G_2-GEOMETRY OF OVERDETERMINED SYSTEMS OF SECOND ORDER

Keizo Yamaguchi

Series #417. June 1998

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#392 T. Tsujishita and H. Watanabe, Monoidal closedness of the category of simulations, 24 pages. 1997.

#393 T. Arase, A remark on the quantale structure of multisets, 10 pages. 1997.

#394 N. H. Bingham and A. Inoue, Extension of the Drasin-Shea-Jordan theorem, 16 pages. 1997.

#395 N. H. Bingham and A. Inoue, Ratio Mercerian theorems with applications to Hankel and Fourier transforms, 30 pages. 1997.

#396 Y. Nishiura and D. Ueyama, A skeleton structure of self-replicating dynamics, 27 pages. 1997.

#397 K. Hirata and K. Sugano, On semisimple extensions of serial rings, 6 pages. 1997.

#398 D. Pei, Singularities of $\mathbb{R}P^2$-valued Gauss maps of surfaces in Minkowski 3-space, 15 pages. 1997.

#399 T. Mikami, Markov marginal problems and their applications to Markov optimal control, 28 pages. 1997.

#400 M. Tsujii, A simple proof for monotonicity of entropy in the quadratic family, 8 pages. 1998.

#401 M. Nakamura and T. Ozawa, Global solutions in the critical Sobolev space for the wave equations with nonlinearity of exponential growth, 9 pages. 1998.

#402 D. Lehmann and T. Suwa, Generalization of variations and Baum-Bott residues for holomorphic foliations on singular varieties, 19 pages. 1998.

#403 T. Nakazi and K. Okubo, ρ-contraction and 2×2 matrix, 6 pages. 1998.

#404 Y. Kohsaka, Free boundary problem for quasilinear parabolic equation with fixed angle of contact to a boundary, 28 pages. 1998.

#405 K. Yokoyama, Global existence of classical solutions to systems of wave equations with critical nonlinearity in three space dimensions, 25 pages. 1998.

#406 F. Hiroshima, Ground states and spectrum of quantum electrodynamics of non-relativistic particles, 58 pages. 1998

#407 N. Kawazumi and T. Uemura, Riemann-Hurwitz formula for Morita-Mumford classes and surface symmetries, 9 pages. 1998.

#408 T. nakazi and K. Okubo, Generalized Numerical Radius And Unitary ρ-Dilation, 12 pages. 1998.

#409 Y. Giga and K. Ito, Loss of convexity of simple closed curves moved by surface diffusion, 16 pages. 1998.

#410 Y. Giga, K. Inui and S. Matsui, On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data, 34 pages. 1998.

#411 S. Izumiya, D. Pei and T. Sano, The lightcone gauss map and the lightcone developable of a spacelike curve in Minkowski 3-space, 16 pages. 1998.

#412 M. Tsujii, Absolutely continuous invarient measures for piecewise real-analytic expanding maps on the plane, 18 pages. 1998.

#413 J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, Milnor classes of local complete intersections, 40 pages. 1998.

#414 T. Suwa, Dual class of a subvariety, 19 pages. 1998.

#415 M. Tsujii, Piecewise expanding maps on the plane with singular ergodic properties, 9 pages. 1998.

#416 T. Yoshida, Categorical aspects of generating functions(I): Exponential formulas and Krull-Schmidt categories, 44 pages. 1998.

G_2-GEOMETRY OF OVERDETERMINED SYSTEMS OF SECOND ORDER

KEIZO YAMAGUCHI

Introduction

The main theme of this paper is "Contact Geometry of Second Order". This topic has its origin in the following paper of E. Cartan.

[C1] *Les systèmes de Pfaff à cinq variables et les équations aux derivèes partielles du second ordre*, Ann. Ec. Normale, 27 (1910), 109-192

In this paper, following the tradition of geometric theory of partial differential equations of 19th century, E.Cartan dealt with the equivalence problem of two classes of second order partial differential equations in two independent variables under "contact transformations". One class consists of overdetermined systems, which are involutive, and the other class consists of single equations of Goursat type, i.e., single equations of parabolic type whose Monge characteristic systems are completely integrable. Especially in the course of the investigation, he found out the following facts: the symmetry algebras (i.e., the Lie algebra of infinitesimal contact transformations) of the following overdetermined system (involutive system) (A) and the single Goursat type equation (B) are both isomorphic with the 14-dimensional exceptional simple Lie algebra G_2.

$$\text{(A)} \qquad \frac{\partial^2 z}{\partial x^2} = \frac{1}{3}\left(\frac{\partial^2 z}{\partial y^2}\right)^3, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^2 z}{\partial y^2}\right)^2.$$

$$\text{(B)} \qquad 9r^2 + 12t^2(rt - s^2) + 32s^3 - 36rst = 0,$$

where

$$r = \frac{\partial^2 z}{\partial x^2}, \qquad s = \frac{\partial^2 z}{\partial x \partial y}, \qquad t = \frac{\partial^2 z}{\partial y^2}$$

are the classical terminology.

Our aim in this paper is to clarify the contents of "Contact Geometry of Second Order" in the course of showing how to recognize the above facts.

§1. Second Order Contact Manifolds

We will here recall the basic facts about the geometry of second order Jet spaces ([Y1], [Y3]).

1.1. Space of Contact Elements (Grassmannian Bundles). The notion of contact manifolds originates from the following space $J(M,n)$ of contact elements: Let M be a C^∞- manifold of dimension $m+n$. We put

$$J(M,n) = \bigcup_{x \in M} J_x, \qquad J_x = \mathrm{Gr}(T_x(M), n),$$

where $\mathrm{Gr}(T_x(M),n)$ denotes the Grassmann manifold consisting of n-dimensional subspaces in $T_x(M)$ (i.e. n-dimensional contact elements to M at x). $J(M,n)$ is endowed with the canonical subbundle C of $T(J(M,n))$ as follows: Let π be the projection of $J(M,n)$ onto M. Each element $u \in J(M,n)$ is a linear subspace of $T_x(M)$ of codimension m, where $x = \pi(u)$. Hence we have a subspace $C(u)$ of codimension m in $T_u(J(M,n))$ by putting

$$C(u) = \pi_*^{-1}(u) \subset T_u(J(M,n)).$$

C is called the canonical system on $J(M,n)$. We have an inhomogeneous Grassmann coordinate system of $J(M,n)$ as follows: Let us fix $u_o \in J(M,n)$ and take a coordinate system $U' : (x_1, \cdots, x_n, z^1, \cdots, z^m)$ of M around $x_o = \pi(u_o)$ such that $dx_1 \wedge \cdots \wedge dx_n \,|_{u_o} \neq 0$. Then we have the coordinate system $(x_1, \cdots, x_n, z^1, \cdots, z^m, p_1^1, \cdots, p_n^m)$ on the neighborhood $U = \{u \in \pi^{-1}(U') \mid \pi(u) = x \in U' \text{ and } dx_1 \wedge \cdots \wedge dx_n \,|_u \neq 0\}$ of u_o by

$$dz^\alpha \,|_u = \sum_{i=1}^{n} p_i^\alpha(u)\ dx_i \,|_u \qquad (\alpha = 1, \cdots, m).$$

Clearly the canonical system C is given in this coordinate system by

$$C = \{\varpi^1 = \cdots = \varpi^m = 0\},$$

where $\varpi^\alpha = dz^\alpha - \sum_{i=1}^n p_i^\alpha\, dx_i$ $(\alpha = 1, \cdots, m)$.

$(J(M,n), C)$ is the (geometric) 1-jet space and especially, in case $m = 1$, is the so-called contact manifold. Let M, $\hat{M}$ be manifolds (of dimension $m+n$) and $\varphi : M \to \hat{M}$ be a diffeomorphism between them. Then φ induces the isomorphism $\varphi_* : (J(M,n), C) \to (J(\hat{M},n), \hat{C})$, i.e., the differential map $\varphi_* : J(M,n) \to J(\hat{M},n)$ is a diffeomorphism sending C onto $\hat{C}$. The reason why the case $m = 1$ is special is explained by the following theorem of Bäcklund (cf. Theorem 1.4 [Y3]).

Theorem 1.1 (Bäcklund). *Let M and $\hat{M}$ be manifolds of dimension $m+n$. Assume $m \geqq 2$. Then, for an isomorphism $\Phi : (J(M,n), C) \to (J(\hat{M},n), \hat{C})$, there exists a diffeomorphism $\varphi : M \to \hat{M}$ such that $\Phi = \varphi_*$.*

In case $m = 1$, it is a well known fact that the group of isomorphisms of $(J(M,n),C)$, i.e., the group of contact transformations, is really larger than the group of diffeomorphisms of M. Therefore, when we consider the geometric 2-jet spaces, the situation differs according to whether the number m of unknown functions is 1 or greater. In case $m = 1$, we should start from a contact manifold (J,C) of dimension $2n+1$, which can be regarded locally as a space of 1-jets for one unknown function by Darboux's theorem. Then we can construct the geometric second order jet space $(L(J),E)$ as follows: We consider the Lagrange-Grassmann bundle $L(J)$ over J consisting of all n-dimensional integral elements of (J,C);

$$L(J) = \bigcup_{u \in J} L_u,$$

where L_u is the Grassmann manifolds of all lagrangian (or legendrian) subspaces of the symplectic vector space $(C(u), d\varpi)$. Here ϖ is a local contact form on J. Let π be the projection of $L(J)$ onto J. Then the canonical system E on $L(J)$ is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \qquad \text{at} \quad v \in L(J).$$

Starting from a canonical coordinate system $(x_1, \cdots, x_n, z, p_1, \cdots, p_n)$ of the contact manifold (J,C), we can introduce a coordinate system (x_i, z, p_i, p_{ij}) $(1 \leqq i \leqq j \leqq n)$ of $L(J)$ such that $p_{ij} = p_{ji}$ and E is defined by

$$E = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\},$$

where $\varpi = dz - \sum_{i=1}^n p_i\, dx_i$, $\varpi_i = dp_i - \sum_{j=1}^n p_{ij}\, dx_j$ $(i = 1, \cdots, n)$. Let (J,C), $(\hat{J},\hat{C})$ be contact manifolds of dimension $2n+1$ and $\varphi : (J,C) \to (\hat{J},\hat{C})$ be a contact diffeomorphism between them. Then φ induces an isomorphism $\varphi_* : (L(J),E) \to (L(\hat{J}),\hat{E})$. Conversely we have (cf. Theorem 3.2 [Y1])

Theorem 1.2. *Let (J,C) and $(\hat{J},\hat{C})$ be contact manifolds of dimension $2n+1$. Then, for an isomorphism $\Phi : (L(J),E) \to (L(\hat{J}),\hat{E})$, there exists a contact diffeomorphism $\varphi : (J,C) \to (\hat{J},\hat{C})$ such that $\Phi = \varphi_*$.*

Our first aim is to formulate the submanifold theory for $(L(J),E)$, which will be given in §4.

1.2. Realization Lemma. We here recall the following Realization Lemma for the Grassmannian construction, which plays the basic role in the discussions of §4 and §5.

Lemma 1.3 (Realization Lemma). *Let R and M be manifolds. Assume that the quadruple (R,D,p,M) satisfies the following conditions :*

(1) *p is a map of R into M of constant rank.*

(2) *D is a differential system on R such that $F =$ Ker p_* is a subbundle of D of codimension r.*

Then there exists a unique map ψ of R into $J(M, r)$ satisfying $p = \pi \cdot \psi$ and $D = \psi_*^{-1}(C)$, where C is the canonical differential system on $J(M, r)$ and $\pi : J(M, r) \to M$ is the projection. Furthermore, let v be any point of R. Then ψ is in fact defined by

$$\psi(v) = p_*(D(v)) \qquad \text{as a point of } Gr\ (T_{p(v)}(M)),$$

and satisfies

$$Ker\ (\psi_*)_v = F(v) \cap Ch(D)(v).$$

where $Ch(D)$ is the Cauchy characteristic system of D (see §2.1 below).

For the proof, see Lemma 1.5 [Y1].

§2. Geometry of Linear Differential Systems (Tanaka Theory)

We will recall here the Tanaka theory for linear differential systems following [T1] and [T2].

2.1. Derived Systems and Characteristic Systems. By a differential system (M, D), we mean a subbundle D of the tangent bundle $T(M)$ of a manifold M of dimension d. Locally D is defined by 1-forms $\omega_1, \dots, \omega_{d-r}$ such that $\omega_1 \wedge \cdots \wedge \omega_{d-r} \neq 0$ at each point, where r is the rank of D;

$$D = \{\, \omega_1 = \cdots = \omega_{d-r} = 0 \,\}.$$

For two differential systems (M, D) and $(\hat{M}, \hat{D})$, a diffeomorphism φ of M onto $\hat{M}$ is called an isomorphism of (M, D) onto $(\hat{M}, \hat{D})$ if the differential map φ_* of φ sends D onto $\hat{D}$.

By the Frobenius theorem, we know that D is completely integrable if and only if

$$d\omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_s} \qquad \text{for } i = 1, \dots, s,$$

or equivalently, if and only if

$$[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}.$$

where $s = d - r$ and $\mathcal{D} = \Gamma(D)$ denotes the space of sections of D.

Thus, for a non-integrable differential system D, we are led to consider the *derived system* ∂D of D, which is defined, in terms of sections, by

$$\partial \mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

Furthermore the *Cauchy characteristic system* Ch(D) of (M, D) is defined at each point $x \in M$ by

$$\begin{aligned}\mathrm{Ch}\,(D)(x) = \{X \in D(x) \mid \\ X \rfloor d\omega_i \equiv 0 \quad (\mathrm{mod}\ \omega_1, \dots, \omega_s) \quad \text{for } i = 1, \dots, s\ \},\end{aligned}$$

When Ch(D) is a differential system (i.e., has constant rank), it is always completely integrable (cf. [Y1]).

Moreover higher derived systems $\partial^k D$ are usually defined successively (cf. $[BCG_3]$) by

$$\partial^k D = \partial(\partial^{k-1} D),$$

where we put $\partial^0 D = D$ for convention.

On the other hand we define the k-th weak derived system $\partial^{(k)} D$ of D inductively by

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}],$$

where $\partial^{(0)} D = D$ and $\partial^{(k)}\mathcal{D}$ denotes the space of sections of $\partial^{(k)} D$. This notion is one of the key point in the Tanaka theory ([T1]).

A differential system (M, D) is called regular, if $D^{-(k+1)} = \partial^{(k)} D$ are subbundles of $T(M)$ for every integer $k \geqq 1$. For a regular differential system (M, D), we have ([T2], Proposition 1.1)

$(S1)$ *There exists a unique integer $\mu > 0$ such that, for all $k \geqq \mu$,*

$$D^{-k} = \cdots = D^{-\mu} \supsetneqq D^{-\mu+1} \supsetneqq \cdots \supsetneqq D^{-2} \supsetneqq D^{-1} = D,$$

$(S2)$ $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$ *for all* $p, q < 0$.

where $\mathcal{D}^p$ denotes the space of sections of D^p. $(S2)$ can be checked easily by induction on q.

Thus $D^{-\mu}$ is the smallest completely integrable differential system, which contains $D = D^{-1}$.

2.2. Symbol Algebras. Let (M, D) be a regular differential system such that $T(M) = D^{-\mu}$. As a first invariant for non-integrable differential systems, we now define the *graded algebra* $\mathfrak{m}(x)$ *associated with a differential system* (M, D) at $x \in M$, which was introduced by N. Tanaka [T2].

We put $\mathfrak{g}_{-1}(x) = D^{-1}(x)$, $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$ $(p < -1)$ and

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

Let ϖ_p be the projection of $D^p(x)$ onto $\mathfrak{g}_p(x)$. Then, for $X \in \mathfrak{g}_p(x)$ and $Y \in \mathfrak{g}_q(x)$, the bracket product $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is defined by

$$[X, Y] = \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x),$$

where $\tilde{X}$ and $\tilde{Y}$ are any element of $\mathcal{D}^p$ and $\mathcal{D}^q$ respectively such that $\varpi_p(\tilde{X}_x) = X$ and $\varpi_q(\tilde{Y}_x) = Y$.

Endowed with this bracket operation, by $(S2)$ above, $\mathfrak{m}(x)$ becomes a nilpotent graded Lie algebra such that $\dim \mathfrak{m}(x) = \dim M$ and satisfies

$$\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)] \qquad \text{for } p < -1.$$

We call $\mathfrak{m}(x)$ the *symbol algebra of* (M, D) at $x \in M$ for short.

Furthermore, let $\mathfrak{m}$ be a FGLA (fundamental graded Lie algebra) of μ-th kind, that is,

$$\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$$

is a nilpotent graded Lie algebra such that

$$\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \qquad \text{for } p < -1.$$

Then (M, D) is called of type $\mathfrak{m}$ if the symbol algebra $\mathfrak{m}(x)$ is isomorphic with $\mathfrak{m}$ at each $x \in M$.

Conversely, given a FGLA $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$, we can construct a model differential system of type $\mathfrak{m}$ as follows: Let $M(\mathfrak{m})$ be the simply connected Lie group with Lie algebra $\mathfrak{m}$. Identifying $\mathfrak{m}$ with the Lie algebra of left invariant vector fields on $M(\mathfrak{m})$, $\mathfrak{g}_{-1}$ defines a left invariant subbundle $D_{\mathfrak{m}}$ of $T(M(\mathfrak{m}))$. By definition of symbol algebras, it is easy to see that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is a regular differential system of type $\mathfrak{m}$. $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is called the standard differential system of type $\mathfrak{m}$. The Lie algebra $\mathfrak{g}(\mathfrak{m})$ of all infinitesimal automorphisms of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ can be calculated algebraically as the prolongation of $\mathfrak{m}$ ([T1], cf. [Y5]). We will discuss in §3 when does $\mathfrak{g}(\mathfrak{m})$ become finite dimensional and simple ?

As an example to calculate symbol algebras, let us show that $(L(J), E)$ is a regular differential system of type $\mathfrak{c}^2(n)$:

$$\mathfrak{c}^2(n) = \mathfrak{c}_{-3} \oplus \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1},$$

where $\mathfrak{c}_{-3} = \mathbb{R}$, $\mathfrak{c}_{-2} = V^*$ and $\mathfrak{c}_{-1} = V \oplus S^2(V^*)$. Here V is a vector space of dimension n and the bracket product of $\mathfrak{c}^2(n)$ is defined accordingly through the pairing between V and V^* such that V and $S^2(V^*)$ are both abelian subspaces of $\mathfrak{c}_{-1}$. This fact can be checked as follows: Let us take a canonical coordinate system $U; (x_i, z, p_i, p_{ij})$ $(1 \leqq i \leqq j \leqq n)$ of $(L(J), E)$. Then we have a coframe $\{\varpi, \varpi_i, dx_i, dp_{ij}\}$ $(1 \leqq i \leqq j \leqq n)$ at each point in U, where $\varpi = dz - \sum_{i=1}^n p_i\, dx_i$, $\varpi_i = dp_i - \sum_{j=1}^n p_{ij}\, dx_j$ $(i = 1, \cdots, n)$. Now take the dual frame $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial p_i}, \frac{d}{dx_i}, \frac{\partial}{\partial p_{ij}}\}$, of this coframe, where

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial z} + \sum_{j=1}^n p_{ij} \frac{\partial}{\partial p_j}$$

is the classical notation. Notice that $\{\frac{d}{dx_i}, \frac{\partial}{\partial p_{ij}}\}$ $(i = 1, \cdots, n)$ forms a free basis of $\Gamma(E)$. Then an easy calculation shows the above fact. Moreover we see that the derived system ∂E of E satisfies the following :

$$\partial E = \{\varpi = 0\} = \pi_*^{-1} C, \qquad \mathrm{Ch}\,(\partial E) = \mathrm{Ker}\, \pi_*.$$

These are the key facts to Theorem 1.2 (cf. Theorem 3.2 [Y1]).

Similarly we see that $(J(M,n), C)$ is a regular differential system of type $\mathfrak{c}^1(n,m)$:

$$\mathfrak{c}^1(n,m) = \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1},$$

where $\mathfrak{c}_{-2} = W$ and $\mathfrak{c}_{-1} = V \oplus W \otimes V^*$ for vector spaces V and W of dimension n and m respectively, and the bracket product of $\mathfrak{c}^1(n,m)$ is defined accordingly through the pairing between V and V^* such that V and $W \otimes V^*$ are both abelian subspaces of $\mathfrak{c}_{-1}$.

2.3. Classification of Symbol Algebras of Lower Dimensions. In this paragraph, following a short passage from Cartan's paper [C1], let us classify FGLAs $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$ such that $\dim \mathfrak{m} \leqq 5$, which gives us the first invariants towards the classification of regular differential system (M, D) such that $\dim M \leqq 5$.

In the case $\dim \mathfrak{m} = 1$ or 2, $\mathfrak{m} = \mathfrak{g}_{-1}$ should be abelian. To discuss the case $\dim \mathfrak{m} \geqq 3$, we further assume that $\mathfrak{g}_{-1}$ is nondegenerate, i.e., $[X, \mathfrak{g}_{-1}] = 0$ implies $X = 0$ for $X \in \mathfrak{g}_{-1}$. This condition is equivalent to say $\mathrm{Ch}\,(D) = \{0\}$ for regular differential system (M, D) of type $\mathfrak{m}$. When $\mathfrak{g}_{-1}$ is degenerate, $\mathrm{Ch}\,(D)$ is non-trivial, hence at least locally, (M, D) induces a regular differential system (X, D^*) on the lower dimensional space X, where $X = M/\mathrm{Ch}\,(D)$ is the leaf space of the foliation on M defined by $\mathrm{Ch}\,(D)$ and D^* is the differential system on X such that $D = p_*^{-1}(D^*)$. Here $p : M \to X = M/\mathrm{Ch}\,(D)$ is the projection. Moreover, for the following discussion, we first observe that the dimension of $\mathfrak{g}_{-2}$ does not exceed $\binom{m}{2}$, where $m = \dim \mathfrak{g}_{-1}$.

In the case $\dim \mathfrak{m} = 3$, we have $\mu \leqq 2$. When $\mu = 2$, $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the contact gradation, i.e., $\dim \mathfrak{g}_{-2} = 1$ and $\mathfrak{g}_{-1}$ is nondegenerate. In the case $\dim \mathfrak{m} = 4$, we see that $\mathfrak{g}_{-1}$ is degenerate when $\mu \leqq 2$. When $\mu = 3$, we have $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-2} = 1$ and $\dim \mathfrak{g}_{-1} = 2$. Moreover it follows that $\mathfrak{m}$ is isomorphic with $\mathfrak{c}^2(1)$ in this case. In the case $\dim \mathfrak{m} = 5$, we have $\dim \mathfrak{g}_{-1} = 4$, 3 or 2. When $\dim \mathfrak{g}_{-1} = 4$, $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the contact gradation. When $\dim \mathfrak{g}_{-1} = 3$, $\mathfrak{g}_{-1}$ is degenerate if $\dim \mathfrak{g}_{-2} = 1$, which implies that $\mu = 2$ and $\dim \mathfrak{g}_{-2} = 2$ in this case. Moreover, when $\mu = 2$, it follows that $\mathfrak{m}$ is isomorphic with $\mathfrak{c}^1(1,2)$. When $\dim \mathfrak{g}_{-1} = 2$, we have $\dim \mathfrak{g}_{-2} = 1$ and $\mu = 3$ or 4. Moreover, when $\mu = 4$, it follows that $\mathfrak{m}$ is isomorphic with $\mathfrak{c}^3(1)$, where $\mathfrak{c}^3(1)$ is the symbol algebra of the canonical system on the third order jet spaces for 1 unknown function (cf. §3 [Y1]).

Summarizing the above discussion, we obtain the following classification of the FGLAs $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$ such that $\dim \mathfrak{m} \leqq 5$ and $\mathfrak{g}_{-1}$ is nondegenerate.

(1) $\dim \mathfrak{m} = 3 \Longrightarrow \mu = 2$

$$\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{c}^1(1) : \text{contact gradation}$$

(2) $\dim \mathfrak{m} = 4 \Longrightarrow \mu = 3$

$$\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{c}^2(1)$$

(3) $\dim \mathfrak{m} = 5$, then $\mu \leqq 4$

(a) $\mu = 4$ $\quad \mathfrak{m} = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{c}^3(1)$

(b) $\mu = 3$ $\quad \mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$
such that $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-1} = 2$ and $\dim \mathfrak{g}_{-2} = 1$

(c) $\mu = 2$ $\quad \mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{c}^1(1,2)$

(d) $\mu = 2$ $\quad \mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{c}^1(2)$: contact gradation

A notable and rather misleading fact is that, once the dimensions of $\mathfrak{g}_p$ are fixed, the Lie algebra structure of $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$ is unique in the above classification list. Moreover, except for the cases (b) and (c), every regular differential system (M, D) of type $\mathfrak{m}$ in the above list is isomorphic with the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$of type $\mathfrak{m}$ by Darboux's theorem (cf. Corollary 6.6 [Y1]). The first non-trivial situation that cannot be analyzed on the basis of Darboux's theorem occurs in the cases (b) and (c) (see [C1], [St]). Regular differential systems of type (b) and (c) are mutually closely related to each other (cf. §6.3 and [C1]). We shall encounter with the type (b) fundamental graded Lie algebra in §6.2 in connection with the root space decomposition of the exceptional simple Lie algebra G_2.

§3. Differential Systems associated with SGLAs

We will classify here the Standard differential systems $(M(\mathfrak{m}), D_{\mathfrak{m}})$ for which the prolongation $\mathfrak{g}(\mathfrak{m})$ becomes finite dimensional and simple ([Y5]). In this section we will solely consider Lie algebras over $\mathbb{C}$ for the sake of simplicity.

3.1. Classification of Gradation of Simple Lie Algebras by Root Systems. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$. Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and choose a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of the root system Φ of $\mathfrak{g}$ relative to $\mathfrak{h}$. Then every $\alpha \in \Phi$ is an (all non-negative or all non-positive) integer coefficient linear combination of elements of Δ and we have the root space decomposition of $\mathfrak{g}$;

$$\mathfrak{g} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha},$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [h, X] = \alpha(h)X \quad \text{for } h \in \mathfrak{h}\}$ is (1-dimensional) root space (corresponding to $\alpha \in \Phi$) and Φ^+ denotes the set of positive roots.

Now let us take a nonempty subset Δ_1 of Δ. Then Δ_1 defines the partition of Φ^+ as in the following and induces the gradation of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ as follows:

$$\Phi^+ = \cup_{p\geqq 0}\Phi_p^+, \qquad \Phi_p^+ = \{\alpha = \sum_{i=1}^{\ell} n_i\alpha_i \mid \sum_{\alpha_i\in\Delta_1} n_i = p\},$$

$$\mathfrak{g}_p = \bigoplus_{\alpha\in\Phi_p^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_0 = \bigoplus_{\alpha\in\Phi_0^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi_0^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{g}_{-p} = \bigoplus_{\alpha\in\Phi_p^+} \mathfrak{g}_{-\alpha},$$

$$[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} \qquad \text{for} \quad p, q \in \mathbb{Z}.$$

Moreover the negative part $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ satisfies the following generating condition :

$$\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \quad \text{for} \quad p < -1$$

We denote the SGLA (simple graded Lie algebra) $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ obtained from Δ_1 in this manner by (X_ℓ, Δ_1), when $\mathfrak{g}$ is a simple Lie algebra of type X_ℓ. Here X_ℓ stands for the Dynkin diagram of $\mathfrak{g}$ representing Δ and Δ_1 is a subset of vertices of X_ℓ. Moreover we have

$$\mu = \sum_{\alpha_i\in\Delta_1} n_i(\theta),$$

where $\theta = \sum_{i=1}^{\ell} n_i(\theta)\,\alpha_i$ is the highest root of Φ^+.

Conversely we have (Theorem 3.12 [Y5])

Theorem 3.1. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ satisfying the generating condition. Let X_ℓ be the Dynkin diagram of $\mathfrak{g}$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with a graded Lie algebra (X_ℓ, Δ_1) for some $\Delta_1 \subset \Delta$. Moreover (X_ℓ, Δ_1) and (X_ℓ, Δ_1') are isomorphic if and only if there exists a diagram automorphism ϕ of X_ℓ such that $\phi(\Delta_1) = \Delta_1'$.*

In the real case, we can utilize the Satake diagram of $\mathfrak{g}$ to describe gradations of $\mathfrak{g}$ (Theorem 3.12 [Y5]).

3.2. Differential Systems associated with SGLAs. By Theorem 3.1, the classification of gradations $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of simple Lie algebras $\mathfrak{g}$ satisfying the generating condition coincides with that of parabolic subalgebras $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ of $\mathfrak{g}$. Accordingly, to each SGLA (X_ℓ, Δ_1), there corresponds a unique R-space $M_{\mathfrak{g}} = G/G'$ (compact simply connected homogeneous complex manifold). Furthermore, when $\mu \geqq 2$, there exists the G-invariant differential system $D_{\mathfrak{g}}$ on $M_{\mathfrak{g}}$, which is induced from $\mathfrak{g}_{-1}$, and $(M(\mathfrak{m}), D_{\mathfrak{m}})$ (Standard differential system of type $\mathfrak{m}$) becomes an open submanifold of $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$. For the Lie algebras of all infinitesimal automorphisms of $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$, hence of $(M(\mathfrak{m}), D_{\mathfrak{m}})$, we have the following theorem (Theorem 5.2 [Y5]).

Theorem 3.2. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ satisfying the generating condition. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ except for the following three cases.*

(1) $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *is of depth* 1 *(i.e.,* $\mu = 1$*).*

(2) $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ *is a (complex) contact gradation.*

(3) $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic with* $(A_\ell, \{\alpha_1, \alpha_i\})$ $(1 < i < \ell)$ *or* $(C_\ell, \{\alpha_1, \alpha_\ell\})$.

Here R-spaces corresponding to the above exceptions (1), (2) and (3) are as follows: (1) correspond to compact irreducible hermitian symmetric spaces. (2) correspond to contact manifolds of Boothby type (Standard contact manifolds), which exist uniquely for each simple Lie algebra other than $\mathfrak{sl}(2, \mathbb{C})$(see §5.1 below). In case of (3), $(J(\mathbb{P}^\ell, i), C)$ corresponds to $(A_\ell, \{\alpha_1, \alpha_i\})$ and $(L(\mathbb{P}^{2\ell-1}), E)$ corresponds to $(C_\ell, \{\alpha_1, \alpha_\ell\})$ $(1 < i < \ell)$, where $\mathbb{P}^\ell$ denotes the ℓ-dimensional complex projective space and $\mathbb{P}^{2\ell-1}$ is the Standard contact manifold of type C_ℓ. Here we note that R-spaces corresponding to (2) and (3) are all Jet spaces of the first or second order.

For the real version of this theorem, we refer the reader to Theorem 5.3 [Y5].

§4. Geometry of PD-manifolds

We will here formulate the submanifold theory for $(L(J), E)$ as the geometry of PD-manifolds ([Y1]).

4.1. ***PD*-manifolds.** Let R be a submanifold of $L(J)$ satisfying the following condition:

$(R.0)$ $\quad p : R \to J$; submersion,

where $p = \pi \mid_R$ and $\pi : L(J) \to J$ is the projection. There are two differential systems $C^1 = \partial E$ and $C^2 = E$ on $L(J)$. We denote by D^1 and D^2 those differential systems on R obtained by restricting these differential systems to R. Moreover we denote by the same symbols those

1-forms obtained by restricting the defining 1-forms $\{\varpi, \varpi_1, \cdots, \varpi_n\}$ of the canonical system E to R. Then it follows from $(R.0)$ that these 1-forms are independent at each point on R and that

$$D^1 = \{\varpi = 0\}, \qquad D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}.$$

In fact $(R; D^1, D^2)$ further satisfies the following conditions:

$(R.1)$ *D^1 and D^2 are differential systems of codimension* 1 *and* $n+1$ *respectively.*

$(R.2)$ $\partial D^2 \subset D^1$.

$(R.3)$ $\mathrm{Ch}\,(D^1)$ *is a subbundle of D^2 of codimension n.*

$(R.4)$ $\mathrm{Ch}\,(D^1)(v) \cap \mathrm{Ch}\,(D^2)(v) = \{0\}$ *at each $v \in R$.*

Conversely these four conditions characterize submanifolds in $L(J)$ satisfying $(R.0)$. In fact we call the triplet $(R; D^1, D^2)$ of a manifold and two differential systems on it a **PD-manifold** if these satisfy the above four conditions $(R.1)$ to $(R.4)$. We have the (local) Realization Theorem for PD-manifolds as follows: From conditions $(R.1)$ and $(R.3)$, it follows that the codimension of the foliation defined by the completely integrable system $\mathrm{Ch}\,(D^1)$ is $2n+1$. Assume that R is regular with respect to $\mathrm{Ch}\,(D^1)$, i.e., the space $J = R/\mathrm{Ch}\,(D^1)$ of leaves of this foliation is a manifold of dimension $2n+1$. Then D^1 drops down to J. Namely there exists a differential system C on J of codimension 1 such that $D^1 = p_*^{-1}(C)$, where $p : R \to J = R/\mathrm{Ch}\,(D^1)$ is the projection. Obviously (J, C) becomes a contact manifold of dimension $2n+1$. Conditions $(R.1)$ and $(R.2)$ guarantees that the image of the following map ι is a legendrian subspace of (J, C):

$$\iota(v) = p_*(D^2(v)) \subset C(u), \qquad u = p(v).$$

Finally the condition $(R.4)$ shows that $\iota : R \to L(J)$ is an immersion by Realization Lemma for (R, D^2, p, J) (see §1.2). Furthermore we have (Corollary 5.4 [Y1])

Theorem 4.1. *Let $(R; D^1, D^2)$ and $(\hat{R}; \hat{D}^1, \hat{D}^2)$ be PD-manifolds. Assume that R and $\hat{R}$ are regular with respect to $Ch\,(D^1)$ and $Ch\,(\hat{D}^1)$ respectively. Let (J, C) and $(\hat{J}, \hat{C})$ be the associated contact manifolds. Then an isomorphism $\Phi : (R; D^1, D^2) \to (\hat{R}; \hat{D}^1, \hat{D}^2)$ induces a contact diffeomorphism $\varphi : (J, C) \to (\hat{J}, \hat{C})$ such that the following commutes;*

$$\begin{array}{ccc} R & \xrightarrow{\ \iota\ } & L(J) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\varphi_*} \\ \hat{R} & \xrightarrow{\ \hat{\iota}\ } & L(\hat{J}). \end{array}$$

By this theorem, the submanifold theory for $(L(J), E)$ is reformulated as the geometry of PD-manifolds.

When $D^1 = \partial D^2$ holds for a PD-manifold $(R; D^1, D^2)$, the geometry of $(R; D^1, D^2)$ reduces to that of (R, D^2) and the Tanaka theory is directly applicable to this case. Concerning about this situation, the following theorem is known under the compatibility condition (C) below:

(C) $\quad p^{(1)} : R^{(1)} \to R$ *is onto.*

where $R^{(1)}$ is the first prolongation of $(R; D^1, D^2)$ (cf. Proposition 5.11 [Y1]).

Theorem 4.2. *Let $(R; D^1, D^2)$ be a PD-manifold satisfying the condition (C) above. Then the following equality holds at each point v of R:*

$$\dim D^1(v) - \dim \partial D^2(v) = \dim Ch(D^2)(v).$$

In particular $D^1 = \partial D^2$ holds if and only if $Ch(D^2) = \{0\}$.

4.2. First Reduction Theorem. When PD-manifold $(R; D^1, D^2)$ admits a non-trivial Cauchy characteristics, i.e.,when rank $\mathrm{Ch}(D^2) > 0$, the geometry of $(R; D^1, D^2)$ is further reducible to the geometry of single differential systems. Here we will be concerned with the local equivalence of $(R; D^1, D^2)$, hence we may assume that R is regular with respect to $\mathrm{Ch}(D^2)$, i.e., the leaf space $X = R/\mathrm{Ch}(D^2)$ is a manifold such that the projection $\rho : R \to X$ is a submersion and there exists a differential system D on X satisfying $D^2 = \rho_*^{-1}(D)$. Then the local equivalence of $(R; D^1, D^2)$ is further reducible to that of (X, D) as in the following : We assume that $(R; D^1, D^2)$ satisfies the condition (C) above and $\mathrm{Ch}(D^2)$ is a subbundle of rank r $(0 < r < n)$. Then, by Theorem 4.2, ∂D^2 is a subbundle of D^1 of codimension r. From (X, D), at least locally, we can construct a PD-manifold $(R(X); D^1_X, D^2_X)$ as follows. $R(X)$ is the collection of hyperplanes v in each tangent space $T_x(X)$ at $x \in X$ which contains the fibre $\partial D(x)$ of the derived system ∂D of D.

$$R(X) = \bigcup_{x \in X} R_x \subset J(X, m-1),$$
$$R_x = \{v \in \mathrm{Gr}(T_x(X), m-1) \mid v \supset \partial D(x)\},$$

where $m = \dim X$. Moreover D^1_X is the canonical system obtained by the Grassmaniann construction and D^2_X is the lift of D. Precisely, D^1_X and D^2_X are given by

$$D^1_X(v) = \nu_*^{-1}(v) \supset D^2_X(v) = \nu_*^{-1}(D(x)),$$

for each $v \in R(X)$ and $x = \nu(v)$, where $\nu : R(X) \to X$ is the projection. Then we have a map κ of R into $R(X)$ given by

$$\kappa(v) = \rho_*(D^1(v)) \subset T_x(X),$$

for each $v \in R$ and $x = \rho(v)$. By Realization Lemma for (R, D^1, ρ, X), κ is a map of constant rank such that

$$\text{Ker } \kappa_* = \text{Ch}(D^1) \cap \text{Ker}\, \rho_* = \text{Ch}(D^1) \cap \text{Ch}(D^2) = \{0\}.$$

Thus κ is an immersion and, by a dimension count, in fact, a local diffeomorphism of R into $R(X)$ such that

$$\kappa_*(D^1) = D^1_X \quad \text{and} \quad \kappa_*(D^2) = D^2_X.$$

Namely $\kappa : (R, D^1, D^2) \to (R(X), D^1_X, D^2_X)$ is a local isomorphism of PD-manifolds. (Precisely, in general, $(R(X), D^1_X, D^2_X)$ becomes a PD-manifold on an open subset.)

Summarizing the above consideration, we obtain the following Reduction Theorem for PD-manifolds admitting non-trivial Cauchy characteristics.

Theorem 4.3. *Let (R, D^1, D^2) and $(\hat{R}; \hat{D}^1, \hat{D}^2)$ be PD-manifolds satisfying the condition (C) such that $Ch(D^2)$ and $Ch(\hat{D}^2)$ are subbundles of rank r $(0 < r < n)$. Assume that R and $\hat{R}$ are regular with respect to $Ch(D^2)$ and $Ch(\hat{D}^2)$ respectively. Let (X, D) and $(\hat{X}, \hat{D})$ be the leaf spaces, where $X = R/Ch(D^2)$ and $\hat{X} = \hat{R}/Ch(\hat{D}^2)$. Let us fix points $v_o \in R$ and $\hat{v}_o \in \hat{R}$ and put $x_o = \rho(v_o)$ and $\hat{x}_o = \hat{\rho}(\hat{v}_o)$. Then a local isomorphism $\psi : (R; D^1, D^2) \to (\hat{R}; \hat{D}^1, \hat{D}^2)$ such that $\psi(v_o) = \hat{v}_o$ induces a local isomorphism $\varphi : (X, D) \to (\hat{X}, \hat{D})$ such that $\varphi(x_o) = \hat{x}_o$ and $\varphi_*(\kappa(x_o)) = \hat{\kappa}(\hat{x}_o)$, and vice versa.*

The involutive system (A) in Introduction is the example of this situation and we have $\dim X = 5$ and rank $D = 2$.

§5. Contact Geometry of Single Equations of Goursat Type

In order to discuss the generalization of the equation (B) in the introduction, we will define single equations of Goursat type and formulate the Reduction Theorems for the contact equivalence of this type of equations.

5.1. Single Equations of Goursat Type. By a single equation (of second order), we mean a hypersurface R of $L(J)$ satisfying the condition $(R.0)$ in §4. Then, by the Cauchy-Kowalevsky theorem, we see that R also satisfies the compatibility condition (C) and the symbol algebra $\mathfrak{s}(v)$ of (R, D^2) at $v \in R$ is a subalgebra of $\mathfrak{c}^2(n)$ such that

$$\mathfrak{s}(v) = \mathfrak{s}_{-3}(v) \oplus \mathfrak{s}_{-2}(v) \oplus \mathfrak{s}_{-1}(v)$$

where $\mathfrak{s}_{-3}(v) = \mathbb{R}$, $\mathfrak{s}_{-2}(v) = V^*$, $\mathfrak{s}_{-1}(v) = V \oplus \mathfrak{f}(v)$ and $\mathfrak{f}(v)$ is a subspace of $S^2(V^*)$ of codimension 1. Let $(\mathfrak{f}(v))^\perp$ be the annihilator of $\mathfrak{f}(v)$ in $S^2(V)$

under the pairing between $S^2(V)$ and $S^2(V^*)$. Then dim $(\mathfrak{f}(v))^\perp = 1$. We say that R is of (weak) parabolic type at v if $(\mathfrak{f}(v))^\perp$ is generated by a symmetric two form of rank 1. When R is defined in a canonical coordinate (x_i, z, p_i, p_{ij}) $(1 \leqq i \leqq j \leqq n)$ by

$$F(x_i, z, p_i, p_{ij}) = 0,$$

then the above condition is equivalent to say that the symmetric matrix $(\frac{\partial F}{\partial p_{ij}}(v))$ has rank 1 (cf. §3.3 [Y1]).

When R is of (weak) parabolic type at each point, (R, D^2) is a regular differential system of type $\mathfrak{s}$:

$$\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1},$$

where $\mathfrak{s}_{-3} = \mathbb{R}$, $\mathfrak{s}_{-2} = V^*$, $\mathfrak{s}_{-1} = V \oplus \mathfrak{f}$ and $\mathfrak{f} \subset S^2(V^*)$ is given by $(\mathfrak{f})^\perp = \langle e^2 \rangle \subset S^2(V)$ for a non-zero vector $e \in V$.

Let $A(\mathfrak{s})$ be the group of graded Lie algebra automorphisms of $\mathfrak{s}$ and E be the 1-dimensional subspace of V spanned by e. Then the annihilator subspace $E^\perp$ of E is an $A(\mathfrak{s})$-invariant subspace of $V^* = \mathfrak{s}_{-2}$. Starting from the 1-dimensional subspace $E = \langle e \rangle$ of V, we can construct the first order covariant system $N(E)$ and the Monge characteristic system $M(E)$ as in the following (For the detail see, §7.3 [Y1]): Let v be any point of R and let $\mathfrak{s}(v)$ be the symbol algebra at v. Take a graded Lie algebra isomorphism ϕ of $\mathfrak{s}(v)$ onto $\mathfrak{s}$. Let $\mathfrak{n}(E)(v)$ denote the linear subspace of $\mathfrak{s}_{-2}(v)$ defined by

$$\mathfrak{n}(E)(v) = \phi^{-1}(E^\perp).$$

Then, since $E^\perp$ is $A(\mathfrak{s})$-invariant, it follows that $\mathfrak{n}(E)(v)$ is well-defined. Let κ_{-2} be the projection of $D^1(v)$ onto $\mathfrak{s}_{-2}(v) = D^1(v)/D^2(v)$. We define the linear subspace $N(E)(v)$ of $D^1(v)$ by setting

$$N(E)(v) = (\kappa_{-2})^{-1}(\mathfrak{n}(E)(v)).$$

Then it follows that the assignment $v \mapsto N(E)(v)$ defines a subbundle $N(E)$ of D^1.

Let $\mathfrak{m}(E)$ denote the linear subspace of $\mathfrak{s}_{-1}$ spanned by linear subspaces $\phi(E)$, $\phi \in A(\mathfrak{s})$, i.e.,

$$\mathfrak{m}(E) = \langle \{\phi(E) \subset \mathfrak{s}_{-1} \mid \phi \in A(\mathfrak{s})\} \rangle.$$

$\mathfrak{m}(E)$ is an $A(\mathfrak{s})$-invariant subspace of $\mathfrak{s}_{-1}$ by construction. Taking a graded Lie algebra isomorphism ϕ of $\mathfrak{s}(v)$ onto $\mathfrak{s}$, let $M(E)(v)$ denotes the linear subspace of $\mathfrak{s}_{-1}(v) = D^2(v)$ defined by

$$M(E)(v) = \phi^{-1}(\mathfrak{m}(E)).$$

It follows that the assignment $v \mapsto M(E)(v)$ defines a subbundle $M(E)$ of D^2. $M(E)(v)$ is the linear subspace of $D^2(v)$ spanned by the Monge characteristic elements corresponding to E.

We say that R is a (single) equation of Goursat type when R is of (weak) parabolic type and its Monge characteristic system $M(E)$ is completely integrable.

Now let us describe the covariant systems $N = N(E)$ and $M = M(E)$ of (R, D^2) in terms of adapted coframes (cf. [Y4]). Let R be a single equation of (weak) parabolic type, i.e., (R, D^2) be a regular differential system of type $\mathfrak{s}$. Let v be any point of R. A coframe, i.e.,a base of 1-forms $\{\varpi, \varpi_a, \omega_a, \varpi_{1\alpha}, \varpi_{\alpha\beta}\}$ $(1 \leqq a \leqq n, 2 \leqq \alpha \leqq \beta \leqq n)$ on a neighborhood U of v in R is called an adapted coframe if it satisfies the following conditions (5.1) and (5.2) :

$$D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}, \tag{5.1}$$

$$\begin{cases} d\varpi \equiv \omega_1 \wedge \varpi_1 + \cdots\cdots + \omega_n \wedge \varpi_n & (\mathrm{mod}\ \varpi), \\ d\varpi_1 \equiv \omega_2 \wedge \varpi_{12} + \cdots + \omega_n \wedge \varpi_{1n} & (\mathrm{mod}\ \varpi, \varpi_1, \cdots, \varpi_n), \\ d\varpi_\alpha \equiv \omega_1 \wedge \varpi_{\alpha 1} + \cdots\cdots + \omega_n \wedge \varpi_{\alpha n} & (\mathrm{mod}\ \varpi, \varpi_1, \cdots, \varpi_n). \end{cases} \tag{5.2}$$

where we understand that $\varpi_{\alpha\beta} = \varpi_{\beta\alpha}$ and $\varpi_{1\alpha} = \varpi_{\alpha 1}$ for $2 \leqq \alpha, \beta \leqq n$. The equalities (5.2) are the structure equations of (R, D^2) in the sense of E. Cartan ([C1], [C2]) and describe the structure of the symbol algebra $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$ of (R, D^2). In terms of an adapted coframe, covariant systems N and M are given by (cf. §3 [Y4])

$$N = \{\varpi = \varpi_1 = 0\},$$
$$M = \{\varpi = \varpi_1 = \cdots = \varpi_n = \omega_\alpha = \varpi_{1\alpha} = 0 \quad (2 \leqq \alpha \leqq n)\}.$$

Then, for the structure of N, we obtain by Cartan's method (cf. §2, §3 [Y4], [Ts])

Proposition 5.1. *Let R be a single equation of Goursat type and let v be any point of R. Then there exists an adapted coframe on a neighborhood of v such that the following equality holds :*

$$d\varpi_1 \equiv \omega_2 \wedge \varpi_{12} + \cdots + \omega_n \wedge \varpi_{1n} \quad (\mathrm{mod}\ \varpi, \varpi_1).$$

Especially, $Ch(N) = M$ on R.

5.2. Reduction Theorems. We now describe the two step reduction procedure for the (contact) equivalence problem of single equations of Goursat type, which explains the link between the exceptional simple Lie algebra G_2 and the equation (B) of Goursat type mentioned in Introduction.

Let $R \subset L(J)$ be a single equation of Goursat type. We consider the (involutive) Grassmann bundle $I(J, 1)$ of codimension 1 over the contact manifold (J, C) :

$$I(J,1) = \bigcup_{u \in J} I_u, \qquad I_u = \mathrm{Gr}\ (C(u), 2n-1),$$

where $C(u) \subset T_u(J)$ is the fibre of the contact distribution. Here we note that each hyperplane in $C(u)$ is an involutive subspace of the symplectic vector space $(C(u), d\varpi)$. In this sense, I_u is the collection of involutive subspaces of codimension 1 in $(C(u), d\varpi)$. On $I(J,1)$, we have two differential systems C^* and N^*, where N^* is the canonical system obtained by the Grassmaniann construction and C^* is the lift of C. More precisely, C^* and N^* are given by

$$C^*(w) = \pi_*^{-1}(C(u)) \supset N^*(w) = \pi_*^{-1}(w),$$

for each $w \in I(J,1)$ and $u = \pi(w) \in J$, where $\pi : I(J,1) \to J$ is the bundle projection.

The first order covariant system N of (R, D^2) induces a map φ of R into $I(J,1)$ by

$$\varphi(v) = p_*(N(v)) \subset C(u),$$

for each $v \in R$ and $u = p(v)$, where $p : R \to J$ is the projection. By Realization Lemma for (R, N, p, J), φ is a map of constant rank such that

$$\mathrm{Ker}\ \varphi_* = \mathrm{Ch}\,(N) \cap \mathrm{Ker}\ p_* = \mathrm{Ch}\,(N) \cap \mathrm{Ch}\,(D^1).$$

By Proposition 5.1, we have

$$\mathrm{rank\ Ker}\ \varphi_* = \frac{1}{2}n(n-1) = \dim S^2(E^\perp).$$

In the rest of this section, we will be concerned with the local equivalence problem for single equations (R, D^2) of Goursat type. Hence we may assume that the image $W = \mathrm{Im}\ \varphi$ is a submanifold of $I(J,1)$. Thus φ is a submersion of R onto W such that $p = q \cdot \varphi$, where q is the restriction of the projection $\pi : I(J,1) \to J$ to W. Here we note that $\dim W = 3n$. Moreover we have two differential systems C_W and N_W on W, which are the restrictions to W of C^* and N^* on $I(J,1)$. Then we have

$$\varphi_*^{-1}(N_W) = N, \qquad \text{and} \qquad \varphi_*^{-1}(C_W) = D^1.$$

We call $(W; C_W, N_W)$ the associated involutive bundle of the single equation R of Goursat type.

Now the local equivalence of (R, D^2) is first reducible to that of the involutive bundle $(W; C_W, N_W)$ as in the following: Locally W is the leaf space of the foliation on R defined by $\mathrm{Ch}\,(N) \cap \mathrm{Ch}\,(D^1)$. Conversely, from $(W; C_W, N_W)$, we can construct a PD-manifold $(R(W); D_W^1, D_W^2)$ as follows. First, by Grassmannian construction, we define

$$R(W) = \bigcup_{w \in W} R_w,$$

$$R_w = \{v \in \mathrm{Gr}(N_W(w), 2n-1) \mid$$
$$v \supset \mathrm{Ch}\,(C_W)(w) \quad \text{and} \quad q_*(v) \in L_u, u = q(w)\},$$

where L_u is the fibre of $L(J)$ at $u \in J$. D^2_W is the canonical system obtained by the Grassmannian construction and D^1_W is the lift of C_W.

Precisely, D^1_W and D^2_W are given by

$$D^1_W(v) = (\varphi_W)^{-1}_*(C_W(w)) \supset D^2_W(v) = (\varphi_W)^{-1}_*(v),$$

for each $v \in R(W)$ and $w = \varphi_W(v)$, where $\varphi_W : R(W) \to W$ is the projection. By definition, we have a map ι_W of $R(W)$ into $L(J)$ given by

$$\iota_W(v) = q_*(v) \in L_u,$$

for each $v \in R(W)$ and $u = q(v) \in J$. Then we note that the image $R^*(W) = \mathrm{Im}\ \iota_W$ has the following description :

$$R^*(W) = \bigcup_{w \in W} R^*_w, \qquad R^*_w = \{v \in L_u \mid v \subset w \subset C(u), \quad u = q(w)\ \}.$$

Namely $R^*(W)$ is the collection of legendrian subspaces of (J, C) contained in involutive subspaces of codimension 1 belonging to $W \subset I(J, 1)$.

Now we have a map κ_1 of R into $R(W)$ given by

$$\kappa_1(v) = \varphi_*(D^2(v)) \subset N_W(w),$$

for each $v \in R$ and $w = \varphi(v)$. By Realization Lemma for (R, D^2, φ, W), κ_1 is a map of constant rank such that

$$\mathrm{Ker}\ \kappa_1 = \mathrm{Ch}\,(D^2) \cap \mathrm{Ker}\ \varphi_* = \mathrm{Ch}\,(N) \cap \mathrm{Ch}\,(D^1) \cap \mathrm{Ch}\,(D^2) = \{0\}.$$

Thus κ_1 is an immersion and, by a dimension count, in fact, a local diffeomorphism of R into $R(W)$ such that

$$(\kappa_1)_*(D^1) = D^1_W \qquad \text{and} \qquad (\kappa_1)_*(D^2) = D^2_W.$$

Namely $\kappa_1 : (R, D^1, D^2) \to (R(W), D^1_W, D^2_W)$ is a local isomorphism of PD-manifolds. (Precisely $(R(W), D^1_W, D^2_W)$ becomes a PD-manifold on an open subset.)

Summarizing, we obtain the following first Reduction Theorem for contact equivalence of single equations of Goursat type.

Theorem 5.2. *Let R and $\hat{R}$ be single equations of Goursat type. Let $(W; C_W, N_W)$ and $(\hat{W}; C_{\hat{W}}, N_{\hat{W}})$ be the associated involutive bundles of R and $\hat{R}$ respectively. Let κ_1 and $\hat{\kappa}_1$ be defined as above. Let us fix points $v_o \in R$ and $\hat{v}_o \in \hat{R}$ and put $w_o = q(v_o)$ and $\hat{w}_o = \hat{q}(\hat{v}_o)$. Then a local isomorphism $\psi : (R, D^2) \to (\hat{R}, \hat{D}^2)$ such that $\psi(v_o) = \hat{v}_o$ induces a local isomorphism $\varphi : (W; C_W, N_W) \to (\hat{W}; C_{\hat{W}}, N_{\hat{W}})$ such that $\varphi(w_o) = \hat{w}_o$ and $\varphi_*(\kappa_1(w_o)) = \hat{\kappa}_1(\hat{w}_o)$, and vice versa.*

By Proposition 5.1, it follows that rank $\mathrm{Ch}\,(N_W) = 1$. Then, similarly as in §4.2, the geometry of $(W; C_W, N_W)$ is further reducible to the geometry of regular differential system of type $\mathfrak{c}^1(n-1,2)$ as follows. We may assume that W is regular with respect to $\mathrm{Ch}\,(N_W)$ so that the leaf space $Y = W/\mathrm{Ch}\,(N_W)$ is a manifold such that the projection $\beta : W \to Y$ is a submersion and there exists a differential system D_N on Y satisfying $N_W = \beta_*^{-1}(D_N)$. Moreover, by Proposition 5.1, (Y, D_N) is a regular differential system of type $\mathfrak{c}^1(n-1,2)$ (cf. Theorem 1.6 [Y3]). From (Y, D_N), we can construct $(W(Y); C_Y, N_Y)$ as follows. $W(Y)$ is the collection of hyperplanes w in each tangent space $T_y(Y)$ at $y \in Y$ which contains the fibre $D_N(y)$ of D_N:

$$W(Y) = \bigcup_{y \in Y} W_y \subset J(Y, 3n-2),$$
$$W_y = \{ w \in \mathrm{Gr}(T_y(Y), 3n-2) \mid w \supset D_N(y) \}.$$

C_Y is the canonical system obtained by the Grassmannian construction and N_Y is the lift of D_N. Precisely C_Y and N_Y are defined by

$$C_Y(w) = \mu_*^{-1}(w) \supset N_Y(w) = \mu_*^{-1}(D_N(y)),$$

for each $w \in W(Y)$ and $y = \mu(w)$, where $\mu : W(Y) \to Y$ is the projection. Then we have a map κ_2 of W into $W(Y)$ given by

$$\kappa_2(w) = \beta_*(C_W(w)) \subset T_y(Y),$$

for each $w \in W$ and $y = \beta(w)$. By Realization Lemma for (W, C_W, β, Y), κ_2 is a map of constant rank such that

$$\mathrm{Ker}\ \kappa_2 = \mathrm{Ch}\,(C_W) \cap \mathrm{Ker}\ \beta_* = \mathrm{Ch}\,(C_W) \cap \mathrm{Ch}\,(N_W) = \{0\}.$$

Thus κ_2 is an immersion and, by a dimension count, in fact, a local diffeomorphism of W into $W(Y)$ such that

$$(\kappa_2)_*(C_W) = C_Y \qquad \text{and} \qquad (\kappa_2)_*(N_W) = N_Y.$$

Namely $\kappa_2 : (W; C_W, N_W) \to (W(Y); C_Y, N_Y)$ is a local isomorphism.

Summarizing, we obtain the second Reduction Theorem for contact equivalence of single equations of Goursat type.

Theorem 5.3. *Let R and $\hat{R}$ be single equations of Goursat type. Let $(W; C_W, N_W)$ and $(\hat{W}; C_{\hat{W}}, N_{\hat{W}})$ be the associated involutive bundles of R and $\hat{R}$ respectively. Assume that W and $\hat{W}$ are regular with respect to $Ch(N_W)$ and $Ch(N_{\hat{W}})$ respectively. Let (Y, D_N) and $(\hat{Y}, \hat{D}_N)$ be the leaf spaces, where $Y = W/Ch(N_W)$ and $\hat{Y} = \hat{W}/Ch(N_{\hat{W}})$. Let us fix points $w_o \in W$ and $\hat{w}_o \in \hat{W}$ and put $y_o = \beta(w_o)$ and $\hat{y}_o = \hat{\beta}(\hat{w}_o)$. Then a local isomorphism $\psi : (W; C_W, N_W) \to (\hat{W}; C_{\hat{W}}, N_{\hat{W}})$ such that $\psi(w_o) = \hat{w}_o$*

induces a local isomorphism $\varphi : (Y, D_N) \to (\hat{Y}, \hat{D}_N)$ *such that* $\varphi(y_o) = \hat{y}_o$ *and* $\varphi_*(\kappa_2(y_o)) = \hat{\kappa}_2(\hat{y}_o)$, *and vice versa.*

Thus, finally, the local contact equivalence problem of single equations R of Goursat type reduces to the equivalence of (Y, D_N), which are regular differential systems of type $\mathfrak{c}^1(n-1, 2)$(cf. [Ts], §3 [Y4]).

§6. G_2-geometry

In view of discussions in §3, §4 and §5, we will here consider the generalization of (A) and (B) to other simple Lie algebras.

6.1. Standard Contact Manifolds. Each simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has the highest root θ. Let Δ_θ denote the subset of Δ consisting of all vertices which are connected to $-\theta$ in the Extended Dynkin diagram of X_ℓ $(\ell \geqq 2)$. This subset Δ_θ of Δ, by the construction in §4, defines a gradation (or a partition of Φ^+), which distinguishes the highest root θ. Then, this gradation (X_ℓ, Δ_θ) turns out to be a contact gradation, which is unique up to conjugacy.

Moreover we have the adjoint (or equivalently coadjoint) representation, which has θ as the highest weight. The R-space $J_\mathfrak{g}$ corresponding to (X_ℓ, Δ_θ) can be obtained as the projectiviation of the (co-)adjoint orbit of G passing through the root vector of θ. By this construction, $J_\mathfrak{g}$ has the natural contact structure $C_\mathfrak{g}$ induced from the symplectic structure as the coadjoint orbit, which corresponds to the contact gradation (X_ℓ, Δ_θ) (cf. [Y5, §4]). Standard contact manifolds $(J_\mathfrak{g}, C_\mathfrak{g})$ were first found by Boothby ([Bo]) as compact simply connected homogeneous complex contact manifolds.

6.2. Gradation of G_2. The Dynkin diagram of G_2 is given as follows:

$$\underset{\alpha_1}{\circledcirc} \Lleftarrow \underset{\alpha_2}{\circledcirc}, \qquad \theta = 3\alpha_1 + 2\alpha_2.$$

In this case, from $\Delta = \{\alpha_1, \alpha_2\}$, we have three choices for Δ_1:

$(G1)$ $\Delta_1 = \{\alpha_1\}$. In this case, we have $\mu = 3$, $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-1} = 2$ and $\dim \mathfrak{g}_{-2} = 1$. Moreover $(M_\mathfrak{g}, D_\mathfrak{g})$ coincides with (X, D) in case of (A).

$(G2)$ $\Delta_1 = \{\alpha_2\}$. In this case, we have the standard contact gradation.

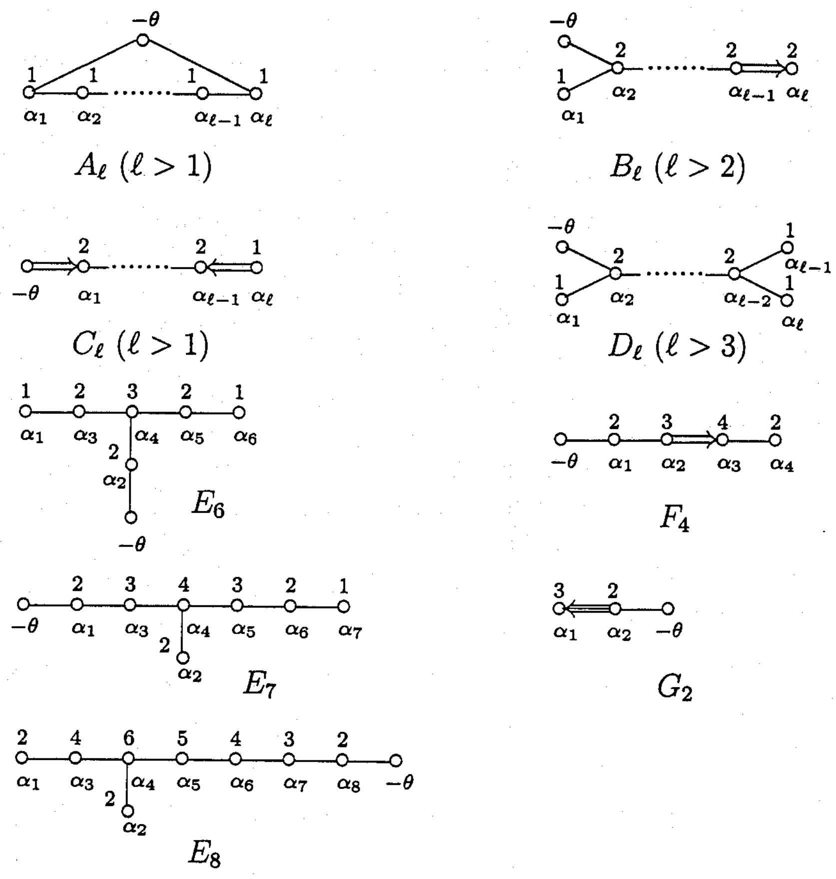
$(G3)$ $\Delta_1 = \{\alpha_1, \alpha_2\}$. In this case, we have $\mu = 5$, $\dim \mathfrak{g}_{-1} = 2$ and $\dim \mathfrak{g}_p = 1$ for others.

Let $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ be the Standard contact manifold of type G_2.If we lift the action of the exceptional group G_2 to $L(J_{\mathfrak{g}})$, then we have the following orbit decomposition:

$$L(J_{\mathfrak{g}}) = O \cup R_1 \cup R_2,$$

where O is the open orbit and R_i is the orbit of codimension i. Here R_1 and R_2 can be considered as the global model of (B) and (A) respectively. Moreover R_2 is compact and is a R-space corresponding to $(G_2, \{\alpha_1, \alpha_2\})$. From this fact, it becomes possible to describe the PD-manifold $(R; D^1, D^2)$ corresponding to (A) in terms of the R-space corresponding to $(G_2, \{\alpha_1, \alpha_2\})$.

Extended Dynkin Diagrams with the coefficient of Highest Root (cf. [Bu])



6.3. G_2-geometry. In the Extended Dynkin diagram, except for A_ℓ type, Δ_θ consists of one simple root α_θ. The coefficient of α_θ in the highest root $\theta = \sum_{i=1}^{\ell} n_i(\theta)\, \alpha_i$ is of course 2. Furthermore, for the exceptional simple Lie algebras, there exists, without exception, a unique simple root α_G next to α_θ such that the coefficient of α_G in the highest root is 3.

For $X_\ell \cong E_6, E_7, E_8, G_2, F_4$, the gradation $(X_\ell, \{\alpha_G\})$ has the following property;

$$\mu = 3, \quad \dim \mathfrak{g}_{-3} = 2 \quad \text{and} \quad \dim \mathfrak{g}_{-1} = 2 \dim \mathfrak{g}_{-2}.$$

Moreover, ignoring the bracket product in $\mathfrak{g}_{-1}$, the bracket product of other part can be expressed in terms of pairing by

$$\mathfrak{g}_{-3} = W, \quad \mathfrak{g}_{-2} = V \quad \text{and} \quad \mathfrak{g}_{-1} = W \otimes V^*.$$

Namely the derived system $(M_\mathfrak{g}, \partial D_\mathfrak{g})$ is a regular differential system of type $\mathfrak{c}^1(r, 2)$ for suitable r, where $(M_\mathfrak{g}, D_\mathfrak{g})$ is the standard differential system of type $(X_\ell, \{\alpha_G\})$. This fact assures us to construct the single equation of Goursat type from the differential system $(Y, D_N) = (M_\mathfrak{g}, \partial D_\mathfrak{g})$, which is the generalization of (B).

Obviously the R-space R_G corresponding to $(X_\ell, \{\alpha_\theta, \alpha_G\})$ is a fibre space over the Standard contact manifold $(J_\mathfrak{g}, C_\mathfrak{g})$ corresponding to $(X_\ell, \{\alpha_\theta\})$. In fact this R_G can be realized as a compact orbit in $L(J_\mathfrak{g})$, which gives the generalization of (A). Moreover, in this case, we have rank $Ch(D^2) = 1$ as a PD-manifold and (X, D) coincides with the R-space corresponding to $(X_\ell, \{\alpha_G\})$.

Remark 6.1 (Classical cases). *In the classical simple Lie algebras, there is no simple root whose coefficient in the highest root is* 3. *However, in B_ℓ and D_ℓ types, there is a set $\{\alpha_1, \alpha_3\}$ of simple roots next to $\alpha_\theta = \alpha_2$ whose sum of coefficients in the highest root is* 3. *In fact, $(B_\ell, \{\alpha_1, \alpha_3\})$ and $(D_\ell, \{\alpha_1, \alpha_3\})$ have the above property;*

$$\mu = 3, \quad \dim \mathfrak{g}_{-3} = 2 \quad \textit{and} \quad \dim \mathfrak{g}_{-1} = 2 \dim \mathfrak{g}_{-2}.$$

and the set $\{\alpha_1, \alpha_3\}$ plays the role of $\{\alpha_G\}$. Hence we have the generalizations of (A) and (B) for simple Lie algebras of type B_ℓ and type D_ℓ.

References

[Bo] W. M. Boothby, *Homogeneous complex contact manifolds*, Proc. Symp. Pure Math., Amer. Math. Soc. **3** (1961), 144–154.

[Bu] N.Bourbaki, *Groupes et algèbles de Lie, Chapitre 4,5 et 6*, Hermann, Paris (1968).

[BCG3] R. Bryant, S. S. Chern, R. B. Gardner, H. Goldschmidt, and P. Griffiths, *Exterior differential systems*, Springer-Verlag, New-York (1986).

[C1] E.Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Ec. Normale **27** (1910), 109–192.

[C2] ——, *Sur les systèmes en involution d'équations aux dérivées partielles du second ordre à une fonction inconnue de trois variables indépendantes*, Bull. Soc. Math. France **39** (1911), 352–443.

[Hu] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York (1972).

[Ko] B. Kostant, *Lie algebra cohomology and generalized Borel-Weil theorem*, Ann. Math. **74** (1961), 329–387.

[Ku] M. Kuranishi, *Lectures on involutive systems of partial differential equations*, Pub.Soc.Mat., São Paulo (1967).

[St] S. Sternberg *Lectures on Differential Geometry*, Prentice-Hall, New Jersey (1964).

[T1] N. Tanaka *On generalized graded Lie algebras and geometric structures I*, J. Math. Soc. Japan **19** (1967), 215–254.

[T2] ——, *On differential systems, graded Lie algebras and pseudo-groups*, J. Math. Kyoto Univ.**10** (1970), 1–82.

[T3] ——, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan. J. Math.**2** (1976), 131–190.

[T4] ——, *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J. **8** (1979), 23–84.

[T5] ——, *On affine symmetric spaces and the automorphism groups of product manifolds*, Hokkaido Math. J. **14** (1985), 277–351.

[Ts] A.Tsuchiya *Geometric theory of partial differential equations of second order*, Lecture Note at Nagoya University (in Japanese) (1981)

[Y1] K.Yamaguchi *Contact geometry of higher order*, Japanese J. of Math. **8** (1982), 109–176.

[Y2] ——, *On involutive systems of second order of codimension 2*, Proc. of Japan Acad. **58**, Ser A, No.7 (1982), 302–305.

[Y3] ——, *Geometrization of Jet bundles*, Hokkaido Math. J. **12** (1983), 27–40.

[Y4] ——, *Typical classes in involutive systems of second order*, Japanese J. Math. **11** (1985), 265–291.

[Y5] ——, *Differential systems associated with simple graded Lie algebras*, Adv. Studies in Pure Math. **22** (1993), 413–494.

Department of Mathematics, Faculty of Science,
Hokkaido University, Sapporo 060, Japan
E-mail:yamaguch@@math.sci.hokudai.ac.jp

CORRECTION TO THE LIST OF NON-VANISHING SECOND COHOMOLOGY

In our previous paper: *Differential Systems Associated with Simple Graded Lie Algebras*, Adv. Studies in Pure Math. **22** (1993), 413–494, the list of the Non-vanishing second Spencer cohomology (Proposition5.5) contains some misprints and omissions. Here we would like to correct the following points:

(I) A_ℓ-type : $(11)^o$ is missing and, (2) and (4) lack the information for $\ell = 3$.

(II) B_ℓ-type : (7) contains a misprint ($\mu = 3$ shuld be replaced by $\mu = 4$).

(III) C_ℓ-type : (3) contains a misprint ($p_{21} = 2$ ($\ell = 2$) should be deleted).

(IV) D_ℓ-type : (3), (5) and (7) lack the information for the case $\ell = 4$.

Corrected version of Proposition 5.5 should be stated as follows:

Proposition 5.5. *Let (X_ℓ, Δ_1) be a simple graded Lie algebra over $\mathbb{C}$ described in §3.4. Then the following are the list of (X_ℓ, Δ_1) and p_{ij} such that $p_{ij} \geqq 0$ holds for the irreducible component $\mathcal{H}^{\sigma_{ij}} \subset C^{p_{ij},2}(\mathfrak{m}, \mathfrak{g})$ of the harmonic space $\mathcal{H}^2 \cong H^2(\mathfrak{m}, \mathfrak{g})$ corresponding to $\sigma_{ij} \in W^0(2)$ in Kostant's theorem.*

(I) *A_ℓ-type ($\ell \geqq 2$).*

(1) $\{\alpha_1\}$ $\quad p_{12} = 2 \quad (\ell = 2)$,
$p_{12} = 1 \quad (\ell \geqq 3)$.

(2) $\{\alpha_2\}$ $\quad p_{21} = p_{23} = 1 \quad (\ell = 3)$,
$p_{21} = 1, \quad p_{23} = 0 \quad (\ell \geqq 4)$.

(3) $\{\alpha_i\}$ $\quad p_{i\,i-1} = p_{i\,i+1} = 0 \quad (2 < i \leqq [\frac{\ell+1}{2}])$.

(4) $\{\alpha_1, \alpha_2\}$ $\quad p_{12} = p_{21} = 3 \quad (\ell = 2)$,
$p_{12} = 1, \quad p_{21} = 2, \quad p_{23} = 0 \quad (\ell = 3)$,
$p_{12} = 1, \quad p_{21} = 2 \quad (\ell \geqq 4)$.

(5) $\{\alpha_1, \alpha_i\}$ $\quad p_{12} = p_{1i} = 0 \quad (2 < i < \ell - 1)$.

(6) $\{\alpha_1, \alpha_{\ell-1}\}$ $\quad p_{12} = p_{1\,\ell-1} = p_{\ell-1\,\ell} = 0 \quad (\ell \geqq 4)$.

(7) $\{\alpha_1, \alpha_\ell\}$ $\quad p_{12} = p_{\ell\,\ell-1} = 0, \quad p_{1\ell} = 1 \quad (\ell \geqq 3)$.

(8) $\{\alpha_2, \alpha_3\}$ $\quad p_{21} = p_{23} = p_{32} = p_{34} = 0 \quad (\ell = 4)$,
$p_{21} = p_{23} = p_{32} = 0 \quad (\ell \geqq 5)$.

(9) $\{\alpha_2, \alpha_i\}$ $\quad p_{21} = 0 \quad (3 < i < \ell - 1)$.

(10) $\{\alpha_2, \alpha_{\ell-1}\}$ $\quad p_{21} = p_{\ell-1\,\ell} = 0 \quad (\ell \geqq 5)$.

(11) $\{\alpha_i, \alpha_{i+1}\}$ $\quad p_{i\,i+1} = p_{i+1\,i} = 0 \quad (2 < i \leqq [\frac{\ell}{2}])$.

$(11)^o$ $\{\alpha_1, \alpha_2, \alpha_i\}$ $\quad p_{12} = 0, \quad p_{21} = 1 \quad (2 < i < \ell)$.

(12) $\{\alpha_1, \alpha_2, \alpha_\ell\}$ $\quad p_{13} = p_{12} = p_{32} = 0, \quad p_{21} = p_{23} = 1 \quad (\ell = 3)$,
$p_{1\ell} = p_{12} = 0, \quad p_{21} = 1 \quad (\ell \geqq 4)$.

(13) $\{\alpha_1, \alpha_i, \alpha_\ell\}$ $\quad p_{1\ell} = 0 \quad (2 < i \leqq [\frac{\ell}{2}])$.

(14) $\{\alpha_1, \alpha_2, \alpha_i, \alpha_j\}$ $\quad p_{21} = 0 \quad (2 < i < j \leqq \ell)$.

(15) $\{\alpha_1, \alpha_2, \alpha_{\ell-1}, \alpha_\ell\}$ $\quad p_{21} = p_{\ell-1\,\ell} = 0$.

(II) *B_ℓ-type ($\ell \geqq 3$).*

(1) $\{\alpha_1\}$ $\quad \mu = 1 \quad p_{12} = 1$.

(2) $\{\alpha_2\}$ $\mu = 2$ $p_{21} = p_{23} = 0$.
(3) $\{\alpha_3\}$ $\mu = 2$ $p_{32} = 2$ $(\ell = 3)$,
$p_{32} = 0$ $(\ell \geqq 4)$.
(4) $\{\alpha_\ell\}$ $\mu = 2$ $p_{\ell\ell-1} = 0$ $(\ell \geqq 4)$.
(5) $\{\alpha_1, \alpha_2\}$ $\mu = 3$ $p_{21} = 0$, $p_{12} = 1$.
(6) $\{\alpha_1, \alpha_3\}$ $\mu = 3$ $p_{32} = 1$ $(\ell = 3)$.
(7) $\{\alpha_2, \alpha_3\}$ $\mu = 4$ $p_{32} = 2$ $(\ell = 3)$,
$p_{32} = 0$ $(\ell \geqq 4)$.
(8) $\{\alpha_1, \alpha_2, \alpha_3\}$ $\mu = 5$ $p_{32} = 1$ $(\ell = 3)$.

(III) *C_ℓ-type* $(\ell \geqq 2)$.
(1) $\{\alpha_\ell\}$ $\mu = 1$ $p_{21} = 2$ $(\ell = 2)$, $p_{\ell\ell-1} = 0$ $(\ell \geqq 3)$.
(2) $\{\alpha_1\}$ $\mu = 2$ $p_{12} = 2$ $(\ell = 2)$, $p_{12} = 1$ $(\ell \geqq 3)$.
(3) $\{\alpha_2\}$ $\mu = 2$ $p_{21} = 1$, $p_{23} = 0$ $(\ell = 3)$,
$p_{21} = 1$ $(\ell \geqq 4)$.
(4) $\{\alpha_{\ell-1}\}$ $\mu = 2$ $p_{\ell-1\ell} = 0$ $(\ell \geqq 4)$.
(5) $\{\alpha_1, \alpha_\ell\}$ $\mu = 3$ $p_{12} = 2$, $p_{21} = 3$ $(\ell = 2)$,
$p_{1\ell} = p_{12} = 0$ $(\ell \geqq 3)$.
(6) $\{\alpha_2, \alpha_\ell\}$ $\mu = 3$ $p_{21} = p_{23} = 0$ $(\ell = 3)$,
$p_{21} = 0$ $(\ell \geqq 4)$.
(7) $\{\alpha_{\ell-1}, \alpha_\ell\}$ $\mu = 3$ $p_{\ell-1\ell} = 0$ $(\ell \geqq 4)$.
(8) $\{\alpha_1, \alpha_2\}$ $\mu = 4$ $p_{12} = 0$, $p_{21} = 2$ $(\ell \geqq 3)$.
(9) $\{\alpha_1, \alpha_2, \alpha_\ell\}$ $\mu = 5$ $p_{21} = 1$.
(10) $\{\alpha_1, \alpha_2, \alpha_i\}$ $\mu = 6$ $p_{21} = 0$ $(2 < i < \ell)$.

(IV) *D_ℓ-type* $(\ell \geqq 4)$.
(1) $\{\alpha_1\}$ $\mu = 1$ $p_{12} = 1$.
(2) $\{\alpha_\ell\}$ $\mu = 1$ $p_{\ell\ell-2} = 0$ $(\ell \geqq 5)$.
(3) $\{\alpha_2\}$ $\mu = 2$ $p_{21} = p_{23} = p_{24} = 0$ $(\ell = 4)$,
$p_{21} = p_{23} = 0$ $(\ell \geqq 5)$.
(4) $\{\alpha_3\}$ $\mu = 2$ $p_{32} = 0$ $(\ell \geqq 5)$.
(5) $\{\alpha_1, \alpha_\ell\}$ $\mu = 2$ $p_{12} = p_{42} = 0$ $(\ell = 4)$,
$p_{12} = 0$ $(\ell \geqq 5)$.
(6) $\{\alpha_1, \alpha_2\}$ $\mu = 3$ $p_{12} = 1$, $p_{21} = 0$.
(7) $\{\alpha_1, \alpha_2, \alpha_\ell\}$ $\mu = 4$ $p_{12} = p_{42} = 0$ $(\ell = 4)$,
$p_{12} = 0$ $(\ell \geqq 5)$.
(8) $\{\alpha_2, \alpha_3\}$ $\mu = 4$ $p_{32} = 0$ $(\ell \geqq 5)$.

(V) *Exceptional types.*
(1) $(E_6, \{\alpha_1\})$, $(E_7, \{\alpha_7\})$ $\mu = 1$ $p_{ij} = 0$, *where* $\{\alpha_i\} = \Delta_1$ *and* $\langle \alpha_i, \alpha_j \rangle \neq 0$.
(2) $(E_6, \{\alpha_2\})$, $(E_7, \{\alpha_1\})$, $(E_8, \{\alpha_8\})$, $(F_4, \{\alpha_1\})$ *and* $(G_2, \{\alpha_2\})$.
Contact gradations: $\mu = 2$ $p_{ij} = 0$, *where* $\{\alpha_i\} = \Delta_\theta$ *and* $\langle \alpha_i, \alpha_j \rangle \neq 0$.
(3) $(G_2, \{\alpha_1\})$ $\mu = 3$ $p_{12} = 3$.
(4) $(G_2, \{\alpha_1, \alpha_2\})$ $\mu = 5$ $p_{12} = 3$.

Acknowledgment

We would like to thank Professor Hajime Sato at Nagoya University for pointing out these omissions and also for providing us the exe. files for computing non-vanishing second Spencer cohomology.