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Series #413. May 1998

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MILNOR CLASSES OF LOCAL COMPLETE INTERSECTIONS

J.-P. BRASSELET, D. LEHMANN, J. SEADE AND T. SUWA

There are several definitions of characteristic classes of singular varieties generalizing the Chern classes of non-singular varieties. Among them are the Chern-Schwartz-MacPherson classes (abbreviated as CSM classes) [Sc1, Ma, BS] and the Fulton-Johnson canonical classes (abbreviated as FJ classes) [FJ, F]. Each one of them is defined in a relevant context and has its own interest and advantage. The main purpose of this article is to give geometric interpretations and better understanding of these classes. This will be performed by comparing them. There are already results in this direction [P, PP1, SS2, Su1] giving an expression of the comparison in terms of Milnor numbers, in particular cases for which the difference lies in dimension 0. It was natural to expect that for higher degrees the comparison should be expressed in terms of "Milnor classes". This appears in [A1] as so called μ-classes and independently in [Y] and the present paper. In fact, we introduce the "virtual class", which coincides with the FJ class when the variety is a local complete intersection, and compare the CSM class and the virtual class. The comparison is done by localizing each class using vector fields, or more generally, frames of the tangent bundle of the non-singular part. On the non-singular part of the variety, the two localizations coincide and on the singular part, we are naturally lead to define the "Milnor class" as the difference of the two. The methods we use are of two kinds : 1) obstruction theory to construct vector fields and frames and 2) differential geometric interpretation of characteristic classes by way of connections for suitable bundles.

Let us first consider the case of a single vector field. Let V be a singular variety. We denote by $\mathrm{Sing}(V)$ its singular set and V_0 the non-singular part. Assume for the moment that V has only isolated singular points and let v be a vector field on V_0 with isolated singular points. We wish to define a local index of v at each of the singular points of v and V. For a singular point of v in V_0, we have the usual Poincaré-Hopf local index. The question is what to do at the singular points of V. On one hand we would like the sum of the local indices to give "the correct number", i.e., the Euler-Poincaré characteristic $\chi(V)$ of V, as in the case of manifolds. The number $\chi(V)$ is the 0-degree term in the CSM class of V and this point of view was introduced by M.-H. Schwartz in [Sc1, Sc2] for "radial vector fields", which are used to define her characteristic classes, with no hypotheses on $\mathrm{Sing}(V)$. These classes are shown in [BS] to coincide with MacPherson's classes

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[Ma]. It is not difficult to extend the results of M.-H. Schwartz to vector fields which are non-radial. We do this below (see also [EG,KT,SS2,Si]), and we call the corresponding index the *generalized Schwartz index.* On the other hand, we would like the local index to be "stable" under deformations. It turns out that this condition is not compatible, in general, with the previous condition of obtaining the "correct" number. The corresponding index was introduced in [Se, GSV, SS1, also GM] when V is a complete intersection near p and this is called the *GSV-index.* The difference between these indices is, up to sign, the Milnor number of V at p, and is independent of the vector field. The definition of the GSV-index requires (a priori) the singularities of V to be isolated. However, a differential geometric index introduced in [LSS], the *virtual index*, is well defined even if the singularities of V are non-isolated, and it coincides with the GSV-index when both are defined. The sum of these virtual indices is in fact the 0-degree term in the virtual class of V, which will be explained below.

More generally, an r-frame $F^{(r)} = \{v_1, \dots . v_r\}$ is a set of r linearly independent vector fields. Let V be a singular variety of dimension n embedded in a complex manifold M of dimension m. We take a triangulation of M compatible with V and $\mathrm{Sing}(V)$. Let S be a compact connected component of $\mathrm{Sing}(V)$ or a compact connected subcomplex of V_0. For an r-frame on the $2p$-skeleton (of the dual cellular decomposition) of a neighborhood of S, $p = n - r + 1$, we define the Schwartz class $\mathrm{Sch}(F^{(r)}, S)$ as a class in the homology $H_{2r-2}(S)$. In the global situation, the sum of the Schwartz classes gives the $(r-1)$-st CSM class of V (Theorem 2.13). In particular, when $r = 1$, $\mathrm{Sch}(v, S)$, $v = v_1$, is the generalized Schwartz index mentioned above and the formula reduces to the one expressing $\chi(V)$ as the sum of these indices. If S consists of a point p in V_0, then the Schwartz index $\mathrm{Sch}(v,p)$ coincides with the usual Poincaré-Hopf index and if V is non-singular, the above formula reduces to the classical Poincaré-Hopf formula.

We recall that, if V is a local complete intersection in M, then the normal bundle of V_0 extends canonically to a vector bundle N_V on V. In this case, the FJ class of V is equal to $c(\tau_V) \frown [V]$, where $\tau_V = TM|_V - N_V$ is the virtual tangent bundle of V and $c(\tau_V) = c(TM|_V) \cdot c(N_V)^{-1}$. In this article, we consider a slightly different situation. Namely, we assume that there is a holomorphic vector bundle E of rank $k = m - n$ over M and a holomorphic section s of E, generically transverse to the zero section, such that $V = s^{-1}(0)$. Thus V is a set theoretic local complete intersection. Note that, if, moreover, the local components of s generate the ideal of functions vanishing on V, then V is a local complete intersection. We set $N_V = E|_V$ and also call $\tau_V = TM|_V - N_V$ the virtual tangent bundle of V. In this situation, we call $c(\tau_V) \frown [V]$ the "total" virtual class of V. Let again S be a compact connected component of $\mathrm{Sing}(V)$ or a compact connected subcomplex of V_0. For an r-frame on the $2p$-skeleton of a neighborhood of S, we define the virtual class $\mathrm{Vir}(F^{(r)}, S)$ as a class in the homology $H_{2r-2}(S)$. In the global situation, the sum of the virtual classes gives the $(r-1)$-st virtual class of V (Theorem 5.9), which is the $(r-1)$-st FJ class of V, if V is a local complete intersection. In particular, when $r=1$, $\mathrm{Vir}(v,S)$ is the virtual index mentioned above.

If S is in V_0, then $\mathrm{Sch}(F^{(r)},S)$ coincides with $\mathrm{Vir}(F^{(r)},S)$. In general, the difference of the two is shown to be independent of the frame $F^{(r)}$ and there is at least one such a frame. Hence we define the $(r-1)$-st Milnor class $\mu_{r-1}(V,S)$ of V at S by

$$\mu_{r-1}(V,S)=(-1)^{n+1}(\mathrm{Sch}(F^{(r)},S)-\mathrm{Vir}(F^{(r)},S)),$$

which is a class in $H_{2r-2}(S)$. In particular, when $k=1$ (i.e., V is a hypersurface) and $r=1$, $\mu_0(V,S)$ is shown to coincide with the generalized Milnor number in [P]. Also, if S consists of a point p and if V is a complete intersection near p, then $\mu_0(V,S)$ coincides with the usual Milnor number [Mi, Ha1]. With these, we have a comparison formula of the Schwartz-MacPherson class and the virtual class (Theorem 7.8). When the singularities of V are isolated, then we recover the formula in [Su1]. A similar formula for hypersurfaces is given in [A2]. When $r=1$, the formula is proved in [SS2] in the case V has only isolated singularities and in [P, PP1] for hypersurfaces with arbitrary singularities.

In this paper, virtual classes and Milnor classes are defined using differential geometric method in homology with real coefficients. This framework is useful for explicit computations (see for instance Theorem 7.13). However, in fact it is possible to define these classes in integral homology by obstruction theory (cf. section 6, where the integrality in degree 0 is proved by different approach).

In section 1 of this article, we briefly review the Poincaré-Hopf indices of vector fields and the Chern classes of almost complex manifolds. In section 2, after reviewing the CSM classes of singular varieties following [Sc1, Sc2, BS], we define the (local) Schwartz indices and classes and prove the Theorem 2.13 mentioned above. This is achieved by noting that, in analogy with the classical case, these classes are the "primary obstruction" for extending to the $2p$-skeleton of V certain "stratified r-frames". We give, in section 3, a differential geometric description of the Schwartz classes, which will be used in later sections.

In section 4 we restrict ourselves to the case where V is the zero set of a holomorphic section s of a rank k vector bundle E over a complex manifold M and s is generically transverse to the zero section of E. We discuss here the characteristic classes for the virtual tangent bundle. In section 5, we recall virtual indices and introduce virtual classes. We study in detail the "0-degree case" in section 6. We give a geometric interpretation of the virtual index, in the spirit of the GSV-index.

In section 7 we introduce the generalized Milnor number and Milnor class of V at each connected component of $\mathrm{Sing}(V)$ and obtain a formula expressing the CSM class of the variety V in terms of the virtual class of V and the Milnor classes at the components of $\mathrm{Sing}(V)$ (Theorem 7.8). We also give a formula for the Schwartz and virtual classes at a non-singular component S of $\mathrm{Sing}(V)$ (Theorem 7.13), which implies an explicit formula for the Milnor class at S (Corollary 7.18).

In section 8 we study the case $k=1$, i.e., when V is a hypersurface in

M. We show that in this case our number $\mu_0(V, S)$ coincides with the generalized Milnor number of [P, PP1]. Finally, section 9 contains several examples and explicit computations in various cases.

We would like to thank Lê Dũng Tráng and S. Yokura for valuable conversations. While working on this article we visited various institutions and we are grateful, in particular, to Institut de Mathématiques de Luminy, Université de Montpellier, Instituto de Mathemáticas UNAM at Cuernavaca and Hokkaido University for their support and hospitality. We also acknowledge the financial support of CNRS, CONACYT and the Ministry of ESC, Japan with gratitude.

After the preparation of the manuscript we learned that, in the hypersurface case, the formula for the Milnor number in [PP1] has been generalized in [PP3] to a formula for the Milnor class, which was conjectured in [Y]. In particular, for a non-singular component of the singular set, our formula in Corollary 7.18 and the one in [PP3] coincide.

1. Background on Poincaré-Hopf indices and Chern classes

We refer to [St] for basics on the material in this section. In sections 1 and 2, all cohomology and homology groups are relative to integral coefficients.

Let M be an oriented C^∞ manifold, possibly with boundary, and let v be a continuous vector field on a part of M. A *singular point* of v is a point where v vanishes.

The Poincaré-Hopf index $\mathrm{PH}(v, p)$ of v at p, isolated singularity of v which is an interior point of M, is the degree of the Gauss map induced by v near p.

If W is a compact manifold with boundary (possibly with corners) in M, of the same dimension as M, and if v is a non-singular vector field on (a neighborhood of) the boundary ∂W of W, we may define the Poincaré-Hopf index $\mathrm{PH}(v, W)$ of v on W in the following way : the vector field v can be extended in the interior of W with a finite number of isolated singularities p_i with index $\mathrm{PH}(v, p_i)$. We define $\mathrm{PH}(v, W)$ as the sum of the $\mathrm{PH}(v, p_i)$'s. This number is an integer independent of the way we extend v to the interior of W, see [St].

Let us consider a triangulation (K) of M compatible with the boundary, $(\hat{K})$ a barycentric subdivision of (K) and (D) the associated cellular dual decomposition of M. In the following, the triangulation (K) we consider will be compatible with some subsets of M. For instance, let S be a compact connected (K)-subcomplex of the interior of M, and denote by $\tilde{T}$ the union of (closed) cells of (D) which are dual of simplices in S. The boundary $\partial\tilde{T}$ is the union of cells in $\tilde{T}$ which do not meet S.

Definition 1.1. In the above situation, we will call *cellular tube* around S in M a neighborhood of S in M union of dual cells (in M) relative to the triangulation of S.

This notion generalizes the notion of tubular neighborhood of a submanifold S, in this case, $\tilde{T}$ is a bundle on S, whose fibers are disks.

Lemma 1.2. *A cellular tube $\tilde{\mathcal{T}}$ around S has the following properties :*
i) $\tilde{\mathcal{T}}$ is a compact neighborhood of S, containing S in its interior and such that $\partial\tilde{\mathcal{T}}$ is a retract of $\tilde{\mathcal{T}} - S$.
ii) $\tilde{\mathcal{T}}$ is a regular neighborhood of S, i.e., $\tilde{\mathcal{T}}$ retracts to S.

Let us denote by $\tilde{U}$ a neighborhood of S in M. If the triangulation is sufficiently "fine", then $\tilde{\mathcal{T}} \subset \tilde{U}$. While it is not necessary to consider this situation in the two first sections, we introduce it now in order to facilitate the comparison with the following sections.

Let v be a continuous vector field on a neighborhood $\tilde{U}$ of S in M, non-singular on $\tilde{U} - S$, then the *Poincaré-Hopf index* of v at S, denoted $\mathrm{PH}(v, S)$, can be defined as $\mathrm{PH}(v, \tilde{\mathcal{T}})$, where $\tilde{\mathcal{T}}$ denotes any cellular tube in $\tilde{U}$ around S. This number $\mathrm{PH}(v, S)$ depends only on the behavior of v near S and not on the choice of the neighborhood $\tilde{U}$, nor on the tube $\tilde{\mathcal{T}}$. Moreover, for this index it does not matter what actually happens on S, we only care what happens around S, but away from S.

Now let M be a compact oriented C^∞ manifold and v a continuous vector field on M, non-singular on the boundary. From the above consideration, we may assume that the set $\mathrm{Sing}(v)$ of singular points of v has only a finite number of components $(S_\alpha)_\alpha$. The theorem of Poincaré-Hopf says that, if M has no boundary, then we have

$$\sum_\alpha \mathrm{PH}(v, S_\alpha) = \chi(M). \tag{1.3}$$

If M has a boundary ∂M, the sum $\sum_\alpha \mathrm{PH}(v, S_\alpha)$ depends only on the behavior of v near ∂M. For example, if v is pointing outwards everywhere on ∂M, then we have the same formula (1.3), if v is pointing inwards everywhere on ∂M, the right hand side is $\chi(M) - \chi(\partial M)$. In particular, if the (real) dimension of M is even (as it will always be the case in this article) and if v is everywhere transverse to ∂M, then we have again the same formula (1.3).

Next we recall the interpretation of the Chern classes via obstruction theory. This can be done in full generality, however, we restrict the discussion to the case where the base space is an almost complex $2m$-manifold M, the tangent bundle TM of M is a complex vector bundle of rank m.

Definition 1.4. An r-field is a set $F^{(r)} = \{v_1, \ldots, v_r\}$ of r continuous vector fields defined on a subset in M. A singular point of $F^{(r)}$ is a point where the vectors (v_i) fail to be linearly independent. A non-singular r-field is also called an r-frame.

Let $W_r(m)$ be the Stiefel manifold of complex r-frames in $\mathbb{C}^m$. Notice that we will use not necessarily orthonormal r-frames, but this does not change the results. We know (see [St]) that $W_r(m)$ is $(2m-2r)$-connected and its first non-zero homotopy group is $\pi_{2m-2r+1}(W_r(m)) \simeq \mathbb{Z}$. The bundle of r-frames on M, denoted by $W_r(TM)$, is the bundle associated with the tangent bundle and whose fiber over $x \in M$ is the set of r-frames in $T_x(M)$ (diffeomorphic to $W_r(m)$). In the following, we fix the notation $q = m - r + 1$.

The Chern class $c^q(M) \in H^{2q}(M)$ is the first possibly non-zero obstruction to construct a section of $W_r(TM)$. Let us recall the standard obstruction theory process. Let σ be a k-cell of the given cell decomposition (D), contained in an open subset $\Omega \subset M$ on which the bundle $W_r(TM)$ is trivialized. If the section $F^{(r)}$ of $W_r(TM)$ is already defined over the boundary of σ, it defines a map :

$$\partial\sigma \simeq S^{k-1} \xrightarrow{F^{(r)}} W_r(TM)|_\Omega \simeq \Omega \times W_r(m) \xrightarrow{pr_2} W_r(m),$$

thus an element of $\pi_{k-1}(W_r(m))$. If $k \leq 2m - 2r + 1$, this homotopy group is zero, so the section $F^{(r)}$ can be extended within σ without singularity. If $k = 2m - 2r + 2 = 2q$, we meet an obstruction. So we can always construct a section $F^{(r)}$ of $W_r(TM)$ over the $(2q-1)$-skeleton of (D). When we try to extend $F^{(r)}$ to the $2q$-skeleton, we have an r-frame on the boundary of each cell σ, which defines an element $I(F^{(r)}, \sigma) \in \pi_{2q-1}(W_r(m)) \simeq \mathbb{Z}$. The generators of $\pi_{k-1}(W_r(m))$ being consistent (see [St]), this defines a cochain

$$\gamma \in C^{2q}(M; \pi_{2q-1}(W_r(m)))$$

by $\gamma(\sigma) = I(F^{(r)}, \sigma)$, for each $2q$-cell σ, and then extend it by linearity. This cochain is actually a cocycle and the cohomology class that it represents is the q-th Chern class $c^q(M)$ of M in $H^{2q}(M)$. It is independent of the various choices involved in its definition. Note that $c^m(M)$ coincides with the Euler class of the underlying real tangent bundle $T_{\mathbb{R}}M$.

We give now an interpretation of the index $I(F^{(r)}, \sigma)$ which will be useful in the following. Let us write $F^{(r)} = (F^{(r-1)}, v_r)$ where the last vector is individualized and suppose that $F^{(r)}$ is already defined on $\partial\sigma$. There is no obstruction to extend the $(r-1)$-frame $F^{(r-1)}|_{\partial\sigma}$ within σ, because the dimension of the obstruction for such an extension is $2(m - (r-1) + 1) = \dim \sigma + 2$.

The $(r-1)$-frame obtained $F^{(r-1)}|_\sigma$ generates a complex subbundle G^{r-1} of rank $(r-1)$ of $TM|_\sigma$, i.e.,

$$TM|_\sigma \simeq G^{r-1} \oplus H^q$$

where H^q is an orthogonal complement of (complex) rank $q = m - (r-1)$. The obstruction to extend the last vector v_r inside σ as a non zero section of H^q is given by an element of $\pi_{2q-1}((\mathbb{C}^q)^*) \simeq \mathbb{Z}$ corresponding to the composition of the map $v_r : \partial\sigma \simeq S^{2q-1} \to H^q|_\Omega$ with the projection on the fibre $(\mathbb{C}^q)^*$. Let us denote by $I_{H^q}(v_r, \sigma)$ the obtained integer.

The obstruction to extend the r-frame $F^{(r)}|_{\partial\sigma}$ inside σ as an r-frame tangent to M is the same as the obstruction to extend the last vector v_r inside σ as a non zero section of H^q. In fact, there is a natural isomorphism $\pi_{2q-1}(W_r(m)) \simeq \pi_{2q-1}((\mathbb{C}^q)^*)$ (for compatible orientations) and by this isomorphism, we have the equality of integers

$$I(F^{(r)}, \sigma) = I_{H^q}(v_r, \sigma)$$

Another choice of $F^{(r-1)}$ gives another choices of v_r and of H^q but all H^q are homotopic and the obtained index is the same.

Remark 1.5. Let us remark that, on a given $2q$-cell σ contained in an open set $\Omega \subset \mathbb{C}^m$, it is possible to proceed in a more precise way. Let us denote by s the $2(r-1)$-simplex of (K) whose dual is σ and by $\hat{s} = s \cap \sigma$ the barycenter of s. Consider, in this point, a complex $(r-1)$-frame $F^{(r-1)}(\hat{s}) = (e_1, e_2, \dots, e_{r-1})$ which generates $T_{\hat{s}}(s)$, i.e., the (real) tangent plane in $\hat{s}$ to s (it has the good dimension). Considering, for $F^{(r-1)}|_\sigma$ an $(r-1)$-frame parallel to $F^{(r-1)}(\hat{s})$, (and considering if necessary a smoothing of the cell σ), the orthogonal complement H^q is then identified with the tangent bundle to σ and, in this case, $I_{H^q}(v_r, \sigma)$ is the usual $I(v_r, \sigma)$.

Suppose now that (L) is a sub-complex of (D), whose realisation $|L|$ is also denoted by L. Assume that we are already given an r-frame $F^{(r)}$ on the $2q$-skeleton of L, denoted by $L^{(2q)}$. The same arguments as before tell that we can always extend $F^{(r)}$ without singularity to $L^{(2q)} \cup D^{(2q-1)}$. If we want to extend this frame to the $2q$-skeleton of (D) we meet an obstruction for each corresponding cell which is not in (L). This gives rise to a cochain which vanishes on L and represents the relative Chern class

$$c^q(M, L; F^{(r)}) \in H^{2q}(M, L),$$

whose image by the natural map in $H^{2q}(M)$ is the usual Chern class, but as a relative class it does depend on the choice of the frame $F^{(r)}$ on L.

If we have two frames $F_1^{(r)}$ and $F_2^{(r)}$ on $L^{(2q)}$, the difference between the corresponding classes is given by the difference cocycle of the frames on L : in the product $L \times I$, suppose $F_1^{(r)}$ is defined at the level $L \times \{0\}$ and $F_2^{(r)}$ is defined at the level $L \times \{1\}$, then the difference cocycle $d(F_1^{(r)}, F_2^{(r)})$ is well defined in

$$H^{2q}(L \times I, L \times \{0\} \cup L \times \{1\}) \simeq H^{2q-1}(L),$$

as the obstruction to the extension of the given sections on the boundary of $L \times I$ ([St] §33.3). As shown in [St], we have the following formula :

$$c^q(M, L; F_2^{(r)}) = c^q(M, L; F_1^{(r)}) + \delta d(F_1^{(r)}, F_2^{(r)}),$$

where $\delta : H^{2q-1}(L) \longrightarrow H^{2q}(M, L)$ is the connecting homomorphism. Also, for three frames $F_1^{(r)}$, $F_2^{(r)}$ and $F_3^{(r)}$ as above, we have

$$(1.6) \qquad d(F_1^{(r)}, F_3^{(r)}) = d(F_1^{(r)}, F_2^{(r)}) + d(F_2^{(r)}, F_3^{(r)}).$$

Let S be a compact (K)-subcomplex of M, and $\tilde{U}$ a neighborhood of S. Let $\tilde{T}$ be a cellular tube in $\tilde{U}$ around S. Take an r-field $F^{(r)}$ defined on $D^{(2q)}$, possibly with singularities. We suppose that the only singularities inside $\tilde{U}$ are located on S. This implies that $F^{(r)}$ has no singularity on $(\partial\tilde{T})^{(2q)}$ so there is a well defined relative Chern class,

$$c^q(\tilde{T},\partial\tilde{T};F^{(r)}) \in H^{2q}(\tilde{T},\partial\tilde{T}).$$

Taking the image of this class by the isomorphism $H^{2q}(\tilde{T},\partial\tilde{T}) \simeq H^{2q}(\tilde{T},\tilde{T}-S)$ followed by the Alexander duality

$$\psi_M : H^{2q}(\tilde{T},\tilde{T}-S) \xrightarrow{\sim} H_{2r-2}(S),$$

we get a class

$$\psi_M(c^q(\tilde{T},\partial\tilde{T};F^{(r)})) \in H_{2r-2}(S),$$

that we call the *Poincaré-Hopf class* $\mathrm{PH}(F^{(r)},S)$ of $F^{(r)}$ at S .

Note that if $\dim S < 2r-2$, then $\mathrm{PH}(F^{(r)},S)=0$.

The relation between the Poincaré-Hopf class of $F^{(r)}$ and the index we defined above is the following :

$$\mathrm{PH}(F^{(r)},S)=\sum I(F^{(r)},\sigma(s))s$$

where the sum runs over the $2r-2$-simplices s of the triangulation of S and $\sigma(s)$ is the dual cell of s (of dimension $2q$). This relation is a consequence of the combinatorial definition of the Alexander duality [Br]. In particular, when $r=1$, then $F^{(1)}=\{v\}$ and $\mathrm{PH}(F^{(1)},S)\in H_0(S)$ is identified with the integer $\mathrm{PH}(v,S)$ previously defined.

Let M be a compact almost complex $2m$-manifold possibly with boundary and let $F^{(r)}$ be an r-field on the $(2q)$-skeleton of M, with singularities located on a compact subcomplex Σ in the interior of M. On the $(2q)$-skeleton of ∂M, we have a well defined r-frame $F^{(r)}$. Let $(S_\alpha)_\alpha$ be the connected components of Σ. Then, from the definitions, we have

$$\text{(1.7)} \qquad \sum_\alpha (i_\alpha)_* \,\mathrm{PH}(F^{(r)},S_\alpha)=c_{r-1}(M;F^{(r)}) \qquad \text{in } H_{2r-2}(M),$$

where $i_\alpha : S_\alpha \hookrightarrow M$ is the inclusion and we set

$$c_{r-1}(M;F^{(r)})=\psi_M(c^q(M,\partial M;F^{(r)}))=c^q(M,\partial M;F^{(r)})\frown[M,\partial M].$$

In particular, the sum of the Poincaré-Hopf classes is determined by the behavior of $F^{(r)}$ near ∂M and does not depend on the extension to the interior of M. Note that we may assume that $F^{(r)}$ is non-singular on $D^{(2q-1)}$. If $r=1$, $F^{(1)}=\{v\}$, and if v is everywhere transverse to ∂M, (1.7) reduces to (1.3).

2. Schwartz indices and classes

The basic references for this section are [Sc1, BS]. As before, all homology and cohomology groups are relative to integral coefficients.

2.1. The Alexander homomorphisms

We now consider a complex analytic n-variety (i.e., a reduced complex analytic space) V embedded in a complex m-manifold M. The singular set of V is denoted $\mathrm{Sing}(V)$ and the regular one $V_0 = V - \mathrm{Sing}(V)$. If V is reducible, we assume it is pure dimensional. If V is compact, we denote by $[V]$ the sum of the fundamental classes of the irreducible components. Denote by $\{V_i\}$ a Whitney stratification of M compatible with V and $\mathrm{Sing}(V)$.

Let S denote a compact connected subset of V either contained in the regular part V_0, or a component of $\mathrm{Sing}(V)$.

We suppose now that the triangulation (K) of M is compatible with the stratification of M and with S, and we denote by (D) a dual cell decomposition of M obtained from a barycentric subdivision of (K). We assume (K) makes M a PL manifold and we endow (K) with an orientation compatible with the orientation of the stratification $\{V_i\}$. This determines an orientation for (D) (see [Br]).

Let $\tilde{U}$ be an open neighborhood of S in M, denote $U = \tilde{U} \cap V$, and assume that $U - S$ is in V_0. According to Lemma 1.2, the union of (D)-cells which are dual of (K)-simplices contained in S is a regular tube $\tilde{\mathcal{T}}$ around S in M, contained in $\tilde{U}$ (taking a subtriangulation if necessary). On the other hand, $\partial\tilde{\mathcal{T}}$ is transverse to V_0. The intersection $\mathcal{T} = \tilde{\mathcal{T}} \cap V$ is no more a cellular tube in V, but satisfies the properties (i) and (ii) of Lemma 1.2. We write $\partial\mathcal{T} = V \cap \partial\tilde{\mathcal{T}}$, this is a hypersurface in V_0.

Definition 2.1. A neighborhood $\mathcal{T}$ of S in V, contained in U and satisfying the properties (i) and (ii) of Lemma 1.2 will be called a *tube* in U around S.

We already used the Alexander isomorphism, for $0 \leq q \leq m$

$$\psi_M : H^{2q}(\tilde{\mathcal{T}}, \tilde{\mathcal{T}} - S) \simeq H^{2q}(\tilde{\mathcal{T}}, \partial\tilde{\mathcal{T}}) \xrightarrow{\sim} H_{2m-2q}(S)$$

when S is a compact subcomplex of the complex m-manifold M. In the situation $S \subset V \subset M$ considered now, there is an Alexander homomorphism (in general not an isomorphism), for any $0 \leq p \leq n = \dim_{\mathbb{C}} V$,

$$\psi_V : H^{2p}(\mathcal{T}, \mathcal{T} - S) \simeq H^{2p}(\mathcal{T}, \partial\mathcal{T}) \longrightarrow H_{2n-2p}(S),$$

defined in the following way : let us denote by $[\mathcal{T}, \partial\mathcal{T}]$ the fundamental class of $\mathcal{T}$ in $H_{2n}(\mathcal{T}, \partial\mathcal{T})$, then

$$\psi_V(c) = r_*(c \frown [\mathcal{T}, \partial\mathcal{T}]),$$

where the cap-product is the internal one

$$H^{2p}(\mathcal{T}, \partial\mathcal{T}) \times H_{2n}(\mathcal{T}, \partial\mathcal{T}) \longrightarrow H_{2n-2p}(\mathcal{T}),$$

and $r_* : H_{2n-2p}(\mathcal{T}) \to H_{2n-2p}(S)$ is induced by the retraction $r : \mathcal{T} \to S$.

The Alexander homomorphism ψ_V corresponds, at the chain level to the composition

$$C^{2p}_{(\hat{K})}(\mathcal{T}, \partial\mathcal{T}) \xrightarrow{\tau} C^{2q}_{(D)}(\tilde{\mathcal{T}}, \partial\tilde{\mathcal{T}}) \xrightarrow{\psi_M} C^{(K)}_{2m-2q}(S),$$

with $2m - 2q = 2n - 2p$, where the map τ is defined by $< \tau(c), \sigma > = < c, \sigma \frown V >$ for a $2q$-cell σ in (D) and the subscripts and superscripts indicate the complex considered (see [Br]). The map τ defines a map

$$\tau : H^{2p}(\mathcal{T}, \partial\mathcal{T}) \to H^{2q}(\tilde{\mathcal{T}}, \partial\tilde{\mathcal{T}})$$

which, via the isomorphisms $H^{2p}(\mathcal{T}, \partial\mathcal{T}) \simeq H^{2p}(\mathcal{T}, \mathcal{T} - S) \simeq H^{2p}(U, U - S)$ in V and the corresponding ones in M, provides a homomorphism, still denoted

$$\tau : H^{2p}(U, U - S) \to H^{2q}(\tilde{U}, \tilde{U} - S) \tag{2.2}$$

and called Thom-Gysin homomorphism. In general, it is neither injective, nor surjective. There is a commutative diagram :

$$\begin{array}{ccccc} H^{2q}(\tilde{U}, \tilde{U} - S) & \xrightarrow[\simeq]{} & H^{2q}(\tilde{\mathcal{T}}, \partial\tilde{\mathcal{T}}) & \xrightarrow[\simeq]{\psi_M} & H_{2m-2q}(S) \\ \uparrow{\scriptstyle \tau} & & \uparrow{\scriptstyle \tau} & & \uparrow{\scriptstyle =} \\ H^{2p}(U, U - S) & \xrightarrow[\simeq]{} & H^{2p}(\mathcal{T}, \partial\mathcal{T}) & \xrightarrow{\psi_V} & H_{2n-2p}(S). \end{array}$$

M.-H. Schwartz proved that, given the Whitney stratification $\{V_i\}$ of M, one can construct a continuous vector field v_0 on M, which is tangent to the strata, with isolated singularities and *radial* in a sense which she made precise. One of the fundamental properties of a radial vector field v_0 is that, if p is an isolated singularity of v_0, then the indices at p of v_0 with respect to M and to the stratum containing p are the same. We refer to [Sc1, Sc2, BS] for details. Applying this construction to the stratification $\{V_i\}$, we obtain a radial vector field tangent to V, i.e., leaving V invariant.

An important fact we use in this article is that, for a compact connected component S of Sing(V), we may consider a compact regular tube (arbitrarily small) $\tilde{\mathcal{T}}$ of S in M and a radial vector field v_0, tangent to V, so that v_0 is non-singular on $\tilde{\mathcal{T}} - S$ and is everywhere pointing outwards $\partial\tilde{\mathcal{T}}$ (it is called "sortant de $\tilde{\mathcal{T}}$" in [Sc2]). Thus, for every radial vector field v_0, the sum of the indices of the singularities of v_0 in $\tilde{\mathcal{T}}$ is equal to

$$\mathrm{PH}(v_0, S) = \chi(S),$$

and is independent of the ambient manifold M. This is a fundamental property of the Schwartz radial vector fields. For simplicity, we call such a vector field v_0

a *radial vector field outbound from* S and a pair $(\tilde{\mathcal{T}}, v_0)$ a *radial pair around* S. Although it is not necessary here, we may assume furthermore that $\partial\tilde{\mathcal{T}}$ is C^∞ [Hi] and, in this case, v_0 is transverse to $\partial\tilde{\mathcal{T}}$.

Recall from section 1 above, that for two continuous vector fields v_1 and v_2, non-singular on $U - S$, the difference $d(v_1, v_2)$ is defined in $H^{2n-1}(\partial\mathcal{T})$. Let $\delta : H^{2n-1}(\partial\mathcal{T}) \to H^{2n}(\mathcal{T}, \partial\mathcal{T})$ be the connecting homomorphism, we denote

$$d_S(v_1, v_2) = \psi_V \delta\, d(v_1, v_2)$$

Definition 2.3. For a continuous vector field v, non-singular on $U - S$, we define the *generalized Schwartz index* of v at S, denoted $\operatorname{Sch}(v, S)$, as follows :

$$\operatorname{Sch}(v, S) = \begin{cases} \operatorname{PH}(v, S), & \text{if } S \text{ is in } V_0, \text{ with PH computed in } V_0 \\ \chi(S) + d_S(v_0, v), & \text{if } S \text{ is in } \operatorname{Sing}(V), \end{cases}$$

where v_0 is a radial vector field outbound from U.

In particular, we have $\operatorname{Sch}(v_0, S) = \chi(S)$ for radial vector fields. From the definition and (1.6) we have, for two vector fields v_1 and v_2 as above,

$$\operatorname{Sch}(v_2, S) = \operatorname{Sch}(v_1, S) + d_S(v_1, v_2). \tag{2.4}$$

For a continuous vector field v whose domain of definition is in the regular part V_0 of V, we denote by $(S_\alpha)_\alpha$ the connected components of $\operatorname{Sing}(V) \cup \operatorname{Sing}(v)$, where $\operatorname{Sing}(v)$ is the singular set of v in V_0. In what follows we assume that each component S_α is either in V_0 or in $\operatorname{Sing}(V)$ and that it admits a neighborhood disjoint from each other. We have the following theorem, which is a generalization of Poincaré-Hopf Theorem and [Sc2]. It extends Theorem 1.2 in [SS2] to the non-isolated singularity case.

Theorem 2.5. *Let V be a compact complex analytic variety embedded in some complex manifold M and v a continuous vector field on the regular part of V. Let S_α be defined as above, then we have*

$$\sum_\alpha \operatorname{Sch}(v, S_\alpha) = \chi(V).$$

Proof. The theorem is proved as Theorem 1.2 in [SS2]. Let $(S_{\alpha'})_{\alpha'}$ be the connected components of $\operatorname{Sing}(V)$. For each α', take an open neighborhood $\tilde{U}_{\alpha'}$ of $S_{\alpha'}$ in M so that it does not intersect with the other components of $\operatorname{Sing}(v)$. Also, for each α', let $(\tilde{\mathcal{T}}_{\alpha'}, v_{\alpha'})$ be a radial pair in $\tilde{U}_{\alpha'}$ around $S_{\alpha'}$. Set $V^* = V - \bigcup_{\alpha'} \operatorname{Int}(\tilde{\mathcal{T}}_{\alpha'})$ and let v_0 be the vector field on (a neighborhood of) $\partial V^* = \bigcup_{\alpha'} \partial\mathcal{T}_{\alpha'}$, $\partial\mathcal{T}_{\alpha'} = \partial\tilde{\mathcal{T}}_{\alpha'} \cap V$, which coincides with $v_{\alpha'}$ on $\partial\mathcal{T}_{\alpha'}$. Then, since V is even real dimensional, we have

$$\chi(V) = \chi(V^*) + \sum_{\alpha'} \chi(\mathcal{T}_{\alpha'}).$$

Thus the theorem follows from

$$\chi(V^*) = \mathrm{PH}(v_0, V^*) = \mathrm{PH}(v, V^*) - \sum_{\alpha'} d_{S_\alpha}(v, v_{\alpha'}) \qquad \text{and}$$
$$\chi(\mathcal{T}_{\alpha'}) = \chi(S_{\alpha'}) = \mathrm{Sch}(v, S_{\alpha'}) + d_{S_\alpha}(v, v_{\alpha'}). \qquad \square$$

2.2. The Schwartz classes

The above considerations can be generalized to the "higher" Schwartz-MacPherson classes as follows. With the previous stratification and triangulation of M, let us remark that the cells of (D) are transverse to each stratum, so that, for every stratum V_i of complex dimension d and every cell of (real) dimension $2q = 2(m - r + 1)$, $V_i \cap \sigma$ is a cell of dimension $2k = 2(d - r + 1)$. In particular $V_i \cap \sigma$ is empty when $d < r - 1$.

For each point $x \in M$, let T_xM be the tangent space of M at x. Let V_i be the stratum that contains x and let T_xV_i be the corresponding tangent space, so that T_xV_i is a subspace of T_xM. Let us denote $\tilde{T}(M) = \bigcup T_xV_i \subset TM$. Thus one has a projection $\pi : \tilde{T}(M) \longrightarrow M$, which is the restriction to $\tilde{T}(M)$ of the usual projection $TM \longrightarrow M$.

Definition 2.6. Let A be a subspace of M. A *stratified* vector field on A is a section of $\tilde{T}(M)$ over A, i.e., a continuous section v of TM on A such that at each point $x \in V_i \cap A$, $v(x)$ is tangent to the stratum V_i. A *stratified r-field* on A is an r-field $F^{(r)} = \{v_1, ..., v_r\}$ consisting of stratifed vector fields $v_1, \dots, v_r$. A stratified r-frame is a non singular stratified r-field.

Let us recall that the obstruction dimension to construct an r-frame tangent to M is $2q = 2(m - r + 1)$, and the index $I(F^{(r)}, \sigma) \in \pi_{2q-1}(W_r(m))$ is well defined for every $2q$-cell σ. The obstruction dimension to construct an r-frame tangent the stratum V_i of (complex) dimension d is $2k = 2(d - r + 1)$, so, if $F^{(r)}$ is a stratified r-field, the index $I(F^{(r)}, V_i \cap \sigma) \in \pi_{2k-1}(W_r(d))$ is well defined, for the same cell σ, by the above considerations of dimensions. In general these two indices are different.

The Schwartz classes are the obstruction to constructing such stratified frames on V. We will denote the r-fields by $F^{(r)} = (F^{(r-1)}, v_r)$, individualizing the last vector field in $F^{(r)}$. Then one knows [Sc1, BS] :

Lemma 2.7. *One can construct on the $(2q-1)$-skeleton $(D)^{(2q-1)}$ of (D) a stratified r-frame, called radial r-frame, $F_0^{(r)} = (F_0^{(r-1)}, v_r)$ which satisfies :*
(i) $F_0^{(r-1)}$ is the restriction to $(D)^{(2q-1)}$ of an $(r-1)$-frame, still denoted by $F_0^{(r-1)}$, defined on the $2q$-skeleton $(D)^{(2q)}$.
(ii) $F_0^{(r)}$ extends to $(D)^{(2q)}$ as an r-field with isolated singularities, which are the singularities of v_r.
(iii) If $F_0^{(r)}$ has singularities in a $2q$-cell σ which intersects several strata of M, then all these singularities are in the stratum V_i of the lowest dimension.

(iv) *The index of* $F_0^{(r)}$ *in* σ : $I(F_0^{(r)}, \sigma)$ *is equal to the index of its restriction along* V_i : $I(F_0^{(r)}, V_i \cap \sigma)$. *If furthermore the complex dimension of* V_i *is* $r-1$ *(the minimal possible value), then* $I(F_0^{(r)}, a) = 1$.

Remark 2.8. In fact, we can take all singularities of $F_0^{(r)}$ as the centers of the cells σ, so that there is only one singular point in each cell.

The radial r-frames $F_0^{(r)}$ are pointing outwards from certain regular neighborhoods of the strata V_i and from the cellular tube $\tilde{\mathcal{T}}_V$ around V, this motivates the terminology.

The radial r-frame $F_0^{(r)}$ determines a $2q$-cochain $\tilde{c}_q \in C^{2q}(\tilde{\mathcal{T}}_V, \partial\tilde{\mathcal{T}}_V)$ on M as follows : If σ is a $2q$-cell that does not intersect V, then $\tilde{c}^q(\sigma) = 0$; If σ intersects V, then

$$\tilde{c}^q(\sigma) = I(F_0^{(r)}, \sigma).$$

Then one extends $\tilde{c}^q$ as a cochain by linearity.

It is proved in [Sc1, BS] that this cochain is actually a cocycle, representing a cohomology class $\tilde{c}^q(V) \in H^{2q}(\tilde{\mathcal{T}}_V, \partial\tilde{\mathcal{T}}_V) \simeq H^{2q}(M, M-V)$. This class does not depend on the choices of the Whitney stratification of M, the triangulations, nor the r-frame $F^{(r)}$, so long as it is radial (see [Sc1] and [Sc3]). This result is also an easy consequence of Theorem 2.10 below.

Definition 2.9. The class $\tilde{c}^q(V) \in H^{2q}(M, M-V)$ is the q-th Schwartz class of V.

Theorem 2.10 [BS]. *For each* $r = 1, \dots, n$ *and* $q = m-r+1$, *the image* $\psi_M \tilde{c}^q(V)$ *of the Schwartz class by the Alexander duality* $\psi_M : H^{2q}(M, M-V) \xrightarrow{\sim} H_{2r-2}(V)$ *is the corresponding MacPherson class of* V *(defined in [Ma]),*

$$c_{r-1}(V) \in H_{2r-2}(V).$$

2.3. The localized Schwartz classes

Now denote by S a compact connected (K)-subcomplex of V such that $S \cap D^{(2q)}$ is either a subset of V_0 or a component of $\mathrm{Sing}(V)$. Let us denote by U a neighborhood of S in V such that $U - S$ still intersects $(D)^{(2q)}$ in V_0. We write $p = n - r + 1$, thus we have $q - p = m - n$.

It follows from 2.7 that there exist stratified r-fields on $(D)^{(2q)} \cap U$ whose singularities are all located on S. Let $F_1^{(r)}$ and $F_2^{(r)}$ be two such r-fields, and let us consider a tube $\mathcal{T}$ in U around S. There is a well defined secondary characteristic class $d(F_1^{(r)}, F_2^{(r)}) \in H^{2p-1}(\partial\mathcal{T})$, called the *difference* and defined as in section 1. Let $\delta : H^{2p-1}(\partial\mathcal{T}) \to H^{2p}(\mathcal{T}, \partial\mathcal{T})$ be the connecting homomorphism and ψ_V : $H^{2p}(\mathcal{T}, \partial\mathcal{T}) \to H_{r-1}(S)$ the Alexander homomorphism, we denote

$$d_S(F_1^{(r)}, F_2^{(r)}) = \psi_V \delta\, d(F_1^{(r)}, F_2^{(r)}).$$

Definition 2.11. For an r-frame $F^{(r)}$ on $(D)^{(2q)} \cap (U - S)$, we define the Schwartz class $\mathrm{Sch}(F^{(r)}, S)$ of $F^{(r)}$ at S to be the class in $H_{2r-2}(S)$ given by :

$$\mathrm{Sch}(F^{(r)}, S) = \begin{cases} \mathrm{PH}(F^{(r)}, S) & \text{if } S \cap (D)^{(2q)} \subset V_0, \\ c_{r-1}(S) + d_S(F_0^{(r)}, F^{(r)}) & \text{if } S \subset \mathrm{Sing}(V), \end{cases}$$

where PH is computed in V_0 and $F_0^{(r)}$ is a radial frame.

In particular, for a radial frame $F_0^{(r)}$, $\mathrm{Sch}(F_0^{(r)}, S)$ is the Chern-Schwartz-MacPherson class $c_{r-1}(S) \in H_{2r-2}(S)$.

From the definition and (1.6), we get, for two r-frames $F_1^{(r)}$ and $F_2^{(r)}$ on $(D)^{(2q)} \cap (U - S)$,

$$\mathrm{Sch}(F_2^{(r)}, S) = \mathrm{Sch}(F_1^{(r)}, S) + d_S(F_1^{(r)}, F_2^{(r)}), \tag{2.12}$$

which generalizes (2.4).

Let us consider now a neighborhood U of $\mathrm{Sing}(V)$ in V. We know already that there exist stratified r-fields on $(D)^{(2q)} \cap U$ whose singularities are all in $\mathrm{Sing}(V)$. Elementary obstruction theory [St] then tells us that every such r-field can be extended to all of $(D)^{(2q)} \cap V_0$ with a singular set which is a subcomplex of V_0. More generally, let Σ be a compact (K)-subcomplex in V_0 disjoint from a neighborhood U_1 of $\mathrm{Sing}(V)$ in V. We denote by (S_α) the connected components of $\mathrm{Sing}(V) \cup \Sigma$ and set $V^* = V - U_1$. Let i_α and ι be the inclusions $S_\alpha \hookrightarrow V$ and $V^* \hookrightarrow V$, respectively. The second one induces a homomorphism ι_* in homology with compact supports. The following theorem follows from (1.7), the Schwartz construction and arguments similar to the ones for Theorem 2.5.

Theorem 2.13. *Let V be a compact complex analytic n-variety embedded in a complex m-manifold M and let Σ be a subcomplex in V_0 as above. For any stratified r-frame $F^{(r)}$ on $(D)^{(2q)} \cap (V_0 - \Sigma)$, $q = m - r + 1$, we have*

$$\sum_\alpha (i_\alpha)_* \mathrm{Sch}(F^{(r)}, S_\alpha) = c_{r-1}(V).$$

Thus, decomposing the previous summation according to the fact that S_α is in $\mathrm{Sing}(V)$ or in Σ, we get :

$$\sum_{S_\alpha \subset \mathrm{Sing}(V)} (i_\alpha)_* \mathrm{Sch}(F^{(r)}, S_\alpha) + \iota_* c_{r-1}(V^*; F^{(r)}) = c_{r-1}(V),$$

where the sum is taken over the connected components of $\mathrm{Sing}(V)$. In particular, for a radial r-frame $F_0^{(r)}$, we have :

$$c_{r-1}(V) = \sum_{S_\alpha \subset \mathrm{Sing}(V)} (i_\alpha)_* c_{r-1}(S_\alpha) + \iota_* c_{r-1}(V^*; F_0^{(r)}).$$

3. Differential geometric viewpoint

In this section, and from now on, all homology and cohomology groups will be with real coefficients; the (co)homology classes previously defined with integral coefficients will be looked in this context.

In general, for a Chern polynomial φ, i.e., a polynomial on the Chern classes, and a connection ∇ for a complex C^∞ vector bundle E, we denote by $\varphi(\nabla)$ the cocycle on the base space which is the image of φ by the Chern-Weil homomorphism associated with ∇. It is a closed form whose class is the characteristic class $\varphi(E)$ of the bundle E with respect to φ. In particular, the class of $c^i(\nabla)$ is the i-th Chern class $c^i(E)$. If $(\nabla_0, \nabla_1, \ldots, \nabla_r)$ is a family of $r+1$ connections for the same vector bundle, $\varphi(\nabla_0, \nabla_1, \ldots, \nabla_r)$ will denote the Bott difference operator [B], so that

$$(3.1) \qquad d\varphi(\nabla_0, \nabla_1, \ldots, \nabla_r) = \sum_{i=0}^{r} (-1)^i \varphi(\nabla_0, \nabla_1, \ldots, \widehat{\nabla}_i, \cdots, \nabla_r).$$

In particular, for $r = 1$, $d\varphi(\nabla_0, \nabla_1) = \varphi(\nabla_1) - \varphi(\nabla_0)$.

Now let $S \subset V \subset M$ as defined in section 2. Let $\tilde{U}$ be a neighborhood of S in M such that $U - S$ is in V_0, with $U = \tilde{U} \cap V$. Suppose we have an r-frame $F^{(r)}$ on $(D)^{(2q)} \cap (U - S)$, $q = m - r + 1$, we may describe the Schwartz class $\mathrm{Sch}(F^{(r)}, S)$ of $F^{(r)}$ at S as follows.

We take a cellular tube $\tilde{\mathcal{T}}$ around S in $\tilde{U}$ with C^∞ boundary $\partial\tilde{\mathcal{T}}$, so that $\mathcal{T} = \tilde{\mathcal{T}} \cap V$ is a tube around S in U. As it is a (D)-subcomplex, $\partial\tilde{\mathcal{T}}$ is transverse to V.

First we consider the case where S is in the regular part V_0 of V. Assume $F^{(r)} = \{v_1, \cdots, v_r\}$ to be an r-frame on $(D)^{(2q)} \cap (U - S)$, let ∇_0 be a connection for TV_0 on U and let ∇ be an $F^{(r)}$-trivial connection for TV_0 on a neighborhood W of $(D)^{(2q)} \cap (U - S)$ in U. Then the image $c^q(\tilde{U}, \tilde{U} - S; F^{(r)})$ of the class $c^p(U, U - S; F^{(r)})$ (defined in section 1) by the Thom-Gysin homomorphism (2.2) is represented by the cocycle

$$(3.2) \qquad \gamma \mapsto \int_{\gamma\cap\mathcal{T}} c^p(\nabla_0) + \int_{\gamma\cap\partial\mathcal{T}} c^p(\nabla_0, \nabla),$$

for a relative $2q$-cycle γ in $\tilde{U}$ modulo $\tilde{U} - S$ consisting of cells in (D) (cf. [Leh]). Recall that the Poincaré-Hopf class $\mathrm{PH}(F^{(r)}, S)$ is given by $\psi_V\, c^p(U, U - S; F^{(r)}) = \psi_M\, c^q(\tilde{U}, \tilde{U} - S; F^{(r)})$, where ψ_V and ψ_M denote the Alexander homomorphism and the Alexander isomorphism. In particular, when $r = 1$, $F^{(1)} = \{v\}$, we have

$$(3.3) \qquad \mathrm{PH}(v, S) = \int_{\mathcal{T}} c^n(\nabla_0) + \int_{\partial\mathcal{T}} c^n(\nabla_0, \nabla).$$

Now we consider the case where S may be in $\mathrm{Sing}(V)$. We first give a differential geometric formula for the difference. Let $F_1^{(r)}$ and $F_2^{(r)}$ be two r-frames on $(D)^{(2q)} \cap (U - S)$. For $i = 1, 2$, let ∇_i be an $F_i^{(r)}$-trivial connection for TV_0 on a neighborhood W of $(D)^{(2q)} \cap (U - S)$ in U $(i = 1, 2)$.

Lemma 3.4. *The difference $\delta d(F_1^{(r)}, F_2^{(r)})$ is an element in $H^{2p}(U, U-S)$ whose image by the Thom Gysin homomorphism τ is represented by the cocycle*

$$\gamma \mapsto \int_{\gamma \cap \partial T} c^p(\nabla_1, \nabla_2),$$

for a relative $2q$-cycle γ in $\tilde{U}$ modulo $\tilde{U} - S$ consisting of cells in (D). In particular, when $r = 1$,

$$d_S(v_1, v_2) = \int_{\partial T} c^n(\nabla_1, \nabla_2).$$

Proof. Note that the lemma follows from (3.1) if S is in V_0. Let T' be another tube around S contained in the interior of T and set $C = T - T'$. Then $\partial C = \partial T - \partial T'$. Let $\tilde{\nabla}$ be a connection for TV_0 on $C \cap W$ which extends simultaneously ∇_1 on $\partial T' \cap W$ and ∇_2 on $\partial T \cap W$. Let $\tilde{\nabla}_1$ an $F_1^{(r)}$-trivial connection for TV_0 on $C \cap W$ which extends ∇_1 on $\partial T' \cap W$. Thus we have $c^p(\tilde{\nabla}_1) = 0$, hence $dc^p(\tilde{\nabla}_1, \tilde{\nabla}) = c^p(\tilde{\nabla})$.

Let us denote by $F^{(r)}$ the r-frame on the intersection of $(D)^{(2q)}$ with a neighborhood of ∂C such that $F^{(r)} = F_1^{(r)}$ on the intersection of $(D)^{(2q)}$ with a neighborhood of $\partial T'$ and $F^{(r)} = F_2^{(r)}$ on the intersection of $(D)^{(2q)}$ with a neighborhood of ∂T. Then we have $\delta d(F_1^{(r)}, F_2^{(r)}) = c^p(C, \partial C; F^{(r)})$. The image of this class by τ is represented by the cocycle $\gamma \to \int_{\gamma \cap C} c^p(\tilde{\nabla})$ for a relative $2q$-cycle γ. By the Stokes formula $\int_{\gamma \cap C} c^p(\tilde{\nabla}) = \int_{\gamma \cap \partial C} c^p(\tilde{\nabla}_1, \tilde{\nabla}) = \int_{\gamma \cap \partial T} c^p(\nabla_1, \nabla_2)$. □

Suppose S is a component of $\mathrm{Sing}(V)$ and let $F_0^{(r)}$ be a radial r-frame on $(D)^{(2q)} \cap (\tilde{U} - S)$. We may assume that $F_0^{(r)}$ is given on a neighborhood W of $(D)^{(2q)} \cap (\tilde{U} - S)$. Let $\tilde{\nabla}_0$ be a connection for TM on $\tilde{U}$ and $\tilde{\nabla}$ an $F_0^{(r)}$-trivial connection for TM on $\tilde{W}$. Then the class $c^q(\tilde{U}, \tilde{U} - S; F_0^{(r)}) \in H^{2q}(\tilde{U}, \tilde{U} - S)$, as defined in section 1, is represented by the cocycle

$$\gamma \mapsto \int_{\gamma \cap \tilde{T}} c^q(\tilde{\nabla}_0) + \int_{\gamma \cap \partial \tilde{T}} c^q(\tilde{\nabla}_0, \tilde{\nabla}) \tag{3.5}$$

for a relative $2q$-cycle γ in $\tilde{U}$ modulo $\tilde{U} - S$ consisting of cells in (D). The Schwartz class of $F_0^{(r)}$ at S is then given by $\mathrm{Sch}(F_0^{(r)}, S) = \psi_M \, c^q(\tilde{U}, \tilde{U} - S; F_0^{(r)})$, where $\psi_M : H^{2q}(\tilde{U}, \tilde{U} - S) \xrightarrow{\sim} H_{2r-2}(S)$ denotes the Alexander duality. In particular, when $r = 1$, $F^{(1)} = \{v_0\}$ with v_0 a radial vector field, and we have

$$\mathrm{Sch}(v_0, S) = \int_{\tilde{T}} c^m(\tilde{\nabla}_0) + \int_{\partial \tilde{T}} c^m(\tilde{\nabla}_0, \tilde{\nabla}). \tag{3.6}$$

For an r-frame $F^{(r)}$ on $(D)^{(2q)} \cap (U - S)$, we may describe its Schwartz class by combining Lemma 3.4 and the above description for $\mathrm{Sch}(F_0^{(r)}, S)$ (cf. Definition 2.10).

4. Characteristic classes for the virtual tangent bundle

As before, let V be a subvariety of dimension n in a complex manifold M of dimension m. Hereafter we assume that there exists a holomorphic vector bundle $E \longrightarrow M$ of rank $k = m - n$ over M and a holomorphic section s of E, generically transverse to the zero section, so that $V = s^{-1}(0)$. Thus V is a set-theoretic local complete intersection and the restriction $E|_{V_0}$ coincides with the (holomorphic) normal bundle N_{V_0} of the regular part V_0 of V. We set $N = E|_V$ and call $TM|_V - N$ the virtual tangent bundle of V. For example, this condition is satisfied in the following cases, with a naturally given bundle E :
(i) V is a hypersurface in M ($k = 1$). In this case, we may take as E the line bundle determined by the divisor V.
(ii) V is a set-theoretic complete intersection in M, i.e., is defined by k equations $f_\lambda = 0$, ($\lambda = 1, \cdots, k$), where the f_λ's denote holomorphic functions globally defined on M. In this case, we may take as E the trivial bundle.
(iii) V is a set-theoretic (projective algebraic) complete intersection in the projective space $\mathbb{CP}^m$. This means V is the zero set of k homogeneous polynomials F_λ, ($\lambda = 1, \cdots, k$). It is only locally a set-theoretic complete intersection in the previous sense while it is globally the intersection of the k algebraic hypersurfaces $F_\lambda = 0$. In this case, we may take as E the bundle $L^{d_1} \oplus \cdots \oplus L^{d_k}$, where L denotes the hyperplane bundle and the d_λ's the degrees of the homogeneous polynomials F_λ defining V (cf. also Remark 7.10 below).

Recall that the total Chern class $c(TM - E) \in H^*(M)$ of the virtual bundle $TM - E$ is given by $c(TM - E) = c(TM) \cdot c(E)^{-1}$. Hence the p-th Chern class $c^p(TM - E)$ is given as the coefficient of t^p in the expansion of

$$(1 + \sum_{i=1}^{m} t^i c^i(TM))\,(1 + \sum_{j=1}^{k} t^j c^j(E))^{-1}.$$

This polynomial may be written as a finite sum

$$c^p(TM - E) = \sum_{\ell} \varphi_\ell^{(p)}(c^1(TM), \ldots, c^m(TM)) \cdot \psi_\ell^{(p)}(c^1(E), \ldots, c^k(E)),$$

for suitable polynomials $\varphi_\ell^{(p)}$ and $\psi_\ell^{(p)}$.

Let ∇ and ∇' be connections for TM and E respectively, defined on some submanifold Ω of M. Denoting by $\nabla^\bullet$ the pair (∇, ∇'), we set

$$c^p(\nabla^\bullet) = \sum_{\ell} \varphi_\ell^{(p)}(\nabla) \cdot \psi_\ell^{(p)}(\nabla'),$$

where the product is the exterior product. Then $c^p(\nabla^\bullet)$ is a closed $2p$-form and defines the class $c^p(TM - E)$ on Ω.

If $(\nabla_0^\bullet, \dots, \nabla_r^\bullet)$ is a family of $r+1$ pairs of connections, $\nabla_j^\bullet = (\nabla_j, \nabla_j')$, we may construct $c^p(\nabla_0^\bullet, \dots, \nabla_r^\bullet)$ satisfying an identity similar to (3.1). There are several ways to construct such forms. One way is to do just as in the case of families of connections ([B]). Namely, let Δ^r be the standard simplex $\sum_{i=0}^r t_i = 1,\ t_i > 0$ in $\mathbb{R}^{r+1}$, and $\pi : \Omega \times \Delta^r \to \Omega$ the natural projection. We set $\tilde{\nabla}^\bullet = (\tilde{\nabla}, \tilde{\nabla}')$, where $\tilde{\nabla}$ and $\tilde{\nabla}'$ denote the connections on $\pi^{-1}(TM)$ and $\pi^{-1}(E)$ which are the usual affine combination of the connections ∇_j and ∇_j', respectively. Then, $c^p(\nabla_0^\bullet, \dots, \nabla_r^\bullet)$ is equal to the integration of $c^p(\tilde{\nabla}^\bullet)$ along the fiber Δ^r of the projection π.

In particular, if $\nabla_1^\bullet = (\nabla_1, \nabla_1')$ and $\nabla_2^\bullet = (\nabla_2, \nabla_2')$ are two such pairs,

$$d\,c^p(\nabla_1^\bullet, \nabla_2^\bullet) = c^p(\nabla_2^\bullet) - c^p(\nabla_1^\bullet). \tag{4.1}$$

Also, if $\nabla_1^\bullet = (\nabla_1, \nabla_1')$, $\nabla_2^\bullet = (\nabla_2, \nabla_2')$ and $\nabla_3^\bullet = (\nabla_3, \nabla_3')$ are three such pairs,

$$d\,c^p(\nabla_1^\bullet, \nabla_2^\bullet, \nabla_3^\bullet) = c^p(\nabla_2^\bullet, \nabla_3^\bullet) - c^p(\nabla_1^\bullet, \nabla_3^\bullet) + c^p(\nabla_1^\bullet, \nabla_2^\bullet). \tag{4.2}$$

Recall that there is an exact sequence of vector bundles on V_0 :

$$0 \longrightarrow TV_0 \longrightarrow TM|_{V_0} \stackrel{\pi}{\longrightarrow} N_{V_0} \longrightarrow 0. \tag{4.3}$$

Let Ω_0 be a subset in $V_0 \cap \Omega$. The pair $\nabla^\bullet = (\nabla, \nabla')$ will be said to be *compatible* on Ω_0 if, on Ω_0, the connection ∇' is obtained from ∇ by passing to the quotient : $\pi \circ \nabla = \nabla' \circ \pi$. This implies that ∇ preserves the subbundle $TV_0|_{\Omega_0}$ of TM. The induced connection for TV_0 will be denoted by ∇^V. Thus the triple $(\nabla^V, \nabla, \nabla')$ is compatible with (4.3) in the sense of [BB] 4.16.

Lemma 4.4. (i) *If $\nabla^\bullet$ is a compatible pair on Ω_0, then $c^p(\nabla^\bullet) = c^p(\nabla^V)$ on Ω_0.*
(ii) *If $\nabla_1^\bullet$ and $\nabla_2^\bullet$ are two compatible pairs on Ω_0, then $c^p(\nabla_1^\bullet, \nabla_2^\bullet) = c^p(\nabla_1^V, \nabla_2^V)$ on Ω_0.*

Proof. (i) is proved as in [BB] 4.22. To prove (ii), let $\varpi : \Omega_0 \times [0,1] \to \Omega_0$ be the projection and let $\tilde{\nabla}$ and $\tilde{\nabla}'$ be connections for $\varpi^* TM$ and $\varpi^* E$, respectively, given by $\tilde{\nabla} = t\nabla_2 + (1-t)\nabla_1$ and $\tilde{\nabla}' = t\nabla_2' + (1-t)\nabla_1'$. Then the pair $\tilde{\nabla}^\bullet = (\tilde{\nabla}, \tilde{\nabla}')$ is compatible on $\Omega_0 \times [0,1]$. Therefore, for the connection $\tilde{\nabla}^V = t\nabla_2^V + (1-t)\nabla_1^V$, we have $c^p(\tilde{\nabla}^V) = c^p(\tilde{\nabla}^\bullet) = \sum_\ell \varphi_\ell^{(p)}(\tilde{\nabla}) \cdot \psi_\ell^{(p)}(\tilde{\nabla}')$. The formula of (ii) is obtained by integration on $[0,1]$. $\square$

When V is compact, letting $r = n - p + 1$, the image of $c^p(TM|_V - N)$ by the Poincaré homomorphism $H^{2p}(V) \to H_{2(r-1)}(V)$ is denoted by

$$c_{r-1}(TM|_V - N) = c^p(TM|_V - N) \frown [V].$$

Remark 4.5. If V is a local complete intersection, i.e., if the local components of the section s defining V generate (locally) the ideal sheaf of holomorphic functions vanishing on V, then $c_{r-1}(TM|_V - N)$ coincides with the $(r-1)$-st component of the "canonical class" of W. Fulton and K. Johnson [F,FJ].

5. Virtual indices and classes

The virtual index of a vector field on a "strong" local complete intersection is introduced in [LSS]. First we recall the definition and fundamental properties in our situation. As in section 4, let V be a subvariety of dimension n in M defined by a holomorphic section, generically transverse to the zero section, of a holomorphic vector bundle E on M and set $N = E|_V$. Given a C^∞ vector field v on V, the idea is to localize the class $c^n(TM|_V - N)$ of the virtual tangent bundle of V at the singular set of v. Note that the holomorphic tangent bundle of a complex manifold is canonically isomorphic, as a real bundle, to the real tangent bundle of the underlying C^∞ manifold. Thus we may think of v as a C^∞ section of TV_0.

Let S be either a compact connected set in V_0 or a compact connected component of $\mathrm{Sing}(V)$. Also let $\tilde{U}$ be a neighborhood of S in M such that $U - S$ is in V_0, $U = \tilde{U} \cap V$. For a C^∞ vector field v non-singular on $U - S$, we define the *virtual index* $\mathrm{Vir}(v, S)$ of v at S as follows. We consider a cellular tube $\tilde{\mathcal{T}}$ as in section 3 so that $\mathcal{T} = \tilde{\mathcal{T}} \cap V$ is a tube around S in U.

Recall that, if S is in V_0, letting ∇_0 be a connection for TV_0 on U and ∇ a v-trivial connection for TV_0 on $U - S$, $\mathrm{PH}(v, S)$ is given by (3.3). If we assume now that S may be in $\mathrm{Sing}(V)$, the above definition of the index still makes sense, if we use the virtual tangent bundle $TM|_V - N$ instead of TV_0, and if we replace ∇_0 by a pair $\nabla_0^\bullet = (\nabla_0, \nabla_0')$ of connections for TM and E on $\tilde{U}$, and ∇ by some v-trivial compatible pair $\nabla^\bullet = (\nabla, \nabla')$ on $U - S$ (v-trivial means that ∇ is v-trivial). Thus we set

$$I(v, \mathcal{T}; \nabla_0^\bullet, \nabla^\bullet) = \int_{\mathcal{T}} c^n(\nabla_0^\bullet) + \int_{\partial\mathcal{T}} c^n(\nabla_0^\bullet, \nabla^\bullet).$$

Lemma 5.1. *The above number $I(v, \mathcal{T}; \nabla_0^\bullet, \nabla^\bullet)$ does not depend on $\nabla_0^\bullet$, $\nabla^\bullet$ or $\mathcal{T}$.*

Proof. If $D_0^\bullet = (D_0, D_0')$ denotes another pair of connections on $\tilde{U}$, then we have, by (4.1), (4.2) and the Stokes formula,

$$I(v, \mathcal{T}; D_0^\bullet, \nabla^\bullet) - I(v, \mathcal{T}; \nabla_0^\bullet, \nabla^\bullet) = \int_{\mathcal{T}} dc^n(\nabla_0^\bullet, D_0^\bullet) - \int_{\partial\mathcal{T}} c^n(\nabla_0^\bullet, D_0^\bullet) = 0.$$

If $D^\bullet = (D, D')$ denotes another v-trivial compatible pair on $U - S$, then, by (4.2), the Stokes formula and Lemma 4.4,

$$I(v, \mathcal{T}; \nabla_0^\bullet, D^\bullet) - I(v, \mathcal{T}; \nabla_0^\bullet, \nabla^\bullet) = \int_{\partial\mathcal{T}} c^n(\nabla^\bullet, D^\bullet) = \int_{\partial\mathcal{T}} c^n(\nabla^V, D^V),$$

which is 0, since both connections ∇^V and D^V for $TV_0|_{U-S}$ are v-trivial.

If $\mathcal{T}'$ denotes another tube around S in U, then we may always assume (taking a third $\mathcal{T}''$ containing both $\mathcal{T}$ and $\mathcal{T}'$ if necessary) that $\mathcal{T}$ is included in the interior of $\mathcal{T}'$. Then, by (4.1), the Stokes formula and Lemma 4.4,

$$I(v, \mathcal{T}'; \nabla_0^\bullet, \nabla^\bullet) - I(v, \mathcal{T}; \nabla_0^\bullet, \nabla^\bullet) = \int_{\mathcal{T}' - \mathcal{T}} c^n(\nabla^\bullet) = \int_{\mathcal{T}' - \mathcal{T}} c^n(\nabla^V),$$

which is 0, since ∇^V is v-trivial. □

Definition 5.2. Using the previous notations, the *virtual index* of v at S, denoted by $\mathrm{Vir}(v, S)$, is defined by $I(v, \mathcal{T}; \nabla_0^\bullet, \nabla^\bullet)$.

Remark 5.3. If S is in the regular part V_0, we may take compatible pairs $\nabla_0^\bullet$ and $\nabla^\bullet$ in the definition of $\mathrm{Vir}(v, S)$. Hence, by Lemma 4.4, $\mathrm{Vir}(v, S) = \mathrm{PH}(v, S)$.

For a continuous vector field v on the regular part $V_0 = V - \mathrm{Sing}(V)$ of V, we denote by $(S_\alpha)_\alpha$ the connected components of $\mathrm{Sing}(V) \cup \mathrm{Sing}(v)$. As in section2, we assume that each component S_α is either in V_0 or in $\mathrm{Sing}(V)$ and that it admits a neighborhood disjoint from each other. Then we have the following theorem, which appears essentially in [LSS].

Theorem 5.4. *Let V be a compact analytic subvariety in a complex manifold M as above and v a C^∞ vector field on the regular part of V. Then we have :*

$$\sum_\alpha \mathrm{Vir}(v, S_\alpha) = c_0(TM|_V - N).$$

Proof. For each α, let $\tilde{U}_\alpha$ be a neighborhood of S_α in M with $\tilde{U}_\alpha \cap \tilde{U}_\beta = \emptyset$, for $\alpha \neq \beta$. Also, let $\nabla_\alpha^\bullet$ be a compatible pair on $\tilde{U}_\alpha$ and $\nabla^\bullet$ a v-trivial compatible pair on $V - \mathrm{Sing}(v)$. If we denote by $\mathcal{T}_\alpha$ a tube around S_α in $\tilde{U}_\alpha \cap V$ for each α, we have

$$\begin{aligned} c^n(TM|_V - N) \frown [V] = & \sum_\alpha \left(\int_{\mathcal{T}_\alpha} c^n(\nabla_\alpha^\bullet) + \int_{\partial \mathcal{T}_\alpha} c^n(\nabla_\alpha^\bullet, \nabla^\bullet) \right) \\ & + \int_{V - \bigcup_\alpha \mathrm{Int}(\mathcal{T}_\alpha)} c^n(\nabla^\bullet). \end{aligned}$$

By Lemma 4.4, $c^n(\nabla^\bullet) = c^n(\nabla^V)$, which is zero, since ∇^V is v-trivial. □

Lemma 5.5. *For vector fields v_1 and v_2 as above, we have*

$$\mathrm{Vir}(v_2, S) = \mathrm{Vir}(v_1, S) + d_S(v_1, v_2).$$

Proof. We have :

$$\mathrm{Vir}(v_2, S) - \mathrm{Vir}(v_1, S) = \int_{\partial \mathcal{T}} \left(c^n(\nabla_0^\bullet, \nabla_2^\bullet) - c^n(\nabla_0^\bullet, \nabla_1^\bullet) \right),$$

where $\nabla_1^\bullet = (\nabla_1, \nabla_1')$ and $\nabla_2^\bullet = (\nabla_2, \nabla_2')$ denote compatible pairs of connections, respectively v_1-trivial and v_2-trivial. This difference is equal to $\int_{\partial \mathcal{T}} c^n(\nabla_1^\bullet, \nabla_2^\bullet)$, which is equal to $\int_{\partial \mathcal{T}} c^n(\nabla_1^V, \nabla_2^V) = d(v_1, v_2)$, by Lemmas 3.4 and 4.4. □

Remark 5.6. Let S and U be as above. If v is a continuous vector field, non-singular on $U - S$, we may define the virtual index $\mathrm{Vir}(v, S)$ by taking a C^∞ approximation of v and the above results still hold. Thus Theorem 5.4 and Lemma 5.5 are valid for continuous vector fields as well.

The above constructions are generalized for r-frames as follows. Let us consider $S \subset V \subset M$ as previously defined, $\tilde{U}$ a neighborhood of S in M such that $\tilde{U} \cap V - S$ is in V_0, $\tilde{\mathcal{T}}$ a cellular tube in $\tilde{U}$ and the tube $\mathcal{T} = \tilde{\mathcal{T}} \cap V$. Let $F^{(r)}$ be an r-frame on $(D)^{(2q)} \cap (U - S)$, we define the *virtual class* $\mathrm{Vir}(F^{(r)}, S)$ of $F^{(r)}$ at S as follows.

Let ∇_0 and ∇'_0 be connections for TM and E, respectively, on $\tilde{U}$ and set $\nabla_0^\bullet = (\nabla_0, \nabla'_0)$. Also, let ∇ and ∇' be connections for TM and E, respectively, on a neighborhood W of $(D)^{(2q)} \cap (U - S)$ in U such that the pair $\nabla^\bullet = (\nabla, \nabla')$ is compatible and ∇ is $F^{(r)}$-trivial. We consider the $2q$-cochain given by

$$\sigma \mapsto \int_{\sigma \cap \mathcal{T}} c^p(\nabla_0^\bullet) + \int_{\sigma \cap \partial \mathcal{T}} c^p(\nabla_0^\bullet, \nabla^\bullet), \tag{5.7}$$

for a $2q$-cell σ in (D). Then by the same arguments as in Lemma 5.1, this cochain does not depend on $\nabla_0^\bullet$, $\nabla^\bullet$ or $\mathcal{T}$, it is a cocycle and defines an element ξ in $H^{2p}(\tilde{U}, \tilde{U} - S)$.

Definition 5.8. We define the *virtual class* $\mathrm{Vir}(F^{(r)}, S)$ of $F^{(r)}$ at S to be the image of ξ by the Alexander isomorphism $\psi_M : H^{2q}(\tilde{U}, \tilde{U} - S) \longrightarrow H_{2r-2}(S)$.

Recall that, if S is in V_0, the Poincaré-Hopf class $\mathrm{PH}(F^{(r)}, S) \in H_{2r-2}(S)$ is dual to the class represented by the cocycle (3.2). In this case, we have (cf. Remark 5.3)

$$\mathrm{Vir}(F^{(r)}, S) = \mathrm{PH}(F^{(r)}, S).$$

Let Σ be a compact (K)-subcomplex in V_0 disjoint from a neighborhood U_1 of $\mathrm{Sing}(V)$ in V. We denote by (S_α) the connected components of $\mathrm{Sing}(V) \cup \Sigma$ and set $V^* = V - U_1$. Let i_α and ι be the inclusions $S_\alpha \hookrightarrow V$ and $V^* \hookrightarrow V$, respectively. The following theorem follows from (1.7) and arguments similar to the ones for Theorem 5.4.

Theorem 5.9. *Let V be a compact complex analytic variety of dimension n embedded in a complex manifold M of dimension m and Σ a subcomplex in V_0 as above. For a non-singular r-frame $F^{(r)}$ on $(D)^{(2q)} \cap (V_0 - \Sigma)$, $q = m - r + 1$, we have*

$$\sum_\alpha (i_\alpha)_* \mathrm{Vir}(F^{(r)}, S_\alpha) = c_{r-1}(TM|_V - N)$$

where the sum is taken over all connected components of $\mathrm{Sing}(V) \cup \Sigma$. *Hence we have*

$$\sum_{S_\alpha \subset \mathrm{Sing}(V)} (i_\alpha)_* \mathrm{Vir}(F^{(r)}, S_\alpha) + \iota_* c_{r-1}(V^*, F^{(r)}) = c_{r-1}(TM|_V - N),$$

The following is proved as in Lemma 5.5.

Lemma 5.10. *For two r-frames $F_1^{(r)}$ and $F_2^{(r)}$ as above, we have*

$$\mathrm{Vir}(F_2^{(r)}, S) = \mathrm{Vir}(F_1^{(r)}, S) + d_S(F_1^{(r)}, F_2^{(r)}).$$

6. Geometric interpretation of the virtual index

Let (V,p) be an isolated complete intersection singularity. An index of a continuous vector field v on V at p has been first defined and studied in [Se] when V is two dimensional, then extended in [GSV] to higher dimensional hypersurfaces and in [SS1] to the general case. This index coincides with the Poincaré-Hopf index $\mathrm{PH}(v', F)$ of the vector field v' obtained on a nearby Milnor fiber F by perturbing v. We call it the *GSV-index* of v at p and denote it by $\mathrm{GSV}(v,p)$. In [LSS], it is proved that, for a vector field v which is the restriction of a holomorphic vector field on the ambient space leaving V invariant, we have :

$$\mathrm{Vir}(v,p) = \mathrm{GSV}(v,p). \tag{6.1}$$

The same proof works for the C^∞ or continuous case [SS2].

In this section we give a similar interpretation of the virtual index in the case where the variety V may have non-isolated singularities, thus extending the definition of the GSV-index to non-isolated singularities. We assume again that V is given as the zero set of a holomorphic section s, generically transverse to the zero section, of a holomorphic vector bundle E on a complex manifold M. Let S be a compact connected component of $\mathrm{Sing}(V)$ and $\tilde{U}$ a neighborhood of S in M disjoint from the other components of $\mathrm{Sing}(V)$.

Definition 6.2. Let s' be a C^∞ section of E on $\tilde{U}$ satisfying the following conditions :
(i) s' coincides with s on the complement of a relatively compact neighborhood $\tilde{U}'$ of S in M, whose closure is in $\tilde{U}$,
(ii) s' is transverse to the zero section of E,
(iii) s' and s are C^∞ homotopic.
Such sections exist by Thom's transversality (essentially, see [O]). We call the zero set F of s' a *C^∞ smoothing of V near S*. A *global C^∞ smoothing* V' of V is defined by replacing, in the previous construction, $\tilde{U}$ by a neighborhood of V in M, $\tilde{U}'$ by the union of mutually disjoint relatively compact neighborhoods of the components of $\mathrm{Sing}(V)$ and F by V', the zero set of s'.

Note that a C^∞ smoothing F is a C^∞ submanifold of real dimension $2n$ in $\tilde{U}$ which coincides with V away from $\tilde{U}'$. Denoting by $T_{\mathbb{R}}Z$ the (real) tangent bundle of F, and considering TM and E are considered as real bundles, we have the exact sequence

$$0 \longrightarrow T_{\mathbb{R}}F \longrightarrow TM|_F \xrightarrow{\nabla s'} E|_F \longrightarrow 0,$$

which coincides with (4.3) on $F \cap (\tilde{U} - \tilde{U}') = V \cap (\tilde{U} - \tilde{U}')$. Let v be a vector field, non-singular on $U - S$, $U = \tilde{U} \cap V$. Thus v can be considered as a non-singular vector field "near the boundary" of F.

Theorem 6.3. *Let V be a subvariety of M as above.*
(i) *If F is a C^∞ smoothing of V near S, then we have :*

$$\mathrm{Vir}(v,S)=\mathrm{PH}(v,F).$$

In particular, we recover the fact that $\mathrm{Vir}(v,S)$ *is an integer. If v is everywhere transverse to ∂F, we have :*

$$\mathrm{Vir}(v,S)=\chi(F).$$

Thus $\chi(F)$ is independent of the smoothing.
(ii) *If V is compact and if V' is a global C^∞ smoothing of V, then :*

$$\chi(V')=c_0(TM|_V-N).$$

In particular, $\chi(V')$ depends only on V and E, but not on the choice of the global C^∞ smoothing V', cf. [PP2].

Proof. To compute $\mathrm{Vir}(v,S)$, we take a cellular tube $\tilde{\mathcal{T}}$ in $\tilde{U}$, as before, so that it contains the closure of $\tilde{U}'$ in its interior. In the definition of $\mathrm{Vir}(v,S)$, we may take the pair of connections $\nabla_0^\bullet$ on $\tilde{U}$ so that it is compatible on F. Set $\mathcal{T}=\tilde{\mathcal{T}}\cap V$ and denote $\tilde{\mathcal{T}}\cap F$ also by F so that $\partial F=\partial\mathcal{T}$. Then, by Lemma 4.4,

$$\mathrm{PH}(v,F)=\int_F c^n(\nabla_0^\bullet)+\int_{\partial F}c^n(\nabla_0^\bullet,\nabla^\bullet).$$

Hence we have

$$\mathrm{Vir}(v,S)-\mathrm{PH}(v,F)=\int_{\mathcal{T}-F}c^n(\nabla_0^\bullet),$$

which is 0, since s and s' are C^∞ homotopic. This proves (i).

Let V' be defined by a section s' satisfying the conditions in Definition 6.2. Let $(S_\alpha)_\alpha$ be the connected components of $\mathrm{Sing}(V)$ and, for each α, $\tilde{U}'_\alpha$ a relatively compact neighborhood of S_α in M disjoint from the others so that s' coincides with s on the complement of $\bigcup_\alpha\tilde{U}'_\alpha$. For each α, we take a compact manifold with boundary $\tilde{\mathcal{T}}_\alpha$ of real dimension $2m$ as before and a radial vector field v_α outbound from S_α so that $\tilde{\mathcal{T}}_\alpha$ contains the closure of $\tilde{U}'_\alpha$ in its interior, $\tilde{\mathcal{T}}_\alpha\cap\tilde{\mathcal{T}}_\beta=\emptyset$, for $\alpha\neq\beta$, and that v_α is non-singular on $\tilde{\mathcal{T}}_\alpha-S_\alpha$.

The zero set of s' in the interior of $\tilde{\mathcal{T}}_\alpha$, denoted by F_α, is a C^∞ smoothing of V near S_α. Then $V^*=V-\bigcup_\alpha\mathrm{Int}(\tilde{\mathcal{T}}_\alpha)=V'-\bigcup_\alpha F_\alpha$, is a C^∞ manifold with boundary $\partial V^*=\bigcup_\alpha\partial\mathcal{T}_\alpha$, $\partial\mathcal{T}_\alpha=\partial\tilde{\mathcal{T}}_\alpha\cap V$. We denote by v_0 a vector field on (a neighborhood of) ∂V^* which is v_α on $\partial\mathcal{T}_\alpha$. Then we have

$$\chi(V')=\mathrm{PH}(v_0,V^*)+\sum_\alpha\mathrm{PH}(v_\alpha,F_\alpha).$$

Part (ii) of the theorem follows therefore from (i) above, Remark 5.3 and Theorem 5.4 (see also Remark 5.6). □

7. Milnor numbers and classes

First we define our generalized Milnor number. Let V be, as before, a subvariety of dimension n in M, defined by a section, generically transverse to the zero section, of a holomorphic vector bundle E on M.

Definition 7.1. The *generalized Milnor number* $\mu(V, S)$ of V at a compact component S of $\mathrm{Sing}(V)$ is defined by

$$\mu(V,S) = (-1)^{n+1}\left(\mathrm{Sch}(v_0,S) - \mathrm{Vir}(v_0,S)\right),$$

where v_0 is a radial vector field outbound from S (recall that $\mathrm{Sch}(v_0, S) = \chi(S)$).

Note that the above definition does not depend on the choice of the radial vector field v_0.

Remark 7.2. If (V,p) is an isolated complete intersection singularity, its Milnor number μ, as defined in [M2, Ha1, Lê, G, Lo], is given by

$$\mu = (-1)^{n+1}(1 - \chi(F)),$$

where F is a Milnor fiber of (V,p). Since a radial vector field v_0 points outwards from p is transverse to the link of (V,p), we have, from (6.1), $\mathrm{Vir}(v_0,p) = \chi(F)$. On the other hand, $\chi(p) = 1$. Hence we have in this case : $\mu(V,p) = \mu$.

The following proposition follows from (2.3) and Lemma 5.5.

Proposition 7.3. *If v is a vector field on V_0, non-singular near but away from S, we have*

$$\mathrm{Sch}(v,S) = \mathrm{Vir}(v,S) + (-1)^{n+1}\mu(V,S).$$

Theorem 7.4. *Let V be a subvariety of a complex manifold M as above. If V is compact, we have :*

$$\chi(V) = c_0(TM|_V - N) + (-1)^{n+1} \sum_{S_\alpha \subset \mathrm{Sing}(V)} \mu(V, S_\alpha),$$

where the sum is taken over the connected components of $\mathrm{Sing}(V)$.

Proof. For each component S of $\mathrm{Sing}(V)$, we take a radial vector field pointing outwards from S, then we can extend these vector fields so that we have a vector field v on V whose singular set $\mathrm{Sing}(v)$ consists of $\mathrm{Sing}(V)$ and a finite number of compact connected sets in V_0. For a component S of $\mathrm{Sing}(v)$ in V_0, we have $\mathrm{Sch}(v,S) = \mathrm{PH}(v,S) = \mathrm{Vir}(v,S)$. Hence the theorem follows from Theorems 2.4 and 5.4. □

From Theorems 6.3 and 7.4 and Proposition 7.3, we have the following :

Proposition 7.5. (i) *If F is a C^∞ smoothing of V near S,*

$$\mu(V,S) = (-1)^{n+1}\left(\chi(S) - \mathrm{PH}(v_0, F)\right),$$

where v_0 is a radial vector field pointing outwards from S. In particular, $\mu(V,S)$ is an integer. Thus if we choose ∂F to be C^∞ and if v_0 is everywhere transverse to ∂F, then

$$\mu(V,S) = (-1)^{n+1}\left(\chi(S) - \chi(F)\right).$$

(ii) *If F is a C^∞ smoothing of V near S, we have, for a continuous vector field v non-singular away from S,*

$$\mathrm{Sch}(v,S) = \mathrm{PH}(v,F) + (-1)^{n+1}\mu(V,S).$$

(iii) *If V is compact and if V' is a global C^∞ smoothing of V,*

$$\chi(V) = \chi(V') + (-1)^{n+1} \sum_{S_\alpha \subset \mathrm{Sing}(V)} \mu(V, S_\alpha),$$

where the sum is taken over the components of Sing(V).

Now we introduce the Milnor classes of V at a compact component of Sing(V).

Definition 7.6. The $(r-1)$-st *Milnor class* $\mu_{r-1}(V,S)$ of V at a compact component S of Sing(V) is defined by

$$\mu_{r-1}(V,S) = (-1)^{n+1}\left(\mathrm{Sch}(F_0^{(r)}, S) - \mathrm{Vir}(F_0^{(r)}, S)\right) \quad \text{in} \quad H_{2r-2}(S),$$

where $F_0^{(r)}$ is a radial r-frame outbound from S (recall that $\mathrm{Sch}(F_0^{(r)}, S) = c_{r-1}(S)$).

Note that the above definition does not depend on the choice of the radial r-frame $F_0^{(r)}$. In fact, the following proposition shows that we may use arbitrary r-frame $F^{(r)}$ on V_0 defined and non-singular on $(D)^{(2q)} \cap (U - S)$, where U is a neighborhood of S in V.

Proposition 7.7. *If $F^{(r)}$ is an r-frame non-singular on $(D)^{(2q)} \cap (U-S)$, where U is a neighborhood of S in V, we have*

$$\mathrm{Sch}(F^{(r)}, S) = \mathrm{Vir}(F^{(r)}, S) + (-1)^{n+1}\mu_{r-1}(V,S).$$

Theorem 7.8. *Let V be a subvariety of a complex manifold M as above. If V is compact, we have, for each $r = 0, ..., n-1$:*

$$c_r(V) = c_r(TM|_V - N) + (-1)^{n+1} \sum_{S_\alpha \subset \mathrm{Sing}(V)} (i_\alpha)_* \mu_r(V, S_\alpha) \quad in \quad H_{2r}(V),$$

where the sum is taken over the connected components of $\mathrm{Sing}(V)$.

In other words, the difference between the total Schwartz-MacPherson class $c_*(V)$ of V and the total virtual class $c_*(TM|V - N)$, regarded in homology, is the sum over the connected components of $\mathrm{Sing}(V)$ of the "total" Milnor classes $\mu_*(V, S) = \oplus_{i=0}^{n-1} \mu_i(V, S)$. In particular, if the singularities of V are isolated points, then the Milnor classes are zero except in degree 0, and we obtain the following result of [Su1] :

Corollary 7.9. *If V has only isolated singularities $p_1, \ldots, p_s$, then*

$$c_*(V) = c_*(TM|_V - N) + (-1)^{n+1} \sum_{i=1}^{s} \mu(V, p_i),$$

where $\mu(V, p_i)$ is the Milnor number of V at p_i.

Remarks 7.10. 1. The above constructions and results, except for the ones in section 6 and Proposition 7.5, are valid for a subvariety V in a complex manifold M for which there is a C^∞ extension $\tilde{N}$ of the normal bundle N_{V_0} of the regular part V_0 of V to a neighborhood of V in M. This is done simply by replacing E by $\tilde{N}$ in sections 4 and 5. The invariants that we obtain depend on the extension $\tilde{N}$, as shown in Example 9.4 below. If V is a local complete intersection (LCI) in M, there is a canonical holomorphic vector bundle N on V and a homomorphism $TM|_V \longrightarrow N$, which extend N_{V_0} and π in (4.3), (see, e.g., [LS], section 2). We call N the "reduced extension" of N_{V_0}. An LCI, V, with a C^∞ extension $\tilde{N}$ of N to a neighborhood of V is called a strong local complete intersection (SLCI) in [LS]. If E and V are as in section 4 above, then V is an SLCI with $\tilde{N} = E$. Note that, even for a hypersurface with isolated singularity, if we take an extension of N_{V_0} other than the reduced one, we may obtain different values for the virtual index and the Milnor number (see Example 9.4). In general, for a subvariety V in a complex manifold M, if N is an extension of N_{V_0} to V and $\tilde{N}$ an extension of N to a neighborhood of V in M, we conjecture that the Milnor classes do not depend on $\tilde{N}$, once we fix N.

2. As mentioned in the introduction, the Milnor class $\mu_{r-1}(V, S)$ may be defined in the integral homology $H_{2r-2}(S, \mathbb{Z})$.

In the rest of this section, we give a formula for the Milnor class at a non-singular component of $\mathrm{Sing}(V)$.

First we make a remark on the differential geometric expression of the Schwartz class in the case V is defined as the zero set of a section, as in the beginning of this section. Let S be a component of $\mathrm{Sing}(V)$ and let $F^{(r)}$ be an r-frame on $(D)^{(2q)} \cap (\tilde{U} - S)$ tangent to V_0, which may not be radial. We may assume that $F^{(r)}$ is given on a neighborhood $\tilde{W}$ of $(D)^{(2q)} \cap (\tilde{U} - S)$. In this situation, we see that the Schwartz class of $F^{(r)}$ at S can be described in the same way as for radial frames (see (3.5) and (3.6)) :

Lemma 7.11. *In the above situation, let $\tilde{\nabla}_0$ be a connection for TM on $\tilde{U}$ and $\tilde{\nabla}$ an $F^{(r)}$-trivial connection for TM on $\tilde{W}$. Then the class $c^q(\tilde{U}, \tilde{U} - S; F^{(r)}) \in H^{2q}(\tilde{U}, \tilde{U} - S)$, as defined in section 1, is represented by the cocycle*

$$\gamma \mapsto \int_{\gamma \cap \tilde{T}} c^q(\tilde{\nabla}_0) + \int_{\gamma \cap \partial \tilde{T}} c^q(\tilde{\nabla}_0, \tilde{\nabla})$$

for a relative $2q$-cycle γ in $\tilde{U}$ modulo $\tilde{U} - S$ consisting of cells in (D).

Recall that the Schwartz class of $F^{(r)}$ at S is given by $\mathrm{Sch}(F^{(r)}, S) = \psi_M\, c^q(\tilde{U}, \tilde{U} - S; F^{(r)})$, where $\psi_M : H^{2q}(\tilde{U}, \tilde{U} - S) \xrightarrow{\sim} H_{2r-2}(S)$ denotes the Alexander duality.

Proof of Lemma 7.11. This is a consequence of (3.5) and the following claim. Let $F_1^{(r)}$ and $F_2^{(r)}$ be two r-frames on $\tilde{W}$ tangent to V_0. For $i = 1, 2$, let $\tilde{\nabla}_i$ be an $F_i^{(r)}$-trivial connection for TM on $\tilde{W}$. Then the difference $\delta d(F_1^{(r)}, F_2^{(r)})$ is an element in $H^{2p}(U, U - S)$ whose image by the Thom Gysin homomorphism τ is represented by the cocycle

$$\gamma \mapsto \int_{\gamma \cap \partial \tilde{T}} c^q(\tilde{\nabla}_1, \tilde{\nabla}_2), \tag{7.12}$$

for a relative $2q$-cycle γ in $\tilde{U}$ modulo $\tilde{U} - S$ consisting of cells in (D). To prove this, consider the exact sequence (4.3). Since the integral in (7.12) does not depend on the connections $\tilde{\nabla}_1$ and $\tilde{\nabla}_2$ satisfying the above conditions, we may assume that each $\tilde{\nabla}_i$ restricts to an $F_i^{(r)}$-trivial connection ∇_i for TV_0 on (a neighborhood of) $(D)^{(2q)} \cap (U - S)$ and that both of them define the same connection ∇^E for $E|_{V_0} = N_{V_0}$ by passing to the quotient. Then we have $c^q(\tilde{\nabla}_1, \tilde{\nabla}_2) = c^p(\nabla_1, \nabla_2) \wedge c^k(\nabla^E)$. Noting that $d\,c^p(\nabla_1, \nabla_2) = 0$ and that γ is transverse to $\partial \tilde{T}$ and V, by the duality (see [Su2] (6.2) and Remark 6.3), we have

$$\int_{\gamma \cap \partial \tilde{T}} c^q(\tilde{\nabla}_1, \tilde{\nabla}_2) = \int_{\gamma \cap \partial T} c^p(\nabla_1, \nabla_2).$$

This proves the claim (cf. Lemma 3.4). □

Now let S be a compact non-singular component of Sing(V) of dimension ℓ, and $\tilde{\rho} : \tilde{U} \longrightarrow S$ a tubular neighborhood of S in M. Suppose that V satisfies the Whitney condition along S (in fact, in the following, it is sufficient if V is locally topologically trivial along the $2(\ell - r + 1)$ skeleton of S). Hence we may suppose that the fibers of $\tilde{\rho}$ are transverse to V and that $\tilde{\rho}$ extends the retraction $\rho : U_0 \longrightarrow S$, where $U_0 = U - S$, $U = \tilde{U} \cap V$. Let $F^{(r-1)}$ be an $(r-1)$-frame on the $2(\ell - r + 1)$ skeleton of S. We denote also by $F^{(r-1)}$ the $(r-1)$-frame on the $2q$ skeleton of $\tilde{U}$ obtained by pulling back $F^{(r-1)}$ by $\tilde{\rho}$, after identification of $\tilde{\rho}^*TS$ with a subbundle of TM transverse to the fibers of $\tilde{\rho}$. We also identify $\rho^*(E|_S)$ with $E|_{U_0}$, and $\tilde{\rho}^*(E|_S)$ with $E|_{\tilde{U}}$. Let v_r be a non-singular vector field on $\tilde{U} - S$ which is tangent to V_0 and to the fibers of $\tilde{\rho}$. Then $F^{(r)} = (F^{(r-1)}, v_r)$ is an r-frame on $(\tilde{U} - S) \cap (D)^{(2q)}$ constructed in the same way as in Remark 1.5. Note that there exists a radial frame with these properties. For a point x in S, let $\tilde{U}_x$ denote the fiber of $\tilde{\rho}$ and set $U_x = \tilde{U}_x \cap V$.

The restriction of v_r to $\tilde{U}_x - \{x\}$ has an isolated singularity at x and we denote simply by $\mathrm{Sch}(v_r, x)$ and $\mathrm{Vir}(v_r, x)$, respectively, its Schwartz index and its virtual index relative to U_x. From the Whitney conditions, we see that they do not depend on x.

Theorem 7.13. *In the above situation, we have*

$$\text{(i)} \qquad \mathrm{Sch}(F^{(r)}, S) = \mathrm{Sch}(v_r, x) \cdot c_{r-1}(S)$$

$$\begin{aligned} \text{(ii)} \qquad \mathrm{Vir}(F^{(r)}, S) =& \mathrm{Vir}(v_r, x) \cdot \left[c(S)c(E)^{-1}\right]^{\ell - r + 1} \frown [S] \\ &+ \mathrm{Sch}(v_r, x) \cdot \left[c(S)c(E)^{-1}c^k(E)\right]^{\ell - r + 1} \frown [S] \\ &+ \sum_{i=1}^{k-1} \left[c(S)c(E)^{-1}\omega^i\right]^{\ell - r + 1} \frown [S], \end{aligned}$$

where ω^i is a $2i$-form on S which is specified in the proof (see (7.17)).

Proof. First, we recall the commutative diagram

$$\text{(7.14)} \qquad \begin{array}{ccc} H^{2q}(\tilde{U}, \tilde{U} - S) & \xrightarrow[\tilde{\rho}_*]{\sim} & H^{2\ell - 2r + 2}(S) \\ \wr \downarrow \psi_M & & \wr \downarrow \\ H_{2r-2}(S) & \xrightarrow{=} & H_{2r-2}(S), \end{array}$$

where the first row is the inverse of the Thom isomorphism, which is given by the integration along the fibers of $\tilde{\rho}$ and the second column is the Poincaré duality.

We consider the exact sequence of vector bundles on $\tilde{U}$:

$$\text{(7.15)} \qquad 0 \to T\tilde{\rho} \to T\tilde{U} \to \tilde{\rho}^*TS \to 0,$$

where $T\tilde{\rho}$ denotes the bundle of vectors in $T\tilde{U}$ tangent to the fibers of $\tilde{\rho}$. Let $\tilde{\nabla}^{\tilde{\rho}}$ be a v_r-trivial connection for $T\tilde{\rho}$ on $\tilde{U}-S$ and ∇^S an $F^{(r-1)}$-trivial connection for TS on a neighborhood of $S^{(2\ell-2r+2)}$. We take a connection $\tilde{\nabla}$ for $T\tilde{U}$ so that $(\tilde{\nabla}^{\tilde{\rho}},\tilde{\nabla},\rho^*\nabla^S)$ is compatible with (7.15). Thus $\tilde{\nabla}$ is $F^{(r)}$-trivial on a neighborhood of $(\tilde{U}-S)^{(2q)}$. Let $\tilde{\nabla}_0^{\tilde{\rho}}$ be a connection for $T\tilde{\rho}$ on $\tilde{U}$. We take a connection $\tilde{\nabla}_0$ for $T\tilde{U}$ so that $(\tilde{\nabla}_0^{\tilde{\rho}},\tilde{\nabla}_0,\tilde{\rho}^*\nabla^S)$ is compatible with (7.15).

By definition, the homology class $\mathrm{Sch}(F^{(r)},S)$ is the Alexander dual of the class in $H^{2q}(\tilde{U},\tilde{U}-S)$ represented by the cocycle in Lemma 7.11. We have

$$c^q(\tilde{\nabla}_0)=\sum_{i+j=q}c^i(\tilde{\nabla}_0^{\tilde{\rho}})\cdot\tilde{\rho}^*c^j(\nabla^S)\quad\text{and}\quad c^q(\tilde{\nabla}_0,\tilde{\nabla})=\sum_{i+j=q}c^i(\tilde{\nabla}_0^{\tilde{\rho}},\tilde{\nabla}^{\tilde{\rho}})\cdot\tilde{\rho}^*c^j(\nabla^S).$$

We consider the dual of the first row in (7.14)

$$H_{2q}(\tilde{U},\tilde{U}-S)\overset{\sim}{\leftarrow}H_{2\ell-2r+2}(S).$$

This shows that every relative $2q$-cycle γ fibers over a $2(\ell-r+1)$ cycle τ of S. Let $\tilde{T}$ be a tube around S in $\tilde{U}$ and set $T=\tilde{T}\cap V$, $\tilde{T}_x=\tilde{T}\cap\tilde{U}_x$ and $T_x=T\cap U_x$. Note that $\tilde{T}$ is a closed tubular neighborhood of S. By the projection formula, we have

$$\begin{aligned}\int_{\gamma\cap\tilde{T}}c^q(\tilde{\nabla}_0)&+\int_{\gamma\cap\partial\tilde{T}}c^q(\tilde{\nabla}_0,\tilde{\nabla})\\&=\left(\int_{\tilde{T}_x}c^{m-\ell}(\tilde{\nabla}_0^{\tilde{\rho}})+\int_{\partial\tilde{T}_x}c^{m-\ell}(\tilde{\nabla}_0^{\tilde{\rho}},\tilde{\nabla}^{\tilde{\rho}})\right)\cdot\int_\tau c^{\ell-r+1}(\nabla^S),\end{aligned}$$

where x is a point in τ. Noting that the first factor in the right hand side is $\mathrm{Sch}(v_r,x)$, we obtain the first formula, in view of (7.14).

To prove the second formula, we consider the commutative diagram of vector bundles on U_0 with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & E|_{U_0} & \xrightarrow{=} & E|_{U_0} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & T\tilde{\rho}|_{U_0} & \longrightarrow & T\tilde{U}|_{U_0} & \longrightarrow & \rho^*TS & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow{=} & & \\
0 & \longrightarrow & T\rho & \longrightarrow & TU_0 & \longrightarrow & \rho^*TS & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}\tag{7.16}$$

,

where $T\rho$ denotes the bundle of vectors in TU_0 tangent to the fibers of ρ. Let ∇^ρ be a v_r-trivial connection for $T\rho$ and ∇^S an $F^{(r-1)}$-trivial connection for TS on a neighborhood of $S^{(2\ell-2r+2)}$. We take a connection ∇^V for TU_0 so that $(\nabla^\rho, \nabla^V, \rho^*\nabla^S)$ is compatible with the third row in (7.16). Thus ∇^V is $F^{(r)}$-trivial on a neighborhood of $(D)^{(2q)} \cap (U - S)$. Let ∇^E be a connection for $E|_S$ and take a connection $\nabla^{\tilde\rho}$ for $T\tilde\rho|_{U_0}$ so that $(\nabla^\rho, \nabla^{\tilde\rho}, \rho^*\nabla^E)$ is compatible with the first column in (7.16). Finally take a connection ∇ for $T\tilde U|_{U_0}$ so that $(\nabla^{\tilde\rho}, \nabla, \rho^*\nabla^S)$ is compatible with the second row. Then $(\nabla^V, \nabla, \rho^*\nabla^E)$ is compatible with the second column. We may extend $\nabla^{\tilde\rho}$ and ∇ to connections $\tilde\nabla^{\tilde\rho}$ and $\tilde\nabla$ for $T\tilde\rho$ and $T\tilde U$, respectively, on $\tilde U - S$ so that $(\tilde\nabla^{\tilde\rho}, \tilde\nabla, \tilde\rho^*\nabla^S)$ is compatible, on $\tilde U - S$, with the exact sequence (7.15). Let $\tilde\nabla_0^{\tilde\rho}$ be a connection for $T\tilde\rho$ on $\tilde U$. We take a connection $\tilde\nabla_0$ for $T\tilde U$ so that $(\tilde\nabla_0^{\tilde\rho}, \tilde\nabla_0, \tilde\rho^*\nabla^S)$ is compatible with (7.15), as before.

With these, $\mathrm{Vir}(F^{(r)}, S)$ is the Alexander dual of the class in $H^{2q}(\tilde U, \tilde U - S)$ represented by the cocycle (5.7) with $\nabla_0^\bullet = (\tilde\nabla_0, \tilde\rho^*\nabla^E)$ and $\nabla^\bullet = (\nabla, \rho^*\nabla^E)$. Here we take, instead of a $2q$-cell σ, a relative $2q$-cycle γ in $\tilde U$ modulo $\tilde U - S$ consisting of cells in (D). In view of the exact sequence (7.15), we have

$$\begin{aligned} c^p(T\tilde U - E) = &\sum_{j>\ell-r+1} [\tilde\rho^* c(S) c(E)^{-1}]^j \cdot c^{p-j}(T\tilde\rho) \\ &+ \left[\tilde\rho^* c(S) c(E)^{-1}\right]^{\ell-r+1} \cdot c^{n-\ell}(T\tilde\rho) \\ &+ \sum_{i=1}^{k} \left[\tilde\rho^* c(S) c(E)^{-1}\right]^{\ell-r-i+1} \cdot c^{n-\ell+i}(T\tilde\rho). \end{aligned}$$

On the other hand, we have

$$c^{n-\ell}(T\tilde\rho) = c^{n-\ell}(T\tilde\rho - E) + \sum_{i>0} c^{n-\ell-i}(T\tilde\rho - E) \cdot c^i(E).$$

From the above two equalities, we get

$$\begin{aligned} c^p(T\tilde U - E) = &\left[\tilde\rho^* c(S) c(E)^{-1}\right]^{\ell-r+1} \cdot c^{n-\ell}(T\tilde\rho - E) \\ &+ \sum_{i=1}^{k} \left[\tilde\rho^* c(S) c(E)^{-1}\right]^{\ell-r-i+1} \cdot c^{n-\ell+i}(T\tilde\rho) + c^p(S, E, T\tilde\rho), \end{aligned}$$

where $c^p(S, E, T\tilde\rho)$ denotes a polynomial in the Chern classes of TS, E and $T\tilde\rho$, homogeneous of degree p, such that each of its monomial involves the Chern classes of TS and E of total degree greater than $\ell - r + 1$. Hence we have

$$\begin{aligned} c^p(\nabla_0^\bullet) = &\left[\tilde\rho^* c(\nabla^S) c(\nabla^E)^{-1}\right]^{\ell-r+1} \cdot c^{n-\ell}(\nabla_0^{\tilde\rho\bullet}) \\ &+ \sum_{i=1}^{k} \left[\tilde\rho^* c(\nabla^S) c(\nabla^E)^{-1}\right]^{\ell-r-i+1} \cdot c^{n-\ell+i}(\tilde\nabla_0^{\tilde\rho}) + c^p(\nabla^S, \nabla^E, \tilde\nabla_0^{\tilde\rho}) \end{aligned}$$

and

$$
\begin{aligned}
c^p(\nabla_0^\bullet,\nabla^\bullet) &= \left[\tilde\rho^* c(\nabla^S)c(\nabla^E)^{-1}\right]^{\ell-r+1}\cdot c^{n-\ell}(\nabla_0^{\tilde\rho\bullet},\nabla^{\tilde\rho\bullet})\\
&\quad+\sum_{i=1}^{k}\left[\tilde\rho^* c(\nabla^S)c(\nabla^E)^{-1}\right]^{\ell-r-i+1}\cdot c^{n-\ell+i}(\tilde\nabla_0^{\tilde\rho},\nabla^{\tilde\rho})\\
&\quad+c^p(\nabla^S,\nabla^E,(\tilde\nabla_0^{\tilde\rho},\nabla^{\tilde\rho})),
\end{aligned}
$$

where $\nabla_0^{\tilde\rho\bullet}=(\tilde\nabla_0^{\tilde\rho},\tilde\rho^*\nabla^E)$ and $\nabla^{\tilde\rho\bullet}=(\nabla^{\tilde\rho},\rho^*\nabla^E)$. For $i=1,\dots,k-1$, we set

$$
\omega^i=\rho_* c^{n-\ell+i}(\tilde\nabla_0^{\tilde\rho})+(\partial\rho)_* c^{n-\ell+i}(\tilde\nabla_0^{\tilde\rho},\nabla^{\tilde\rho}), \tag{7.17}
$$

where ρ_* and $(\partial\rho)_*$ denote, respectively, the integration along the fibers of $\rho|_{\mathcal{T}}$ and $\rho|_{\partial\mathcal{T}}$. Thus ω^i is a $2i$-form on S. Again, a relative $2q$-cycle γ fibers over a $2(\ell-r+1)$ cycle τ in S. By the projection formula and duality ([Su2] Theorem 6.1, Remark 6.3), we have

$$
\begin{aligned}
&\int_{\gamma\cap\mathcal{T}}c^p(\nabla_0^\bullet)+\int_{\gamma\cap\partial\mathcal{T}}c^p(\nabla_0^\bullet,\nabla^\bullet)\\
&=\left(\int_{\mathcal{T}_x}c^{n-\ell}(\nabla_0^{\tilde\rho\bullet})+\int_{\partial\mathcal{T}_x}c^{n-\ell}(\nabla_0^{\tilde\rho\bullet},\nabla^{\tilde\rho\bullet})\right)\cdot\int_\tau\left[c(\nabla^S)c(\nabla^E)^{-1}\right]^{\ell-r+1}\\
&\quad+\left(\int_{\tilde{\mathcal{T}}_x}c^{m-\ell}(\tilde\nabla_0^{\tilde\rho})+\int_{\partial\tilde{\mathcal{T}}_x}c^{m-\ell}(\tilde\nabla_0^{\tilde\rho},\tilde\nabla^{\tilde\rho})\right)\cdot\int_\tau\left[c(\nabla^S)c(\nabla^E)^{-1}c^k(\nabla^E)\right]^{\ell-r+1}\\
&\quad+\sum_{i=1}^{k-1}\int_\tau\left[c(S)c(E)^{-1}\omega^i\right]^{\ell-r+1}.
\end{aligned}
$$

Noting that, in the right hand side above, the first factor of the first term is $\mathrm{Vir}(v_r,x)$ and the first factor in the second term is $\mathrm{Sch}(v_r,x)$, we obtain the second formula. □

Corollary 7.18. *In the above situation,*

$$
\begin{aligned}
\mu_{r-1}(V,S)=&(-1)^\ell\mu(V\cap H,x)\cdot\left[c(S)c(E)^{-1}\right]^{\ell-r+1}\frown[S]\\
&+(-1)^{n+1}\left[c(S)c(E)^{-1}\sum_{i=1}^{k-1}\left(\mathrm{Sch}(v_r,x)\cdot c^i(E)-\omega^i\right)\right]^{\ell-r+1}\frown[S],
\end{aligned}
$$

where H is an $m-\ell$ dimensional plane transverse to S in M. In particular, when $k=1$, we have

$$
\mu_{r-1}(V,S)=(-1)^\ell\mu(V\cap H,x)\cdot\left[c(S)c(E)^{-1}\right]^{\ell-r+1}\frown[S].
$$

Also, if $r = \ell + 1$ (k arbitrary),

$$\mu_{r-1}(V,S) = (-1)^\ell \mu(V \cap H, x) \cdot [S].$$

Remarks 7.19. 1. In Theorem 7.13 and Corollary 7.18, each ω^i may not be a closed form. However, the sum $\sum_{i=1}^{k-1} \left[c(\nabla^S)\, c(\nabla^E)^{-1}\right]^{\ell-r+1-i} \omega^i$ is closed and defines a cohomology class of S. It would be an interesting problem to give a geometric interpretation of this invariant. Note that these forms do not occur when $k = 1$ (i.e., V hypersurface).

2. When $k = 1$, a formula is proved in [PP3] which gives the "global" Milnor class of V (with arbitrary singularities) as the sum of contributions from each stratum of a stratification of V. The contribution from a non-singular component of Sing(V) is given as in the formula above.

8. Case of hypersurfaces

The classical Milnor number of a hypersurface at an isolated singular point [M2] has been generalized to the case of non-isolated singularities by A. Parusiński [P] in the following way.

Recall that a hypersurface V in M is always defined by a holomorphic section s of a holomorphic line bundle E. If we set $N = E|_V$, there is a canonical vector bundle homomorphism $\pi : TM|_V \longrightarrow N$ which extends the one in (4.3) (see, e.g., [LS]). Note that Sing(V) coincides with the set of points in V where π fails to be surjective. Now let ∇' be a connection for E of type $(1,0)$. This means that in the decomposition $\nabla' = \nabla^{(1,0)} + \nabla^{(0,1)}$ of ∇' into the $(1,0)$ and $(0,1)$ components, we have $\nabla^{(0,1)} = \bar\partial$. Since s is holomorphic, we have $\nabla' s = \nabla^{(1,0)} s$, which is a C^∞ section t of $T^*M \otimes E$. Write $\tilde\pi : TM \longrightarrow E$ the corresponding bundle homomorphism. The restriction of $\tilde\pi$ to V coincides with the bundle homomorphism π above. The set of zeros of t in M, denoted by Sing(t), is equal to the set of points in M where $\tilde\pi$ fails to be surjective.

Lemma 8.1 ([P]) **.** *Let S be a connected component of* Sing(V)*. Then S is also a connected component of* Sing(t)*.*

Let S be a compact component of Sing(V) and $\tilde U$ a neighborhood of S in M disjoint from the other components of Sing(t). Parusiński defines the Milnor number $\tilde\mu_S(V)$ to be the intersection number in $\tilde U$ of the section t of $T^*M \otimes E$ with the zero section (note that the bundle $T^*M \otimes E$ has fiber and base of the same complex dimension $n+1$). Thus it is described as follows.

Let ∇_0 be an arbitrary connection for TM on $\tilde U$ and $\nabla_0^\bullet = (\nabla_0, \nabla')$ the corresponding pair of connections on $\tilde U$. There exists a connection ∇ for TM on $\tilde U - S$, such that the pair $\nabla^\bullet = (\nabla, \nabla')$ is compatible, i.e., such that $\nabla' \circ \tilde\pi = \tilde\pi \circ \nabla$; furthermore the restriction of $\nabla^\bullet$ to V_0 is compatible. Denote by D_0 the connection $\nabla_0^* \otimes \nabla'$ for $T^*M \otimes E$ on $\tilde U$ and by D the connection $\nabla^* \otimes \nabla'$ for $T^*M \otimes E$ on $\tilde U - S$.

Since the pair $\nabla^\bullet$ is compatible, the connection D is t-trivial (both properties are in fact equivalent). Let $\tilde{\mathcal{T}}$ be a $2(n+1)$ real dimensional compact manifold in $\tilde{U}$ with boundary, containing S in its interior. Then we have

$$\check{\mu}_S(V) = \int_{\tilde{\mathcal{T}}} c^{n+1}(D_0) + \int_{\partial\tilde{\mathcal{T}}} c^{n+1}(D_0, D).$$

Theorem 8.2. *We have : $\check{\mu}_S(V) = \mu(V, S)$.*

Proof. First, from the two identities

$$c^{n+1}(D_0) = \sum_{i=0}^{n+1}(-1)^i c^1(\nabla')^{n+1-i} \wedge c^i(\nabla_0),$$
$$c^n(\nabla_0^\bullet) = \sum_{i=0}^{n}(-1)^{n-i} c^1(\nabla')^{n-i} \wedge c^i(\nabla_0),$$

we get $c^{n+1}(D_0) = (-1)^n \left(c^1(\nabla') \wedge c^n(\nabla_0^\bullet) - c^{n+1}(\nabla_0)\right)$. Similarly, we have

$$c^{n+1}(D_0, D) = (-1)^n \left(c^1(\nabla') \wedge c^n(\nabla_0^\bullet, \nabla^\bullet) - c^{n+1}(\nabla_0, \nabla)\right).$$

Now let $(\tilde{R}, v_0)$ be a radial pair around S and take $\tilde{R}$ as $\tilde{\mathcal{T}}$. Also, let F be a C^∞ smoothing of V near S (with $\partial F = \partial\mathcal{T}$). Since $\tilde{\pi}|_F$ is surjective, we get by duality,

$$\int_{\tilde{\mathcal{T}}} c^1(\nabla') \wedge c^n(\nabla_0^\bullet) = \int_F c^n(\nabla_0^\bullet) \quad \text{and} \quad \int_{\partial\tilde{\mathcal{T}}} c^1(\nabla') \wedge c^n(\nabla_0^\bullet, \nabla^\bullet) = \int_{\partial F} c^n(\nabla_0^\bullet, \nabla^\bullet).$$

Thus we have

$$\begin{aligned}\int_{\tilde{\mathcal{T}}} c^1(\nabla') \wedge c^n(\nabla_0^\bullet) + \int_{\partial\tilde{\mathcal{T}}} c^1(\nabla') \wedge c^n(\nabla_0^\bullet, \nabla^\bullet) &= \int_F c^n(\nabla_0^\bullet) + \int_{\partial F} c^n(\nabla_0^\bullet, \nabla^\bullet) \\ &= \int_{\mathcal{T}} c^n(\nabla_0^\bullet) + \int_{\partial\mathcal{T}} c^n(\nabla_0^\bullet, \nabla^\bullet) = \mathrm{Vir}(v_0, S).\end{aligned}$$

On the other hand, we may assume that ∇ is v_0-trivial, hence :

$$\int_{\tilde{\mathcal{T}}} c^{n+1}(\nabla_0) + \int_{\partial\tilde{\mathcal{T}}} c^{n+1}(\nabla_0, \nabla) = \mathrm{PH}(v_0, \tilde{\mathcal{T}}) = \chi(\tilde{\mathcal{T}}) = \chi(S). \qquad \square$$

If M is compact, we have :

$$\begin{aligned}c^{n+1}(T^*M \otimes E) \frown [M] + (-1)^n \chi(M) &= (-1)^n c^1(E) \cdot c^n(TM - E) \frown [M] \\ &= (-1)^n c^n(TM - E) \frown [V].\end{aligned}$$

Hence, from Theorems 7.4 and 8.2, we have following :

Corollary 8.3 ([P]). *With the previous hypothesis, and if furthermore M is compact, and $\tilde{\pi}$ surjective everywhere off V, then :*

$$\sum_{S_\alpha \subset \mathrm{Sing}(V)} \check{\mu}_{S_\alpha}(V) = (-1)^{n+1}\chi(V) + c^{n+1}(T^*M \otimes N) \frown [M] + (-1)^n \chi(M).$$

where the sum is taken over the components of $\mathrm{Sing}(V)$.

Remark 8.4. The method of [P] does not directly extend to the higher codimensional case, first because Lemma 8.1 does not extend, and secondly because M and the fiber of $T^*M \otimes E$ have no more the same dimension.

9. Examples

Example 9.1. Let $F = (f_1, \dots, f_k)$ be a family of k quasi-homogeneous polynomials in $n+k$ variables of the same weights $(d_1, \dots, d_{n+k})$ and respective weighted degrees $r_1, \dots, r_k$: this means that $X(f_\lambda) = r_\lambda f_\lambda$, $(\lambda = 1, \dots, k)$, where X is a vector field on $\mathbb{C}^{n+k}$ given by $X = \sum_{i=1}^{n+k} \frac{z_i}{d_i} \frac{\partial}{\partial z_i}$. Assume furthermore :
(i) The point $0 \in \mathbb{C}^{n+k}$ is an isolated singularity of $V = F^{-1}(0)$,
(ii) the sequence $(z_1, \dots, z_n, f_1, \dots, f_k)$ is regular,
(iii) the natural projection $(z_1, \dots, z_n, z_{n+1}, \dots, z_{n+k}) \to (z_1, \dots, z_n)$ induces by restriction to $F^{-1}(0) - \{0\}$ an N-fold covering, where $N = \prod_{\lambda=1}^k d_{n+\lambda} r_\lambda$.

By [LSS], $\mathrm{Vir}(X, 0) = \left[\frac{\prod_{i=1}^{n+k}(t+d_i))}{\prod_{\lambda=1}^{k}(t+\frac{1}{r_\lambda})}\right]_n$, where $\left[\cdots\right]_n$ denotes the coefficient of t^n in the power series expansion of $\left[\cdots\right]$ in t. Since X is radial outbound from 0, the Schwartz index $\mathrm{Sch}(X, 0)$ is equal to 1, and thus the Milnor number of V at 0 is given by

$$\mu_0(V, 0) = (-1)^n \left(\left[\frac{\prod_{i=1}^{n+k}(t + d_i))}{\prod_{\lambda=1}^{k}(t + \frac{1}{r_\lambda})} \right]_n - 1 \right).$$

Here are some special cases :

a) Assume that all r_λ are equal to 1. Denoting by σ_i the i-th elementary symmetric function of $n+k$ variables, the Milnor number is equal to

$$\mu_0(V, 0) = \sum_{i=n+1}^{n+k} \binom{i-1}{n} \sigma_i(d_1 - 1, \dots, d_{n+k} - 1).$$

In fact, we have $\mathrm{Vir}(X, 0) = \frac{\Phi^{(n)}(0)}{n!}$ with $\Phi(t) = \frac{\prod_{i=1}^{n+k}(t+d_i)}{(1+t)^k}$. By setting $s = 1 + t$ and $\Psi(s) = \Phi(t)$ and computing the n-th derivative of $\Psi(s)$, we get the above formula.
We remark that :

1) For $k = 1$, we recover the usual formula for the Milnor number of quasi-homogeneous functions ([MO]).
2) In the case of functions given by

$$f_\lambda(z_1, \dots, z_{n+k}) = \sum_{i=1}^{n+k} a_{\lambda i}\, z_i^{d_i}$$

such that all the k-minors of the $k \times (n+k)$ matrix $(a_{\lambda i})$ are non-zero, this formula has been proved by different methods, computing the homology of the Milnor fiber in [H2], and using methods of local algebra in [G].

b) Assume that $k = 2$ and that P and Q are homogeneous polynomials of respective degree ℓ and m. According to [LSS] section 4, we have, for $H = \sum_{i=1}^{n+2} z_i \frac{\partial}{\partial z_i}$:

$$\mathrm{Vir}(H, 0) = \ell m \sum_{j=0}^{n} (-1)^j \binom{n+2}{n-j} \frac{\ell^{j+1} - m^{j+1}}{\ell - m},$$

while $\mathrm{Sch}(H, p_0)$ is equal to 1. Hence

$$\mu_0(V, 0) = (-1)^n \ell m \sum_{j=0}^{n} (-1)^j \binom{n+2}{n-j} \left(\frac{\ell^{j+1} - m^{j+1}}{\ell - m} - 1 \right).$$

In particular, for $\ell = m$, we get

$$\mu_0(V, 0) = (\ell - 1)^{n+1} \left(\ell(n+1) + 1 \right).$$

In fact, for $\Phi(t) = \sum_{i=2}^{n+2} \binom{n+2}{i}(i-1)t^{i-2}$, we have $\Phi(-\ell) = \frac{1}{\ell^2}\left((-1)^n \mu_0(V, 0) + 1\right)$. It is easy to check that $\Phi(t) = \frac{d}{dt}\left(\frac{(1+t)^{n+2}-1}{t}\right)$ and from the value of $\Phi(-\ell)$, we get the above formula for $\mu_0(V, 0)$. In particular, for $\ell = 2$, we recover the value $\mu_0(V, 0) = 2n + 3$ given in [Lo] p.78, for $P(z_1, \dots, z_{n+2}) = \sum_{i=1}^{n+2} z_i^2$ and $Q(z_1, \dots, z_{n+2}) = \sum_{i=1}^{n+2} \lambda_i z_i^2$, the λ_i's being distinct complex numbers.

We may apply the above to compute $\chi(V)$, where V denotes this time the cone in $\mathbb{CP}^{n+2}$ defined by P and Q. Thus V has an isolated singular point at $p_0 = [0, \dots, 0, 1]$ and we have $\chi(V) = c^n(\tau) \frown [V] + (-1)^{n+1} \mu_0(V, p_0)$. If γ denotes the Chern class $c^1(L)$ of the hyperplane bundle L (the dual to the tautological line bundle on $\mathbb{CP}^{n+2}$), the virtual tangent bundle τ of V is equal to the restriction to V of $(n+3)L - L^\ell - L^m$, so that

$$c^n(\tau) \frown [V] = \ell m \left[\frac{(1+\gamma)^{n+3}}{(1+\ell\gamma)(1+m\gamma)} \right]_n.$$

Using this and the above formula for the Milnor number, we get $\chi(V)$. Taking for instance $n = 2$, we get $\chi(V) = 1 + \ell m(4 - (\ell + m))$.

Example 9.2. Take for M the projective space $\mathbb{CP}^3$ with homogeneous coordinates $[X, Y, Z, T]$, and let V be the complex analytic curve defined by $X^2 - YT = 0$ and $Z^2 - XY = 0$ in $\mathbb{CP}^3$. It is easy to check that the origin $p_0 = [0,0,0,1]$ is the only singular point of the curve. The vector field $R = 2x\frac{\partial}{\partial x} + 4y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}$ (with respect to the affine coordinates $(x, y, z) = \left(\frac{X}{T}, \frac{Y}{T}, \frac{Z}{T}\right)$ in the affine space $T \neq 0$) is tangent to V, radial outbound from p_0; thus $\mathrm{Sch}(R, p_0) = 1$. On the other hand, $\mathrm{Vir}(R, p_0) = -1$; in fact, using the formula in p.186 of [LSS], we have : $\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 3$, $\mu_1 = 4$, $\mu_2 = 6$, and $\ell = 2$ if we remark that the sequence $(x, x^2 - y, z^2 - xy)$ is regular. We get therefore : $\mu_0(V, p_0) = -(-1-1) = 2$. Since the virtual tangent bundle τ is the restriction to V of $4L - L^2 - L^2$, we get $c^1(\tau) = 0$, hence $\chi(V) = 2$.

Note that we can recover geometrically this value for $\chi(V)$ using the ramified covering $[u, v] \mapsto [X, Y, Z, T] = [u^2v^2, u^4, u^3v, v^4]$ from $\mathbb{CP}^1$ (with homogeneous coordinates $[u, v]$) onto V.

Example 9.3. Take for M the projective space $\mathbb{CP}^4$ with homogeneous coordinates $[X, Y, Z, T, U]$, and let V be the cone over the curve of Example 9.2, i.e., the complex analytic surface defined by $X^2 - YT = 0$ and $Z^2 - XY = 0$ in $\mathbb{CP}^4$.

If we set $\omega = [0,0,0,0,1]$, $p_0 = [0,0,0,1,0]$ and $p = [0,1,0,0,0]$, it is easy to check that the singular set of V is the complex projective line $S = (\omega p_0)$ defined by $X = Y = Z = 0$. Let S' be the complex projective line $S' = (\omega p)$ defined by $X = Z = T = 0$, Γ the intersection of V with the hyperplane $U = 0$, and $\Sigma = S \cup S' \cup \Gamma$. The vector fields

$$H = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} \quad \text{and} \quad R = 2x\frac{\partial}{\partial x} + 4y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}$$

(with respect to the affine coordinates $(x, y, z, t) = \left(\frac{X}{U}, \frac{Y}{U}, \frac{Z}{U}, \frac{T}{U}\right)$ in the affine space $U \neq 0$) are tangent to V, and extend naturally to the hyperplane at infinity $U = 0$. For any complex number a, define $R_a = R + aH$.

Computations for $r = 1$:

The vector field H (resp.R_{-4}) vanishes at ω (resp. p_0) and along Γ (resp. S'). Thus, $\mathrm{Sing}(H) = \Gamma \cup \{\omega\}$ and $\mathrm{Sing}(R_{-4}) = S' \cup \{p_0\}$ are connected, without being included neither in $\mathrm{Sing}(V)$ nor in V_0. For all other values of a, the only singular point of R_a on $V - S$ is p. Thus $\mathrm{Sing}(R_a)$ has two components which are S and $\{p\}$. All R_a $(a \neq -4)$ are radial outbound from p, while all R_a such that $a \neq -2, -3, -4$ are radial outbound from S. Thus $\chi(V) = \chi(S) + \chi(p) = 2 + 1 = 3$, $\mathrm{Sch}(R_a, S) = 2$ and $\mathrm{Sch}(R_a, p) = 1$.

On the other hand, the virtual tangent bundle τ to V is equal to the restriction to V of $5L - L^2 - L^2$, hence $c^2(\tau) \frown [V] = 8$. Since the point p is regular, $\mathrm{Vir}(R_a, p) = \mathrm{Sch}(R_a, p) = 1$ for $a \neq -4$. We deduce therefore $\mathrm{Vir}(R_a, S) = 8 - 1 = 7$, and $\mu_0(V, S) = 7 - 2 = 5$.

Computations for $r = 2$, first method (using general definitions) :

Take the stratification of $\mathbb{CP}^4$ with strata $\{\omega\}$, $S-\{\omega\}$, $V_0 = V-S$, $\mathbb{CP}^4-V$. Assume a smooth triangulation $(\hat{K})$ in $\mathbb{CP}^4$ to be adapted to this stratification, and such that $\Sigma = S \cup S' \cup \Gamma$ be a subcomplex of $(\hat{K})$. Let (D) be the cellular structure dual to the barycentric subdivision K of $(\hat{K})$: the 5 skeleton $(D)^{(5)}$ does not intersect Σ, while $(D)^{(6)}$ intersects Σ at a finite number of isolated points, all different from ω, p, p_0.

Let $v_1 = H + 2R$ and let v_2 be equal to $(H + R)$ close to S and radially extended to a neighborhood of V in M. Observe that v_1 and v_2 are linearly independant off Σ, that v_1 vanishes only at the three points ω, p, and p_0, while v_2 vanishes also on Σ. Let $F^{(2)} = (v_1, v_2)|_{(D)^{(6)}}$: it is a stratified 2 field which is actually a radial 2 frame. In fact,

(i) v_1 and v_2 are linearly independants off Σ, but $(D)^{(5)}$ does not intersect Σ : $F^{(2)}$ is therefore a 2 frame on $(D)^{(5)}$.

(ii) v_1 vanishes only at ω, p, and p_0. However, these three points do not belong to $(D)^{(6)}$: v_1 is therefore non-singular on $(D)^{(6)}$.

(iii) v_2 is a radial extension in the sense of [Sc2]. It has for singular points two kinds of points : first the three points ω, p, and p_0 not belonging to $(D)^{(6)}$, and also the isolated points of $(D)^{(6)} \cap \Sigma$. One can check that the index at any isolated points of $(D)^{(6)} \cap \Sigma$ with respect to some 6 cell of (D) dual to some 2 simplex of $K \cap \Sigma$ is allways 1 (actually, it is the index of the projection on the complementary part of the sub space generated by v_1 in TV_0 for points of $S' \cup \Gamma$, in TM for points of S).

(iv) It is possible to refine (K), thus (D), so that if some 6 cell of (D) contains ω, p, or p_0, it does not contain any other singular point of v_2, and it does not intersect simultaneously $S - \{\omega, p_0\}$, $S' - \{\omega, p\}$ and $\Gamma - \{p, p_0\}$: thus condition (iv) of Lemma 2.7 will also be satisfied.

Since the above index of v_2 is everywhere 1, $c_1(V) = [S] + [S'] + [\Gamma]$, or, equivalently, since S and S' are homologuous in $H_2(V)$, $c_1(V) = 2[S] + [\Gamma]$ (cf. [BFK]). Notice however that the intersection of $S' \cup \Gamma$ with S is off $(D)^{(6)}$, so that we have in fact $\mathrm{Sch}(F^{(2)}, S) = [S] \in H_2(S)$, while $\mathrm{Sch}(F^{(2)}, V_*) = [S'] + [\Gamma]$ in the 2 homology of V_* with closed support. On the other hand, the virtual class $c^1(\tau)$ is given by γ in $H^2(V)$. Since the image of γ by the Poincaré homomorphism is $[\Gamma]$, we get $c_1(\tau) = [\Gamma]$. Thus, $\mathrm{Vir}(F^{(2)}, V_*) = \mathrm{Sch}(F^{(2)}, V_*) = [S'] + [\Gamma]$, while $\mathrm{Vir}(F^{(2)}, S) = -[S]$. Finally, we get $\mu_1(V, S) = -2[S]$.

Computations for $r = 2$, second method (using Corollary 7.18) :

Observe that $\ell = r - 1$ and that the point ω is not in $S^{(0)}$. We get therefore $\mu_1(V, S) = -\mu_0(\Gamma, p_0) \cdot [S]$. By Example 9.2, $\mu_0(\Gamma, p_0) = 2$, thus $\mu_1(V, S) = -2[S]$.

Example 9.4. Take for M the projective space $\mathbb{CP}^2$ with homogeneous coordinates $[X, Y, Z]$, and let C be the curve defined by $X^3 - Y^2 Z = 0$. The curve C may be seen also as an irreducible component of the curve C' defined by $Y(X^3 - Y^2 Z) = 0$. The origin $p_0 = [0, 0, 1]$ is the only singular point of both C and C'. The normal bundle of the regular part C_0 of C coincides with the normal bundle of the regular part of

C'. It may extend to M as L^3 and as L^4. We get therefore two possible virtual tangent bundles τ with two possible values for $c^1(\tau) \frown [C]$ which are, respectively, equal to $3(3-3) = 0$ and $3(3-4) = -3$. Since $\chi(C) = 2$, we have two possible Milnor numbers of C at p_0, which are 2 and 5. The reduced extension provides the first of these numbers, which is the usual number. Note that we must choose $E = L^3$ if we wish C to be the zero set of a holomorphic section of E.

Example 9.5. Take for M the projective space $\mathbb{CP}^4$ with homogeneous coordinates $[X, Y, Z, T, U]$ and for V the algebraic set of pure dimension two defined by

$$(a_1X^2 + a_2Y^2)U^2 + a_3Z^4 + a_4T^4 = 0, \quad (b_1X^2 + b_2Y^2)U^2 + b_3Z^4 + b_4T^4 = 0.$$

First, we have $c^2(\tau) \frown [V] = 288$. Now we assume that all numbers $D_{i,j} = a_ib_j - a_jb_i$ $(i < j)$ are different from zero. Denote by ω the point $[0, 0, 0, 0, 1]$. Since $D_{3,4} \neq 0$, the set $V \cap (U = 0)$ of points "at infinity" is the projective line $L = (p_X p_Y)$ joining $p_X = [1, 0, 0, 0, 0]$ and $p_Y = [0, 1, 0, 0, 0]$. Since $D_{i,j} \neq 0$ $(i < j)$, $\mathrm{Sing}(V)$ has two components, which are ω and L.

Computations for r=1 : The vector field $v = \frac{1}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + \frac{1}{4}\left(z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right)$, defined for $U \neq 0$ (with $(x, y, z, t) = \left(\frac{X}{U}, \frac{Y}{U}, \frac{Z}{U}, \frac{T}{U}\right)$), extends at infinity, and is tangent to V. It is expressed as $v = -\frac{1}{2}u'\frac{\partial}{\partial u} - \frac{1}{4}\left(z'\frac{\partial}{\partial z'} + t'\frac{\partial}{\partial t'}\right)$, for $X \neq 0$ (with $(y', z', t', u') = \left(\frac{Y}{X}, \frac{Z}{X}, \frac{T}{X}, \frac{U}{X}\right)$), and similarly for $Y \neq 0$. The restriction to V of this vector field does not vanish off $\mathrm{Sing}(V)$. Since this vector field is radial outbound from ω, and radial inbound to L, $\mathrm{Sch}(v, \omega)$ and $\mathrm{Sch}(v, L)$ are, respectively, equal to 1 and 2. We deduce : $\chi(V) = 1 + 2 = 3$. By Example 9.1, $\mu_0(V, \omega) = 3^1(4 + 4) + 3^2(4 - 1) = 51$, hence

$$\mathrm{Vir}(v, \omega) = \mu_0(V, \omega) + 1 = 52, \qquad \mathrm{Vir}(v, L) = c^2(\tau) \frown [V] - \mathrm{Vir}(v, \omega) = 236,$$

and

$$\mu_0(V, L) = \mathrm{Vir}(v, L) - \mathrm{Sch}(v, L) = 234.$$

Computations for $r = 2$: According to Corollary 7.18, since $\ell = r - 1$, we get :

$$\mu_1(V, L) = -\mu_0(\Gamma, p_Y) \cdot [L],$$

where Γ denotes the curve $V \cap (X = 0)$ in $\mathbb{CP}^3$ with coordinates (U, Y, Z, T). This Γ is defined locally, near its intersecting point p_Y with L by quasi-homogeneous polynomials of the type

$$a_2(u')^2 + a_3(z')^4 + a_4(t')^4 = 0, \qquad b_2(u')^2 + b_3(z')^4 + b_4(t')^4 = 0.$$

Hence by Example 9.1, we get $\mu_0(\Gamma, p_Y) = 33$, thus $\mu_1(V, L) = -33[L]$.

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