ON CLASSICAL SOLUTIONS OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH NONNEGATIVE INITIAL DENSITIES

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Abstract. We study the Navier-Stokes equations for compressible barotropic fluids in a bounded or unbounded domain Ω of $\mathbf{R}^3$. We first prove the local existence of solutions (ρ, u) in $C([0, T_*]; (\rho^\infty + H^3(\Omega)) \times (D_0^1 \cap D^3)(\Omega))$ under the assumption that the data satisfies a natural compatibility condition. Then deriving the smoothing effect of the velocity u in $t > 0$, we conclude that (ρ, u) is a classical solution in $(0, T_{**}) \times \Omega$ for some $T_{**} \in (0, T_*]$. For these results, the initial density needs not be bounded below away from zero and may vanish in an open subset (*vacuum*) of Ω.

1. Introduction

The motion of a viscous compressible barotropic fluid in a domain Ω of $\mathbf{R}^3$ can be described by the Naiver-Stokes equations

$$\rho_t + \operatorname{div}(\rho u) = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{1.1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + L\,u + \nabla p = \rho f \quad \text{in} \quad (0,T)\times\Omega, \tag{1.2}$$

$$Lu = -\mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u, \quad p = p(\rho) \tag{1.3}$$

and the initial and boundary conditions

$$(\rho,\, u)|_{t=0} = (\rho_0,\, u_0) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{1.4}$$

$$\rho(t,x) \to \rho^\infty, \quad u(t,x) \to 0 \quad \text{as} \quad |x| \to \infty, \quad (t,x) \in (0,T)\times\Omega. \tag{1.5}$$

Here we denote by ρ, p and u the unknown density, pressure and velocity fields of the fluid, respectively. f denotes a given external force and the constants μ, λ are the viscosity coefficients. We assume that the pressure $p = p(\rho)$ is a smooth function of the density ρ and the viscosity coefficients μ and λ satisfy the natural physical restrictions $\mu > 0$ and $3\lambda + 2\mu \geq 0$ so that $L = -\mu\Delta - (\lambda+\mu)\nabla\text{div}$ is a strongly elliptic operator. Moreover, $(0,T)\times\Omega$ is the time-space domain for the evolution of the fluid, where T is a finite positive number and Ω is either a bounded domain in $\mathbf{R}^3$ with smooth boundary or a usual unbounded domain such as the whole space $\mathbf{R}^3$, the half space $\mathbf{R}^2 \times \mathbf{R}_+$ and an exterior domain with smooth boundary. Of course, if Ω is a bounded domain (or the whole space), then the condition (1.5) at

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infinity (or the boundary condition in (1.4) respectively) is unnecessary and should be neglected.

In this paper, we study the initial boundary value problem (simply IBVP) (1.1)-(1.5) with nonnegative initial densities.

Under the crucial assumption that the initial density ρ_0 is bounded below away from zero, the first existence results for the IBVP (1.1)-(1.5) were obtained by Nash [20], Itaya [13] and Tani [24]. They applied a fixed point argument or the method of successive approximations in Hölder spaces to prove the local (in time) existence of classical solutions even for more general heat-conducting fluid models. Then using delicate energy methods in Sobolev spaces, Matsumura and Nishida showed in their pioneering papers [18, 19] that the classical solutions exist globally in time provided that the data are small in some sense. See also the papers [6, 12, 23, 27, 28, 29] for some further local or global results in case of positive densities.

On the other hand, the existence of weak or strong solutions has been proved in rather recent works even for the general case of nonnegative initial densities. In fundamental works [16, 17], Lions developed an existence theory of global (in time) weak solutions to the IBVP (1.1)-(1.5). Then Lions' theory has been improved by several authors to deduce more general results; see [7, 8, 9, 10, 14, 15] for details. The very recent papers [2, 3, 4] by Choe and the authors are devoted to establishing some local existence results on strong solutions. Among other things, we showed in [2, 3] (see also the paper [21] by Salvi and Straškraba) that if the initial data ρ_0, u_0 satisfy the regularity condition

$$\rho_0 - \rho^\infty \in H^2, \quad \rho^\infty \in \overline{\mathbf{R}}_+, \quad \rho_0 \geq 0 \quad \text{in} \quad \Omega, \quad u_0 \in D_0^1 \cap D^2 \tag{1.6}$$

and the compatibility condition

$$Lu_0 + \nabla p(\rho_0) = \rho_0^{\frac{1}{2}} g_1 \quad \text{in} \quad \Omega \quad \text{for some} \quad g_1 \in L^2, \tag{1.7}$$

then there exist a small time $T_* \in (0, T)$ and a unique strong solution (ρ, u) to the IBVP (1.1)-(1.5) such that

$$\begin{gathered}\rho - \rho^\infty \in C([0,T_*]; H^2), \quad u \in C([0,T_*]; D_0^1 \cap D^2) \cap L^2(0,T_*; D^3), \\ \rho_t \in C([0,T_*]; H^1), \quad u_t \in L^2(0,T_*; D_0^1) \quad \text{and} \quad \sqrt{\rho} u_t \in L^\infty(0,T_*; L^2).\end{gathered} \tag{1.8}$$

Throughout this paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev spaces.

$$\begin{aligned}
&L^r = L^r(\Omega), \quad D^{k,r} = \{v \in L^1_{loc}(\Omega) : |v|_{D^{k,r}} < \infty\}, \\
&W^{k,r} = L^r \cap D^{k,r}, H^k = W^{k,2}, \quad D^k = D^{k,2}, \\
&D_0^1 = \{v \in L^6(\Omega) : |v|_{D_0^1} < \infty \ \text{and} \ v = 0 \ \text{on} \ \partial\Omega\}, \\
&H_0^1 = L^2 \cap D_0^1, \quad |v|_{D^{k,r}} = |\nabla^k v|_{L^r} \quad \text{and} \quad |v|_{D_0^1} = |\nabla v|_{L^2}.
\end{aligned}$$

Then it follows from the classical Sobolev embedding results that

$$|v|_{L^6} \leq C|v|_{D_0^1}, \quad |v|_{L^\infty} \leq C|v|_{W^{1,4}} \quad \text{and} \quad |v|_{L^\infty} \leq C|v|_{D_0^1 \cap D^2}.$$

Hereafter we use the obvious notation

$$|\cdot|_{X\cap Y} = |\cdot|_X + |\cdot|_Y \quad \text{for (semi-)normed spaces} \quad X, Y$$

and C denotes a generic positive constant depending only on the fixed constants μ, λ, T and the norms of $p = p(\cdot)$ and f. We also denote by H^{-1} the dual space of H_0^1 with $< \cdot, \cdot >$ being the dual paring of H^{-1} and H_0^1. A detailed study of homogeneous Sobolev spaces may be found in Galdi's book [11].

The main purpose of this paper is to prove the local existence of *classical solutions* to the IBVP (1.1)-(1.5) with nonnegative initial densities. First we prove the existence of solutions in $C([0,T_*];(\rho^\infty + H^3) \times (D_0^1 \cap D^3))$ under a stronger compatibility condition than (1.7) on the data.

Theorem 1.1. *Assume that*

$$\rho_0 - \rho^\infty \in H^3, \quad \rho^\infty \in \overline{\mathbf{R}}_+, \quad \rho_0 \geq 0 \quad \textit{in} \quad \Omega, \quad u_0 \in D_0^1 \cap D^3,$$
$$f \in L^2(0,T;H^2), \quad f_t \in L^2(0,T;L^2) \quad \textit{and} \quad p = p(\cdot) \in C^3(\overline{\mathbf{R}}_+). \tag{1.9}$$

Assume further that the data ρ_0, u_0, f satisfy the compatibility condition

$$Lu_0 + \nabla p(\rho_0) = \rho_0 \left(f(0) + g_2\right) \ \textit{for some}\ g_2 \in D_0^1 \ \textit{with}\ \sqrt{\rho_0}\, g_2 \in L^2. \tag{1.10}$$

Then there exist a small time $T_* \in (0,T)$ and a unique strong solution (ρ, u) to the IBVP (1.1)-(1.5) *such that*

$$\rho - \rho^\infty \in C([0,T_*];H^3), \quad u \in C([0,T_*];D_0^1 \cap D^3) \cap L^2(0,T_*;D^4),$$
$$u_t \in L^\infty(0,T_*;D_0^1) \cap L^2(0,T_*;D^2) \quad \textit{and} \quad \sqrt{\rho} u_t \in L^\infty(0,T_*;L^2). \tag{1.11}$$

Remark 1.2. *From the continuity equation* (1.1)*, it follows immediately that*

$$\rho_t \in C([0,T_*];H^2) \quad \textit{and} \quad \rho_{tt} \in L^\infty(0,T_*;L^2) \cap L^2(0,T_*;H^1).$$

Note that the hypotheses of Theorem 1.1 imply (1.6) and (1.7) with $g_1 = \sqrt{\rho_0}(f(0) + g_2) \in L^2$. Hence the existence of a unique local solution (ρ, u) with the regularity (1.8) was already proved in [2, 3] and our new theorem shows that (ρ, u) has some additional regularity if the data satisfy a stronger compatibility condition (1.10). It is easy to show that (1.10) is also necessary for the existence of solutions with the regularity (1.11). In fact, let (ρ, u) be a solution to the IBVP (1.1)-(1.5) with the regularity (1.11). Then since $u_t \in L^\infty(0,T_*;D_0^1)$ and $\sqrt{\rho}u_t \in L^\infty(0,T_*;L^2)$, there is a sequence $\{t_k\}$, $t_k \to 0$, such that $u_t(t_k) \rightharpoonup g$ in D_0^1 for some $g \in D_0^1$ with $\sqrt{\rho}(0)g \in L^2$. Hence letting $t = t_k \to 0$ in the momentum equation (1.2), we readily obtain

$$Lu(0) + \nabla p(\rho(0)) = \rho(0)(f(0) - u(0) \cdot \nabla u(0) - g),$$

which implies then that

$$Lu(0) + \nabla p(\rho(0)) = \rho(0)(f(0) + g_2),$$

where $g_2 = -u(0) \cdot \nabla u(0) - g$. Noting that $\rho(0) = \rho_0$, $u(0) = u_0$, $g_2 \in D_0^1$ and $\sqrt{\rho}(0)g_2 \in L^2$, we conclude that the compatibility condition (1.10) is necessary for the existence of solutions with the regularity (1.11).

In case that ρ_0 has a positive lower bound and u_0 has the additional integrability condition $u_0 \in L^2$, Theorem 1.1 can be proved applying the method of successive approximations or a fixed point argument as in [1, 13, 18, 24, 29]. Our proof of the theorem is based on the method of successive approximations, whose general strategy may be described as follows. First we consider a linearized problem for the IBVP (1.1)-(1.5) and solve it successively to construct a sequence of approximate solutions. Then we derive some uniform bounds for approximate solutions and finally prove the convergence of the sequence to a solution to the original nonlinear problem. A detailed proof of Theorem 1.1 following this strategy is provided in Section 4.

Next, we prove the existence of classical solutions to the IBVP (1.1)-(1.5). Let (ρ, u) be a solution to (1.1)-(1.5) satisfying the regularity in Theorem 1.1. Then in view of the Sobolev embedding results, we have

$$(\rho, u) \in C([0, T_*]; C^1(\overline{\Omega})) \quad \text{and} \quad \rho_t \in C([0, T_*] \times \overline{\Omega}), \tag{1.12}$$

which implies that (ρ, u) satisfies (1.1), (1.3), (1.4) and (1.5) in a classical sense. But in order to conclude that (1.2) is satisfied in a classical sense, we need to prove further regularity of u. In case that ρ_0 is bounded below away from zero, that is, $\delta = \inf_\Omega \rho_0 > 0$, it follows from (1.12) that $\rho \geq \frac{1}{2}\delta > 0$ on $[0, T_{**}] \times \overline{\Omega}$ for some $T_{**} \in (0, T_*]$ and the momentum equation (1.2) can be rewritten as

$$u_t + \rho^{-1} L u = f - u \cdot \nabla u - \rho^{-1} \nabla p(\rho)$$

in $(0, T_{**}) \times \Omega$. Hence by virtue of the smoothing effect of solutions of parabolic equations, we deduce that $(\nabla^2 u, u_t) \in C((0, T_{**}] \times \overline{\Omega})$ and (ρ, u) is a classical solution of (1.2) in $(0, T_{**}) \times \Omega$. For details, see Lemma 2.4 in the next section and the paper [18] by Matsumura and Nishida. However the smoothing effect of the velocity u in $t > 0$ is not obvious for the general case of nonnegative initial densities because (1.2) is no more parabolic in the region where the density vanishes.

Nevertheless, using the same method as in the proof of Theorem 1.1, we can prove the following result.

Theorem 1.3. *In addition to* (1.9) *and* (1.10)*, we assume that*

$$\begin{gathered} t^{\frac{1}{2}} f \in L^\infty(0, T; H^2), \quad t^{\frac{1}{2}} f_t \in L^\infty(0, T; L^2), \quad t^{\frac{1}{2}} f_{tt} \in L^2(0, T; H^{-1}), \\ t f_t \in L^\infty(0, T; H^1), \quad t f_{tt} \in L^2(0, T; L^2), \\ t^{\frac{3}{2}} f_{tt} \in L^\infty(0, T; L^2) \quad \text{and} \quad t^{\frac{3}{2}} f_{ttt} \in L^2(0, T; H^{-1}). \end{gathered} \tag{1.13}$$

Then there exist a small time $T_* \in (0, T)$ and a unique strong solution (ρ, u) to the IBVP (1.1)-(1.5) *such that*

$$\begin{gathered} \rho - \rho^\infty \in C([0, T_*]; H^3), \quad u \in C([0, T_*]; D_0^1 \cap D^3) \cap L^2(0, T_*; D^4), \\ u_t \in L^\infty(0, T_*; D_0^1) \cap L^2(0, T_*; D^2), \quad \sqrt{\rho} u_{tt} \in L^2(0, T_*; L^2); \\ t^{\frac{1}{2}} u \in L^\infty(0, T_*; D^4), \quad t^{\frac{1}{2}} u_t \in L^\infty(0, T_*; D^2), \quad t^{\frac{1}{2}} u_{tt} \in L^2(0, T_*; D_0^1), \\ t^{\frac{1}{2}} \sqrt{\rho} u_{tt} \in L^\infty(0, T_*; L^2); \qquad t u_t \in L^\infty(0, T_*; D^3), \\ t u_{tt} \in L^\infty(0, T_*; D_0^1) \cap L^2(0, T_*; D^2), \quad t \sqrt{\rho} u_{ttt} \in L^2(0, T_*; L^2); \\ t^{\frac{3}{2}} u_{tt} \in L^\infty(0, T_*; D^2), \quad t^{\frac{3}{2}} u_{ttt} \in L^2(0, T_*; D_0^1), \quad t^{\frac{3}{2}} \sqrt{\rho} u_{ttt} \in L^\infty(0, T_*; L^2). \end{gathered} \tag{1.14}$$

Let (ρ, u) be a solution of the compressible Navier-Stokes equations (1.1)-(1.3) with the regularity (1.14). Then it is easy to show that (ρ, u) is indeed a classical solution of (1.1)-(1.3) in $(0, T_*] \times \Omega$. First, using the standard embedding results

$$L^2(0, T_*; H^1) \cap W^{1,2}(0, T_*; H^{-1}) \hookrightarrow C([0, T_*]; L^2)$$

and

$$L^\infty(0, T_*; H^1) \cap W^{1,2}(0, T_*; H^{-1}) \hookrightarrow C([0, T_*]; L^q)$$

for any $2 \leq q < 6$, we deduce from (1.13) and (1.14) that

$$t^{\frac{1}{2}} f \in C([0, T_*]; W^{1,4}) \quad \text{and} \quad t u_t \in C([0, T_*]; D_0^1 \cap D^2).$$

On the other hand, by virtue of the continuity equation (1.1), we can rewrite the momentum equation (1.2) as

$$\rho u_t + \rho u \cdot \nabla u + Lu + \nabla p(\rho) = \rho f \qquad \text{in} \quad (0, T_*) \times \Omega,$$

which implies that for each $t \in (0, T_*]$, $u = u(t) \in D_0^1 \cap D^3$ is a solution of the elliptic system

$$Lu = \rho(f - u_t - u \cdot \nabla u) - \nabla p(\rho) \equiv F \qquad \text{in} \quad \Omega.$$

Note that $tF \in C([0, T_*]; W^{1,4})$. Hence it follows from the elliptic regularity result in [3] that

$$t\nabla^2 u \in C([0, T_*]; W^{1,4}).$$

Therefore, in view of the Sobolev embedding results, we conclude that

$$(u_t,\, \nabla^2 u) \in C((0, T_*] \times \overline{\Omega})$$

and so (ρ, u) is a classical solution of (1.1)-(1.3) in $(0, T_*] \times \Omega$.

We have considered the Navier-Stokes equations (1.1)-(1.3) for general barotropic compressible fluids including *isentropic fluids* as an important special class. An isentropic viscous compressible fluid is governed by the Navier-Stokes equations (1.1)-(1.3) with the density-pressure law $p = p(\cdot)$ given by

$$p = A\rho^\gamma \qquad \text{for some constants} \quad A > 0, \;\; \gamma > 1. \tag{1.15}$$

Note that (1.15) defines a C^3-function on $\overline{\mathbf{R}}_+$ if and only if $\gamma = 2$ or $\gamma \geq 3$. Hence Theorem 1.1 and Theorem 1.3 can be used to deduce the corresponding existence results for the isentropic equations (1.1)-(1.3) and (1.15) only in case when $\gamma = 2$ or $\gamma \geq 3$. But in some physical situations, the case $1 < \gamma < 2$ is most important: for instance, $\gamma = \frac{5}{3}$ in case of monatomic gases like Helium and Neon. The final goal of the paper is to prove the existence of classical solutions of the isentropic compressible Navier-Stokes equations (1.1)-(1.3) and (1.15) for general $\gamma > 1$.

Theorem 1.4. *Assume that the data ρ_0, u_0, f satisfy the regularity condition*

$$\begin{aligned} &(\rho_0 - \rho^\infty, p_0 - p^\infty) \in H^3, \quad \rho^\infty \in \overline{\mathbf{R}}_+, \quad \rho_0 \geq 0 \quad \text{in} \quad \Omega, \\ &u_0 \in D_0^1 \cap D^3, \quad f \in L^2(0, T; H^2) \quad \text{and} \quad f_t \in L^2(0, T; L^2) \end{aligned}$$

and the compatibility condition

$$Lu_0 + \nabla p_0 = \rho_0\,(f(0) + g_2) \;\; \text{for some } g_2 \in D_0^1 \text{ with } \sqrt{\rho_0}\, g_2 \in L^2,$$

where

$$p_0 = A\rho_0^\gamma \qquad \text{and} \qquad p^\infty = A(\rho^\infty)^\gamma.$$

Then there exist a small time $T_* \in (0, T)$ and a unique strong solution (ρ, p, u) to the IBVP (1.1)-(1.5) *and* (1.15) *such that*

$$\begin{aligned} &(\rho - \rho^\infty,\, p - p^\infty) \in C([0, T_*]; H^3), \quad u \in C([0, T_*]; D_0^1 \cap D^3) \cap L^2(0, T_*; D^4), \\ &\quad u_t \in L^\infty(0, T_*; D_0^1) \cap L^2(0, T_*; D^2) \quad \text{and} \quad \sqrt{\rho} u_t \in L^\infty(0, T_*; L^2). \end{aligned}$$

Moreover, if the external force f satisfies the additional regularity (1.13)*, then the velocity u satisfies* (1.14) *with T_* replaced by some $T_{**} \in (0, T_*]$ and so (ρ, p, u) is a classical solution of* (1.1)-(1.3) *and* (1.15) *in* $(0, T_{**}) \times \Omega$.

If $\gamma = 2$ or $\gamma \geq 3$, then Theorem 1.4 is just a reformulation of Theorem 1.1 and Theorem 1.3 because

$$\rho - \rho^\infty \in C([0, T_*]; H^3) \quad \text{implies that} \quad p - p^\infty \in C([0, T_*]; H^3). \tag{1.16}$$

But (1.16) fails to hold for general $\gamma > 1$ and in fact, one major difficulty in proving Theorem 1.4 is to show that $p - p^\infty \in C([0, T_*]; H^3)$. Our proof relies heavily on the observation that since ρ satisfies (1.1) and (1.4), the pressure $p = A\rho^\gamma$ is a solution to the linear hyperbolic problem

$$p_t + u \cdot \nabla p + \gamma p \, \mathrm{div} u = 0 \quad \text{in} \quad (0, T) \times \Omega \quad \text{and} \quad p|_{t=0} = p_0 \quad \text{in} \quad \Omega,$$

provided that u is regarded as a known vector field. Hence assuming that u is sufficiently regular, we can deduce from a standard regularity theory of hyperbolic equations that if $p_0 - p^\infty \in H^3$, then $p - p^\infty \in C([0, T_*]; H^3)$. A detailed proof of Theorem 1.4 is given in the final section.

The main results in this paper are Theorem 1.3 and Theorem 1.4 which are both local existence results on classical solutions. It is then a fundamental question to ask whether the solutions exist globally in time. A negative answer was obtained by Xin [30] for the case that the spatial domain Ω is the whole space $\mathbf{R}^3$. He showed that there is no global classical solution to the Cauchy problem for the isentropic compressible Navier-Stokes equations with compactly supported initial density and velocity. On the other hand, Choe and the second author [5] obtained a global existence result on radially symmetric strong solutions of the isentropic compressible Navier-Stokes equations in bounded and unbounded annular domains. Hence it is very likely that the methods in this paper and [5] can be combined to prove the global existence of radially symmetric classical solutions with nonnegative densities. This issue will be studied in a separated paper.

The rest of this paper is organized as follows. Section 2 is devoted to a study of a linearized problem. We provide some existence and regularity results for a linear transport equation and a linear parabolic system. In Section 3, we derive some a priori estimates for solutions to the linearized problem. Applying the method of successive approximations based on these estimates, we prove Theorem 1.1 in Section 4. Finally, the proofs of Theorem 1.3 and Theorem 1.4 are given in Section 5 and Section 6, respectively.

2. Existence and regularity on solutions of linear equations

In this section, we obtain some existence and regularity results on solutions of a linear transport equation and a linear parabolic system, which are necessary to prove all the main theorems in the paper.

2.1. **A linear transport equation.** First, we consider the following linear hyperbolic problem

$$\rho_t + v \cdot \nabla \rho + \rho \, \mathrm{div} v = 0 \quad \text{in} \quad (0, T) \times \Omega \quad \text{and} \quad \rho(0) = \rho_0 \quad \text{in} \quad \Omega, \tag{2.1}$$

where v is a known vector field in $(0, T) \times \Omega$ such that

$$v \in C([0, T]; D_0^1 \cap D^m) \cap L^2(0, T; D^{m+1}) \quad \text{for some integer} \quad m \geq 2.$$

Following the arguments in [2], we prove

Lemma 2.1. *Assume that* $\rho_0 - \rho^\infty \in H^m$, $\rho^\infty \in \overline{\mathbf{R}}_+$ *and* $\rho_0 \geq 0$ *in* Ω. *Then*
(i) *there exists a unique solution* ρ *to the problem* (2.1) *such that*

$$\rho - \rho^\infty \in C([0,T]; H^m) \quad \text{and} \quad \rho_t \in C([0,T]; H^{m-1}),$$

(ii) *the solution* ρ *satisfies the following estimate*

$$|\rho(t) - \rho^\infty|_{H^m} \leq (|\rho_0 - \rho^\infty|_{H^m} + \rho^\infty) \exp\left(C \int_0^t |v(s)|_{D_0^1 \cap D^{m+1}} \, ds \right)$$

for $0 \leq t \leq T$ *and finally,*
(iii) *the solution* ρ *is represented by the formula*

$$\rho(t,x) = \rho_0(\, U(0,t,x)\,) \exp\left[- \int_0^t \operatorname{div} v(s, U(s,t,x)\,)\, ds \right], \tag{2.2}$$

where $U \in C([0,T] \times [0,T] \times \overline{\Omega})$ *is the solution to the initial value problem*

$$\begin{cases} \frac{\partial}{\partial t} U(t,s,x) = v(t, U(t,s,x)\,), & 0 \leq t \leq T, \\ U(s,s,x) = x, & 0 \leq s \leq T, \;\; x \in \overline{\Omega}. \end{cases} \tag{2.3}$$

Proof. To begin with, we construct sequences $\{\rho_0^k\}$ and $\{v^k\}$ of more regular scalar and vector fields such that

$$\begin{aligned} &\rho_0^k - \rho^\infty \in H^m \cap C^{m+1}(\overline{\Omega}), \\ &v^k \in L^2(0,T; D_0^1 \cap D^{m+1}) \cap C^{m+1}([0,T] \times \overline{\Omega}), \\ &|\rho_0^k - \rho_0|_{H^m} + |v^k - v|_{L^2(0,T;D_0^1 \cap D^{m+1})} \to 0 \quad \text{as} \quad k \to \infty. \end{aligned} \tag{2.4}$$

For this purpose, we first recall that H^{m+3} and $L^2(0,T;H^{m+2})$ are dense in H^m and $L^2(0,T;H^m)$, respectively. Then since $\rho_0 - \rho^\infty \in H^m$ and $g = \nabla v \in L^2(0,T;H^m)$, there exist sequences $\{\rho_0^k\}$ in $\rho^\infty + H^{m+3}$ and $\{g^k\}$ in $L^2(0,T;H^{m+2})$ such that $\rho_0^k - \rho^\infty \to \rho_0 - \rho^\infty$ in H^m and $g^k \to g$ in $L^2(0,T;H^m)$ as $k \to \infty$.

For a.e. $t \in (0,T)$, let $w^k = w^k(t) \in D_0^1$ be the unique weak solution to the elliptic boundary value problem

$$\Delta w^k = \operatorname{div} g^k \quad \text{in} \quad \Omega \quad \text{and} \quad w^k = 0 \quad \text{on} \quad \partial\Omega.$$

It is obvious that $w^k \in L^2(0,T;D_0^1)$ and $|w^k(t) - v(t)|_{D_0^1} \leq |g^k(t) - g(t)|_{L^2}$ for a.e. $t \in (0,T)$. Then by virtue of the elliptic regularity result in [3], we deduce that $w^k \in L^2(0,T;D_0^1 \cap D^{m+3})$ and

$$\begin{aligned} |w^k(t) - v(t)|_{D_0^1 \cap D^{m+1}} &\leq C \left(|\operatorname{div} g^k(t) - \operatorname{div} g(t)|_{H^{m-1}} + |w^k(t) - v(t)|_{D_0^1} \right) \\ &\leq C |g^k(t) - g(t)|_{H^m} \end{aligned}$$

for a.e. $t \in (0,T)$. Hence it follows that $w^k \to v$ in $L^2(0,T;D_0^1 \cap D^{m+1})$ as $k \to \infty$. Therefore, recalling that $C^\infty([0,T];D_0^1 \cap D^{m+3})$ is dense in $L^2(0,T;D_0^1 \cap D^{m+3})$, we conclude that there exists a sequence $\{v^k\}$ in $C^\infty([0,T];D_0^1 \cap D^{m+3})$ such that $v^k \to v$ in $L^2(0,T;D_0^1 \cap D^{m+1})$ as $k \to \infty$. In view of the Sobolev embedding results

$$H^{m+3} \hookrightarrow C^{m+1}(\overline{\Omega}) \quad \text{and} \quad D_0^1 \cap D^{m+3} \hookrightarrow C^{m+1}(\overline{\Omega}),$$

we complete the proof of (2.4). To treat the case of unbounded domains, we also need a cut-off procedure. Assuming that Ω is an unbounded domain such as the whole space, the half space and an exterior domain, we choose a sufficiently large integer $R_0 > 1$ so that

$$\mathbf{R}^3 \setminus \Omega \subset B_{R_0/2} \quad \text{if} \quad \mathbf{R}^3 \setminus \Omega \subset\subset \mathbf{R}^3,$$

where for each $R > 0$, B_R denotes the open ball of radius R centered at the origin: $B_R = \{x \in \mathbf{R}^3 : |x| < R\}$. Then taking a cut-off function $\varphi \in C_c^\infty(B_1)$ such that $\varphi = 1$ in $B_{1/2}$, we define ρ_0^R and v^R by

$$\rho_0^R(x) = \rho^\infty + \varphi(x/R)\,(\rho_0(x) - \rho^\infty) \quad \text{and} \quad v^R(t,x) = \varphi(x/R)v(t,x)$$

for $(t,x) \in [0,T] \times \Omega$ and $R > R_0$. Note that $\rho_0^R = \rho^\infty$ and $v^R = 0$ in $(0,T) \times (\Omega \setminus \Omega_R)$, where $\Omega_R = \Omega \cap B_R$. Moreover, it is easy to show that

$$|\rho_0^R - \rho_0|_{H^m} + |v^R - v|_{L^2(0,T;D_0^1 \cap D^{m+1})} \to 0 \quad \text{as} \quad R \to \infty.$$

Hence applying this cut-off technique to ρ_0^k and v^k for each $k \geq 1$, we may assume without loss of generality that if Ω is an unbounded domain, then

$$\rho_0^k(x) = \rho^\infty \quad \text{and} \quad v^k(t,x) = 0 \quad \text{for} \quad t \in [0,T],\ \ x \in \Omega \setminus \Omega_{R_k}, \tag{2.5}$$

where $\{R_k\}$ is a sequence such that $R_0 < R_1 < R_2 < \cdots$ and $R_k \to \infty$.

Now we consider the following regularized problem

$$\rho_t + v^k \cdot \nabla\rho + \rho\,\mathrm{div} v^k = 0 \quad \text{in} \quad (0,T) \times \Omega \quad \text{and} \quad \rho(0) = \rho_0^k \quad \text{in} \quad \Omega \tag{2.6}$$

for each $k \geq 1$. Then since $\rho_0^k \in C^{m+1}(\overline{\Omega})$, $v^k \in C^{m+1}([0,T] \times \overline{\Omega})$ and $v^k = 0$ on $[0,T] \times \partial\Omega$, it follows from the standard hyperbolic theory that there exists a unique solution $\rho^k \in C^{m+1}([0,T] \times \overline{\Omega})$ to the problem (2.6) and the solution ρ^k can be represented by

$$\rho^k(t,x) = \rho_0^k(U^k(0,t,x)) \exp\left[-\int_0^t \mathrm{div}\, v^k(s, U^k(s,t,x))\, ds\right], \tag{2.7}$$

where $U^k \in C^{m+1}([0,T] \times [0,T] \times \overline{\Omega})$ is the solution to the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} U^k(t,s,x) = v^k(t, U^k(t,s,x)\,), & 0 \leq t \leq T, \\ U^k(s,s,x) = x, \quad 0 \leq s \leq T, & x \in \overline{\Omega}. \end{cases} \tag{2.8}$$

It should be noted from (2.5) that if Ω is an unbounded domain, then

$$U^k(t,s,x) = x \quad \text{and} \quad \rho^k(t,x) = \rho^\infty \quad \text{for} \quad t,s \in [0,T],\ \ x \in \Omega \setminus \Omega_{R_k}.$$

We will prove that the sequence $\{\rho^k\}$ converges to a solution of the original problem. To show this, we first observe that

$$\begin{aligned} &|U^k(t,s,x) - U^l(t,s,x)| \\ &\quad \leq \int_s^t \left|v^k(\tau, U^k(\tau,s,x)\,) - v^l(\tau, U^l(\tau,s,x)\,)\right|\, d\tau \\ &\quad \leq \int_s^t |v^k(\tau) - v^l(\tau)|_{L^\infty}\, d\tau + \int_s^t |\nabla v^l(\tau)|_{L^\infty} |U^k(\tau,s,x) - U^l(\tau,s,x)|\, d\tau. \end{aligned}$$

Then in view of Gronwall's inequality, we have

$$\begin{aligned}&|U^k(t,s,x)-U^l(t,s,x)|\\&\leq\left(\int_0^T|v^k(\tau)-v^l(\tau)|_{L^\infty}\,d\tau\right)\exp\left(\int_0^T|\nabla v^l(\tau)|_{L^\infty}\,d\tau\right)\\&\leq C\left(\int_0^T|v^k(\tau)-v^l(\tau)|_{D_0^1\cap D^2}\,d\tau\right)\exp\left(C\int_0^T|v^l(\tau)|_{D_0^1\cap D^3}\,d\tau\right)\end{aligned}$$

for each $s,t\in[0,T]$ and $x\in\overline{\Omega}$, and thus

$$|U^k-U^l|_{C([0,T]\times[0,T]\times\overline{\Omega})}\to 0\qquad\text{as}\quad k,l\to\infty. \tag{2.9}$$

Hence it follows from the well-known embedding result $H^2\hookrightarrow C^{0,\frac{1}{2}}$ that

$$\begin{aligned}&\int_0^T\left|\operatorname{div}v(s,U^k(s,t,x))-\operatorname{div}v(s,U^l(s,t,x))\right|\,ds\\&\leq C\int_0^T|\nabla v(s)|_{H^2}\left|U^k(s,t,x)-U^l(s,t,x)\right|^{\frac{1}{2}}\,ds\to 0\qquad\text{as}\quad k,l\to\infty\end{aligned}$$

uniformly in $(t,\,x)\in[0,\,T]\times\overline{\Omega}$. Therefore, observing that

$$\begin{aligned}&\int_0^t\left|\operatorname{div}v^k(s,U^k(s,t,x))-\operatorname{div}v^l(s,U^l(s,t,x))\right|\,ds\\&\leq\int_0^T\left(|\operatorname{div}v^k(s)-\operatorname{div}v(s)|_{L^\infty}+|\operatorname{div}v^l(s)-\operatorname{div}v(s)|_{L^\infty}\right)\,ds\\&\quad+\int_0^T\left|\operatorname{div}v(s,U^k(s,t,x))-\operatorname{div}v(s,U^l(s,t,x))\right|\,ds,\end{aligned}$$

we deduce from (2.7) that

$$|\rho^k-\rho^l|_{C([0,T]\times\overline{\Omega})}\to 0\quad\text{as}\quad k,l\to\infty.$$

This proves the existence of a limit ρ in $C([0,T]\times\overline{\Omega})$ such that

$$\rho^k\to\rho\quad\text{in}\quad C([0,T]\times\overline{\Omega})\quad\text{as}\quad k\to\infty. \tag{2.10}$$

It is easy to show that ρ is a weak solution to the original problem (2.1).

To prove the higher regularity of ρ, we derive uniform estimates for ρ^k in higher norms. Multiplying the equation in (2.6) with $\rho=\rho^k$ by $\rho^k-\rho^\infty$ and integrating over Ω, we have

$$\frac{d}{dt}\int|\rho^k-\rho^\infty|^2\,dx\leq C\int|\operatorname{div}v^k|\left(|\rho^k-\rho^\infty|+\rho^\infty\right)|\rho-\rho^\infty|\,dx$$

and thus

$$\frac{d}{dt}|\rho^k-\rho^\infty|_{L^2}^2\leq C|\nabla v^k|_{L^\infty}|\rho^k-\rho^\infty|_{L^2}^2+C\rho^\infty|\rho^k-\rho^\infty|_{L^2}|\nabla v^k|_{L^2}. \tag{2.11}$$

Let α be a multi-index with $1\leq|\alpha|=\alpha_1+\alpha_2+\alpha_3\leq m$. Then taking the differential operator D^α to (2.6), we have

$$\begin{aligned}&(D^\alpha\rho^k)_t+v^k\cdot\nabla(D^\alpha\rho^k)\\&\quad=v^k\cdot\nabla(D^\alpha\rho^k)-D^\alpha(v^k\cdot\nabla\rho^k)-D^\alpha(\rho^k\operatorname{div}v^k)\equiv F_\alpha^k.\end{aligned}$$

Multiplying this by $D^\alpha \rho$ and integrating over Ω, we obtain

$$\frac{d}{dt}\int |D^\alpha \rho^k|^2\,dx \le C\int \left(|\mathrm{div}\, v^k||D^\alpha\rho^k|^2 + |F^k_\alpha||D^\alpha\rho^k|\right)\,dx$$

and thus

$$\frac{d}{dt}|D^\alpha\rho^k|^2_{L^2} \le C\left(|\mathrm{div}\, v^k|_{L^\infty}|D^\alpha\rho^k|^2_{L^2} + |F^k_\alpha|_{L^2}|D^\alpha\rho^k|_{L^2}\right). \tag{2.12}$$

But since

$$|v^k\cdot\nabla(D^\alpha\rho^k) - D^\alpha(v^k\cdot\nabla\rho^k)| \le C\sum_{l=1}^{|\alpha|}\left|\nabla^{|\alpha|+1-l}v^k\right|\left|\nabla^l\rho^k\right|,$$

it follows from Hölder and Sobolev inequalities that

$$\sup_{1\le|\alpha|\le m}|v^k\cdot\nabla(D^\alpha\rho^k) - D^\alpha(v^k\cdot\nabla\rho^k)|_{L^2} \le C|v^k|_{D^1_0\cap D^{m+1}}|\nabla\rho^k|_{H^{m-1}}.$$

A similar calculation also shows that

$$\sup_{1\le|\alpha|\le m}|D^\alpha(\rho^k\,\mathrm{div}\, v^k)|_{L^2} \le C|v^k|_{D^1_0\cap D^{m+1}}\left(|\nabla\rho^k|_{H^{m-1}} + |\rho^k|_{L^\infty}\right).$$

Hence from (2.11) and (2.12), it follows that

$$\frac{d}{dt}|\rho^k-\rho^\infty|^2_{H^m} \le C|v^k|_{D^1_0\cap D^{m+1}}|\rho^k-\rho^\infty|^2_{H^m} + C\rho^\infty|v^k|_{D^1_0\cap D^{m+1}}|\rho^k-\rho^\infty|_{H^m}.$$

Therefore, in view of Gronwall's inequality, we conclude that

$$\begin{aligned}|\rho^k(t)-\rho^\infty|_{H^m} \le &\left(|\rho^k_0-\rho^\infty|_{H^m} + C\rho^\infty\int_0^t|v^k(s)|_{D^1_0\cap D^{m+1}}\,ds\right)\\ &\times\exp\left(C\int_0^t|v^k(s)|_{D^1_0\cap D^{m+1}}\,ds\right)\end{aligned} \tag{2.13}$$

for each $t\in[0,T]$. As a consequence of (2.10) and (2.13), we deduce that

$$\rho^k-\rho^\infty \overset{*}{\rightharpoonup} \rho-\rho^\infty \quad \text{in} \quad L^\infty(0,T;H^m) \quad \text{as} \quad k\to\infty.$$

Moreover since $\rho_t = -\mathrm{div}\,(\rho v)\in L^\infty(0,T;H^{m-1})$, it follows from a classical embedding result (see [26] for instance) that $\rho-\rho^\infty\in C([0,T];H^{m-1})\cap C([0,T];H^m-\mathit{weak})$. To prove the strong time-continuity of $\rho-\rho^\infty$ in H^m, we observe that for each fixed $t\in[0,T]$, $\rho^k(t)-\rho^\infty\to\rho(t)-\rho^\infty$ weakly in H^m. Hence from (2.13), it follows immediately that

$$\begin{aligned}|\rho(t)-\rho^\infty|_{H^m} \le &\left(|\rho_0-\rho^\infty|_{H^m} + C\rho^\infty\int_0^t|v(s)|_{D^1_0\cap D^{m+1}}\,ds\right)\\ &\times\exp\left(C\int_0^t|v(s)|_{D^1_0\cap D^{m+1}}\,ds\right)\end{aligned} \tag{2.14}$$

for each $t\in[0,T]$. In particular, we have

$$\limsup_{t\to+0}|\rho(t)-\rho^\infty|_{H^m} \le |\rho_0-\rho^\infty|_{H^m},$$

which implies that $\rho-\rho^\infty$ is right-continuous in H^m at $t=0$. Since the equation in (2.1) is invariant under the reflections and translations in time, we conclude that $\rho-\rho^\infty\in C([0,T];H^m)$. It also follows from (2.1) that $\rho_t\in C([0,T];H^{m-1})$. It is easy to prove the uniqueness of solutions in this regularity class. This completes the proof of (i). The estimate in (ii) follows immediately from (2.14). Hence it remains to show (iii). By virtue of the regularity of v, we can prove the uniqueness of a solution U in $C([0,T] \times [0,T] \times \overline{\Omega})$ to the problem (2.3), whose existence is guaranteed by (2.8) and (2.9). Finally, from (2.7), (2.9) and (2.10), we obtain the representation formula (2.2) for the solution ρ. □

2.2. **A linear parabolic system.** Next, let Ω be a bounded domain in $\mathbf{R}^3$ with smooth boundary, and we consider the following linear parabolic problem

$$\begin{cases} \rho u_t + Lu = F \quad \text{in} \quad (0,T)\times\Omega, \\ \\ u(0) = u_0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \end{cases} \tag{2.15}$$

where ρ is a known scalar field in $(0,T)\times\Omega$ such that

$$\rho \in C([0,T];H^3), \quad \rho_t \in C([0,T];H^2) \quad \text{and} \quad \rho \geq \delta \quad \text{on} \quad [0,T]\times\overline{\Omega} \tag{2.16}$$

for some constant $\delta > 0$. Recall that $L = -\mu\Delta - (\lambda+\mu)\nabla\text{div}$ is a strongly elliptic operator (see [3] for instance). Then applying a standard method such as a semi-discrete Galerkin method or the method of continuity, we can prove the following existence and regularity results on solutions to the linear parabolic problem (2.15). See also the papers [27, 28, 29] for similar results.

Lemma 2.2. (i) *Assume that* $u_0 \in H_0^1$ *and* $F \in L^2(0,T;L^2)$*. Then there exists a unique strong solution* u *to the problem* (2.15) *such that*

$$u \in C([0,T];H_0^1) \cap L^2(0,T;H^2) \quad \text{and} \quad u_t \in L^2(0,T;L^2).$$

(ii) *If* $u_0 \in H_0^1 \cap H^2$*,* $F \in L^\infty(0,T;L^2)$ *and* $F_t \in L^2(0,T;H^{-1})$*, then the solution* u *satisfies*

$$u \in L^\infty(0,T;H^2), \quad u_t \in L^2(0,T;H_0^1) \quad \text{and} \quad u_{tt} \in L^2(0,T;H^{-1}).$$

(iii) *Finally, if* $u_0 \in H_0^1 \cap H^3$*,* $F \in L^\infty(0,T;H^1)$*,* $F_t \in L^2(0,T;L^2)$ *and* $u_t(0) = \rho(0)^{-1}(F(0) - Lu_0) \in H_0^1$*, then the solution* u *also satisfies*

$$u \in L^\infty(0,T;H^3), \quad u_t \in L^2(0,T;H^2) \quad \text{and} \quad u_{tt} \in L^2(0,T;L^2).$$

Remark 2.3. *Let* u *be the solution obtained in the result* (iii) *of* Lemma 2.2*. Then by virtue of a standard embedding result, we have*

$$u \in C([0,T];H^2) \quad \text{and} \quad u_t \in C([0,T];H_0^1).$$

Moreover, it follows from an elliptic regularity result that if $F \in L^2(0,T;H^2)$ *in addition, then* u *also satisfies*

$$u \in L^2(0,T;H^4) \quad \text{and so} \quad u \in C([0,T];H^3).$$

Standard arguments based on Lemma 2.2 enable us to prove the *smoothing effect* of the solution u for positive time $t > 0$, provided that ρ and F are sufficiently regular in $t > 0$. Throughout this paper, we denote

$$L^r_{loc}((0,T];X) = \bigcap_{\tau>0} L^r(\tau,T;X)$$

for $1 \leq r \leq \infty$ and a Banach space X.

Lemma 2.4. *Let $u_0 \in H_0^1$ and $F \in L^2(0,T;L^2)$. Assume in addition to* (2.16) *that*

$$\rho_{tt} \in L^\infty_{loc}((0,T];L^2), \quad \rho_{ttt} \in L^2_{loc}((0,T];H^{-1}), \quad F \in L^\infty_{loc}((0,T];H^2),$$
$$F_t \in L^\infty_{loc}((0,T];H^1), \quad F_{tt} \in L^\infty_{loc}((0,T];L^2) \quad \textit{and} \quad F_{ttt} \in L^2_{loc}((0,T];H^{-1}).$$

Then there exists a unique solution u to the problem (2.15) *such that*

$$u \in C([0,T];H_0^1) \cap L^2(0,T;H^2), \quad u_t \in L^2(0,T;L^2);$$
$$u \in L^\infty_{loc}((0,T];H^4), \quad u_t \in L^\infty_{loc}((0,T];H_0^1 \cap H^3), \quad u_{tt} \in L^\infty_{loc}((0,T];H_0^1 \cap H^2),$$
$$u_{ttt} \in L^\infty_{loc}((0,T];L^2) \cap L^2_{loc}((0,T];H_0^1) \quad \textit{and} \quad u_{tttt} \in L^2_{loc}((0,T];H^{-1}).$$

Proof. The result (i) of Lemma 2.2 guarantees the existence of a unique solution u with the regularity

$$u \in C([0,T];H_0^1) \cap L^2(0,T;H^2) \quad \text{and} \quad u_t \in L^2(0,T;L^2).$$

We prove the additional regularity of u using a standard iterative argument (see [25] for instance). Let t_0 be a fixed small time in $(0,T)$.

(a) Since $u \in L^2(0,T;H_0^1 \cap H^2)$, we can choose a time t_1 in $(0,t_0)$ such that $u(t_1) \in H_0^1 \cap H^2$. Then the result (ii) yields that $u \in L^\infty(t_1,T;H_0^1 \cap H^2)$ and $u_t \in L^2(t_1,T;H_0^1)$. Moreover, since $F \in L^2(t_1,T;H^1)$, it follows from the elliptic regularity result that $u \in L^2(t_1,T;H^3)$.

(b) There is a time $t_2 \in (t_1,t_0)$ such that $u(t_2) \in H_0^1 \cap H^3$ and $u_t(t_2) \in H_0^1$. In view of the result (iii), we deduce that

$$u \in L^\infty(t_2,T;H^3), \quad u_t \in L^2(t_2,T;H^2) \quad \text{and} \quad u_{tt} \in L^2(t_2,T];L^2).$$

(c) There is a time $t_3 \in (t_2,t_0)$ such that $u_t(t_3) \in H_0^1 \cap H^2$. Note that $w = u_t$ is the unique solution to the problem

$$\text{(2.17)} \qquad \begin{cases} \rho w_t + Lw = G & \text{in} \quad (t_3,T) \times \Omega, \\ w(t_3) = u_t(t_3) \quad \text{in} \quad \Omega, & w = 0 \quad \text{on} \quad (t_3,T) \times \partial\Omega, \end{cases}$$

where $G = F_t - \rho_t u_t$. Note that $G \in L^2(t_3,T;H^1)$ and $G_t \in L^2(t_3,T;H^{-1})$. Hence it follows from the result (ii) that

$$w \in L^\infty(t_3,T;H^2), \quad w_t \in L^2(t_3,T;H_0^1) \quad \text{and} \quad w_{tt} \in L^2(t_3,T;H^{-1}).$$

Moreover, using the elliptic regularity result again, we deduce that

$$u \in L^\infty(t_3,T;H^4) \quad \text{and} \quad w \in L^2(t_3,T;H^3).$$

(d) There is $t_4 \in (t_3,t_0)$ such that $w(t_4) \in H_0^1 \cap H^3$ and $w_t(t_4) \in H_0^1$. Note that $G = F_t - \rho_t w \in L^\infty(t_4,T;H^1)$ and $G_t \in L^2(t_4,T;L^2)$. Hence it follows from (iii) that

$$w \in L^\infty(t_4,T;H^3), \quad w_t \in L^2(t_4,T;H^2) \quad \text{and} \quad w_{tt} \in L^2(t_4,T;L^2).$$

(e) There is a time $t_5 \in (t_4,t_0)$ such that $w_t(t_5) \in H_0^1 \cap H^2$ and $v = w_t$ is the unique solution to the problem

$$\text{(2.18)} \qquad \begin{cases} \rho v_t + Lv = H & \text{in} \quad (t_5,T) \times \Omega, \\ w(t_5) = u_t(t_5) \quad \text{in} \quad \Omega, & w = 0 \quad \text{on} \quad (t_5,T) \times \partial\Omega, \end{cases}$$

where $H = G_t - \rho_t w_t$. Since $H \in L^\infty(t_5, T; L^2)$ and $H_t \in L^2(t_5, T; H^{-1})$, it follows from (ii) that

$$v \in L^\infty(t_5, T; H^2), \quad v_t \in L^2(t_5, T; H_0^1) \quad \text{and} \quad v_{tt} \in L^2(t_5, T; H^{-1}).$$

Observing that $0 < t_1 < t_2 < t_3 < t_4 < t_5 < t_0$ and t_0 can be chosen to be arbitrarily small, we complete the proof of Lemma 2.4. □

3. A priori estimates for the linearized problem

To prove Theorem 1.1, we consider the following linearized problem

$$\rho_t + \operatorname{div}(\rho v) = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{3.1}$$

$$\rho u_t + Lu + \nabla p = \rho(f - v\cdot\nabla v) \quad \text{in} \quad (0,T)\times\Omega, \tag{3.2}$$

$$(\rho,\, u)|_{t=0} = (\rho_0,\, u_0) \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{3.3}$$

$$\rho(t,x) \to \rho^\infty, \quad u(t,x) \to 0 \quad \text{as} \quad |x| \to \infty, \quad (t,x) \in (0,T)\times\Omega, \tag{3.4}$$

where v is a known vector field in $(0,T)\times\Omega$ such that

$$v \in C([0,T]; D_0^1 \cap D^3) \cap L^2(0,T; D^4), \quad v_t \in L^\infty(0,T; D_0^1) \cap L^2(0,T; D^2). \tag{3.5}$$

Recall again that $Lu = -\mu\Delta u - (\lambda+\mu)\nabla\operatorname{div} u$ and $p = p(\rho)$.

First, from the lemmas in Section 2, we obtain an existence result for positive initial densities.

Lemma 3.1. *Let Ω be a bounded domain in $\mathbf{R}^3$ with smooth boundary. In addition to* (1.9) *and* (3.5)*, we assume that $\rho_0 \geq \delta$ in Ω for some constant $\delta > 0$ and $f(0) - v(0)\cdot\nabla v(0) - \rho_0^{-1}(Lu_0 + \nabla p(\rho_0)) \in H_0^1$. Then there exists a unique solution (ρ, u) to the linearized problem* (3.1), (3.2) *and* (3.3) *such that*

$$\begin{aligned} &\rho \in C([0,T]; H^3), \quad \rho_t \in C([0,T]; H^2),\\ &u \in C([0,T]; H_0^1 \cap H^3) \cap L^2(0,T; H^4),\\ &u_t \in C([0,T]; H_0^1) \cap L^2(0,T; H^2),\\ &u_{tt} \in L^2(0,T; L^2) \quad \text{and} \quad \rho \geq \underline{\delta} \quad \text{on} \quad [0,T]\times\overline{\Omega} \end{aligned} \tag{3.6}$$

for some constant $\underline{\delta} > 0$.

Proof. The existence and regularity of a unique solution ρ to the linear hyperbolic problem (3.1) and (3.3) were already proved in Lemma 2.1. To prove the remaining part of the lemma, let us define F by $F = -\nabla p(\rho) + \rho(f - v\cdot\nabla v)$. Then by virtue of (1.9), (3.5) and the regularity of ρ, we can easily show that $F \in L^2(0,T;H^2)$ and $F_t \in L^2(0,T;L^2)$. Moreover since $\rho_0^{-1}(F(0) - Lu_0) \in H_0^1$, Lemma 2.2 and Remark 2.3 allow us to deduce the existence and regularity of a unique solution u to the linear parabolic problem (3.2) and (3.3). This completes the proof of Lemma 3.1. □

Assume that $\rho_0, u_0, v, f,\ p = p(\cdot)$ and Ω satisfy the hypotheses of Lemma 3.1. Then it follows from Lemma 3.1 that there exists a unique strong solution (ρ, u) to the linear problem (3.1), (3.2) and (3.3) satisfying the regularity (3.6). The purpose of this section is to derive some *local (in time) a priori estimates* for (ρ, u) which are independent of the lower bound δ of ρ_0 and size of the domain Ω. Let us choose a constant $c_0 > 1$ so that

$$1 + \rho^\infty + |\rho_0 - \rho^\infty|_{H^3} + |u_0|_{D_0^1} + |\sqrt{\rho_0}\, g_2|_{L^2} + |g_2|_{D_0^1} < c_0,$$

where $g_2 = \rho_0^{-1}(Lu_0 + \nabla p(\rho_0)) - f(0) = -v(0)\cdot\nabla v(0) - u_t(0)$, and assume that

$$
\begin{aligned}
|v(0)|_{D_0^1\cap D^3} &\le 1 + c_1,\\
\sup_{0\le t\le T_*} |v(t)|_{D_0^1} + \int_0^{T_*} |v(t)|_{D^2}^2\,dt &\le 1 + c_2,\\
\sup_{0\le t\le T_*} |v(t)|_{D^2} + \int_0^{T_*} \left(|v_t(t)|_{D_0^1}^2 + |v(t)|_{D^3}^2\right) dt &\le 1 + c_3, \qquad (3.7)\\
\operatorname*{ess\,sup}_{0<t<T_*}\left(|v_t(t)|_{D_0^1} + |v(t)|_{D^3}\right) + \int_0^{T_*} \left(|v_t(t)|_{D^2}^2 + |v(t)|_{D^4}^2\right) dt &\le 1 + c_4
\end{aligned}
$$

for some time $T_* \in (0,T)$ and constants c_i's with $1 < c_0 \le c_1 \le c_2 \le c_3 \le c_4$. The constants c_i's, $1 \le i \le 4$, and T_* will be determined later and depend only on c_0 and the parameters of C. Throughout this and next two sections, we denote by C a generic positive constant depending only on the fixed constants μ, λ, T, $|p|_{C^3(\overline{\mathbf{R}}_+)}$ and the norm of f. Moreover, $M = M(\cdot)$ denotes an increasing continuous function from $[1,\infty)$ to $[1,\infty)$ which is independent of δ and the size of Ω.

Lemma 3.2.

$$
|\rho(t)|_{L^\infty} + |\rho(t) - \rho^\infty|_{H^3} \le Cc_0, \quad |p(t) - p^\infty|_{H^3} \le M(c_0), \quad |\rho_t(t)|_{H^1} \le Cc_3^2,
$$
$$
|p_t(t)|_{H^1} \le M(c_0)c_3^2, \quad \int_0^t |\rho_{tt}(s)|_{L^2}^2\,ds \le Cc_3^8, \quad \int_0^t |p_{tt}(s)|_{L^2}^2\,ds \le M(c_0)c_3^8,
$$
$$
|\rho_t(t)|_{H^2} \le Cc_4^2, \quad |p_t(t)|_{H^2} \le M(c_0)c_4^2 \quad \text{and} \quad \inf_\Omega \rho(t) \ge C^{-1}\delta
$$

for $0 \le t \le \min(T_*, T_1)$, *where* $T_1 = (1 + c_4)^{-1}$ *and* $p^\infty = p(\rho^\infty)$.

Proof. From Lemma 2.1, we recall that

$$
|\rho(t) - \rho^\infty|_{H^3} \le (|\rho_0 - \rho^\infty|_{H^3} + \rho^\infty)\exp\left(C\int_0^t |v(s)|_{D_0^1\cap D^4}\,ds\right)
$$

and

$$
\inf_\Omega \rho(t) \ge \left(\inf_\Omega \rho_0\right)\exp\left(-C\int_0^t |v(s)|_{D_0^1\cap D^4}\,ds\right)
$$

for $0 \le t \le T$. Hence observing that

$$
\int_0^t |v(s)|_{D_0^1\cap D^4}\,ds \le t^{\frac12}\left(\int_0^t |v(s)|_{D_0^1\cap D^4}^2\,ds\right)^{\frac12} \le C(1+c_4)t + C((1+c_4)t)^{\frac12},
$$

we obtain the desired estimate for ρ. Then the esimates for ρ_t, ρ_{tt}, p, p_t and p_{tt} follow immediately from the quations $\rho_t = -\operatorname{div}(\rho v)$ and $p = p(\rho)$. □

Lemma 3.3.

$$
|u(t)|_{D_0^1}^2 + \int_0^t |u(s)|_{D^2}^2\,ds \le M(c_0)
$$

for $0 \le t \le \min(T_*, T_2)$, *where* $T_2 = (1 + c_4)^{-4} < T_1$.

Proof. Multiplying the equation (3.2) by u_t and integrating over Ω, we obtain

$$\begin{aligned}&\int \rho|u_t|^2\,dx+\frac{1}{2}\frac{d}{dt}\int \mu|\nabla u|^2+(\lambda+\mu)(\mathrm{div}u)^2\,dx\\&=-\int \nabla p\cdot u_t\,dx+\int \rho(f-v\cdot\nabla v)\cdot u_t\,dx.\end{aligned}\tag{3.8}$$

Using Lemma 3.2 together with (3.7), we can estimate the second term of the right hand side in (3.8) as follows:

$$\begin{aligned}\int \rho(f-v\cdot\nabla v)\cdot u_t\,dx&\le|\rho|_{L^\infty}^{\frac{1}{2}}|f-v\cdot\nabla v|_{L^2}|\sqrt{\rho}u_t|_{L^2}\\&\le C|\rho|_{L^\infty}\left(|f|_{L^2}^2+|v|_{D_0^1\cap D^2}^4\right)+\frac{1}{2}|\sqrt{\rho}u_t|_{L^2}^2\\&\le Cc_0c_3^4+\frac{1}{2}|\sqrt{\rho}u_t|_{L^2}^2.\end{aligned}$$

To estimate the first term, we observe that

$$\begin{aligned}-\int \nabla p\cdot u_t\,dx&=\int (p-p^\infty)\,\mathrm{div}u_t\,dx\\&=\frac{d}{dt}\int (p-p^\infty)\,\mathrm{div}u\,dx-\int p_t\,\mathrm{div}u\,dx,\end{aligned}$$

$$\int (p-p^\infty)\,\mathrm{div}u\,dx\le C|p(\rho)-p^\infty|_{L^2}^2+\frac{\mu}{4}|\nabla u|_{L^2}^2\le M(c_0)+\frac{\mu}{4}|\nabla u|_{L^2}^2$$

and

$$-\int p_t\,\mathrm{div}u\,dx\le|p_t|_{L^2}^2+|\nabla u|_{L^2}^2\le M(c_0)c_3^4+|\nabla u|_{L^2}^2.$$

Hence integrating (3.8) in time over $(0,t)$, we have

$$\begin{aligned}&\int_0^t|\sqrt{\rho}u_t(s)|_{L^2}^2\,ds+|\nabla u(t)|_{L^2}^2\\&\le M(c_0)\left(1+|\nabla u_0|_{L^2}^2\right)+M(c_0)c_3^4t+C\int_0^t|\nabla u(s)|_{L^2}^2\,ds\end{aligned}$$

for $0\le t\le\min(T_*,T_1)$. Therefore, in view of Gronwall's inequality, we conclude that

$$\int_0^t|\sqrt{\rho}u_t(s)|_{L^2}^2\,ds+|\nabla u(t)|_{L^2}^2\le M(c_0)\quad\text{for}\quad 0\le t\le\min(T_*,T_2),$$

where $T_2=(1+c_3)^{-4}<T_1$. Moreover, since for each $t\in(0,T)$, $u=u(t)\in D_0^1\cap D^2$ is a solution of the elliptic system

$$Lu=-\nabla p+\rho(f-v\cdot\nabla v)-\rho u_t\quad\text{in}\quad\Omega,$$

it follows from the elliptic regularity result in [3] that

$$\begin{aligned}|u|_{D^2}&\le C\left(|-\nabla p+\rho(f-v\cdot\nabla v)-\rho u_t|_{L^2}+|u|_{D_0^1}\right)\\&\le M(c_0)\left(1+c_3^2+|\sqrt{\rho}u_t|_{L^2}\right)\end{aligned}$$

and thus

$$\int_0^t|u(s)|_{D^2}^2\,ds\le M(c_0)\quad\text{for}\quad 0\le t\le\min(T_*,T_2).$$

This completes the proof of Lemma 3.3. □

Lemma 3.4.

$$|\sqrt{\rho}u_t(t)|_{L^2}^2 + |u(t)|_{D^2}^2 + \int_0^t \left(|u_t(s)|_{D_0^1}^2 + |u(s)|_{D^3}^2\right) ds \le M(c_1)c_2^{\frac{3}{2}}c_3^{\frac{1}{2}}$$

for $0 \le t \le \min(T_*, T_3)$*, where* $T_3 = (1+c_4)^{-9} < T_2$.

Proof. We differentiate (3.2) with respect to t and have

$$\rho u_{tt} + Lu_t + \nabla p_t = \rho(f - v\cdot\nabla v)_t + \rho_t(f - v\cdot\nabla v - u_t). \tag{3.9}$$

Multiplying this by u_t and integrating over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2\,dx + \int \mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div} u_t)^2\,dx\\ &= \int \left(-\nabla p_t + \rho(f - v\cdot\nabla v)_t + \rho_t(f - v\cdot\nabla v - \frac{1}{2}u_t)\right)\cdot u_t\,dx.\end{aligned} \tag{3.10}$$

To estimate each term in the right hand side of (3.10), we follow the arguments in [2, 3, 4]; we first apply the standard inequalities such as Hölder, Sobolev and Young's inequalities and then use Lemma 3.2.

$$-\int \nabla p_t\cdot u_t\,dx = \int p_t\,\mathrm{div}\,u_t\,dx \le C|p_t|_{L^2}^2 + \frac{\mu}{8}|\nabla u_t|_{L^2}^2 \le M(c_0)c_3^4 + \frac{\mu}{8}|\nabla u_t|_{L^2}^2,$$

$$\int \rho f_t\cdot u_t\,dx \le |f_t|_{L^2}|\rho|_{L^\infty}^{\frac{1}{2}}|\sqrt{\rho}u_t|_{L^2} \le |f_t|_{L^2}^2 + Cc_0|\sqrt{\rho}u_t|_{L^2}^2,$$

$$\begin{aligned}-\int \rho(v\cdot\nabla v)_t\cdot u_t\,dx &\le C|\rho|_{L^\infty}^{\frac{1}{2}}|v_t|_{D_0^1}|v|_{D_0^1}|\sqrt{\rho}u_t|_{L^3}\\ &\le C|\rho|_{L^\infty}^{\frac{3}{4}}|v_t|_{D_0^1}|v|_{D_0^1}|\sqrt{\rho}u_t|_{L^2}^{\frac{1}{2}}|\nabla u_t|_{L^2}^{\frac{1}{2}}\\ &\le \eta^{-2}C|\rho|_{L^\infty}^3|v|_{D_0^1}^4|\sqrt{\rho}u_t|_{L^2}^2 + \eta|v_t|_{D_0^1}^2 + \frac{\mu}{8}|\nabla u_t|_{L^2}^2\\ &\le \eta^{-2}Cc_2^7|\sqrt{\rho}u_t|_{L^2}^2 + \eta|v_t|_{D_0^1}^2 + \frac{\mu}{8}|\nabla u_t|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}\int \rho_t(f - v\cdot\nabla v)\cdot u_t\,dx &\le C|\rho_t|_{H^1}\left(|f|_{L^2} + |v|_{D_0^1\cap D^2}^2\right)|\nabla u_t|_{L^2}\\ &\le Cc_3^4\left(|f|_{L^2}^2 + c_3^4\right) + \frac{\mu}{8}|\nabla u_t|_{L^2}^2\\ &\le Cc_3^8 + \frac{\mu}{8}|\nabla u_t|_{L^2}^2\end{aligned}$$

and finally

$$\begin{aligned}-\int \rho_t\left(\frac{1}{2}|u_t|^2\right)dx &= \int \mathrm{div}(\rho v)\left(\frac{1}{2}|u_t|^2\right)dx\\ &\le \int \rho|v||u_t||\nabla u_t|\,dx \le C|\rho|_{L^\infty}^{\frac{3}{4}}|v|_{D_0^1}|\sqrt{\rho}u_t|_{L^2}^{\frac{1}{2}}|\nabla u_t|_{L^2}^{\frac{3}{2}}\\ &\le Cc_2^7|\sqrt{\rho}u_t|_{L^2}^2 + \frac{\mu}{8}|\nabla u_t|_{L^2}^2.\end{aligned}$$

Here $\eta \in (0,1)$ is a small number. Substituting these estimates into (3.10) and taking $\eta = (1+c_3)^{-1}$, we have

$$\begin{aligned}&\frac{d}{dt}\int \rho|u_t|^2\,dx + \mu\int |\nabla u_t|^2\,dx\\ &\le M(c_0)\left(|f_t|_{L^2}^2 + c_3^8\right) + Cc_3^9|\sqrt{\rho}u_t|_{L^2}^2 + (1+c_3)^{-1}|v_t|_{D_0^1}^2\end{aligned}\tag{3.11}$$

for $0 \le t \le \min(T_*, T_2)$. On the other hand, since

$$u_t \in C([0,T];H_0^1) \quad \text{and} \quad u_t(0) = -v(0)\cdot\nabla v(0) - g_2,$$

it follows that

$$|\sqrt{\rho}u_t(0)|_{L^2} + |u_t(0)|_{D_0^1} \le Cc_1^3.\tag{3.12}$$

Hence integrating (3.11) over $(0,t)$, we also have

$$|\sqrt{\rho}u_t(t)|_{L^2}^2 + \int_0^t |\nabla u_t(s)|_{L^2}^2\,ds \le M(c_1)\left(1 + c_3^8 t\right) + Cc_3^9\int_0^t |\sqrt{\rho}u_t(s)|_{L^2}^2\,ds.$$

Therefore, in view of Gronwall's inequality, we conclude that

$$|\sqrt{\rho}u_t(t)|_{L^2}^2 + \int_0^t |u_t(s)|_{D_0^1}^2\,ds \le M(c_1) \quad \text{for} \quad 0 \le t \le \min(T_*, T_3),$$

where $T_3 = (1+c_4)^{-9} < T_2$. Moreover, since for each $t \in (0,T)$, $u = u(t) \in D_0^1 \cap D^3$ is a solution of the elliptic system

$$Lu = -\nabla p + \rho(f - v\cdot\nabla v) - \rho u_t \quad \text{in} \quad \Omega,$$

it follows from the elliptic regularity result in [3] that

$$\begin{aligned}|u(t)|_{D^2} &\le M(c_1)\left(1 + |v\cdot\nabla v|_{L^2}\right)\\ &\le M(c_1)\left(1 + |v|_{D_0^1}^{\frac{3}{2}}|v|_{D_0^1\cap D^2}^{\frac{1}{2}}\right) \le M(c_1)c_2^{\frac{3}{2}}c_3^{\frac{1}{2}}\end{aligned}$$

and

$$\int_0^t |u(s)|_{D^3}^2\,ds \le M(c_1)\int_0^t \left(1 + |v(s)|_{D_0^1\cap D^2}^4 + |u_t(s)|_{D_0^1}^2\right)\,ds \le M(c_1)$$

for $0 \le t \le \min(T_*, T_3)$. This completes the proof of Lemma 3.4. □

Lemma 3.5.

$$|u_t(t)|_{D_0^1}^2 + |u(t)|_{D^3}^2 + \int_0^t \left(|\sqrt{\rho}u_{tt}(s)|_{L^2}^2 + |u_t(s)|_{D^2}^2 + |u(s)|_{D^4}^2\right)\,ds \le M(c_1)c_3^{12}$$

for $0 \le t \le \min(T_*, T_3)$.

Proof. Multiplying (3.9) by u_{tt} and integrating over Ω, we have

$$\begin{aligned}&\int \rho|u_{tt}|^2\,dx + \frac{1}{2}\frac{d}{dt}\int \mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div}u_t)^2\,dx\\ &= \int \left(-\nabla p_t + \rho(f - v\cdot\nabla v)_t + \rho_t(f - v\cdot\nabla v - u_t)\right)\cdot u_{tt}\,dx.\end{aligned}\tag{3.13}$$

We can estimate the first two terms in the right hand side of (3.13) as follows:

$$
\begin{aligned}
-\int \nabla p_t \cdot u_{tt}\,dx = \int p_t \mathrm{div} u_{tt}\,dx &= \frac{d}{dt}\int p_t \mathrm{div} u_t\,dx - \int p_{tt}\mathrm{div} u_t\,dx \\
&\le \frac{d}{dt}\int p_t \mathrm{div} u_t\,dx + |p_{tt}|_{L^2}^2 + |\nabla u_t|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\int \rho (f - v\cdot\nabla v)_t \cdot u_{tt}\,dx &\le C|\rho|_{L^\infty}^{\frac{1}{2}} \left(|f_t|_{L^2} + |v|_{D_0^1\cap D^2}|v_t|_{D_0^1}\right)|\sqrt{\rho}u_{tt}|_{L^2} \\
&\le Cc_0\left(|f_t|_{L^2}^2 + c_3^2|v_t|_{D_0^1}^2\right) + \frac{1}{2}|\sqrt{\rho}u_{tt}|_{L^2}^2.
\end{aligned}
$$

To estimate the last term, we observe that

$$
\begin{aligned}
&\int \rho_t\,(f - v\cdot\nabla v)\cdot u_{tt}\,dx \\
&= \frac{d}{dt}\int \rho_t\,(f - v\cdot\nabla v)\cdot u_t\,dx - \int \rho_{tt}\,(f - v\cdot\nabla v)\cdot u_t\,dx \\
&\quad - \int \rho_t\,(f - v\cdot\nabla v)_t \cdot u_t\,dx
\end{aligned}
$$

and

$$
-\int \rho_t u_t \cdot u_{tt}\,dx = -\frac{d}{dt}\int \rho_t \left(\frac{1}{2}|u_t|^2\right)\,dx + \int \rho_{tt}\left(\frac{1}{2}|u_t|^2\right)\,dx.
$$

Then by virtue of Lemma 3.2, we obtain

$$
\begin{aligned}
-\int \rho_{tt}\,(f - v\cdot\nabla v)\cdot u_t\,dx &\le C|\rho_{tt}|_{L^2}\left(|f|_{H^1} + |v|_{D_0^1\cap D^2}^2\right)|\nabla u_t|_{L^2} \\
&\le Cc_3^4|\rho_{tt}|_{L^2}^2 + |\nabla u_t|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
-\int \rho_t\,(f - v\cdot\nabla v)_t \cdot u_t\,dx &\le C|\rho_t|_{L^3}\left(|f_t|_{L^2} + |v|_{D_0^1\cap D^2}|v_t|_{D_0^1}\right)|\nabla u_t|_{L^2} \\
&\le Cc_3^4\left(|f_t|_{L^2}^2 + c_3^2|v_t|_{D_0^1}^2\right) + |\nabla u_t|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\int \rho_{tt}\left(\frac{1}{2}|u_t|^2\right)\,dx &= -\int \mathrm{div}(\rho_t v + \rho v_t)\left(\frac{1}{2}|u_t|^2\right)\,dx \\
&\le \int \left(|\rho_t||v| + \rho|v_t|\right)|u_t||\nabla u_t|\,dx \\
&\le Cc_3^3|\nabla u_t|_{L^2}^2 + Cc_0^{\frac{3}{4}}|v_t|_{D_0^1}|\sqrt{\rho}u_t|_{L^2}^{\frac{1}{2}}|\nabla u_t|_{L^2}^{\frac{3}{2}} \\
&\le Cc_3^3|\nabla u_t|_{L^2}^2 + (1+c_3)^{-1}|v_t|_{D_0^1}^2|\sqrt{\rho}u_t|_{L^2}|\nabla u_t|_{L^2} \\
&\le Cc_3^3|u_t|_{D_0^1}^2 + (1+c_3)^{-1}|v_t|_{D_0^1}^2\left(|\sqrt{\rho}u_t|_{L^2}^2 + |\nabla u_t|_{L^2}^2\right).
\end{aligned}
$$

Substituting all the above estimates into (3.13), we have

$$\begin{aligned}
&\int \rho|u_{tt}|^2\,dx + \frac{d}{dt}\int \mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div}u_t)^2\,dx\\
&\le \frac{d}{dt}\int \left(2p_t\mathrm{div}u_t + 2\rho_t(f - v\cdot\nabla v)\cdot u_t - \rho_t|u_t|^2\right)\,dx\\
&\quad + C\left(|p_{tt}|^2_{L^2} + c_3^4|\rho_{tt}|^2_{L^2} + c_3^4|f_t|^2_{L^2} + c_3^6|v_t|^2_{D_0^1} + c_3^3|u_t|^2_{D_0^1}\right)\\
&\quad + |v_t|^2_{D_0^1}|\sqrt{\rho}u_t|^2_{L^2} + (1+c_3)^{-1}|v_t|^2_{D_0^1}|\nabla u_t|^2_{L^2}
\end{aligned} \tag{3.14}$$

for $0 \le t \le \min(T_*, T_3)$. Now let us define a function Λ by

$$\begin{aligned}
\Lambda(t) = &\int \left(\mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div}u_t)^2\right)(t)\,dx\\
&- \int \left(2p_t\mathrm{div}u_t + 2\rho_t(f - v\cdot\nabla v)\cdot u_t - \rho_t|u_t|^2\right)(t)\,dx.
\end{aligned}$$

Then it follows from Lemma 3.2, Lemma 3.4 and (3.12) that

$$\begin{aligned}
|\Lambda| &\le C\left(|\nabla u_t|^2_{L^2} + |p_t|^2_{L^2} + |\rho_t|^2_{L^3}|f - v\cdot\nabla v|^2_{L^2} + |\rho|^3_{L^\infty}|v|^4_{D_0^1}|\sqrt{\rho}u_t|^2_{L^2}\right)\\
&\le C|\nabla u_t|^2_{L^2} + M(c_1)c_3^8,\\
\Lambda &\ge C^{-1}|\nabla u_t|^2_{L^2} - M(c_1)c_3^8 \quad \text{and} \quad |\Lambda(0)| \le M(c_1)c_3^8.
\end{aligned}$$

Hence integrating (3.14) over $(0,t)$ and using Lemma 3.2 and Lemma 3.4, we deduce that

$$\begin{aligned}
&\int_0^t |\sqrt{\rho}u_{tt}(s)|^2_{L^2}\,ds + |\nabla u_t(t)|^2_{L^2}\\
&\qquad \le M(c_1)c_3^{12} + \int_0^t C(1+c_3)^{-1}|v_t|^2_{D_0^1}|\nabla u_t(s)|^2_{L^2}\,ds
\end{aligned}$$

for $0 \le t \le \min(T_*, T_3)$. Therefore, in view of Gronwall's inequality, we conclude that

$$\int_0^t |\sqrt{\rho}u_{tt}(s)|^2_{L^2}\,ds + |u_t(t)|^2_{D_0^1} \le M(c_1)c_3^{12}$$

for $0 \le t \le \min(T_*, T_3)$. Moreover, since $Lu = -\nabla p + \rho(f - v\cdot\nabla v - u_t)$ in Ω, it follows from the elliptic regularity result that

$$\int_0^t |u_t(s)|^2_{D^2}\,ds + |u(t)|^2_{D^3} \le M(c_1)c_3^{12} \qquad \text{for} \quad 0 \le t \le \min(T_*, T_3).$$

This completes the proof of Lemma 3.5. □

From Lemma 3.2–Lemma 3.5, it follows that

$$\begin{aligned}
|u(t)|_{D_0^1} + \int_0^t |u(s)|^2_{D^2}\,ds &\le M(c_1),\\
|u(t)|_{D^2} + \int_0^t \left(|u_t(s)|^2_{D_0^1} + |u(s)|^2_{D^3}\right)\,ds &\le M(c_1)c_2^{\frac{3}{2}}c_3^{\frac{1}{2}},\\
|u_t(t)|_{D_0^1} + |u(t)|_{D^3} + \int_0^t \left(|u_t(s)|^2_{D^2} + |u(s)|^2_{D^4}\right)\,ds &\le M(c_1)c_3^{12},\\
|\rho(t) - \rho^\infty|_{H^3} + |\rho_t(t)|_{H^2} + |\sqrt{\rho}u_t(t)|_{L^2} &\le M(c_1)c_3^{12}
\end{aligned}$$

for $0 \le t \le \min(T_*, T_3)$. Here $M = M(\cdot)$ is a fixed increasing continuous function on $[1, \infty)$ which depends only on the parameters of C. Therefore, defining the constants c_i's and T_* by

$$c_1 = M(c_0), \quad c_2 = M(c_1), \quad c_3 = c_2^5, \quad c_4 = c_2 c_3^{12} \tag{3.15}$$

and

$$T_* = \min(T, T_3) \quad \text{with} \quad T_3 = (1 + c_4)^{-9}, \tag{3.16}$$

we conclude that

$$\begin{aligned}
\sup_{0\le t\le T_*} |u(t)|_{D_0^1} + \int_0^{T_*} |u(t)|_{D^2}^2\,dt &\le c_2,\\
\sup_{0\le t\le T_*} |u(t)|_{D^2} + \int_0^{T_*} \left(|u_t(t)|_{D_0^1}^2 + |u(t)|_{D^3}^2\right)dt &\le c_3,\\
\operatorname*{ess\,sup}_{0\le t\le T_*} \left(|u_t(t)|_{D_0^1} + |u(t)|_{D^3}\right) + \int_0^{T_*} \left(|u_t(t)|_{D^2}^2 + |u(t)|_{D^4}^2\right)dt &\le c_4,\\
\operatorname*{ess\,sup}_{0\le t\le T_*} \left(|\rho(t) - \rho^\infty|_{H^3} + |\rho_t(t)|_{H^2} + |\sqrt{\rho}u_t(t)|_{L^2}\right) &\le c_4.
\end{aligned} \tag{3.17}$$

4. Proof of Theorem 1.1

Let (ρ_0, u_0, f) be a given data satisfying the hypotheses of Theorem 1.1. To prove the existence, we construct a sequence $\{(\rho^k, u^k)\}_{k\ge1}$ of approximate solutions solving the linearized problem (3.1)–(3.4) successively. First, let $F \in C([0,\infty); H^1) \cap L^2(0,\infty; H^2)$ be the solution of the heat equation $F_t - \Delta F = 0$ in $(0,\infty)\times\Omega$ with $F(0) = -\nabla p(\rho_0) + \rho_0(f(0) + g_2) \in H^1$. Then since $u_0 \in D_0^1 \cap D^3$ and $F(0) - Lu_0 = 0 \in D_0^1$, we can easily show that there exists a unique solution $w = u^0 \in C([0,\infty); D_0^1 \cap D^3) \cap L^2(0,\infty; D^4)$ to the following linear parabolic problem

$$w_t + Lw = F \quad \text{in} \quad (0,\infty)\times\Omega \quad \text{and} \quad w(0) = u_0 \quad \text{in} \quad \Omega.$$

It is also easy to show that

$$\begin{aligned}
&\sup_{0\le t\le 1} \left(|u^0(t)|_{D_0^1\cap D^3} + |u_t^0(t)|_{D_0^1}\right) + \int_0^1 \left(|u_t^0(t)|_{D^2}^2 + |u^0(t)|_{D^4}^2\right)dt\\
&\qquad \le C\left(1 + |F(0)|_{H^1}^2 + |u_0|_{D_0^1\cap D^3}^2\right).
\end{aligned} \tag{4.1}$$

Let us define c_0 by

$$c_0 = 2 + \rho^\infty + |\rho_0 - \rho^\infty|_{H^3} + |u_0|_{D_0^1} + |\sqrt{\rho_0}\, g_2|_{L^2} + |g_2|_{D_0^1},$$

and we choose the positive constants c_1, c_2, c_3, c_4 and T_* as in (3.15) and (3.16), which are dependent only on c_0 and the parameters of C. Then since $u_0 \in D_0^1 \cap D^3$ is a solution to the elliptic system

$$Lu_0 = F(0) = -\nabla p(\rho_0) + \rho_0(f(0) + g_2) \qquad \text{in} \quad \Omega$$

and

$$|F(0)|_{H^1} = |-\nabla p(\rho_0) + \rho_0(f(0) + g_2)|_{H^1} \le M(c_0), \tag{4.2}$$

it follows from the elliptic regularity result in [3] that

$$|u_0|_{D_0^1\cap D^3} \le C\left(|F(0)|_{H^1} + |u_0|_{D_0^1}\right) \le M(c_0). \tag{4.3}$$

By virtue of (3.15), (4.1), (4.2) and (4.3), we may assume without loss of generality that

$$\sup_{0\le t\le T_*}\left(|u^0(t)|_{D_0^1\cap D^3} + |u_t^0(t)|_{D_0^1}\right) + \int_0^{T_*}\left(|u_t^0(t)|_{D^2}^2 + |u^0(t)|_{D^4}^2\right)dt \le c_1. \tag{4.4}$$

The construction of the sequence $\{(\rho^k, u^k)\}_{k\ge1}$ is based on the following key lemma to the proof of Theorem 1.1.

Lemma 4.1. *Let v be a vector field satisfying the regularity* (3.5) *with T replaced by T_*. Assume further that v satisfies the following estimate*

$$\begin{aligned} |v(0)|_{D_0^1\cap D^3} &\le c_1,\\ \sup_{0\le t\le T_*}|v(t)|_{D_0^1} + \int_0^{T_*}|v(t)|_{D^2}^2\,dt &\le c_2,\\ \sup_{0\le t\le T_*}|v(t)|_{D^2} + \int_0^{T_*}\left(|v_t(t)|_{D_0^1}^2 + |v(t)|_{D^3}^2\right)dt &\le c_3,\\ \operatorname*{ess\,sup}_{0\le t\le T_*}\left(|v_t(t)|_{D_0^1} + |v(t)|_{D^3}\right) + \int_0^{T_*}\left(|v_t(t)|_{D^2}^2 + |v(t)|_{D^4}^2\right)dt &\le c_4. \end{aligned} \tag{4.5}$$

Then there exists a unique solution (ρ, u) to the linearized problem (3.1)–(3.4) *satisfying the estimate* (3.17) *as well as the regularity*

$$\begin{aligned} &\rho-\rho^\infty \in C([0,T_*];H^3), \quad u \in C([0,T_*];D_0^1\cap D^3)\cap L^2(0,T_*;D^4),\\ &u_t \in L^\infty(0,T_*;D_0^1)\cap L^2(0,T_*;D^2) \quad \textit{and} \quad \sqrt{\rho}u_t \in L^\infty(0,T_*;L^2). \end{aligned} \tag{4.6}$$

Proof. Let $R_0 > 1$ be a sufficiently large number so that

$$\Omega \subset B_{R_0/2} \quad \text{if} \quad \Omega \subset\subset \mathbf{R}^3; \qquad \mathbf{R}^3\setminus\Omega \subset B_{R_0/2} \quad \text{if} \quad \mathbf{R}^3\setminus\Omega \subset\subset \mathbf{R}^3,$$

and we define

$$\begin{gathered} \varphi^R(x) = \varphi(x/R), \quad g_2^R(x) = \varphi^R(x)g_2(x),\\ v^R(t,x) = \varphi^R(x)v(t,x) \quad \text{and} \quad f^R(t,x) = \varphi^R(x)f(t,x) \end{gathered}$$

for $(t,x) \in [0,T_*]\times\Omega$, where $\varphi \in C_c^\infty(B_1)$ is a smooth cut-off function such that $\varphi = 1$ in $B_{1/2}$. Note that if $\Omega \subset\subset \mathbf{R}^3$, then $g_2^R = g_2$, $v^R = v$ and $f^R = f$ for each $R > R_0$ and otherwise, they are supported in Ω_R or $[0,T_*]\times\Omega_R$, where $\Omega_R = \Omega\cap B_R$ [1].

For each $R > R_0$, let $u_0^R \in H_0^1(\Omega_R)\cap H^3(\Omega_R)$ be a unique solution to the elliptic boundary value problem

$$Lu_0^R = F_0^R \quad \text{in} \quad \Omega_R \qquad \text{and} \qquad u_0^R = 0 \quad \text{on} \quad \partial\Omega_R, \tag{4.7}$$

where

$$F_0^R = -\nabla p(\rho_0^R) + \rho_0^R\left(f^R(0) + g_2^R\right) \quad \text{and} \quad \rho_0^R = \rho_0 + R^{-3}.$$

[1] If Ω is the half space $\mathbf{R}^2\times\mathbf{R}_+$, then the non-smooth domain Ω_R should be replaced by a smooth domain $\tilde\Omega_R$ such that $\Omega_R \subset \tilde\Omega_R \subset \Omega_{2R}$.

Then we extend u_0^R to Ω by defining zero outside Ω_R. We will show that

$$u_0^R \to u_0 \quad \text{in} \quad D_0^1(\Omega) \quad \text{as} \quad R \to \infty. \tag{4.8}$$

To do this, we first observe that

$$Lu_0 = -\nabla p(\rho_0) + \rho_0 \left(f(0) + g_2\right) \equiv F_0 \quad \text{in} \quad \Omega. \tag{4.9}$$

From (4.7) and (4.9), it follows that $L\left(u_0^R - u_0\right) = F_0^R - F_0$ in Ω_R. Hence noting that $u_0^R \in H_0^1(\Omega_R)$, we obtain

$$\begin{aligned} &\int_{\Omega_R} \mu |\nabla u_0^R|^2 + (\lambda+\mu)(\mathrm{div} u_0^R)^2 \, dx \\ &= \int_{\Omega_R} \mu \nabla u_0 : \nabla u_0^R + (\lambda+\mu)\, \mathrm{div} u_0 \, \mathrm{div} u_0^R \, dx + \int_{\Omega_R} (F_0^R - F_0)\cdot u_0^R \, dx. \end{aligned} \tag{4.10}$$

The second term of the right hand side in (4.10) is bounded by

$$\begin{aligned} \int_{\Omega_R} (F_0^R - F_0)\cdot u_0^R \, dx \le & \int_{\Omega_R} |p(\rho_0^R) - p(\rho_0)||\nabla u_0^R| \, dx \\ & + R^{-3} \int_{\Omega_R} \left(|f(0)| + |g_2|\right) |u_0^R| \, dx \\ & + \int_{\Omega_R} \rho_0 \left(\varphi^R - 1\right) \left(f(0) + g_2\right) \cdot u_0^R \, dx, \end{aligned}$$

while

$$\int_{\Omega_R} |p(\rho_0^R) - p(\rho_0)||\nabla u_0^R| \, dx \le R^{-\frac{3}{2}} M(c_0) |\nabla u_0^R|_{L^2},$$

$$R^{-3} \int_{\Omega_R} \left(|f(0)| + |g_2|\right) |u_0^R| \, dx \le C R^{-1} \left(|f(0)|_{H^1} + |g_2|_{D_0^1} \right) |\nabla u_0^R|_{L^2}$$

and

$$\begin{aligned} &\int_{\Omega_R} \rho_0 \left(\varphi^R - 1\right) \left(f(0) + g_2\right) \cdot u_0^R \, dx \\ &= \int_{\Omega_R} \left(\varphi^R - 1\right) \left(Lu_0 + \nabla p(\rho_0)\right) \cdot u_0^R \, dx \\ &\le C \int_{\Omega_R} \left(|\nabla \varphi^R||u_0^R| + |\varphi^R - 1||\nabla u_0^R| \right) \left(|\nabla u_0| + |p(\rho_0) - p(\rho^\infty)| \right) dx \\ &\le M(c_0) \left(|\nabla u_0|_{L^2(\Omega \setminus \Omega_{R/2})} + |\rho_0 - \rho^\infty|_{L^2(\Omega\setminus\Omega_{R/2})} \right) |\nabla u_0^R|_{L^2}. \end{aligned}$$

Hence from (4.10), it follows that

$$|u_0^R|_{D_0^1(\Omega)} \le C |u_0|_{D_0^1(\Omega)} + o(1) \quad \text{and} \quad \int_\Omega (F_0^R - F_0)\cdot u_0^R \, dx = o(1) \tag{4.11}$$

where $o(1)$ denotes a function of R which tends to zero as $R \to \infty$. This means that there exists a sequence $\{R_j\}$, $R_j \to \infty$, such that $\{u_0^{R_j}\}$ converges weakly in $D_0^1(\Omega)$ to a limit u_0^∞. It is easy to show that $Lu_0^\infty = Lu_0$ in $D^{-1}(\Omega)$, where $D^{-1}(\Omega)$ denotes the dual space of $D_0^1(\Omega)$. Hence it follows that $u_0^\infty = u_0$ in Ω and $\{u_0^{R_j}\}$ converges weakly in $D_0^1(\Omega)$ to u_0. Then by virtue of (4.10) and (4.11), we deduce that $\{u_0^{R_j}\}$ converges strongly to u_0 in $D_0^1(\Omega)$. Since the above argument also shows that every subsequence of $\{u_0^R\}$ has a subsequence converging in $D_0^1(\Omega)$

to the same limit u_0, we conclude that the whole sequence $\{u_0^R\}$ converges to u_0 in $D_0^1(\Omega)$ as $R \to \infty$, which proves (4.8).

We are now ready to prove Lemma 4.1. To prove the existence, we consider the following initial boundary value problem

$$\rho_t + \operatorname{div}(\rho v^R) = 0 \quad \text{in} \quad (0, T_*) \times \Omega_R, \tag{4.12}$$

$$\rho u_t + Lu + \nabla p(\rho) = \rho(f^R - v^R \cdot \nabla v^R) \quad \text{in} \quad (0, T_*) \times \Omega_R, \tag{4.13}$$

$$(\rho,\, u)|_{t=0} = (\rho_0^R,\, u_0^R) \quad \text{in} \quad \Omega_R \quad \text{and} \quad u = 0 \quad \text{on} \quad (0, T_*) \times \partial\Omega_R. \tag{4.14}$$

Since $\rho_0^R \geq R^{-3} > 0$ in Ω_R, it follows from Lemma 3.1 that for each $R > R_0$, there exists a unique strong solution $(\rho, u) = (\rho^R, u^R)$ to the problem (4.12), (4.13) and (4.14). It is easy to show that

$$\begin{aligned} &|v^R - v|_{C([0,T_*];D_0^1 \cap D^3)} + |(v^R)_t - v_t|_{L^\infty(0,T_*;D_0^1) \cap L^2(0,T_*;D^2)} \to 0 \\ &\text{and} \quad |\sqrt{\rho_0}^R g_2^R - \sqrt{\rho_0} g_2|_{L^2} + |g_2^R - g_2|_{D_0^1} \to 0 \qquad \text{as} \quad R \to \infty. \end{aligned}$$

Combining this, (4.5) and (4.8), we deduce that there exists a large number $R_1 > R_0$ such that for all $R > R_1$, v_R satisfies the estimate (3.7) with the spatial domain being Ω_R and

$$\begin{aligned} 1 + (\rho^\infty + R^{-3}) + |\rho_0^R - (\rho^\infty + R^{-3})|_{H^3(\Omega_R)} & \\ + |u_0^R|_{D_0^1(\Omega_R)} + |\sqrt{\rho_0^R}\, g_2^R|_{L^2(\Omega_R)} + |g_2^R|_{D_0^1(\Omega_R)} &< c_0. \end{aligned}$$

Therefore, from the results in Section 3, we conclude that for each $R > R_1$, the solution (ρ^R, u^R) satisfies the estimate (3.17) with the domain being Ω_R. We extend (ρ^R, u^R) by defining zero outside Ω_R. Then by virtue of the uniform estimate (3.17) on R, we deduce that there exists a sequence $\{R_j\}$, $R_j \to \infty$, such that $\{(\rho^{R_j}, u^{R_j})\}$ converges in a weak or weak-$*$ sense to a limit (ρ, u). Moreover, since (ρ, u) also satisfies (3.17) with the domain being Ω_R for each $R > R_1$, it follows that

$$\begin{aligned} &\rho - \rho^\infty \in L^\infty(0, T_*; H^3), \quad u \in L^\infty(0, T_*; D_0^1 \cap D^3) \cap L^2(0, T_*; D^4), \\ &u_t \in L^\infty(0, T_*; D_0^1) \cap L^2(0, T_*; D^2) \quad \text{and} \quad \sqrt{\rho} u_t \in L^\infty(0, T_*; L^2). \end{aligned} \tag{4.15}$$

We will show that (ρ, u) is a solution to the original problem (3.1)-(3.4). It is obvious that (ρ, u) satisfies the boundary conditions in (3.3) and (3.4). Let $R > R_1$ be a fixed large number. Then since for all sufficiently large j, (ρ^{R_j}, u^{R_j}) satisfies the uniform estimate (3.17) with the domain being Ω_R, it follows from a standard compactness result (see [22] for instance) that a subsequence of $\{(\rho^{R_j}, u^{R_j})\}$ converges to (ρ, u) in $C([0, T_*]; H^1(\Omega_R))$. Using this result together with (4.8), we can show that (ρ, u) satisfies the equations (3.1) and (3.2) in $(0, T_*) \times \Omega_R$ and $(\rho(0), u(0)) = (\rho_0, u_0)$ in Ω_R. Since R can be arbitrarily large, we have proved the existence of a solution (ρ, u) to the original problem (3.1)-(3.4) satisfying the regularity (4.15). The uniqueness of solutions with this regularity is easily proved. Hence it remains to prove the time-continuity of the solution (ρ, u). First, from a classical embedding result, we deduce that $u \in C([0, T_*]; D_0^1 \cap D^3)$. Then the time-continuity of ρ follows immediately from Lemma 2.1. This completes the proof of Lemma 4.1. □

We turn to the proof of Theorem 1.1. We first observe that by virtue of (4.4), the vector field $v = u^0$ satisfies the hypotheses of Lemma 4.1. Hence it follows from Lemma 4.1 that there exists a unique strong solution $(\rho, u) = (\rho^1, u^1)$ to the linearized problem (3.1)–(3.4) with $v = u^0$, which satisfies the regularity estimate (3.17). Then an obvious inductive argument allows us to construct approximate solutions (ρ^k, u^k) for all $k \geq 1$: assuming that u^{k-1} was defined for $k \geq 1$, let (ρ^k, u^k) be the unique solution to the problem (3.1)–(3.4) with $v = u^{k-1}$. Then since $u^k(0) = u_0$ for each $k \geq 0$, it follows from Lemma 4.1 that there exists a constant $\tilde{C} > 1$ such that

$$\sup_{0\leq t\leq T_*} \left(|\rho^k(t) - \rho^\infty|_{H^3} + |\rho_t(t)|_{H^2} + |u^k(t)|_{D_0^1\cap D^3}\right) \leq \tilde{C},$$

$$\text{(4.16)}\quad \operatorname{ess}\sup_{0\leq t\leq T_*} \left(|u_t^k(t)|_{D_0^1} + |\sqrt{\rho}^k u_t^k(t)|_{L^2}\right) + \int_0^{T_*} \left(|u_t^k(t)|_{D^2}^2 + |u(t)|_{D^4}^2\right) dt \leq \tilde{C}$$

for all $k \geq 1$. Throughout the proof, we denote by $\tilde{C}$ a generic positive constant depending only on c_0 and the parameters of C, but independent of k.

From now on, we show that the full sequence $\{(\rho^k, u^k)\}$ of approximate solutions converges to a solution to the original problem (1.1)-(1.5) in a strong sense. To do this, let us define

$$\overline{\rho}^{k+1} = \rho^{k+1} - \rho^k, \quad \overline{u}^{k+1} = u^{k+1} - u^k \quad \text{and} \quad p^k = p(\rho^k).$$

Then from the equation (3.1), we derive

$$\overline{\rho}_t^{k+1} + \operatorname{div}(\overline{\rho}^{k+1} u^k) + \operatorname{div}(\rho^k \overline{u}^k) = 0. \tag{4.17}$$

Multiplying this by $\overline{\rho}^{k+1}$ and integrating over Ω, we obtain

$$\begin{aligned} &\frac{d}{dt}\int |\overline{\rho}^{k+1}|^2\, dx \\ &\leq C\int |\nabla u^k||\overline{\rho}^{k+1}|^2 + (\,|\nabla\rho^k||\overline{u}^k| + \rho^k|\nabla\overline{u}^k|\,)|\overline{\rho}^{k+1}|\, dx \\ &\leq C|\nabla u^k|_{L^\infty}|\overline{\rho}^{k+1}|_{L^2}^2 + C\left(|\nabla\rho^k|_{H^1} + |\rho^k|_{L^\infty}\right)|\nabla\overline{u}^k|_{L^2}|\overline{\rho}^{k+1}|_{L^2}. \end{aligned}$$

Hence it follows from the uniform bound (4.16) that

$$\frac{d}{dt}|\overline{\rho}^{k+1}|_{L^2}^2 \leq \eta^{-1}\tilde{C}|\overline{\rho}^{k+1}|_{L^2}^2 + \eta|\nabla\overline{u}^k|_{L^2} \tag{4.18}$$

for $0 \leq t \leq T_*$, where $\eta \in (0, 1)$ is a small number.

In case that $\rho^\infty = 0$, we need an estimate for $|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}}$ in addition to (4.18). Multiplying (4.17) by $sgn(\overline{\rho}^{k+1})|\overline{\rho}^{k+1}|^{\frac{1}{2}}$ and integrating over Ω, we get

$$\begin{aligned} &\frac{d}{dt}\int |\overline{\rho}^{k+1}|^{\frac{3}{2}}\, dx \\ &\leq C\int |\nabla u^k||\overline{\rho}^{k+1}|^{\frac{3}{2}} + (\,|\nabla\rho^k||\overline{u}^k| + \rho^k|\nabla\overline{u}^k|\,)|\overline{\rho}^{k+1}|^{\frac{1}{2}}\, dx \\ &\leq C|\nabla u^k|_{L^\infty}|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + C|\rho^k|_{H^1}|\nabla\overline{u}^k|_{L^2}|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}}^{\frac{1}{2}}. \end{aligned}$$

Hence multiplying this by $|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}}^{\frac{1}{2}}$ and using (4.16), we have

$$\frac{d}{dt}|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}}^2 \leq \eta^{-1}\tilde{C}|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}}^2 + \eta|\nabla\overline{u}^k|_{L^2}^2 \tag{4.19}$$

for $0 \leq t \leq T_*$.

Next from the equation (3.2), we derive

$$\begin{aligned}\rho^{k+1}\,\overline{u}_t^{\,k+1} + \rho^{k+1}\,u^k\cdot\nabla\overline{u}^{\,k+1} + L\overline{u}^{\,k+1} + \nabla(p^{k+1}-p^k)\\ = \overline{\rho}^{\,k+1}(f - u_t^k - u^{k-1}\cdot\nabla u^{k-1})\\ +\rho^{k+1}\left(u^k\cdot\nabla\overline{u}^{\,k+1} - \overline{u}^{\,k}\cdot\nabla u^k - u^{k-1}\cdot\nabla\overline{u}^{\,k}\right).\end{aligned}$$

Multiplying this by $\overline{u}^{\,k+1}$, integrating over Ω and using the equation (3.1) with $(\rho, v) = (\rho^{k+1}, u^k)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \rho^{k+1}|\overline{u}^{\,k+1}|^2\,dx + \mu\int|\nabla\overline{u}^{\,k+1}|^2\,dx\\ &\le C\int|\overline{\rho}^{\,k+1}|\,|u_t^k||\overline{u}^{\,k+1}|\,dx + C\int|p^{k+1}-p^k||\nabla\overline{u}^{\,k+1}|\,dx\\ &+C\int|\overline{\rho}^{\,k+1}|\,|f-\,u^{k-1}\cdot\nabla u^{k-1}||\overline{u}^{\,k+1}|\,dx\\ &+C\int\rho^{k+1}\left(|u^k||\nabla\overline{u}^{\,k+1}| + |\overline{u}^{\,k}|\,|\nabla u^k| + |u^{k-1}||\nabla\overline{u}^{\,k}|\right)|\overline{u}^{\,k+1}|\,dx.\end{aligned}\tag{4.20}$$

Using the uniform bound (4.16), we can estimate the last three integrals of the right hand side in (4.20) as follows:

$$C\int|p^{k+1}-p^k||\nabla\overline{u}^{\,k+1}|\,dx \le \tilde{C}|\overline{\rho}^{\,k+1}|_{L^2}^2 + \frac{\mu}{10}|\nabla\overline{u}^{\,k+1}|_{L^2}^2,$$

$$\begin{aligned}&C\int|\overline{\rho}^{\,k+1}|\,|f-\,u^{k-1}\cdot\nabla u^{k-1}||\overline{u}^{\,k+1}|\,dx\\ &\le C|\overline{\rho}^{\,k+1}|_{L^2}|f-\,u^{k-1}\cdot\nabla u^{k-1}|_{H^1}|\nabla\overline{u}^{\,k+1}|_{L^2}\\ &\le \tilde{C}|\overline{\rho}^{\,k+1}|_{L^2}^2 + \frac{\mu}{10}|\nabla\overline{u}^{\,k+1}|_{L^2}^2,\end{aligned}$$

$$C\int\rho^{k+1}|u^k||\nabla\overline{u}^{\,k+1}||\overline{u}^{\,k+1}|\,dx \le \tilde{C}|\sqrt{\rho^{k+1}}\,\overline{u}^{\,k+1}|_{L^2}^2 + \frac{\mu}{10}|\nabla\overline{u}^{\,k+1}|_{L^2}^2$$

and

$$\begin{aligned}&C\int\rho^{k+1}\left(|\overline{u}^{\,k}|\,|\nabla u^k| + |u^{k-1}||\nabla\overline{u}^{\,k}|\right)|\overline{u}^{\,k+1}|\,dx\\ &\le C|\rho^{k+1}|_{L^\infty}^{\frac{1}{2}}\left(|u^k|_{D_0^1\cap D^2} + |u^{k-1}|_{D_0^1\cap D^2}\right)|\sqrt{\rho^{k+1}}\,\overline{u}^{\,k+1}|_{L^2}|\nabla\overline{u}^{\,k}|_{L^2}\\ &\le \eta^{-1}\tilde{C}|\sqrt{\rho^{k+1}}\,\overline{u}^{\,k+1}|_{L^2}^2 + \eta|\nabla\overline{u}^{\,k}|_{L^2}^2.\end{aligned}$$

For the case that $\rho^\infty = 0$ or $\Omega\subset\subset\mathbf{R}^3$, the first integral is readily bounded by

$$C|\overline{\rho}^{\,k+1}|_{L^{\frac{3}{2}}}|u_t^k|_{D_0^1}|\nabla\overline{u}^{\,k+1}|_{L^2} \le \tilde{C}|\overline{\rho}^{\,k+1}|_{L^{\frac{3}{2}}}^2 + \frac{\mu}{10}|\nabla\overline{u}^{\,k+1}|_{L^2}^2.$$

For the remaining case, we assume that Ω is an unbounded domain and $\rho^\infty > 0$. Then since $\rho_0 - \rho^\infty \in H^2$ and $H^2 \hookrightarrow C_0$, where C_0 is the space of all continuous functions on $\overline{\Omega}$ vanishing at infinity, we can choose a sufficiently large number $R > 1$ (of course, independent of k) so that

$$\frac{3}{4}\rho^\infty \le \rho_0(x) \le \frac{5}{4}\rho^\infty \qquad \text{for} \quad x\in\Omega\setminus B_{R/2}.\tag{4.21}$$

On the other hand, it follows from Lemma 2.1 that

$$\rho^{k+1}(t,x) = \rho_0(U^{k+1}(0,t,x)) \exp\left[-\int_0^t \operatorname{div} u^k(s, U^{k+1}(s,t,x)\,)\,ds\right], \tag{4.22}$$

where $U^{k+1} = U^{k+1}(t,s,x)$ is the solution to the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} U^{k+1}(t,s,x) = u^k(t, U^{k+1}(t,s,x)\,), \qquad 0 \le t \le T_*, \\ U^{k+1}(s,s,x) = x, \qquad 0 \le s \le T_*, \;\; x \in \overline{\Omega}. \end{cases}$$

In view of (4.16), we deduce that

$$\int_0^t \left|\operatorname{div} u^k(s, U^{k+1}(s,t,x)\,)\right| ds \le \int_0^t |\nabla u^k|_{L^\infty}\, ds \le \tilde{C} t \le \ln 2$$

and

$$\begin{aligned} \left|U^{k+1}(0,t,x) - x\right| &= \left|U^{k+1}(0,t,x) - U^{k+1}(t,t,x)\right| \\ &\le \int_0^t \left|u^k(\tau, U^{k+1}(\tau,t,x))\right| d\tau \le \tilde{C} t \le \frac{R}{2} \end{aligned}$$

for all (t,x) in $[0,T_1]\times\Omega$, where T_1 is a small positive time in $(0,T_*)$ which depends only on T_*, R and the parameters of $\tilde{C}$. In particular, note that if $0 \le t \le T_1$ and $x \in \Omega \setminus B_R$, then $U^{k+1}(0,t,x) \in \Omega \setminus B_{R/2}$. Hence it follows immediately from (4.21) and (4.22) that

$$\frac{3}{8}\rho^\infty \le \rho^{k+1}(t,x) \le \frac{5}{2}\rho^\infty \qquad \text{for} \quad (t,x) \in [0,T_1]\times(\Omega\setminus B_R). \tag{4.23}$$

Using this result, we can estimate the first integral in the right hand of (4.20) as follows:

$$\begin{aligned} C\int_{\Omega\cap B_R} |\overline{\rho}^{\,k+1}|\,|u_t^k||\overline{u}^{\,k+1}|\,dx &\le C|\overline{\rho}^{\,k+1}|_{L^2}|u_t^k|_{D_0^1}|\nabla\overline{u}^{\,k+1}|_{L^2} \\ &\le \tilde{C}|\overline{\rho}^{\,k+1}|_{L^2}^2 + \frac{\mu}{10}|\nabla\overline{u}^{\,k+1}|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} C\int_{\Omega\setminus B_R} |\overline{\rho}^{\,k+1}|\,|u_t^k||\overline{u}^{\,k+1}|\,dx &\le \frac{C}{\sqrt{\rho^\infty}}\int |\overline{\rho}^{\,k+1}|\,|\sqrt{\rho}^k u_t^k||\overline{u}^{\,k+1}|\,dx \\ &\le \tilde{C}|\overline{\rho}^{\,k+1}|_{L^2}^2 + \frac{\mu}{10}|\nabla\overline{u}^{\,k+1}|_{L^2}^2. \end{aligned}$$

Therefore, substituting all the estimates into (4.20), we deduce that

$$\begin{aligned} &\frac{d}{dt}|\sqrt{\rho^{k+1}}\,\overline{u}^{\,k+1}(t)|_{L^2}^2 + \mu|\nabla\overline{u}^{\,k+1}(t)|_{L^2}^2 \\ &\qquad \le \eta^{-1}\tilde{C}\varphi^{k+1}(t) + 2\eta|\nabla\overline{u}^{\,k}(t)|_{L^2}^2 \end{aligned} \tag{4.24}$$

for $0 \le t \le T_1$, where

$$\varphi^{k+1}(t) = \begin{cases} |\sqrt{\rho^{k+1}}\,\overline{u}^{\,k+1}(t)|_{L^2}^2 + |\overline{\rho}^{\,k+1}(t)|_{L^2}^2, & \text{if} \quad \rho^\infty > 0 \\ |\sqrt{\rho^{k+1}}\,\overline{u}^{\,k+1}(t)|_{L^2}^2 + |\overline{\rho}^{\,k+1}(t)|_{L^{\frac{3}{2}}\cap L^2}^2, & \text{otherwise.} \end{cases}$$

By virtue of (4.18), (4.19) and (4.24), we deduce that

$$\frac{d}{dt}\varphi^{k+1}(t) + \mu\,\psi^{k+1}(t) \le \eta^{-1}\tilde{C}\varphi^{k+1}(t) + 4\eta\psi^k(t) \tag{4.25}$$

for $0 \le t \le T_1$, where $\psi^{k+1}(t) = |\nabla \overline{u}^{\,k+1}(t)|^2_{L^2}$. Note that $\varphi^{k+1}(0) = 0$. Hence integrating (4.25) over $(0,t)$, we have

$$\varphi^{k+1}(t) + \mu\int_0^t \psi^{k+1}(s)\,ds \le 4\eta\int_0^t \psi^k(s)\,ds + \eta^{-1}\tilde{C}\int_0^t \varphi^{k+1}(s)\,ds,$$

which implies, in view of of Gronwall's inequality, that

$$\varphi^{k+1}(t) + \int_0^t \psi^{k+1}(s)\,ds \le \eta\tilde{C}\exp(\eta^{-1}\tilde{C}t)\left(\int_0^t \psi^k(s)\,ds\right). \tag{4.26}$$

Choosing $\eta > 0$ and then $T_2 > 0$ so small that

$$\eta\tilde{C} \le \frac{1}{4}, \quad T_2 < T_1 \quad \text{and} \quad \exp(\eta^{-1}\tilde{C}T_2) < 2,$$

we deduce from (4.26) that

$$\sum_{k=1}^{\infty}\left(\sup_{0\le t\le T_2}\varphi^{k+1}(t) + \int_0^{T_2}\psi^{k+1}(t)\,dt\right) \le \tilde{C}\int_0^{T_2}\psi^1(t)\,dt < \infty.$$

Therefore, we conclude that the sequence $\{(\rho^k, u^k)\}$ converges in a strong sense to a limit (ρ, u) satisfying the regularity estimate (4.16) with T_* replaced by T_2. Adapting the proof of Lemma 4.1, we can show that (ρ, u) is a solution to the original IBVP(1.1)-(1.5) with T replaced by T_2. This completes the proof of the existence. The proof of the uniqueness is similar to (indeed easier than) the proof of the convergence and so omitted. We have completed the proof of Theorem 1.1. □

5. Proof of Theorem 1.3

To prove Theorem 1.3, we follow basically the same methods as in the proof of Theorem 1.1. Hence we consider the linearized problem (3.1)–(3.4) with a known vector field v such that

$$\begin{gathered} v \in C([0,T]; D_0^1 \cap D^3) \cap L^2(0,T;D^4), \quad v_t \in L^\infty(0,T;D_0^1) \cap L^2(0,T;D^2), \\ t^{\frac{1}{2}}v \in L^\infty(0,T;D^4), \ \ t^{\frac{1}{2}}v_t \in L^\infty(0,T;D_0^1\cap D^2), \ \ t^{\frac{1}{2}}v_{tt} \in L^2(0,T;D_0^1), \\ tv_t \in L^\infty(0,T;D_0^1\cap D^3), \quad tv_{tt} \in L^\infty(0,T;D_0^1) \cap L^2(0,T;D^2), \\ t^{\frac{3}{2}}v_{tt} \in L^\infty(0,T;D_0^1\cap D^2) \quad \text{and} \quad t^{\frac{3}{2}}v_{ttt} \in L^2(0,T;D_0^1). \end{gathered} \tag{5.1}$$

For positive initial densities and bounded domains, we have the following existence and regularity results for the linearized problem.

Lemma 5.1. *Let Ω be a bounded domain in $\mathbf{R}^3$ with smooth boundary. Assume that ρ_0, u_0, f, v and $p = p(\cdot)$ satisfy the condition* (5.1) *as well as the hypotheses of*

Theorem 1.3. *If in addition, $\rho_0 \geq \delta$ in Ω for some constant $\delta > 0$, then there exists a unique solution (ρ, u) to the linearized problem* (3.1)*,* (3.2) *and* (3.3) *such that*

$$\rho \in C([0,T]; H^3), \quad \rho_t \in C([0,T]; H^2), \quad \rho_{tt} \in L^\infty(0,T; L^2) \cap L^2(0,T; H^1),$$
$$\rho_{tt} \in L^\infty_{loc}((0,T]; H^1), \quad \rho_{ttt} \in L^2_{loc}((0,T]; L^2),$$
$$u \in C([0,T]; H^1_0 \cap H^3) \cap L^2(0,T; H^4), \quad u_t \in C([0,T]; H^1_0) \cap L^2(0,T; H^2),$$
$$u_{tt} \in L^2(0,T; L^2), \quad u \in L^\infty_{loc}((0,T]; H^4), \quad u_t \in L^\infty_{loc}((0,T]; H^3),$$
$$u_{tt} \in L^\infty_{loc}((0,T]; H^1_0 \cap H^2), \quad u_{ttt} \in L^\infty_{loc}((0,T]; L^2) \cap L^2_{loc}((0,T]; H^1_0),$$
$$u_{tttt} \in L^2_{loc}((0,T]; H^{-1}) \quad \text{and} \quad \rho \geq C^{-1}\delta \quad \text{on} \quad [0,T] \times \overline{\Omega}.$$

Proof. The existence of a unique solution $\rho \in C([0,T]; H^3)$ to the linear hyperbolic problem (3.1) and (3.3) was already proved in Lemma 2.1. Then the remaining regularity of ρ can be derived easily from (3.1) and (5.1).

Next, if we define F by $F = -\nabla p(\rho) + \rho(f - v \cdot \nabla v)$, then

$$F \in L^2(0,T; H^2), \quad F_t \in L^2(0,T; L^2); \quad F \in L^\infty_{loc}((0,T]; H^2),$$
$$F_t \in L^\infty_{loc}((0,T]; H^1), \quad F_{tt} \in L^\infty_{loc}((0,T]; L^2) \quad \text{and} \quad F_{ttt} \in L^2_{loc}((0,T]; H^{-1}).$$

Moreover, we observe that $\rho_0^{-1}(F(0) - Lu_0) = -v(0) \cdot \nabla v(0) - g_2 \in H^1_0$. Hence Lemma 2.2, Remark 2.3 and Lemma 2.4 allow us to deduce the existence and regularity of a unique solution u to the linear parabolic problem (3.2) and (3.3). This completes the proof of Lemma 5.1. □

Let (ρ, u) be a solution to the linearized problem (3.1), (3.2) and (3.3) with the data ρ_0, u_0, f, v and $p = p(\cdot)$ satisfying the hypotheses of Lemma 5.1. We will prove some local a priori estimates for (ρ, u) which are independent of the lower bound δ of ρ_0 and the size of the domain Ω.

Let us choose a constant $c_0 > 1$ so that

$$1 + \rho^\infty + |\rho_0 - \rho^\infty|_{H^3} + |u_0|_{D^1_0} + |\sqrt{\rho_0}\, g_2|_{L^2} + |g_2|_{D^1_0} < c_0.$$

Note that $g_2 = \rho_0^{-1}(Lu_0 + \nabla p(\rho_0)) - f(0) = -v(0) \cdot \nabla v(0) - u_t(0)$. Moreover, we assume that

$$
\begin{aligned}
|v(0)|_{D^1_0 \cap D^3} &\leq 1 + c_1,\\
\sup_{0 \leq t \leq T_*} |v(t)|_{D^1_0} + \int_0^{T_*} |v(t)|^2_{D^2}\, dt &\leq 1 + c_2,\\
\sup_{0 \leq t \leq T_*} |v(t)|_{D^2} + \int_0^{T_*} \left(|v_t(t)|^2_{D^1_0} + |v(t)|^2_{D^3}\right) ds &\leq 1 + c_3,\\
\operatorname*{ess\,sup}_{0 \leq t \leq T_*} \left(|v_t(t)|_{D^1_0} + |v(t)|_{D^3}\right) + \int_0^{T_*} \left(|v_t(t)|^2_{D^2} + |v(t)|^2_{D^4}\right) dt &\leq 1 + c_4,\\
\operatorname*{ess\,sup}_{0 \leq t \leq T_*} \left(t^{\frac{1}{2}}|v_t(t)|_{D^2} + t^{\frac{1}{2}}|v(t)|_{D^4}\right) + \int_0^{T_*} t|v_{tt}(t)|^2_{D^1_0}\, dt &\leq 1 + c_5,\\
\operatorname*{ess\,sup}_{0 \leq t \leq T_*} \left(t|v_{tt}(t)|_{D^1_0} + t|v_t(t)|_{D^3}\right) + \int_0^{T_*} t^2|v_{tt}(t)|^2_{D^2}\, dt &\leq 1 + c_6,\\
\operatorname*{ess\,sup}_{0 \leq t \leq T_*} \left(t^{\frac{3}{2}}|v_{tt}(t)|_{D^2}\right) + \int_0^{T_*} t^3|v_{ttt}(t)|^2_{D^1_0}\, dt &\leq 1 + c_6.
\end{aligned}
\tag{5.2}
$$

for some time $T_* \in (0, T)$ and constants c_i's with $1 < c_0 \le c_1 \le c_2 \le c_3 \le c_4 \le c_5 \le c_6$, which depend only on c_0 and the parameters of C.

Adapting the proofs of Lemma 3.2-Lemma 3.5, we can prove

Lemma 5.2.

$$|\rho(t)|_{L^\infty} + |\rho(t) - \rho^\infty|_{H^3} \le Cc_0, \quad |p(t) - p^\infty|_{H^3} \le M(c_0),$$
$$|\rho_t(t)|_{H^1} \le Cc_3^2, \quad |p_t(t)|_{H^1} \le M(c_0)c_3^2, \quad \int_0^t |\rho_{tt}(s)|_{L^2}^2\, ds \le Cc_3^8,$$
$$\int_0^t |p_{tt}(s)|_{L^2}^2\, ds \le M(c_0)c_3^8, \quad |\rho_t(t)|_{H^2} \le Cc_4^2, \quad |p_t(t)|_{H^2} \le M(c_0)c_4^2,$$
$$|\rho_{tt}(t)|_{L^2} \le Cc_4^4, \quad |p_{tt}(t)|_{L^2} \le M(c_0)c_4^4, \quad \int_0^t |\rho_{tt}(s)|_{H^1}^2\, ds \le Cc_4^8,$$
$$\int_0^t |p_{tt}(s)|_{H^1}^2\, ds \le M(c_0)c_4^8, \quad \int_0^t s|\rho_{ttt}(s)|_{L^2}^2\, ds \le Cc_5^{12},$$
$$\int_0^t s|p_{ttt}(s)|_{L^2}^2\, ds \le M(c_0)c_5^{12} \quad \textit{and} \quad \inf_\Omega \rho(t) \ge C^{-1}\delta$$

for $0 \le t \le \min(T_*, T_1)$, *where* $T_1 = (1 + c_4)^{-1}$ *and* $p^\infty = p(\rho^\infty)$.

Lemma 5.3.

$$|u(t)|_{D_0^1}^2 + \int_0^t |u(s)|_{D^2}^2\, ds \le M(c_0)$$

for $0 \le t \le \min(T_*, T_2)$, *where* $T_2 = (1 + c_4)^{-4} < T_1$.

Lemma 5.4.

$$|\sqrt{\rho} u_t(t)|_{L^2}^2 + |u(t)|_{D^2}^2 + \int_0^t (|u_t(s)|_{D_0^1}^2 + |u(s)|_{D^3}^2)\, ds \le M(c_1)c_2^{\frac{3}{2}} c_3^{\frac{1}{2}},$$
$$|u_t(t)|_{D_0^1}^2 + |u(t)|_{D^3}^2 + \int_0^t (|\sqrt{\rho} u_{tt}(s)|_{L^2}^2 + |u_t(s)|_{D^2}^2 + |u(s)|_{D^4}^2)\, ds \le M(c_1)c_3^{12}$$

for $0 \le t \le \min(T_*, T_3)$, *where* $T_3 = (1 + c_4)^{-9} < T_2$.

Using the same methods as in the proof of Lemma 5.4, we can derive estimates for higher regularity in positive time.

Lemma 5.5.

$$t|\sqrt{\rho} u_{tt}(t)|_{L^2}^2 + t|u_t(t)|_{D^2}^2 + t|u(t)|_{D^4}^2 + \int_0^t s|u_{tt}(s)|_{D_0^1}^2\, ds \le M(c_1)c_4^{12}$$

for $0 \le t \le \min(T_*, T_4)$, *where* $T_4 = (1 + c_5)^{-9} \le T_3$.

Proof. We differentiate (3.9) with respect to t again and derive

$$\begin{aligned} \rho u_{ttt} + Lu_{tt} = -\nabla p_{tt} + \rho\,(f - v \cdot \nabla v)_{tt} &+ 2\rho_t\,(f - v \cdot \nabla v - u_t)_t \\ &+ \rho_{tt}\,(f - v \cdot \nabla v - u_t)\,. \end{aligned} \tag{5.3}$$

Multiplying this by u_{tt} and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_{tt}|^2\,dx + \int \mu|\nabla u_{tt}|^2 + (\lambda+\mu)(\mathrm{div}u_{tt})^2\,dx \\
&\quad = \int p_{tt}\mathrm{div}u_{tt}\,dx + \int \rho\,(f - v\cdot\nabla v)_{tt}\cdot u_{tt}\,dx \\
&\qquad +2\int \rho_t\,(f - v\cdot\nabla v)_t\cdot u_{tt}\,dx + \int \rho_{tt}\,(f - v\cdot\nabla v)\cdot u_{tt}\,dx \\
&\qquad -\frac{3}{2}\int \rho_t|u_{tt}|^2\,dx - \int \rho_{tt}u_t\cdot u_{tt}\,dx.
\end{aligned} \tag{5.4}$$

Following the same arguments as in the derivation of (3.11) from (3.10), we can estimate each term of the right hand side of (5.4) as follows:

$$\int p_{tt}\mathrm{div}u_{tt}\,dx \le C|p_{tt}|_{L^2}^2 + \frac{\mu}{12}|\nabla u_{tt}|_{L^2}^2,$$

$$\int \rho f_{tt}\cdot u_{tt}\,dx \le |f_{tt}|_{H^{-1}}|\rho u_{tt}|_{H_0^1} \le Cc_0^2|f_{tt}|_{H^{-1}}^2 + |\sqrt{\rho}u_{tt}|_{L^2}^2 + \frac{\mu}{12}|\nabla u_{tt}|_{L^2}^2,$$

$$\begin{aligned}
-\int \rho\,(v\cdot\nabla v)_{tt}\cdot u_{tt}\,dx &\le C|\rho|_{L^\infty}^{\frac{1}{2}}\left(|v|_{D_0^1\cap D^2}|v_{tt}|_{D_0^1} + |v_t|_{D_0^1}|v_t|_{D_0^1\cap D^2}\right)|\sqrt{\rho}u_{tt}|_{L^2} \\
&\le \eta^{-1}Cc_3^3|\sqrt{\rho}u_{tt}|_{L^2}^2 + \eta|v_{tt}|_{D_0^1}^2 + c_4^2|v_t|_{D_0^1\cap D^2}^2,
\end{aligned}$$

$$\begin{aligned}
2\int \rho_t\,(f - v\cdot\nabla v)_t\cdot u_{tt}\,dx &\le C|\rho_t|_{L^3}\left(|f_t|_{L^2} + |v|_{D_0^1\cap D^2}|v_t|_{D_0^1}\right)|\nabla u_{tt}|_{L^2} \\
&\le Cc_3^4\left(|f_t|_{L^2}^2 + c_4^4\right) + \frac{\mu}{12}|\nabla u_{tt}|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
\int \rho_{tt}\,(f - v\cdot\nabla v)\cdot u_{tt}\,dx &\le C|\rho_{tt}|_{L^2}\left(|f|_{H^1} + |v|_{D_0^1\cap D^2}^2\right)|\nabla u_{tt}|_{L^2} \\
&\le Cc_3^4|\rho_{tt}|_{L^2}^2 + \frac{\mu}{12}|\nabla u_{tt}|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
-\frac{3}{2}\int \rho_t|u_{tt}|^2\,dx &= \frac{3}{2}\int \mathrm{div}(\rho v)|u_{tt}|^2\,dx \\
&\le 3\int \rho|v||u_{tt}||\nabla u_{tt}|\,dx \le Cc_3^3|\sqrt{\rho}u_{tt}|_{L^2}^2 + \frac{\mu}{12}|\nabla u_{tt}|_{L^2}^2
\end{aligned}$$

and finally

$$
\begin{aligned}
&-\int \rho_{tt} u_t \cdot u_{tt}\, dx \\
&\quad \le \int \left(|\rho_t||v| + \rho|v_t|\right)\left(|u_t||\nabla u_{tt}| + |\nabla u_t||u_{tt}|\right) dx \\
&\le C|\rho_t|_{L^3}|v|_{D_0^1\cap D^2}|u_t|_{D_0^1}|\nabla u_{tt}|_{L^2} + C|\rho|_{L^\infty}^{\frac34}|v_t|_{D_0^1}|u_t|_{D_0^1}^{\frac12}|\sqrt{\rho}u_t|_{L^2}^{\frac12}|\nabla u_{tt}|_{L^2} \\
&\qquad + C|\rho|_{L^\infty}^{\frac34}|v_t|_{D_0^1}|u_t|_{D_0^1}|\sqrt{\rho}u_{tt}|_{L^2}^{\frac12}|\nabla u_{tt}|_{L^2}^{\frac12} \\
&\le C|\rho_t|_{L^3}^2|v|_{D_0^1\cap D^2}^2|u_t|_{D_0^1}^2 + C|\rho|_{L^\infty}^{\frac32}|v_t|_{D_0^1}^2|u_t|_{D_0^1}|\sqrt{\rho}u_t|_{L^2} \\
&\qquad + C|\rho|_{L^\infty}^{\frac32}|v_t|_{D_0^1}^2|u_t|_{D_0^1}^2 + C|\sqrt{\rho}u_{tt}|_{L^2}^2 + \frac{\mu}{12}|\nabla u_{tt}|_{L^2}^2 \\
&\le Cc_4^7|u_t|_{D_0^1}^2 + |\sqrt{\rho}u_t|_{L^2}^2 + C|\sqrt{\rho}u_{tt}|_{L^2}^2 + \frac{\mu}{12}|\nabla u_{tt}|_{L^2}^2.
\end{aligned}
$$

Substituting all the estimates into (5.4) and taking $\eta = (1+c_5)^{-1}$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int \rho|u_{tt}|^2\, dx + \mu\int |\nabla u_{tt}|^2\, dx \\
&\le C\left(c_0^2|f_{tt}|_{H^{-1}}^2 + c_3^4|f_t|_{L^2}^2 + c_4^8 + |p_{tt}|_{L^2}^2 + c_3^4|\rho_{tt}|_{L^2}^2 + c_4^2|v_t|_{D_0^1\cap D^2}^2\right) \\
&\qquad + C\left(c_4^7|u_t|_{D_0^1}^2 + |\sqrt{\rho}u_t|_{L^2}^2\right) + (1+c_5)^{-1}|v_{tt}|_{D_0^1}^2 + Cc_5^4|\sqrt{\rho}u_{tt}|_{L^2}^2
\end{aligned} \tag{5.5}
$$

for $0 \le t \le \min(T_*, T_3)$. From Lemma 5.2 and Lemma 5.4, we observe that

$$
t^{\frac12}|f_{tt}(t)|_{H^{-1}} \in L^2(0,T), \quad t^{\frac12}|v_{tt}(t)|_{D_0^1} \in L^2(0,T_*)
$$

and all the remaining terms in the right hand side of (5.5) are integrable in $(0, \min(T_*, T_3))$. Hence multiplying (5.5) by t and integrating over $(\tau, \bar{t})$, we obtain

$$
\begin{aligned}
&\bar{t}|\sqrt{\rho}u_{tt}(\bar{t})|_{L^2}^2 + \mu\int_\tau^{\bar{t}} t|\nabla u_{tt}(t)|_{L^2}^2\, dt \\
&\qquad \le M(c_1)c_4^{12} + \tau|\sqrt{\rho}u_{tt}(\tau)|_{L^2}^2 + \int_\tau^{\bar{t}} Cc_5^4 t|\sqrt{\rho}u_{tt}(t)|_{L^2}^2\, dt
\end{aligned}
$$

for $0 < \tau \le \bar{t} \le \min(T_*, T_3)$. By virtue of Gronwall's inequality, we deduce that

$$
t|\sqrt{\rho}u_{tt}(t)|_{L^2}^2 + \int_\tau^t s|\nabla u_{tt}(s)|_{L^2}^2\, ds \le M(c_1)\left(c_4^{12} + \tau|\sqrt{\rho}u_{tt}(\tau)|_{L^2}^2\right) \tag{5.6}
$$

for $0 < \tau \le t \le \min(T_*, T_4)$, where $T_4 = (1+c_5)^{-9} \le T_3$. On the other hand, since $\sqrt{\rho}u_{tt} \in L^2(0,T;L^2)$, it follows (see also Remark 5 in [1]) that there is a sequence $\{\tau_k\}$ of positive times such that

$$
\tau_k \to 0 \quad \text{and} \quad \tau_k|\sqrt{\rho}u_{tt}(\tau_k)|_{L^2}^2 \to 0 \qquad \text{as} \quad k \to \infty.
$$

Therefore, letting $\tau = \tau_k \to 0$ in (5.6), we conclude that

$$
t|\sqrt{\rho}u_{tt}(t)|_{L^2}^2 + \int_0^t s|u_{tt}(s)|_{D_0^1}^2\, ds \le M(c_1)c_4^{12}
$$

for $0 < t \le \min(T_*, T_4)$. Moreover, since

$$
Lu = -\nabla p + \rho(f - v\cdot\nabla v - u_t)
$$

and

$$Lu_t = -\nabla p_t + \rho(f - v\cdot\nabla v - u_t)_t + \rho_t(f - v\cdot\nabla v - u_t),$$

it follows from the elliptic regularity result that

$$t|u_t(t)|_{D^2}^2 + t|u(t)|_{D^4}^2 \le M(c_1)c_4^{12} \qquad \text{for} \quad 0\le t\le \min(T_*, T_4).$$

This completes the proof of Lemma 5.5. □

Lemma 5.6.

$$t^2|u_{tt}(t)|_{D_0^1}^2 + t^2|u_t(t)|_{D^3}^2 + \int_0^t s^2\left(|\sqrt{\rho}u_{ttt}(s)|_{L^2}^2 + |u_{tt}(s)|_{D^2}^2\right)ds \le M(c_1)c_5^{17}$$

for $0\le t\le \min(T_*, T_4)$.

Proof. Multiplying (5.3) by u_{ttt} and integrating over Ω, we have

$$\begin{aligned}\int \rho|u_{ttt}|^2\,dx + \frac{1}{2}\frac{d}{dt}\int \mu|\nabla u_{tt}|^2 + (\lambda+\mu)(\mathrm{div} u_{tt})^2\,dx \\ = \int\left(-\nabla p_{tt} + \rho\left(f - v\cdot\nabla v\right)_{tt}\right)\cdot u_{ttt}\,dx \\ + \int 2\rho_t\left(f - v\cdot\nabla v - u_t\right)_t\cdot u_{ttt}\,dx \\ + \int \rho_{tt}\left(f - v\cdot\nabla v - u_t\right)\cdot u_{ttt}\,dx.\end{aligned} \tag{5.7}$$

We easily estimate the first term of the right hand side in (5.7) as follows.

$$\begin{aligned}-\int\nabla p_{tt}\cdot u_{ttt}\,dx &= \frac{d}{dt}\int p_{tt}\mathrm{div}u_{tt}\,dx - \int p_{ttt}\mathrm{div}u_{tt}\,dx \\ &\le \frac{d}{dt}\int p_{tt}\mathrm{div}u_{tt}\,dx + |p_{ttt}|_{L^2}^2 + |u_{tt}|_{D_0^1}^2\end{aligned}$$

and

$$\begin{aligned}&\int\rho(f - v\cdot\nabla v)_{tt}\cdot u_{ttt}\,dx \\ &\le C|\rho|_{L^\infty}^{\frac{1}{2}}|(f - v\cdot\nabla v)_{tt}|_{L^2}|\sqrt{\rho}u_{ttt}|_{L^2} \\ &\le C|\rho|_{L^\infty}^{\frac{1}{2}}(|f_{tt}|_{L^2} + |v|_{D_0^1\cap D^2}|v_{tt}|_{D_0^1} + |v_t|_{D_0^1}|v_t|_{D_0^1\cap D^2})|\sqrt{\rho}u_{ttt}|_{L^2} \\ &\le Cc_4^3\left(|f_{tt}|_{L^2}^2 + |v_{tt}|_{D_0^1}^2 + |v_t|_{D_0^1\cap D^2}^2\right) + \frac{1}{2}|\sqrt{\rho}u_{ttt}|_{L^2}^2.\end{aligned}$$

To estimate the second term, we observe that

$$\begin{aligned}&\int\rho_t\left(f - v\cdot\nabla v\right)_t\cdot u_{ttt}\,dx \\ &= \frac{d}{dt}\int\rho_t\left(f - v\cdot\nabla v\right)_t\cdot u_{tt}\,dx - \int\rho_{tt}\left(f - v\cdot\nabla v\right)_t\cdot u_{tt}\,dx \\ &\quad - \int\rho_t\left(f - v\cdot\nabla v\right)_{tt}\cdot u_{tt}\,dx\end{aligned}$$

and

$$-\int\rho_t u_{tt}\cdot u_{ttt}\,dx = -\frac{d}{dt}\int\rho_t\left(\frac{1}{2}|u_{tt}|^2\right)dx + \int\rho_{tt}\left(\frac{1}{2}|u_{tt}|^2\right)dx.$$

But in view of Lemma 5.2, we obtain

$$
\begin{aligned}
-\int \rho_{tt}\left(f - v\cdot\nabla v\right)_t\cdot u_{tt}\,dx &\le C|\rho_{tt}|_{L^2}\left(|f_t|_{H^1} + |v|_{D_0^1\cap D^2}|v_t|_{D_0^1\cap D^2}\right)|u_{tt}|_{D_0^1}\\
&\le Cc_4^5\left(|f_t|_{H^1}^2 + |v_t|_{D_0^1\cap D^2}^2 + |u_{tt}|_{D_0^1}^2\right),
\end{aligned}
$$

$$
\begin{aligned}
-\int \rho_t\left(f - v\cdot\nabla v\right)_{tt}\cdot u_{tt}\,dx &\le C|\rho_t|_{L^3}|\left(f - v\cdot\nabla v\right)_{tt}|_{L^2}|u_{tt}|_{D_0^1}\\
&\le Cc_4^4\left(|f_{tt}|_{L^2}^2 + |v_{tt}|_{D_0^1}^2 + |v_t|_{D_0^1\cap D^2}^2 + |u_{tt}|_{D_0^1}^2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int \rho_{tt}\left(\frac{1}{2}|u_{tt}|^2\right)\,dx &= -\int \operatorname{div}((\rho v)_t)\left(\frac{1}{2}|u_{tt}|^2\right)\,dx\\
&\le \int \left(|\rho_t||v| + \rho|v_t|\right)|u_{tt}||\nabla u_{tt}|\,dx\\
&\le Cc_3^3|u_{tt}|_{D_0^1}^2 + C|v_t|_{D_0^1\cap D^2}^2|\sqrt{\rho}u_{tt}|_{L^2}^2.
\end{aligned}
$$

Similarly, we can estimate the last term as follows.

$$
\begin{aligned}
&\int \rho_{tt}\left(f - v\cdot\nabla v\right)\cdot u_{ttt}\,dx\\
&= \frac{d}{dt}\int \rho_{tt}\left(f - v\cdot\nabla v\right)\cdot u_{tt}\,dx - \int \left(\rho_{tt}\left(f - v\cdot\nabla v\right)\right)_t\cdot u_{tt}\,dx\\
&\le \frac{d}{dt}\int \rho_{tt}\left(f - v\cdot\nabla v\right)\cdot u_{tt}\,dx\\
&\qquad + Cc_4^5\left(|\rho_{ttt}|_{L^2}^2 + |f_t|_{H^1}^2 + |v_t|_{D_0^1\cap D^2}^2 + |u_{tt}|_{D_0^1}^2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int \rho_{tt}u_t\cdot u_{ttt}\,dx\\
&= -\frac{d}{dt}\int \rho_{tt}u_t\cdot u_{tt}\,dx + \int \left(\rho_{ttt}u_t + \rho_{tt}u_{tt}\right)\cdot u_{tt}\,dx\\
&= -\frac{d}{dt}\int \rho_{tt}u_t\cdot u_{tt}\,dx - \int \operatorname{div}((\rho v)_{tt})u_t\cdot u_{tt}\,dx + \int \rho_{tt}|u_{tt}|^2\,dx\\
&\le -\frac{d}{dt}\int \rho_{tt}u_t\cdot u_{tt}\,dx + Cc_4^5\left(|u_t|_{D_0^1\cap D^2}^2 + |u_{tt}|_{D_0^1}^2\right)\\
&\qquad + C\left(|v_t|_{D_0^1\cap D^2}^2 + |v_{tt}|_{D_0^1}^2\right)\left(|\sqrt{\rho}u_t|_{L^2}^2 + |u_t|_{D_0^1}^2 + |\sqrt{\rho}u_{tt}|_{L^2}^2\right).
\end{aligned}
$$

Substituting all the above estimates into (5.7), we have

$$
\begin{aligned}
&\frac{1}{2}\int \rho|u_{ttt}|^2\,dx + \frac{1}{2}\frac{d}{dt}\int \mu|\nabla u_{tt}|^2 + (\lambda+\mu)(\operatorname{div}u_{tt})^2\,dx\\
&\le \frac{d}{dt}\Lambda_1 + Cc_4^5\left(|p_{ttt}|_{L^2}^2 + |\rho_{ttt}|_{L^2}^2 + |f_{tt}|_{L^2}^2 + |f_t|_{H^1}^2\right)\\
&\quad + Cc_4^5\left(|v_{tt}|_{D_0^1}^2 + |v_t|_{D_0^1\cap D^2}^2 + |u_t|_{D_0^1\cap D^2}^2 + |u_{tt}|_{D_0^1}^2\right)\\
&\quad + C\left(|v_t|_{D_0^1\cap D^2}^2 + |v_{tt}|_{D_0^1}^2\right)\left(|\sqrt{\rho}u_t|_{L^2}^2 + |u_t|_{D_0^1}^2 + |\sqrt{\rho}u_{tt}|_{L^2}^2\right)
\end{aligned}
$$

for $0 \le t \le \min(T_*, T_4)$, where

$$\Lambda_1(t) = \int (p_{tt}\mathrm{div}u_{tt} + 2\rho_t(f - v\cdot\nabla v)_t \cdot u_{tt} + \rho_{tt}(f - v\cdot\nabla v)\cdot u_{tt})(t)\,dx \\ - \int \left(\rho_t|u_{tt}|^2 + \rho_{tt}u_t.\cdot u_{tt}\right)(t)\,dx.$$

Hence if we multiply this by t^2 and integrate over $(\tau, \bar{t})$, then by virtue of the previous lemmas, we deduce that

$$\begin{aligned}(5.8)\qquad &\frac{1}{2}\int_\tau^{\bar{t}} t^2|\sqrt{\rho}u_{ttt}(t)|_{L^2}^2\,dt + \frac{\mu}{2}\bar{t}^2|\nabla u_{tt}(\bar{t})|_{L^2}^2 \\ &\quad\le M(c_1)c_5^{17} + C\tau^2|\nabla u_{tt}(\tau)|_{L^2}^2 + \left|\bar{t}^2\Lambda_1(\bar{t})\right| \\ &\quad\quad + \left|\tau^2\Lambda_1(\tau)\right| + C\int_\tau^{\bar{t}} t|\Lambda_1(t)|\,dt\end{aligned}$$

for $0 < \tau \le \bar{t} \le \min(T_*, T_4)$. It is easy to show that

$$|\Lambda_1(t)| \le t^{-1}M(c_1)c_4^{14} + \frac{\mu}{4}|\nabla u_{tt}(t)|_{L^2}^2 \qquad \text{for}\quad 0 \le t \le \min(T_*, T_4).$$

Therefore, recalling that

$$\int_0^t s|\nabla u_{tt}(s)|_{L^2}^2\,ds \le M(c_1)c_4^{12} \qquad \text{for}\quad 0 \le t \le \min(T_*, T_4)$$

and

$$\tau_k^2|\nabla u_{tt}(\tau_k)|_{L^2}^2 \to 0 \qquad \text{for some sequence}\quad \{\tau_k\}\quad \text{with}\quad \tau_k \to 0,$$

we conclude from (5.8) that

$$\int_0^t s^2|\sqrt{\rho}u_{ttt}(s)|_{L^2}^2\,ds + t^2|\nabla u_{tt}(t)|_{L^2}^2 \le M(c_1)c_5^{17}$$

for $0 \le t \le \min(T_*, T_4)$. Then in view of the elliptic regularity result, we complete the proof of Lemma 5.6. □

Lemma 5.7.

$$t^3|\sqrt{\rho}u_{ttt}(t)|_{L^2}^2 + t^3|u_{tt}(t)|_{D^2}^2 + \int_0^t s^3|u_{ttt}(s)|_{D_0^1}^2\,ds \le M(c_1)c_5^{22}$$

for $0 \le t \le \min(T_*, T_5)$, *where* $T_5 = (1 + c_6)^{-9} \le T_4$.

Proof. Differentiating (5.3) with respect to t again, we derive

$$\begin{aligned}(5.9)\qquad \rho u_{tttt} + Lu_{ttt} &= -\nabla p_{ttt} + \rho\,(f - v\cdot\nabla v)_{ttt} + 3\rho_t\,(f - v\cdot\nabla v - u_t)_{tt} \\ &\quad + 3\rho_{tt}\,(f - v\cdot\nabla v - u_t)_t + \rho_{ttt}\,(f - v\cdot\nabla v - u_t)\,.\end{aligned}$$

Multiplying this by u_{ttt} and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_{ttt}|^2\,dx + \int \mu|\nabla u_{ttt}|^2 + (\lambda+\mu)(\mathrm{div}u_{ttt})^2\,dx \\
&\quad= \int \left(p_{ttt}\mathrm{div}u_{ttt} + \rho\,(f - v\cdot\nabla v)_{ttt}\cdot u_{tt}\right)\,dx \\
&\qquad+3\int \rho_t\,(f - v\cdot\nabla v)_{tt}\cdot u_{ttt}\,dx - \frac{5}{2}\int \rho_t|u_{ttt}|^2\,dx \\
&\qquad+3\int \rho_{tt}\,(f - v\cdot\nabla v)_{t}\cdot u_{ttt}\,dx - 3\int \rho_{tt}u_{tt}\cdot u_{ttt}\,dx \\
&\qquad+\int \rho_{ttt}\,(f - v\cdot\nabla v)\cdot u_{ttt}\,dx - \int \rho_{ttt}u_{t}\cdot u_{ttt}\,dx.
\end{aligned} \tag{5.10}$$

Using Lemma 5.2 and Lemma 5.4 , we can estimate each term of the right hand side of (5.10) as follows:

$$\int p_{ttt}\mathrm{div}u_{ttt}\,dx \le C|p_{ttt}|^2_{L^2} + \frac{\mu}{16}|\nabla u_{ttt}|^2_{L^2},$$

$$\int \rho f_{ttt}\cdot u_{ttt}\,dx \le |f_{ttt}|_{H^{-1}}|\rho u_{ttt}|_{D^1_0} \le Cc_0^2|f_{ttt}|^2_{H^{-1}} + |\sqrt{\rho}u_{ttt}|^2_{L^2} + \frac{\mu}{16}|\nabla u_{ttt}|^2_{L^2},$$

$$\begin{aligned}
&-\int \rho\,(v\cdot\nabla v)_{ttt}\cdot u_{ttt}\,dx \\
&\quad\le C|\rho|^{\frac{1}{2}}_{L^\infty}\left(|v|_{D^1_0\cap D^2}|v_{ttt}|_{D^1_0} + |v_t|_{D^1_0\cap D^2}|v_{tt}|_{D^1_0}\right)|\sqrt{\rho}u_{ttt}|_{L^2} \\
&\quad\le \eta^{-1}Cc_3^3|\sqrt{\rho}u_{ttt}|^2_{L^2} + \eta|v_{ttt}|^2_{D^1_0} + |v_t|^2_{D^1_0\cap D^2}|v_{tt}|^2_{D^1_0},
\end{aligned}$$

$$\begin{aligned}
&3\int \rho_t\,(f - v\cdot\nabla v)_{tt}\cdot u_{ttt}\,dx \\
&\quad\le C|\rho_t|_{L^3}\left(|f_{tt}|_{L^2} + |v|_{D^1_0\cap D^2}|v_{tt}|_{D^1_0} + |v_t|_{D^1_0}|v_t|_{D^1_0\cap D^2}\right)|\nabla u_{ttt}|_{L^2} \\
&\quad\le Cc_4^6\left(|f_{tt}|^2_{L^2} + |v_{tt}|^2_{D^1_0} + |v_t|_{D^1_0\cap D^2}\right) + \frac{\mu}{16}|\nabla u_{tt}|^2_{L^2},
\end{aligned}$$

$$\begin{aligned}
-\frac{5}{2}\int \rho_t|u_{ttt}|^2\,dx = \frac{5}{2}\int \mathrm{div}(\rho v)|u_{ttt}|^2\,dx &\le C\int \rho|v||u_{ttt}||\nabla u_{ttt}|\,dx \\
&\le Cc_3^3|\sqrt{\rho}u_{ttt}|^2_{L^2} + \frac{\mu}{16}|\nabla u_{ttt}|^2_{L^2},
\end{aligned}$$

$$\begin{aligned}
&3\int \rho_{tt}\,(f - v\cdot\nabla v)_{t}\cdot u_{ttt}\,dx \\
&\quad\le C|\rho_{tt}|_{L^2}\left(|f_t|_{H^1} + |v|_{D^1_0\cap D^2}|v_t|_{D^1_0\cap D^2}\right)|\nabla u_{ttt}|_{L^2} \\
&\quad\le Cc_4^{10}\left(|f_t|^2_{H^1} + |v_t|^2_{D^1_0\cap D^2}\right) + \frac{\mu}{16}|\nabla u_{tt}|^2_{L^2},
\end{aligned}$$

$$
\begin{aligned}
-3&\int \rho_{tt} u_{tt}\cdot u_{ttt}\,dx\\
&\le C\int \left(|\rho_t||v|+\rho|v_t|\right)\left(|u_{tt}||\nabla u_{ttt}|+|\nabla u_{tt}||u_{ttt}|\right)\,dx\\
&\le C|\rho_t|_{L^3}|v|_{D_0^1\cap D^2}|u_{tt}|_{D_0^1}|\nabla u_{ttt}|_{L^2}+C|\rho|_{L^\infty}^{\frac34}|v_t|_{D_0^1}|u_{tt}|_{D_0^1}^{\frac12}|\sqrt{\rho}u_{tt}|_{L^2}^{\frac12}|\nabla u_{ttt}|_{L^2}\\
&\qquad +C|\rho|_{L^\infty}^{\frac34}|v_t|_{D_0^1}|u_{tt}|_{D_0^1}|\sqrt{\rho}u_{ttt}|_{L^2}^{\frac12}|\nabla u_{ttt}|_{L^2}^{\frac12}\\
&\le Cc_4^7|u_{tt}|_{D_0^1}^2+|\sqrt{\rho}u_{tt}|_{L^2}^2+C|\sqrt{\rho}u_{ttt}|_{L^2}^2+\frac{\mu}{16}|\nabla u_{ttt}|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
\int \rho_{ttt}\left(f-v\cdot\nabla v\right)\cdot u_{ttt}\,dx &\le C|\rho_{ttt}|_{L^2}\left(|f|_{H^1}+|v|_{D_0^1\cap D^2}^2\right)|\nabla u_{ttt}|_{L^2}\\
&\le Cc_3^4|\rho_{ttt}|_{L^2}^2+\frac{\mu}{16}|\nabla u_{ttt}|_{L^2}^2
\end{aligned}
$$

and finally

$$
\begin{aligned}
-&\int \rho_{ttt} u_t\cdot u_{ttt}\,dx\\
&\le C\int \left(|\rho_{tt}||v|+|\rho_t||v_t|+\rho|v_{tt}|\right)\left(|u_t||\nabla u_{ttt}|+|\nabla u_t||u_{ttt}|\right)\,dx\\
&\le C\left(|\rho_{tt}|_{L^2}|v|_{D_0^1\cap D^2}+|\rho_t|_{H^1}|v_t|_{D_0^1}\right)|u_t|_{D_0^1\cap D^2}|\nabla u_{ttt}|_{L^2}\\
&\quad +C|\rho|_{L^\infty}^{\frac34}|v_{tt}|_{D_0^1}|\sqrt{\rho}u_t|_{L^2}^{\frac12}|u_t|_{D_0^1}^{\frac12}|\nabla u_{ttt}|_{L^2}\\
&\quad +C|\rho|_{L^\infty}^{\frac34}|v_{tt}|_{D_0^1}|u_t|_{D_0^1}|\sqrt{\rho}u_{ttt}|_{L^2}^{\frac12}|\nabla u_{ttt}|_{L^2}^{\frac12}\\
&\le Cc_4^{10}|u_t|_{D_0^1\cap D^2}^2+M(c_1)c_3^{10}|v_{tt}|_{D_0^1}^2+C|\sqrt{\rho}u_{ttt}|_{L^2}^2+\frac{\mu}{16}|\nabla u_{ttt}|_{L^2}^2.
\end{aligned}
$$

Substituting all the estimates into (5.10) and choosing $\eta=(1+c_6)^{-1}$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int \rho|u_{ttt}|^2\,dx+\mu\int|\nabla u_{ttt}|^2\,dx\\
&\le Cc_4^{10}\left(|p_{ttt}|_{L^2}^2+|\rho_{ttt}|_{L^2}^2+|f_{ttt}|_{H^{-1}}^2+|f_{tt}|_{L^2}^2+|f_t|_{H^1}^2\right)\\
&\quad +M(c_1)c_4^{10}\left(|v_t|_{D_0^1\cap D^2}^2+|v_{tt}|_{D_0^1}^2+|v_t|_{D_0^1\cap D^2}^2|v_{tt}|_{D_0^1}^2\right)+(1+c_6)^{-1}|v_{ttt}|_{D_0^1}^2\\
&\quad +Cc_4^{10}\left(|u_{tt}|_{D_0^1}^2+|u_t|_{D_0^1\cap D^2}^2+|\sqrt{\rho}u_{tt}|_{L^2}^2\right)+Cc_6^4|\sqrt{\rho}u_{ttt}|_{L^2}^2
\end{aligned}
$$

for $0\le t\le\min(T_*,T_4)$. Hence multiplying this by t^3, integrating over $(0,\bar t)$ and using Lemma 5.2-Lemma 5.5, we deduce that

$$
\begin{aligned}
\bar t^3|\sqrt{\rho}u_{ttt}(\bar t)|_{L^2}^2&+\int_0^{\bar t} t^3|u_{ttt}(t)|_{D_0^1}^2\,dt\\
&\le M(c_1)c_5^{22}+\int_0^{\bar t} Cc_6^4t^3|\sqrt{\rho}u_{ttt}(t)|_{L^2}^2\,dt
\end{aligned}
$$

for $0\le\bar t\le\min(T_*,T_4)$. Therefore, in view of Gronwall's inequality, we conclude that

$$
t|\sqrt{\rho}u_{ttt}(t)|_{L^2}^2+\int_0^t s^3|u_{ttt}(s)|_{D_0^1}^2\,ds\le M(c_1)c_5^{22}
$$

for $0 \le t \le \min(T_*, T_5)$, where $T_5 = (1 + c_6)^{-9} \le T_4$. Then by virtue of the elliptic regularity result, we complete the proof of Lemma 5.7. □

Combining all the previous lemmas, we obtain

$$\begin{aligned}
|u(t)|_{D_0^1} + \int_0^t |u(s)|_{D^2}^2\, ds &\le M(c_1),\\
|u(t)|_{D^2} + \int_0^t \Big(|u_t(s)|_{D_0^1}^2 + |u(s)|_{D^3}^2\Big)\, ds &\le M(c_1)c_2^{\frac{3}{2}}c_3^{\frac{1}{2}},\\
|u_t(t)|_{D_0^1} + |u(t)|_{D^3} + \int_0^t \big(|u_t(s)|_{D^2}^2 + |u(s)|_{D^4}^2\big)\, ds &\le M(c_1)c_3^{12},\\
t^{\frac{1}{2}}|u_t(t)|_{D^2} + t^{\frac{1}{2}}|u(t)|_{D^4} + \int_0^t s|u_{tt}(s)|_{D_0^1}^2\, ds &\le M(c_1)c_4^{12},\\
t|u_{tt}(t)|_{D_0^1} + t|u_t(t)|_{D^3} + \int_0^t s^2|u_{tt}(t)|_{D^2}^2\, dt &\le M(c_1)c_5^{22},\\
t^{\frac{3}{2}}|u_{tt}(t)|_{D^2} + \int_0^t s^3|u_{ttt}(s)|_{D_0^1}^2\, dt &\le M(c_1)c_5^{22},\\
|\rho(t) - \rho^\infty|_{H^3} + |\rho_t(t)|_{H^2} + |\sqrt{\rho}u_t(t)|_{L^2} + \int_0^t |\sqrt{\rho}u_{tt}(s)|_{L^2}^2\, ds &\le M(c_1)c_5^{22},\\
t^{\frac{1}{2}}|\sqrt{\rho}u_{tt}|_{L^2} + t^{\frac{3}{2}}|\sqrt{\rho}u_{ttt}(t)|_{L^2} + \int_0^t s^2|\sqrt{\rho}u_{ttt}(s)|_{L^2}^2\, ds &\le M(c_1)c_5^{22}.
\end{aligned}$$

for $0 \le t \le \min(T_*, T_5)$, where $T_5 = (1+c_6)^{-9}$. Therefore, if we define the constants c_i and $T_* \in (0, 1)$ by

$$\begin{gathered}
c_1 = M(c_0), \quad c_2 = M(c_1), \quad c_3 = c_2^5, \quad c_4 = c_2c_3^{12},\\
c_5 = c_2c_4^{12}, \quad c_6 = c_2c_5^{22} \quad \text{and} \quad T_* = \min(T, (1 + c_6)^{-9}),
\end{gathered}$$

then we conclude that

$$\begin{aligned}
\sup_{0\le t\le T_*} |u(t)|_{D_0^1} + \int_0^{T_*} |u(t)|_{D^2}^2\, dt &\le c_2,\\
\sup_{0\le t\le T_*} |u(t)|_{D^2} + \int_0^{T_*} \Big(|u_t(t)|_{D_0^1}^2 + |u(t)|_{D^3}^2\Big)\, ds &\le c_3,\\
\operatorname*{ess\,sup}_{0\le t\le T_*} \Big(|u_t(t)|_{D_0^1} + |u(t)|_{D^3}\Big) + \int_0^{T_*} \big(|u_t(t)|_{D^2}^2 + |u(t)|_{D^4}^2\big)\, dt &\le c_4,\\
\operatorname*{ess\,sup}_{0\le t\le T_*} \Big(t^{\frac12}|u_t(t)|_{D^2} + t^{\frac12}|u(t)|_{D^4}\Big) + \int_0^{T_*} t|u_{tt}(t)|_{D_0^1}^2\, dt &\le c_5,\\
\operatorname*{ess\,sup}_{0\le t\le T_*} \Big(t|u_{tt}(t)|_{D_0^1} + t|u_t(t)|_{D^3}\Big) &\le c_6,\\
\operatorname*{ess\,sup}_{0\le t\le T_*} \Big(t^{\frac32}|u_{tt}(t)|_{D^2}\Big) + \int_0^{T_*} t^3|u_{ttt}(t)|_{D_0^1}^2\, dt &\le c_6,\\
\operatorname*{ess\,sup}_{0\le t\le T_*} (|\rho(t)-\rho^\infty|_{H^3} + |\rho_t(t)|_{H^2} + |\sqrt{\rho}u_t(t)|_{L^2}) + \int_0^t |\sqrt{\rho}u_{tt}(s)|_{L^2}^2\, ds &\le c_6,\\
\operatorname*{ess\,sup}_{0\le t\le T_*} \Big(t^{\frac12}|\sqrt{\rho}u_{tt}|_{L^2} + t^{\frac32}|\sqrt{\rho}u_{ttt}(t)|_{L^2}\Big) + \int_0^t s^2|\sqrt{\rho}u_{ttt}(s)|_{L^2}^2\, ds &\le c_6.
\end{aligned}$$

By virtue of these *a priori estimates*, we can prove the existence and regularity of a unique local classical solution (ρ, u) to the original nonlinear problem following exactly the same arguments as in the proof of Theorem 1.1. We omit the details. This completes the proof of Theorem 1.3.

6. Proof of Theorem 1.4

To prove Theorem 1.4, we consider the following initial boundary value problem

$$\rho_t + \operatorname{div}(\rho u) = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{6.1}$$

$$p_t + u\cdot\nabla p + \gamma p \operatorname{div} u = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{6.2}$$

$$(\rho u)_t + \operatorname{div}(\rho u\otimes u) + Lu + \nabla p = \rho f \quad \text{in} \quad (0,T)\times\Omega, \tag{6.3}$$

$$(\rho,\, p,\, u)|_{t=0} = (\rho_0,\, p_0,\, u_0) \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{6.4}$$

$$(\rho,\, p,\, u)(t,x) \to (\rho^\infty,\, p^\infty,\, 0) \quad \text{as} \quad |x|\to\infty, \quad (t,x)\in(0,T)\times\Omega, \tag{6.5}$$

where the known data ρ_0, p_0, u_0 and f satisfy

$$\begin{aligned}
&(\rho_0-\rho^\infty, p_0-p^\infty)\in H^3, \quad \rho^\infty\in\overline{\mathbf{R}}_+, \quad p^\infty\in\mathbf{R}, \quad \rho_0\ge 0 \quad \text{in} \quad \Omega,\\
&u_0\in D_0^1\cap D^3, \quad f\in L^2(0,T;H^2) \quad \text{and} \quad f_t\in L^2(0,T;L^2)
\end{aligned}\tag{6.6}$$

and

$$Lu_0 + \nabla p_0 = \rho_0\,(f(0)+g_2) \;\text{ for some }\; g_2\in D_0^1 \;\text{ with }\; \sqrt{\rho_0}\,g_2\in L^2. \tag{6.7}$$

Theorem 1.4 is an immediate corollary of the following result.

Theorem 6.1. *Assume that the data ρ_0, p_0, u_0, f satisfy* (6.6) *and* (6.7). *Then there exist a small time $T_*\in(0,T)$ and a unique strong solution (ρ,p,u) to the*

IBVP(6.1)-(6.5) *such that*

$$(\rho-\rho^\infty,\, p-p^\infty)\in C([0,T_*];H^3),\quad u\in C([0,T_*];D_0^1\cap D^3)\cap L^2(0,T_*;D^4),$$
$$u_t\in L^\infty(0,T_*;D_0^1)\cap L^2(0,T_*;D^2)\quad and\quad \sqrt{\rho}u_t\in L^\infty(0,T_*;L^2).$$

Moreover, if the external force f satisfies the additional regularity (1.13)*, then the velocity u satisfies* (1.14) *with T_* replaced by some $T_{**}\in(0,T_*]$ and so (ρ,p,u) is a classical solution of* (6.1)-(6.3) *in* $(0,T_{**})\times\Omega$.

Proof of Theorem 1.4 from Theorem 6.1. Let (ρ_0,u_0,f) be a given data satisfying the hypotheses of Theorem 1.4. Then Theorem 6.1 guarantees the existence of a unique solution (ρ,p,u) to the IBVP(6.1)-(6.5) with the initial data (ρ_0,p_0,u_0), where $p_0=A\rho_0^\gamma$ and $p^\infty=A(\rho^\infty)^\gamma$.

To prove Theorem 1.4, we have only to show that $p=A\rho^\gamma$. Let us denote $\overline{p}=p-A\rho^\gamma$. Then using (6.1), (6.2) and (6.4) together with the fact that $\gamma>1$, we deduce that

$$\begin{cases}\overline{p}_t+u\cdot\nabla\overline{p}+\gamma\overline{p}\,\mathrm{div}u=0\quad\text{in}\quad(0,T_*)\times\Omega,\\[2ex] \overline{p}(0)=0\quad\text{in}\quad\Omega,\quad \overline{p}\in C([0,T_*];H^1).\end{cases}$$

Hence by virtue of a standard energy method, we easily conclude that $\overline{p}=0$ in $(0,T_*)\times\Omega$. This completes the proof of Theorem 1.4. □

Finally, we turn to the proof of Theorem 6.1. For this purpose, we follow the same strategy as in the previous sections. Let us consider the following uncoupled linearized problem

$$\rho_t+\mathrm{div}\,(\rho v)=0\qquad\text{in}\quad(0,T)\times\Omega,\tag{6.8}$$
$$p_t+v\cdot\nabla p+\gamma p\,\mathrm{div}v=0\qquad\text{in}\quad(0,T)\times\Omega,\tag{6.9}$$
$$\rho u_t+Lu+\nabla p=\rho(f-v\cdot\nabla v)\qquad\text{in}\quad(0,T)\times\Omega,\tag{6.10}$$
$$(\rho,\,p,\,u)|_{t=0}=(\rho_0,\,p_0,\,u_0)\quad\text{in}\quad\Omega,\qquad u=0\quad\text{on}\quad(0,T)\times\partial\Omega,\tag{6.11}$$
$$(\rho,\,p,\,u)(t,x)\to(\rho^\infty,\,p^\infty,\,0)\quad\text{as}\quad|x|\to\infty,\quad(t,x)\in(0,T)\times\Omega,\tag{6.12}$$

where v is a known vector field such that

$$v\in C([0,T];D_0^1\cap D^3)\cap L^2(0,T;D^4),\quad v_t\in L^\infty(0,T;D_0^1)\cap L^2(0,T;D^2).\tag{6.13}$$

Note that the proof of Lemma 2.1 can be used without any essential change to deduce the corresponding result for the linear hyperbolic problem (6.9), (6.11) and (6.12). Hence adapting the proof of Lemma 3.1, we can prove

Lemma 6.2. *Let Ω be a bounded domain in $\mathbf{R}^3$ with smooth boundary. In addition to* (6.6)*,* (6.7) *and* (6.13)*, we assume that $\rho_0\geq\delta$ in Ω for some constant $\delta>0$. Then there exists a unique solution (ρ,p,u) to the linearized problem* (6.8)–(6.11) *such that*

$$\begin{gathered}\rho,\,p\in C([0,T];H^3),\quad \rho_t,\,p_t\in C([0,T];H^2),\\ u\in C([0,T];H_0^1\cap H^3)\cap L^2(0,T;H^4),\\ u_t\in C([0,T];H_0^1)\cap L^2(0,T;H^2),\\ u_{tt}\in L^2(0,T;L^2)\quad and\quad \rho\geq\underline{\delta}\quad on\quad[0,T]\times\overline{\Omega}.\end{gathered}\tag{6.14}$$

for some constant $\underline{\delta}>0$.

Moreover, from Lemma 2.1 and its proof, it follows that

$$|\rho(t)-\rho^\infty|_{H^3} \le (|\rho_0-\rho^\infty|_{H^3}+\rho^\infty)\exp\left(C\int_0^t |v(s)|_{D_0^1\cap D^4}\,ds\right),$$

$$|p(t)-p^\infty|_{H^3} \le (|p_0-p^\infty|_{H^3}+|p^\infty|)\exp\left(C\int_0^t |v(s)|_{D_0^1\cap D^4}\,ds\right)$$

and

$$\inf_\Omega \rho(t) \ge \left(\inf_\Omega \rho_0\right)\exp\left(-C\int_0^t |v(s)|_{D_0^1\cap D^4}\,ds\right)$$

for $0\le t\le T$. Here we denote by C a generic positive constants depending only on the fixed constants μ, λ, T, γ and the norm of f.

Hence adapting the proof of Lemma 4.1, we can also prove the key lemma.

Lemma 6.3. *Let us choose a constant $c_0>1$ so that*

$$1+\rho^\infty+|p^\infty|+|(\rho_0-\rho^\infty,\,p_0-p^\infty)|_{H^3}+|u_0|_{D_0^1}+|\sqrt{\rho}_0\,g_2|_{L^2}+|g_2|_{D_0^1}<c_0.$$

Then there exist positive constants $T_*\in(0,T)$ and c_i's, depending only on c_0 and the parameters of C, with the following property:

If v is a vector field satisfying the regularity (6.13) *with T replaced by T_* and the estimate*

$$\begin{aligned}
|v(0)|_{D_0^1\cap D^3} &\le c_1,\\
\sup_{0\le t\le T_*}|v(t)|_{D_0^1}+\int_0^{T_*}|v(t)|_{D^2}^2\,dt &\le c_2,\\
\sup_{0\le t\le T_*}|v(t)|_{D^2}+\int_0^{T_*}\left(|v_t(t)|_{D_0^1}^2+|v(t)|_{D^3}^2\right)dt &\le c_3,\\
\operatorname*{ess\,sup}_{0\le t\le T_*}\left(|v_t(t)|_{D_0^1}+|v(t)|_{D^3}\right)+\int_0^{T_*}\left(|v_t(t)|_{D^2}^2+|v(t)|_{D^4}^2\right)dt &\le c_4,
\end{aligned}$$

then there exists a unique solution (ρ,p,u) to the linearized problem (6.8)–(6.12) *satisfying the regularity*

$$\begin{aligned}
(\rho-\rho^\infty,\,p-p^\infty)\in C([0,T_*];H^3),\quad u\in C([0,T_*];D_0^1\cap D^3)\cap L^2(0,T_*;D^4),\\
u_t\in L^\infty(0,T_*;D_0^1)\cap L^2(0,T_*;D^2)\quad \text{and}\quad \sqrt{\rho}u_t\in L^\infty(0,T_*;L^2)
\end{aligned}$$

and the estimate

$$\begin{aligned}
\sup_{0\le t\le T_*}|u(t)|_{D_0^1}+\int_0^{T_*}|u(t)|_{D^2}^2\,dt &\le c_2,\\
\sup_{0\le t\le T_*}|u(t)|_{D^2}+\int_0^{T_*}\left(|u_t(t)|_{D_0^1}^2+|u(t)|_{D^3}^2\right)dt &\le c_3,\\
\operatorname*{ess\,sup}_{0\le t\le T_*}\left(|u_t(t)|_{D_0^1}+|u(t)|_{D^3}\right)+\int_0^{T_*}\left(|u_t(t)|_{D^2}^2+|u(t)|_{D^4}^2\right)dt &\le c_4,\\
\operatorname*{ess\,sup}_{0\le t\le T_*}\left(|(\rho-\rho^\infty,\,p-p^\infty)(t)|_{H^3}+|(\rho_t,p_t)(t)|_{H^2}+|\sqrt{\rho}u_t(t)|_{L^2}\right) &\le c_4.
\end{aligned}$$

The first part of Theorem 6.1 can be deduced from this key lemma following the same arguments as in the proof of Theorem 1.1. Combining this idea and the proof of Theorem 1.3, we can also prove the remaining part of the theorem. We omit its details.

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