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Contemporary Mathematics

Some Special Bounded Homomorphisms Of A Uniform Algebra

Takahiko Nakazi

ABSTRACT. Let $L(H)$ be the algebra of all bounded linear operators on a Hilbert space H and let A be a uniform algebra. In this paper we study the following questions. When is a unital bounded homomorphism Φ of A in $L(H)$ completely bounded ? When is the norm $||\Phi||$ of Φ equal to the completely bounded norm $||\Phi||_{cb}$? In some special cases we answer this question. Suppose Φ is ρ-contractive $(0 < \rho < \infty)$ where Φ is contractive if $\rho = 1$. We show that if A is a Dirichlet algebra or $\dim A/\ker\Phi = 2$ then Φ has a ρ-dilation. If Φ is a ρ-contractive homomorphism then $||\Phi|| = \max(1, \rho)$ and if it has a ρ-dilation then $||\Phi||_{cb} = \max(1, \rho)$. Moreover we give a new example of a hypo-Dirichlet algebra in which a unital contractive homomorphism has a contractive dilation.

1. Introduction

Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. Let H be a complex Hilbert space and $L(H)$ the algebra of all bounded linear operators on H. $I = I_H$ is the identity operator in H. An algebra homomorphism $f \to \Phi(f)$ of A in $L(H)$, which satisfies

$$\Phi(1) = I \text{ and } ||\Phi(f)|| \leq \gamma ||f||_\infty$$

for some positive constant $\gamma \geq 1$, is called a unital bounded homomorphism of A. If $\gamma = 1$, it is called a unital contractive homomorphism.

For a subspace B of A, let $M_n(B)$ denote the set of $n \times n$ matrices with entries from B. For a map $\phi : B \to L(H)$, we obtain maps $\phi_n : M_n(B) \to M_n(L(B))$ via the formula

$$\phi_n((a_{ij})) = (\phi(a_{ij})).$$

If ϕ is a bounded map, then each ϕ_n will be bounded, and when $\sup_n ||\phi_n||$ is finite, we call ϕ a completely bounded map of B in $L(H)$. We write

$$||\phi||_{cb} = \sup_n ||\phi_n||.$$

The following problem is natural and important.

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PROBLEM . *Suppose Φ is a unital bounded homomorphism of A.*

I. *When is Φ completely bounded ?*

II. *When is Φ completely bounded and $\|\Phi\| = \|\Phi\|_{cb}$?*

A unital contractive homomorphism $v \to \tilde{\Phi}(v)$ of $C(X)$ on a Hilbert space K is called a contractive dilation of the unital contractive homomorphism $f \to \Phi(f)$ of A on H if H is a Hilbert subspace of K and

$$\Phi(f) = P\tilde{\Phi}(f) \mid H \quad (f \in A)$$

where P is the orthogonal projection of K onto H. If Φ has a contractive dilation then Φ is completely contractive and hence $\|\Phi\| = \|\Phi\|_{cb}$. If Φ is completely contractive then Φ has a contractive dilation. This is well known (see [18, Corollary 6.7]).

If A is a uniform algebra and the uniform closure of $A + \bar{A}$, that is, $[A + \bar{A}]$ has finite codimension n in $C(X)$ then A is called a n-hypo-Dirichlet algebra and it is called a Dirichlet algebra when $[A + \bar{A}] = C(X)$, that is, $n = 0$.

If $\dim H < \infty$, Φ is completely bounded for arbitrary uniform algebra A (see [18, Exercises 3.11]). If $\dim H = \infty$ and A is the disc algebra, then there exists a unital bounded homomorphism Φ which is not completely bounded. This was recently shown by G.Pisier [19]. If A is a n-hypo-Dirichlet algebra and Φ is a unital contractive homomorphism then Φ is completely bounded. This was shown by R.G.Douglas and V.I.Paulsen [6]. However we don't know whether Φ is completely contractive or not. They are known solutions for Problem I.

Now we will give known solutions for Problem II when Φ is contractive. If A is the disc algebra then there exists a contractive dilation. This is a famous theorem of B.Sz.-Nagy [10]. T.Ando [2] generalized this to the bidisc algebra. However S.K.Parrot [17] gave an example of Φ which does not have a contractive dilation in the polydisc algebra for $n \geq 3$. If A is a Dirichlet algebra then there exists a contractive dilation (cf. [7]). For a n-hypo-Dirichlet algebra with $n \neq 0$, we don't know whether there exists a contractive dilation or does not. The polydisc algebra for $n \geq 2$ is not a n-hypo-Dirichlet algebra. If A is an annulus algebra, that is, a rational function algebra on an annulus, then there exists a contractive dilation. This was shown by J.Agler [1]. An annulus algebra is a 1-hypo-Dirichlet algebra. If $\mathcal{A}$ is the disc algebra and $A = \{f \in \mathcal{A} \ ; \ f(0) = f(1)\}$, then A is also a 1-hypo-Dirichlet algebra. The author [12] proved that Φ has a contractive dilation for this example. Even if $\dim H < \infty$, by an example of S.K.Parrot [17] Φ may not have a contractive dilation for some uniform algebra. The author and the late K.Takahashi [14], and Che-Chen Chu [5] showed that if $\dim H \leq 2$, Φ has a contractive dilation for an arbitrary uniform algebra.

Now we will give more concrete problems than Problem II.

PROBLEM . *Suppose Φ is a unital bounded homomorphism of A.*

(II-*a*) *Suppose $\|\Phi\| \leq 1$. When A is a n-hypo-Dirichlet algebra and $n \geq 1$, does Φ have a contractive dilation ?*

(II-*b*) *Under what conditions on Φ which is $\|\Phi\| > 1$, is Φ completely bounded with $\|\Phi\| = \|\Phi\|_{cb}$ when A is a n-hypo-Dirichlet algebra and $n \geq 0$, or $\dim H \leq 2$?*

In this paper, we study Problem (II-*a*) and (II-*b*). In §2, we give a new example of a 1-hypo-Dirichlet algebra in which a unital contractive homomorphism has a contractive dilation. In §3, we define a ρ-contractive homomorphism Φ and a ρ-dilation of Φ for $0 < \rho < \infty$. If Φ is a ρ-contractive homomorphism then $\|\Phi\| \leq$

$\max(1,\rho)$ and if it has a ρ-dilation then $\|\Phi\|_{cb} \leq \max(1,\rho)$. In §4, we introduce a δ-homomorphism of A for $-\infty < \delta < 1$. This homomorphism is bounded. In fact, we show more, that is, 'δ-homomorphism' is equivalent to '$\rho = 1/(1-\delta)$-contractive homomorphism'. In §5, we show that a ρ-contractive homomorphism has a ρ-dilation when A is a Dirichlet algebra. In §6, we consider Problem II under conditions on Φ, that is, $\dim A/\ker\Phi = 2$ or a hypothesis on $\ker\Phi$.

2. Third example of a hypo-Dirichlet algebra for Problem II-a

For a n-hypo-Dirichlet algebra with $n \neq 0$, we know only two examples ([1], [11]) in which a unital contractive homomorphism has a contractive dilation, that is, Problem (II-a). They are 1-hypo-Dirichlet algebras. In this section, we give a new example which is also a 1-hypo-Dirichlet algebra. In the proof of Theorem 2.1, a theorem of T.Ando [2] is used essentially. Unfortunately we could not generalize Theorem 2.1 to $A = \{f \in \mathcal{A}\ ;\ f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0\}$.

THEOREM 2.1. *Let $\mathcal{A}$ be the disc algebra and $A = \{f \in \mathcal{A}\ ;\ f'(0) = 0\}$. If Φ is a unital contractive homomorphism of A then Φ has a contractive dilation or equivalently Φ is a completely contractive.*

PROOF. Since $A = \mathbf{C} + z^2\mathcal{A}$, $A_0 = \{f \in A\ ;\ f(0) = 0\} = z^2\mathcal{A}$. A_0 has two generators, that is, A_0 is generated by z^2 and z^3 beause $2\ell \pm 1$ can be written as the form $2n+3m$. Let $\Phi(z^2) = S$ and $\Phi(z^3) = T$ then $ST = TS$, $\|S\| \leq 1$ and $\|T\| \leq 1$. By a well known theorem of T.Ando [2], there exist two commuting operators U and V on a Hilbert space K with $H \subset K$ such that

$$S^nT^m = PU^nV^m|H$$

for all nonnegative integers n and m where P is an orthogonal projection from K to H. Any polynomial f in A_0 is written as the following :

$$f = a_{10}z^2 + a_{01}z^3 + \sum_{j,\ell\geq 1} a_{j\ell}z^{2j}z^{3\ell}$$

and so

$$\begin{aligned}\Phi(f) &= a_{10}S + a_{01}T + \sum_{j,\ell\geq 1} a_{j\ell}S^jT^\ell \\ &= P(a_{10}U + a_{01}V + \sum_{j,\ell\geq 1} a_{j\ell}U^jV^\ell)|H\end{aligned}$$

By a theorem of C.R.Putnam and B.Fuglede [20, Corollary 1.19], $U^*V = VU^*$. Hence if we set $\tilde{\Phi}((z^2)^j(z^3)^\ell) = U^jV^\ell$ for any integers j and ℓ then $\tilde{\Phi}$ is a unital contractive homomorphism of $C(X)$ in $L(K)$ and $\Phi = P\tilde{\Phi}|H$ on A_0. Thus $\tilde{\Phi}$ is a contractive dilation of Φ. □

3. ρ-Contractive homomorphism

A bounded linear operator T on H is said to be of class C_ρ if there exists a unitary operator U (called a unitary ρ-dilation) on a Hilbert space $K \supset H$ such that $T^n = \rho PU^n|H$ for $n = 1,2,\cdots$ where P is an orthogonal projection from K to H. B.Sz.-Nagy [9] showed that if T is a contraction then it is of class C_1. If the numerical radius of T is less than equal to one then it is of class C_2 [3]. If T is of class C_ρ then $\|T\| \leq \max(1,\rho)$.

Suppose Φ is a unital algebra homomorphism of A in $L(H)$ and $0 < \rho < \infty$. When $\Phi(f)$ is of class C_ρ for any f in A with $\|f\|_\infty \leq 1$, Φ is called a ρ-contractive homomorphism of A. A 1-contractive homomorphism is equivalent to a contractive homomorphism. A 2-contractive homomorphism Φ is equivalent to that

$$\sup_{\substack{f\in A \\ \|f\|_\infty \leq 1}} \sup_{\substack{y\in H \\ \|y\|=1}} |\langle \Phi(f)y, y\rangle| = 1.$$

If Φ is ρ-contractive then $\|\Phi\| \leq \max(1, \rho)$. We will study Problem (II-b) when Φ is ρ-contractive.

A unital contractive homomorphism $v \to \tilde{\Phi}(v)$ on a Hilbert space K is called a ρ-dilation of the unital bounded homomorphism $f \to \Phi(f)$ of A on H if H is a Hilbert subspace of K and

$$\Phi(f) = \rho P \tilde{\Phi}(f)|H \quad (f \in A_\tau)$$

where P is the orthogonal projection of K onto H, A_τ is the kernel of τ in $M(A)$ and $0 < \rho < \infty$.

If Φ has a ρ-dilation then Φ is a unital completely bounded map. However the converse is not true even for the disc algebra A and $\dim H = 2$. This may be well known. Suppose $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, then $f \to \Phi(f) = f(T)$ is a unital completely bounded homomorphism but it has not a ρ-dilation for any ρ [15]. If Φ is a unital completely bounded homomorphism, then Φ/ρ is completely contractive but it is not unital. However if Φ has a ρ-dilation then the following is true.

PROPOSITION 3.1. *For a unital bounded homomorphism Φ, Φ has a ρ-dilation with respect to τ if and only if $\Phi(f) = \rho\Phi_0(f)$ $(f \in A_\tau)$ where Φ_0 is a unital completely contractive map on A, equivalently Φ_0 has a contractive dilation.*

PROOF. For the 'only if' part, put $\Phi_0(f) = P\tilde{\Phi}(f)|H$ $(f \in A)$. Then Φ_0 has a contractive dilation $\tilde{\Phi}$ and hence it is completely contractive on A. The 'if' part follows from a theorem of W.Arveson (cf.[18, Corollary 6.7]). □

4. δ-homomorphism

For $-\infty < \delta < 1$, Φ is called a δ-homomorphism for τ in $M(A)$ if Φ is a unital algebra homomorphism of A and $Re\Phi(f) \geq 0$ whenever f in A, $\tau(f) = 1$ and $Ref \geq \delta$. In this section we show that Φ is a δ-homomorphism for some τ if and only if Φ is a $\rho = 1/(1-\delta)$-contractive homomorphism.

PROPOSITION 4.1. *If a unital algebra homomorphism Φ has a ρ-dilation for τ, then Φ is a $\delta = \left(1 - \frac{1}{\rho}\right)$-homomorphism for τ.*

PROOF. Suppose that $\Phi(h) = \rho P\tilde{\Phi}(h)|H$ for $h \in A_\tau$. If $f \in A$, $\tau(f) = 1$ and $Ref \geq 1 - \frac{1}{\rho}$, since $\tau(f - \tau(f)) = 0$,

$$\Phi(f) - \tau(f)I = \rho P\tilde{\Phi}(f)|H - \rho\tau(f)I$$

and so

$$\Phi(f) = \rho\left\{P\tilde{\Phi}(f)|H + \tau(f)\left(\frac{1}{\rho} - 1\right)I\right\}.$$

Since $Ref \geq 1 - \frac{1}{\rho}$, $\tilde{\Phi}(Ref) \geq 1 - \frac{1}{\rho}$ and so

$$Re\Phi(f) \geq \rho \left\{ P\tilde{\Phi}(Ref)|H + \left(\frac{1}{\rho} - 1\right) I \right\} \geq 0.$$

□

PROPOSITION 4.2. *Suppose $\delta \neq 0$. Φ is a δ-homomorphism for τ if and only if for any h in A_τ with $\|h\|_\infty < 1$,*

$$|\langle \Phi(h)y, y\rangle| \leq \frac{1}{2|\delta|}\|y\|^2 + \frac{2\delta - 1}{2|\delta|}\|\Phi(h)y\|^2 \quad (y \in H)$$

PROOF. Suppose Φ is a δ-homomorphism for τ. Put

$$f = (1-\delta)\frac{1+h}{1-h} + \delta$$

where $h \in A_\tau$ and $\|h\|_\infty < 1$ then $f \in A$, $Ref \geq \delta$ and $\tau(f) = 1$. For $x \in H$, put $x = (I - \Phi(h))y$ then

$$\begin{aligned} & Re\langle \Phi(f)x, x\rangle \\ = \;& (1-\delta)Re\left\langle \frac{I + \Phi(h)}{I - \Phi(h)}x, x \right\rangle + \delta\|x\|^2 \\ = \;& (1-\delta)Re\langle (I + \Phi(h))y, (I - \Phi(h))y\rangle + \delta\langle (I - \Phi(h))y, (I - \Phi(h))y\rangle \\ = \;& (1-\delta)(\|y\|^2 - \|\Phi(h)y\|^2) + \delta(\|y\|^2 + \|\Phi(h)y\|^2 - 2Re\langle \Phi(h)y, y\rangle) \\ = \;& \|y\|^2 + (2\delta - 1)\|\Phi(h)y\|^2 - 2\delta\langle \Phi(h)y, y\rangle. \end{aligned}$$

By hypothesis on Φ,

$$2\delta Re\langle \Phi(h)y, y\rangle \leq \|y\|^2 + (2\delta - 1)\|\Phi(h)y\|^2$$

and so

$$|\langle \Phi(h)y, y\rangle| \leq \frac{1}{2|\delta|}\|y\|^2 + \frac{2\delta - 1}{2|\delta|}\|\Phi(h)y\|^2.$$

The proof is reversible. In fact, if $f \in A$, $Ref \geq \delta$ and $\tau(f) = 1$ then

$$f = (1-\delta)\frac{1+h}{1-h} + \delta$$

for some $h \in A_\tau$ with $\|h\|_\infty \leq 1$. Hence if we put for $0 < \varepsilon < 1$

$$f_\varepsilon = (1-\delta)\frac{1+\varepsilon h}{1-\varepsilon h} + \delta$$

then $f_\varepsilon \to f$ uniformly as $\varepsilon \to 1$, $\|\varepsilon h\|_\infty \leq 1$ and $\tau(f_\varepsilon) = 1$. Since $Re\langle \Phi(f_\varepsilon)x, x\rangle \geq 0$, as $\varepsilon \to 1$, $Re\langle \Phi(f)x, x\rangle \geq 0$ for any $x \in H$. □

THEOREM 4.3. *Φ is a δ-homomorphism for some (or any) τ in $M(A)$ if and only if Φ is a $\rho = 1/(1-\delta)$-contractive homomorphism.*

PROOF. [16, Theorem 2] and Proposition 4.2 imply the theorem, or we can show this by the proof of Proposition 4.2 and [11, Theorem 11.1]. □

5. Condition A

In this section, we show that a ρ-contractive homomorphism has a ρ-dilation when A is a Dirichlet algebra. This is a generalization of a theorem of C.Foias and I.Suciu [7] for $\rho = 1$ and a theorem of B.Sz.Nagy and C.Foias (cf. [11]) for the disc algebra. They give solutions for Problem (II-b).

THEOREM 5.1. *Let A be a Dirichlet algebra and $0 < \rho < \infty$. If Φ is a ρ-contractive homomorphism of A in $L(H)$ then for any τ in $M(A)$ it has a ρ-dilation.*

PROOF. Put $\Phi'(h) = \dfrac{1}{\rho}\Phi(h) - \tau(h)\left(\dfrac{1}{\rho} - 1\right) I$ for $h \in A$. By Theorem 4.3 if Φ is ρ-contractive then Φ is a $\delta = \left(1 - \dfrac{1}{\rho}\right)$ - homorphism for any $\tau \in M(A)$. Hence if $Reh \geq 0$ then $Re\Phi'(h) \geq 0$. If we extend Φ' to $\tilde{\Phi} : A + \bar{A} \to L(H)$ by $\tilde{\Phi}(f + \bar{g}) = \Phi'(f) + \Phi'(g)^*$, then $\tilde{\Phi} : C(X) \to L(H)$ is positive because A is a Dirichlet algebra. By the dilation theorem of M.A.Naimark (cf. [21, Theorem 7.5]) there exists a Hilbert space K, an orthogonal projection $P : K \to H$ and a multiplicative linear map $u \to \tilde{\tilde{\Phi}}(u)$ of $C(X)$ in $L(K)$, which satisfies $\tilde{\tilde{\Phi}}(1) = I_K$, $\|\tilde{\tilde{\Phi}}(u)\| \leq \|u\|_\infty$, $u \in C(X)$ and $\Phi'(f) = P\tilde{\tilde{\Phi}}(f)|H$ for $f \in A$. If $f \in A_\tau$ then $\Phi'(f) = \dfrac{1}{\rho}\Phi(f)$ and so

$$\Phi(f) = \rho P\tilde{\tilde{\Phi}}(f)|H.$$

□

PROPOSITION 5.2. *Let A be an arbitrary uniform algebra and $0 < \rho < \infty$. Suppose Φ is a ρ-contractive homomorphism of A in $L(H)$. If $A/\ker\Phi$ is isometrically isomorphic to $\mathcal{A}/\mathcal{J}$ where $\mathcal{A}$ is a Dirichlet algebra on some compact Hausdorff space Y and $\mathcal{J}$ is a closed ideal in $\mathcal{A}$, then Φ has a ρ-dilation for any τ in $M(A)$ with $\tau = 0$ on $\ker\Phi$.*

PROOF. Let ϕ be an isometric isomorphism from $\mathcal{A}/\mathcal{J}$ onto $A/\ker\Phi$. For each $f \in \mathcal{A}$, we will write $\phi(f + \mathcal{J}) = \phi(f) + \ker\Phi$ where $\phi(f) \in \mathcal{A}$. Moreover we will write Φ again for the map : $f + \ker\Phi \to \Phi(f)$. Put $\Psi = \Phi\circ\phi$, then Ψ is a unital homomorphism of $\mathcal{A}/\mathcal{J}$ in $L(H)$. We will write Ψ again for the map : $f \to \Psi(f + \mathcal{J})$, then $\mathcal{J} = \ker\Psi$. Since we may assume that τ is a complex homomorphism on $A/\ker\Phi$ by [8, Theorem 6.2 in Chapter I], $\tau\circ\phi$ is a complex homomorphism on $\mathcal{A}/\mathcal{J}$ and so we may assume that $\tau\circ\phi \in M(\mathcal{A})$. If $f \in \mathcal{A}_{\tau\circ\phi}$. and $\|f\|_\infty \leq 1$, then $\phi(f) \in A_\tau$ and $\|\phi(f) + \mathcal{J}\| \leq 1$. By hypothesis, $\Phi(\phi(f))$ is of class C_ρ and so $\Psi(f) = \Phi\circ\phi(f)$ is of class C_ρ for $f \in \mathcal{A}_{\tau\circ\phi}$ with $\|f\|_\infty \leq 1$. Hence Ψ is a ρ-contractive homomorphism of $\mathcal{A}$ in $L(H)$ with respect to $\tau\circ\phi$. Since $\mathcal{A}$ is a Dirichlet algebra, by Proposition 3.1 and Theorem 5.1 $\Psi = \rho\Psi_0$ on $\mathcal{A}_{\tau\circ\phi}$ where Ψ_0 is a unital completely contractive map on $\mathcal{A}$. Put $\Phi_0 = \Psi_0\circ\phi^{-1}$ then Φ_0 is a unital completely contractive map on A and $\Phi = \rho\Phi_0$ on A_τ. Proposition 3.1 implies the theorem. □

Let A be a n-hypo-Dirichlet algebra and let N_τ be the set of all representing measures of τ in $M(A)$. Then $\dim N_\tau = n$ and there exists a core measure m of N_τ (cf. [8, p106]). Then by [8, Theorem 5.1 in Chapter IV], there is a constant $c > 0$ such that $\nu \leq cm$ for all ν in N_τ. Hence if h is the Radon-Nikodym derivative of ν with respect to m then $\nu = hdm$. Set $N_\tau^m = \{h : \nu = hdm$ and

$\nu \in N_\tau\}$, then N_τ^m is a subset of $L^\infty(m)$. Thus N_τ^m can be considered as a subset of $L^\infty(m)$. Many important n-hypo-Dirichlet algebras satisfy a natural condition on $N_\tau^m : N_\tau^m \subset C(X)$. The author showed [13] that if $N_\tau^m \subset C(X)$ then a unital contractive homomorphism Φ of A has a ρ-dilation with respect to τ. It is a long standing open question whether we can choose $\rho = 1$. The motivation of our study in this paper is in this open question. The following Proposition 5.4 implies that if Φ is ρ-contractive for enough small $\rho > 0$ then Φ has a 1-dilation.

LEMMA 5.3. *Let A be a n-hypo-Dirichlet algebra and $N_\tau^m \subset C(X)$. Then there exists a positive linear map T from $C(X)$ to $[A + \bar{A}]$ such that $T(f) = f$ $(f \in A_\tau)$ and $T(1)$ is a positive constant ≥ 1.*

PROOF. This is proved in the proof of [13, Theorem]. □

PROPOSITION 5.4. *Let A be a n-hypo-Dirichlet algebra and $N_\tau^m \subset C(X)$ for some τ in $M(A)$ where m is a core measure of N_τ. If Φ is a ρ-contractive homomorphism of A in $L(H)$ then it has a ρ'-dilation where $\rho' = \rho\, T(1)$ and T is a map in Lemma 5.3.*

PROOF. Suppose Φ is ρ-contractive. Then Φ is a $\delta = \left(1 - \frac{1}{\rho}\right)$-homomorphism for τ by Theorem 4.3. Put

$$\Phi'(f) = \frac{1}{\rho}\Phi(f) - \tau(f)\left(\frac{1}{\rho} - 1\right) I \quad (f \in A),$$

then by the proof of Theorem 5.1, there exists an extension $\tilde{\Phi}'$ of Φ' on $[A + \bar{A}]$ and $\tilde{\Phi}'$ is positive on it. Then $T(1)^{-1}\tilde{\Phi}' \circ T$ is a positive map from $C(X)$ to $L(H)$ and $T(1)^{-1}\tilde{\Phi}' \circ T(1) = I$. By the dilation theorem of M.A.Naimark (cf. [21, Theorem 7.5] there exists a Hilbert space K, an orthogonal projection P from K to H and a multiplicative linear map $u \to \tilde{\Phi}(u)$ of $C(X)$ in $L(K)$, which satisfies $\tilde{\Phi}(1) = I_K$, $\|\tilde{\Phi}(u)\| \leq \|u\|_\infty$, $u \in C(X)$ and $\tilde{\Phi}' \circ T(u) = T(1)P\tilde{\Phi}(u)|H$. Because $T(f) = f$ $(f \in A_\tau)$,

$$\Phi(f) = \rho\, T(1)P\tilde{\Phi}(f)|H \quad (f \in A_\tau).$$

Suppose $\rho' = \rho\, T(1)$. □

6. Condition on Φ

In this section, under conditions on Φ we consider Problem II. We consider ρ-contractive homomorphisms when $A/ker\Phi$ is of two dimension. The author and the late K.Takahashi [13] showed that Φ has a 1-dilation when $\rho = 1$. We generalize it to any ρ. Proposition 6.3 is a generalization of a result of G.Misra [8] which was shown for a rational uniform algebra on the complex plane and $\rho = 1$.

For x, y in $M(A)$ and a bounded point derivation δ at x, let

$$\sigma_A(x, y) = \sup\{|f(y)|\ ;\ f(x) = 0, f \in A \text{ and } \|f\|_\infty \leq 1\}$$

and

$$\omega_A(x, \delta) = \sup\{|\delta(f)|\ ;\ f(x) = 0, f \in A \text{ and } \|f\|_\infty \leq 1\}.$$

THEOREM 6.1. *Let A be an arbitrary uniform algebra. If Φ satisfies one of the following conditions (1), (2) and (3), then Φ is completely bounded and $\|\Phi\| = \|\Phi\|_{cb}$.*

1. $\|\Phi(f)^2\| = \|\Phi(f)\|^2 \quad (f \in A)$

2. *E is an interpolation set in X and* $\ker\Phi = \{f \in A\ ;\ f = 0 \text{ on } E\}$
3. $\|\Phi\| \le 1, \ker\Phi = \{f \in A\ ;\ f = 0 \text{ on } E\}$ *for some finite set* $E \subset M(A)$ *and* $\sigma_A(x,y) = 1$ *for any x, y in E with $x \neq y$.*

PROOF. (1) If $\|\Phi(f)^2\| = \|\Phi(f)\|^2$ $(f \in A)$ then the closure of $\Phi(A)$ is regarded as a uniform algebra. By [18, Theorem 3.8], $\|\Phi\| = \|\Phi\|_{cb}$.

(2) Since E is an interpolation set in $X, A/\ker\Phi$ is isometrically isomorphic to a subalgebra of $C(E)$ and hence $\Phi(A) \subseteq C(E)$. Again by [18, Theorem 3.8], $\|\Phi\| = \|\Phi\|_{cb}$.

(3) Since $E = \{x_1, \cdots, x_n\}$ is a finite set, $\dim A/\ker\Phi = n < \infty$ and $\Phi(A) = \{\sum_{j=1}^{n} a_j P_j\ ;\ a_j \in \mathbf{C}, P_i P_j = \delta_{ij} P_j \text{ and } j = 1, \cdots, n\}$. Suppose $\Phi(f_j) = P_j$ and $f_j \in A$. Then $f_i f_j = \delta_{ij} f_j$ and $f_i(x_j) = \delta_{ij}$. Since $\sigma_A(x_i, x_j) = \delta_{ij}$, there exist $\{g_n^{(i)}\}_{n=1}^{\infty}$ in A such that $g_n^{(i)}(x_i) \to 1$ $(n \to \infty)$, $g_n^{(i)}(x_j) = 0$ $(j \neq i)$ and $\|g_n^{(i)}\|_\infty \le 1$. Since $A/\ker\Phi = \{\sum_{i=1}^{n} a_i f_i + \ker\Phi\ ;\ a_i \in \mathbf{C} \text{ and } i = 1, \cdots, n\}$, $g_n^{(i)} - a_{in} f_i \in \ker\Phi$ and $\Phi(g_n^{(i)}) = a_{in} P_i$. Since Φ is contractive, $|a_{in}|\|P_i\| \le 1$ and so $\{a_{in}\}$ is bounded. Hence there exists a subsequence $\{a_{in(j)}\}$ such that $a_{in(j)} \to a_i$ as $j \to \infty$ for each i. Then $\lim_{j\to\infty} g_{n(j)}^{(i)} - a_i f_i \in \ker\Phi$ and $\lim_{j\to\infty} g_{n(j)}^{(i)}(x_i) = 1$. Therefore $a_i = 1$ and $\|f_i + \ker\Phi\| \le 1$, and P_i is selfadjoint for $i = 1, \cdots, n$ because Φ is contractive. Thus $\Phi(A)$ is a commutative C^*-algebra. By [18, Theorem 3.8], $\|\Phi\| = \|\Phi\|_{cb}$. □

THEOREM 6.2. *Suppose Φ is a ρ-contractive homomorphism of A. If $A/\ker\Phi$ is of two dimension then Φ has a ρ-dilation for any τ in $M(A)$ with $\tau = 0$ on $\ker\Phi$.*

PROOF. Suppose $\ker\Phi = \{f \in A\ ;\ f(x) = f(y) = 0\}$ where $x, y \in M(A)$ with $x \neq y$. By [13, Lemma 1 and its proof],

$$\Phi(f) = \begin{pmatrix} f(x)I_{H_1} & (f(x) - f(y))C \\ 0 & f(y)I_{H_2} \end{pmatrix} \text{ on } H = H_1 \oplus H_2$$

for all $f \in A$ where C is a bounded linear operator from H_2 to H_1, and

$$A/\ker\Phi = \{f(x)f_1 + f(y)f_2 + \ker\Phi\ ;\ f \in A\}$$

where $f_1(x) = f_2(y) = 1$ and $f_1(y) = f_2(x) = 0$. By [13, Lemma 3], if $\|C\|^2 + 1 = 1/\sigma_A(x,y)^2$ then $\|\Phi(f)\| = \|f + \ker\Phi\|$ for all $f \in A$.

For any $x, y \in M(A)$, there exist a Dirichlet algebra $\mathcal{A}$ and $s, t \in M(\mathcal{A})$ such that $\sigma_A(x,y) = \sigma_{\mathcal{A}}(s,t)$. In fact, we can choose the disc algebra $\mathcal{A}$. Suppose

$$\Psi(F) = \begin{pmatrix} F(s)I_{H_1} & (F(s) - F(t))B \\ 0 & F(t)I_{H_2} \end{pmatrix}$$

for all $F \in \mathcal{A}$ where B is a bounded linear operator from H_2 to H_1. Then

$$\mathcal{A}/\ker\Psi = \{F(s)F_1 + F(t)F_2 + \ker\Psi\ ;\ F \in \mathcal{A}\}$$

where $F_1(s) = F_2(t)$ and $F_1(t) = F_2(s)$, and $\|\Phi(f)\| = \|\Psi(F)\|$ whenever $f(x) = F(s)$ and $f(y) = F(t)$, and $B = C$. If $B = C$ and $\|C\|^2 + 1 = 1/\sigma_A(x,y)^2$, then $\|B\|^2 + 1 = 1/\sigma_{\mathcal{A}}(s,t)^2$ and so $\|f + \ker\Phi\| = \|\Phi(f)\| = \|\Psi(f)\| = \|f + \ker\Psi\|$. Hence for given Φ, we can find a unital homomorphism Ψ on $\mathcal{A}$ such that $A/\ker\Phi \cong \mathcal{A}/\ker\Psi$. By Proposition 5.2, Φ has a ρ-dilation for any $\tau \in M(A)$ with $\tau = 0$ on $\ker\Phi$.. If $\ker\Phi$ is not the above form, then $\ker\Phi = \{f \in A\ ;\ f(x) = \delta(f) = 0\}$

where $x \in M(A)$ and δ is a bounded point derivation at x. By [13, Lemma 1 and its proof],

$$\Phi(f) = \begin{pmatrix} f(x)I_{H_1} & \delta(f)C \\ 0 & f(x)I_{H_2} \end{pmatrix} \text{ on } H = H_1 \oplus H_2$$

for all $f \in A$ where C is a bounded linear operator from H_2 to H_1, and

$$A/\ker\Phi = \{f(x)1 + \delta(f)f_0 + \ker\Phi \ ; \ f \in A\}$$

where $f_0(x) = 0$ and $\delta(f_0) = 1$. By [13,Lemma 3], if $\|C\| = 1/\omega_A(x,\delta)$ then $\|\Phi(f)\| = \|f + \ker\Phi\|$ for all $f \in A$. As in the first part of the proof, by Proposition 5.2 we can show that Φ has a ρ-dilation for any $\tau \in M(A)$. □

If $\dim H = 2$, then an algebra homomorphism Φ has the following form :

$$\Phi_1(f) = \begin{pmatrix} f(x) & c(f(x) - f(y)) \\ 0 & f(y) \end{pmatrix}$$

where $x, y \in M(A)$ and $x \neq y$ or

$$\Phi_2(f) = \begin{pmatrix} f(x) & c\delta(f) \\ 0 & f(x) \end{pmatrix}$$

where $x \in M(A)$ and δ is a bounded point derivation at x.

PROPOSITION 6.3. *Suppose Φ is a unital bounded homomorphism of A in $L(H)$ and $\dim H = 2$.*

1. *When $\Phi = \Phi_1, \Phi$ is a ρ-contractive homomorphism if and only if*
$$\begin{aligned}&(1 + |c|^2)|\rho\zeta(f(x) - f(y))|^2 \\ \leq\ & |\{\rho + (1-\rho)\overline{f(x)}\bar{\zeta}\}\{\rho + (1-\rho)f(y)\zeta\} - \overline{f(x)}f(y)|\zeta|^2|^2\end{aligned}$$
for any $f \in A$ with $\|f\|_\infty \leq 1$ and any $\zeta \in D$.
2. *When $\Phi = \Phi_2, \Phi$ is a ρ-contractive homomorphism if and only if*
$$\begin{aligned}&|c|^2|\delta(f)|^2 \\ \leq\ & (\rho - 2)|f(x)|^2 + 2(1-\rho)|f(x)| + \rho\end{aligned}$$
for any $f \in A$ with $\|f\|_\infty \leq 1$.

PROOF. The author and Okubo [15] gave a necessary and sufficient condition for that a triangle 2×2 matrix is of class C_ρ. By [15, Theorem] $\Phi_1(f)$ is of class C_ρ if and only if

$$\begin{aligned}&|c|^2|f(x) - f(y)|^2 + |f(x) - f(y)|^2 \\ \leq\ & \inf_{\zeta \in D}\left|\frac{\{\rho + (1-\rho)\overline{f(x)}\bar{\zeta}\}\{\rho + (1-\rho)f(y)\zeta\} - \overline{f(x)}f(y)|\zeta|^2}{\rho\zeta}\right|^2\end{aligned}$$

and by [15, Remark] $\Phi_2(f)$ is of class C_ρ if and only if

$$|c|^2|\delta(f)|^2 \leq (\rho - 2)|f(x)|^2 + 2(1-\rho)|f(x)| + \rho.$$

□

In Proposition 6.3, suppose $\rho = 1$. $\Phi = \Phi_1$ is a 1-contractive homomorphism if and only if $|c|^2 \leq (1 - |f(x)|^2)(1 - |f(y)|^2)/|f(x) - f(y)|^2$ for any $f \in A$ with $\|f\|_\infty \leq 1$. This implies [9, Theorem 1.1]. $\Phi = \Phi_2$ is a 1-contractive homomorphism if and only if $|c|^2 \leq (1 - |f(x)|^2)/|\delta(f)|^2$ for any $f \in A$ with $\|f\|_\infty \leq 1$. Suppose $\rho = 2$. $\Phi = \Phi_1$ is a 2-contractive homomorphism if and only if $1 + |c|^2 \leq$

$\inf_{\zeta \in D} \left| \frac{2 - (\overline{f(x)}\bar{\zeta} + f(x)\zeta)}{\zeta(f(x) - f(y))} \right|^2$ for any $f \in A$ with $\|f\|_\infty \leq 1$.$\Phi = \Phi_2$ is a 2-contractive homomorphism if and only if $|c|^2 \leq 2(1 - |f(x)|)/|\delta(f)|^2$ for any $f \in A$ with $\|f\|_\infty \leq 1$.

References

[1] J.Agler *Rational dilation on an annulus*, Ann. of Math., **121** (1985), 537–564.
[2] T.Ando *On a pair of commutative contractions*, Acta Sci. Math., **24** (1963), 88–90.
[3] C.A.Berger*A strange dilation theorem, Notices Amer. Math. Soc.*, **12** (1965), 590.
[4] F.Bonsal and J.Duncan *Complete normed algebras*, Springer-Verlag, New York and Berlin, 1973.
[5] Che-Chen Chu *Finite dimensional representation of a function algebra*, Ph.D. thesis in the University of Houston, 1992.
[6] R.G.Douglas and V.I.Paulsen *Completely bounded maps and hypo-Dirichlet algebras*, Acta Sci. Math., **50** (1986), 143–157.
[7] C.Foias and I.Suciu *Szegő-measures and spectral theory in Hilbert spaces*, Rev. Roum. Math. Pures. et Appl. **11** (1966), 147–159.
[8] T.Gamelin *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, New Jersey, 1969.
[9] G.Misra *Curvature inequalities and extremal properties of bundle shifts*, J. Operator Theory **11** (1984), 305–317.
[10] B.Sz.-Nagy *Sue les contractions de e'spase de Hilbert*, Acta Sci. Math. **15** (1953), 87–92.
[11] B.Sz.-Nagy and C.Foias *Harmonic Analysis of Operators on Hilbert Space*, North Holland, New York, 1970.
[12] T.Nakazi *A spectral dilation of some non-Dirichlet algebra*, Acta Sci. Math., **53** (1989), 119–122
[13] T.Nakazi *ρ-dilations and hypo-Dirichlet algebras*, Acta Sci. Math., **56** (1992), 175–181.
[14] T.Nakazi and K.Takahashi *Two-dimensional representations of uniform algebras*, Proc. Amer. Math. Soc. **123** (1995), 2777–2784.
[15] T.Nakazi and K.Okubo *ρ-Contraction and 2 × 2 matrix*, Linear Algebra Appl. **283** (1998), 165–169.
[16] T.Nakazi and K.Okubo *Generalized numerical radius and unitary ρ-Dilation*, to appear in Math. Japonica.
[17] S.Parrot *Unitary dilations for commuting contractions*, Pacific J. Math., **34** (1970), 481–490.
[18] V.I.Paulsen *Completely Bounded Maps and Dilations*, Pitman Research Notes in Math. **146**, Longman, Wiley, New York, 1986.
[19] G.Pisier *A polynomially bounded operator on Hilbert space which is not similar to a contraction*, J.Amer. Math. Soc. **10** (1997), 351–369.
[20] H.Radjavi and P.Rosenthal *Invariant Subspaces*, Springer-Verlag, New York, 1973.
[21] I.Suciu *Function Algebras*, Editura Academiej Republicii Socialiste Romania, Bucuresti, 1973.

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