.$$

Since $-X' + Y'\sin\theta + Z'\cos\theta = \langle \gamma', v \rangle = 0$, we have $-X'^2 + Y'^2 + Z'^2 = 0$. On the other hand, $-X'^2 + Y'^2 + Z'^2 = \langle \gamma', \gamma' \rangle = 1$. This is a contradiction. Hence H is (p) versal. □

We now give the proof of Theorem 2.2.

Proof of Theorem 2.2 For the proof of assertion (1), we consider the singular set associated with the lightcone height function H denoted by S_H. By Proposition 3.2, we have $S_H = \{(s,v) \in I \times S^1_+ \mid v = \gamma(s) \pm e(s)\}$. We also consider the canonical projection $\pi : I \times S^1_+ \longrightarrow S^1_+$ and we can identify $\pi|S_H$ and the hyperbolic lightcone Gauss map HG^+_γ. By the assumption and Proposition 3.1, h_v has the A_2-singularity at s_0 if and only if $\kappa_g^2(s_0) = 1$. It follows from Proposition 5.2 that H is the (p)versal unfolding of h at s_0. Therefore Theorem 5.1,(1) asserts that $\pi|S_H$ has a fold point at s_0.

For the proof of the assertions (2) and (3), we consider the hyperbolic timelike height H^T functions on curves. By Proposition 3.1, the discriminant set of H^T is

$$\mathfrak{B}_{H^T} = \left\{ \pm \frac{1}{\sqrt{\kappa_g^2(s) - 1}} (\kappa_g(s)\gamma(s) + e(s)) \; |\kappa^2(s) > 1 \right\}.$$

The hyperbolic evolute of γ in H^2_+ is a part of this set. By Theorem 5.1 and Proposition 5.1, the discriminant set $\mathfrak{B}_{H^T}$ at $u_0 = \pm \frac{1}{\sqrt{\kappa_g^2(s_0) - 1}} (\kappa_g(s_0)\gamma(s_0) + e(s_0))$is locally diffeomorphic to to the cusp if $\kappa_g'(s_0) = 0$, otherwise it is diffeomorphic to the line. If we consider the hyperbolic spacelike height function H^S, we can prove the remaining assertions of the theorem. □

6. Generic properties of hyperbolic plane curves

In this section we consider the notion of hyperbolic Monge-Taylor maps for curves in H^2_+analogous to the ordinary notion of Monge-Taylor maps for curves in Euclidean plane (cf., [3]). For any regular curve $\gamma : I \longrightarrow H^2_+$, we choose γ, t, e as a pseudo-orthonormal frame of $\mathbb{R}^3_1$ along γ. Applying some Lorentz transformation, we may assume that $\gamma(t_0) = (1,0,0)$. Then the coordinates η, ζ, ξ of $\gamma(t)$ relative to axes γ, t, e are functions of t:$\eta(t) = \langle \gamma(t), \gamma(t_0) \rangle, \zeta(t) = \langle \gamma(t), t(t_0) \rangle, \xi(t) = \langle \gamma(t), e(t_0) \rangle, \eta = f_{2,t}(\zeta)$ and $\xi = f_{1,t}(\zeta)$ with $f_{2,t}(0) = 1$ and $f_{1,t}(0) = 0$. Since $\gamma(t) \in H^2_+$, we have the relation $f_{2,t} = \sqrt{\zeta^2 + f_{1,t}^2 + 1}$. Let $V_k (k \leq 2)$ be the space of polynomials in a single variable ζ of degree $\leq k$ and ≥ 2. So we can identify V_k with $\mathbb{R}^{k-1}$. If we have a function $\xi = f(\zeta)$ with $f(0) = 0$, then we have an element

$$j^k f(0) = a_2\zeta^2 + a_3\zeta^3 + \cdots + a_k\zeta^k \in V_k,$$

where $a_i = \frac{d^i f}{\zeta^i}(0)$. The hyperbolic Monge-Taylor map (of order k) $\mu_\gamma : I \longrightarrow V_k$ is defined to be $\mu_\gamma(t) = j^k f_{1,t}(0)$.

We are showing that there are sufficiently many deformations of our original curve γ to give all of deformations of the hyperbolic Monge-Taylor map μ_γ. Let P_k denote the set of maps $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the form $\psi(x,y) = (\psi_1(x,y), \psi_2(x,y))$ where each ψ_i is a polynomial in x and y of degree $\leq k$, so P_k can be thought as a Euclidean space $\mathbb{R}^N$ with $N = (k+1)(k+2)$.

We now define the canonical projection $\pi_{2,3} : H^2_+ \longrightarrow \mathbb{R}^2$ by $\pi_{2,3}(x_1, x_2, x_3) = (x_2, x_3)$. It is clear that $\pi_{2,3}$ is a diffeomorphism. Let $\delta : I \longrightarrow \mathbb{R}^2$ be a regular curve, then we have the hyperbolic plane curve given by $\pi_{2,3}^{-1} \circ \delta$. We call it *the hyperbolic lift* of δ and denote it $\widehat{\delta}$. To simplify matters we assume that the curve $\gamma(I)$ is compact, i.e. $I = S^1$. By using the compactness $\gamma(S^1)$ there is a neighbourhood of $1_{\mathbb{R}^2}$ in P_k with the property that if $\psi \in U$ then $\widehat{\psi \circ \pi_{2,3} \circ \gamma}$ is a regular curve in H^2_+. We can prove the following theorem by exactly the same arguments as the proof of ([3], Theorem 9.5).

Theorem 6.1. *Let Q be a manifold in $V_k = \mathbb{R}^{k-1}$. For some open set $U_1 \subset U$ containing the identity map the map $\mu : S^1 \times U_1 \longrightarrow V_k$ defined by $\mu(t,\psi) = \mu_{\widehat{\psi\circ\pi_{2,3}\circ\gamma}}(t)$ is transverse to Q.*

By the direct calculations, we have the following lemmas. The calculations are rather long and tedious so we omit details.

Lemma 6.2. *Let $\gamma : I \longrightarrow H^2_+$ be a space curve with $\gamma(t_0) = (1,0,0)$, $\gamma(t) = (f_{1,t}(\zeta), \zeta, f_{2,t}(\zeta))$ and $f_{2,t}(\zeta) = a_2(t)\zeta^2 + a_3(t)\zeta^3 + \cdots$. Then*
(1) $\kappa_g(t_0) = 2a_2(t_0)$, $\kappa_g'(t_0) = 6a_3(t_0)$ *and* $\kappa_g''(t_0) = 24a_4(t_0) - 6a_2(t_0)(4a_2^2(t_0) - 1)$.
(2) (a) $\kappa_g(t_0) = \kappa_g'(t_0)$ *if and only if* $a_2(t_0) = a_3(t_0) = 0$.
(b) $\kappa_g'(t_0) = \kappa_g''(t_0) = 0$ *if and only if* $a_3(t_0) = 4a_4(t_0) - a_2(t_0)(4a_2^2(t_0) - 1) = 0$.

We now define 7 sets in V_4 as follows:

$$\begin{aligned} Q_1 &= \{(a_1,a_2,a_3,a_4) \in V_4 \mid a_3 = 0\},\\ Q_2 &= \{(a_1,a_2,a_3,a_4) \in V_4 \mid a_2 = a_3 = 0\},\\ Q_3 &= \{(a_1,a_2,a_3,a_4) \in V_4 \mid a_3 = 4a_4 - a_2(4a_2^2 - 1) = 0\},\\ Q_4^\pm &= \{(a_1,a_2,a_3,a_4) \in V_4 \mid a_2 = \pm 1\},\\ Q_5^\pm &= \{(a_1,a_2,a_3,a_4) \in V_4 \mid a_2 = \pm 1, a_3 = 0\}. \end{aligned}$$

Then Q_1 and $Q_4^\pm$ are codimension one submanifolds and Q_2, Q_3 and $Q_5^\pm$ are codimension two submanifolds respectively. We can use the above submanifolds and Theorem 6.1 for the proof of Theorem 2.1 exactly the same way as the proof of Corollary 9.7 in [3]. So we omit the detail here.

7. Drawing pictures on the Poincaré's disk

In this section we describe how we can draw the picture of the hyperbolic evolute of a curve in the Poincaré's disk. We now consider the Poincaré's disk $D = \{(x,y) \mid x^2 + y^2 < 1\}$ with the hyperbolic metric

$$ds^2 = \frac{4(dx^2 + dy^2)}{1 - x^2 - y^2}.$$

It has been known that there is the canonical isometric diffeomorphism

$$\Phi : D \longrightarrow H^2_+ \; ; \; \Phi(x,y) = \left(\frac{1 + x^2 + y^2}{1 - x^2 - y^2}, \frac{2x}{1 - x^2 - y^2}, \frac{2y}{1 - x^2 - y^2}\right).$$

The inverse mapping of Φ is given by

$$\Psi : H^2_+ \longrightarrow D \; ; \; \Psi(x_1, x_2, x_3) = \left(\frac{x_2}{x_1 + 1}, \frac{x_3}{x_1 + 1}\right).$$

If we consider a regular curve $\gamma : I \longrightarrow D$, then we have the lift $\Phi \circ \gamma : I \longrightarrow H^2_+$ of the curve γ. By the previous sections, we have the hyperbolic evolute $HE_{\Phi\circ\gamma}$. We denote that $J = \{s \in I \mid \kappa_g^2(s) > 1\}$, then J is an open interval in I. If we consider the restriction $\gamma|J$ of γ, we have the hyperbolic evolute $HE_{\Phi\circ\gamma|J} : J \longrightarrow H^2+$ in H^2_+.

Therefore, we have the hyperbolic evolute

$$\Psi \circ HE_{\Phi\circ\gamma|J} : J \longrightarrow D$$

of $\gamma|J$. We can draw the picture of $\Psi \circ HE_{\Phi\circ\gamma|J}$ by a computer graphics. The following is a program written by using Mathematica:

```
hevolute[{y1_,y2_}]:=
  Module[{Phi,phi,d1,d1st,Mi,d2,d2nd,determ,Mo,x,Psi},
    Phi[1]:=(1+y1^(2)+y2^(2))/(1-y1^(2)-y2^(2)) /.t -> tt;
    Phi[2]:=2*y1/(1-y1^(2)-y2^(2)) /.t -> tt;
    Phi[3]:=2*y2/(1-y1^(2)-y2^(2)) /.t -> tt; phi:={Phi[1],Phi[2],Phi[3]};
    Do[d1[i]=D[Phi[i],tt],{i,3}]; d1st:={d1[1],d1[2],d1[3]};
    Mi=-d1[1]*d1[1]+d1[2]*d1[2]+d1[3]*d1[3];
    Do[d2[i]=D[Phi[i],{tt,2}],{i,3}]; d2nd:={d2[1],d2[2],d2[3]};
    determ=Det[{phi,d1st,d2nd}];
    Mo[1]=-Phi[2]*d1[3]+Phi[3]*d1[2];
    Mo[2]=Phi[3]*d1[1]-Phi[1]*d1[3];
    Mo[3]=Phi[1]*d1[2]-Phi[2]*d1[1];
    Do[x[i]=(Mi^(-3)*determ^(2)-1)^(-1/2)*(
        Mi^(-3/2)*determ*Phi[i]+Mi^(-1/2)*Mo[i]),{i,3}];
    Psi[1]=x[2]/(x[1]+1) /.  tt->t ;
    Psi[2]=x[3]/(x[1]+1) /.  tt->t ;
    {Psi[1],Psi[2]}];
ellipse[t_]:={1/7*Cos[t],1/12*Sin[t]};
Show[
  ParametricPlot[Evaluate[
    Table[hevolute[ellipse[t]+{s,0}],{s,0,1,1/5}]],{t,0,2*Pi},
    DisplayFunction -> Identity],
  ParametricPlot[Evaluate[
    Table[ellipse[t]+{s,0},{s,0,1,1/5}]],{t,0,2*Pi},
    DisplayFunction -> Identity],
  ParametricPlot[Evaluate[
    Table[hevolute[ellipse[t]+{0,s}],{s,0,1,1/5}]],{t,0,2*Pi},
    DisplayFunction -> Identity],
  ParametricPlot[Evaluate[
    Table[ellipse[t]+{0,s},s,0,1,1/5]],{t,0,2*Pi},
    DisplayFunction -> Identity],
  ParametricPlot[Evaluate[
    Table[hevolute[ellipse[t]+{s,s},{s,0,1,1/5}]],{t,0,2*Pi},
    DisplayFunction -> Identity],
  ParametricPlot[Evaluate[
    Table[ellipse[t]+{s,s},s,0,1,1/5]],{t,0,2*Pi},
    DisplayFunction -> Identity],
  ParametricPlot[Evaluate[{Cos[t],Sin[t]}],t,0,2*Pi,
    PlotStyle -> Dashing[{0.02}], DisplayFunction -> Identity],
  DisplayFunction -> $DisplayFunction,
  AspectRatio -> Automatic,
  Axes -> None]
```

Fig.1 is the picture of a family of ellipse and these heperbolic evolute in the Poincaré's disk. The dotted circle denotes the boundary of the Poincaré's disk.

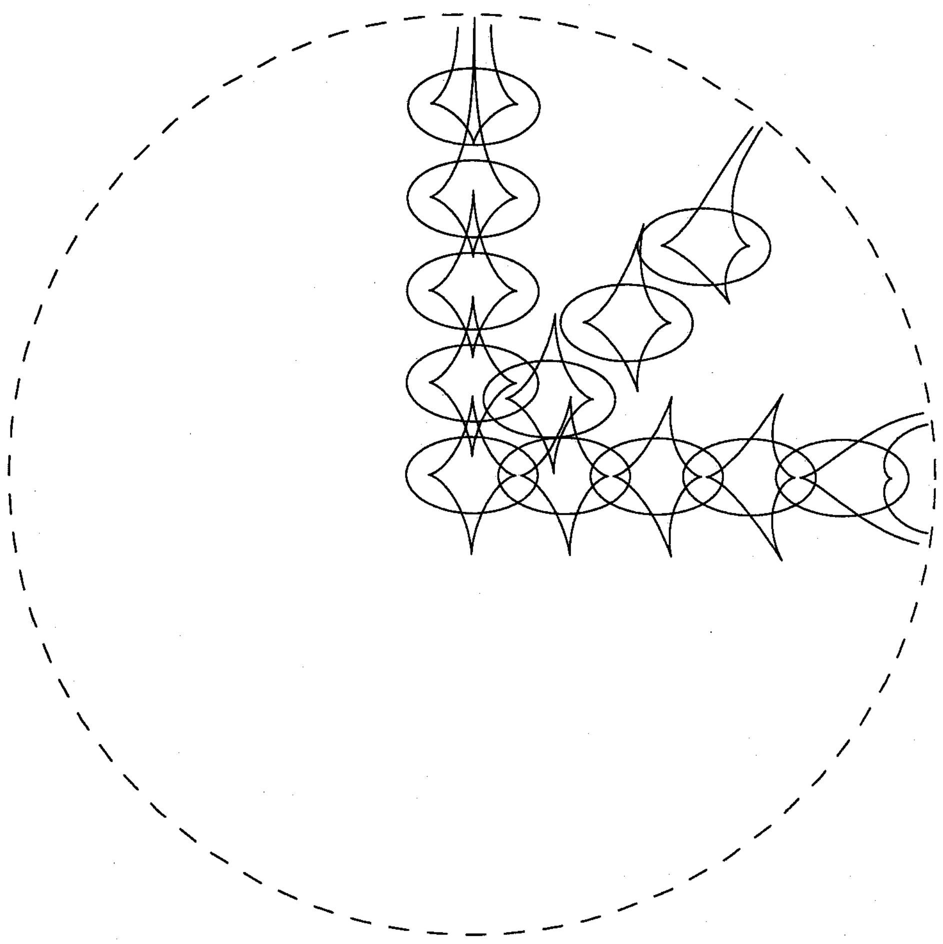

Fig. 1

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