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Generalized Numerical Radius And Unitary ρ-Dilation

by

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Abstract. In this paper, we study an operator A on a Hilbert space H which satisfies one of the following inequalities :
For some λ with $0 \leq \lambda \leq 1$

$$|(Ay, y)| \leq \lambda \|y\|^2 + (1 - \lambda)\|Ay\|^2 \quad (y \in H)$$

or

$$\lambda \|Ay\|^2 + (1 - \lambda)|(Ay, y)| \leq \|y\|^2 \quad (y \in H).$$

These two inequalities can be regarded as special cases of generalized numerical ranges. If A has a ρ-dilation with $\rho > 0$, then it satisfies one of them. We show that the operator radii $w_\rho(A)$ of A are calculated using $|(Ay, y)|$ and $\|Ay\|$. Several applications are given.

§1. Introduction

According to Sz.-Nagy and Foias [7], a bounded linear operator A on a complex Hilbert space H is said to be of class $\mathcal{C}_\rho$ with $\rho > 0$ if there exists a unitary operator U on some Hilbert space K such that K contains H as a subspace and such that

$$A^n = \rho P_H U^n | H \quad \text{for } n = 1, 2, \cdots,$$

where P_H is the orthogonal projection of K onto H. For $\rho = 2$ it is known (see [7, Chapter I, Proposition 11.2]) that A is of class $\mathcal{C}_2$ if and only if its numerical radius of A :

$$w(T) = \sup\{|(Ay, y)| : \|y\| \leq 1\}$$

is not greater than one. Of course, A is of class $\mathcal{C}_1$ if and only if the usual norm $\|A\|$ of A is not greater than one. Holbrook [3] introduced the operator radii $w_\rho(A)$ of an operator A, relative to $\mathcal{C}_\rho$, by the formula :

$$w_\rho(A) = \inf\{\gamma \ ; \ \gamma > 0, \gamma^{-1} A \in \mathcal{C}_\rho\}.$$

Then $w_1(A) = \|A\|$, $w_2(A) = w(A)$, and $\lim_{\rho \to \infty} w_\rho(A) = r(A)$: the spectral radius of A.

Let S be a positive bounded operator on H. We define $V_S^+(A)$ and $v_S(A)$ as

$$V_S^+(A) = \{(Ay, y) \ ; \ y \in H, (Sy, y) = 1\}.$$

and

$$v_S(A) = \sup\{|(Ay, y)| \ ; \ y \in H, (Sy, y) = 1\}.$$

$V_S^+(A)$ is defined and studied in [4] for a self-adjoint S. If S is the identity operator I on H, then $V_S^+(A)$ is the numerical range of A and $v_S(A) = w(A)$. In Section 2, when $0 < \rho \leq 2$ and $\rho \neq 1$, we show that A is of class $\mathcal{C}_\rho$ if and only if

$$S = \frac{\rho}{2|\rho - 1|} I + \frac{\rho - 2}{2|\rho - 1|} |A|^2$$

is nonnegative and $v_S(A) \leq 1$. As results, several corollaries are given. In Section 3, when $0 < \rho \leq 2$ and $\rho \neq 1$, we give two formulae for $w_\rho(A)$ using $|(Ay, y)|$ and $\|Ay\|$. In Section 4, we try to generalize results in Sections 2 and 3 for $2 < \rho < \infty$.

A result in this paper is the following : For $0 < \rho \leq 2$ and $\rho = 2/(\lambda + 1)$,

$$\sup_{\|y\|=1} \{\lambda \|Ay\|^2 + |1 - \lambda| \cdot |(Ay, y)|\} \leq 1$$

if and only if $w_\rho(A) \leq 1$ (see Corollay 2). This shows the following as $A/w_\rho(A)$. If $w_\rho(A) \leq 1$, then

$$\sup_{\|y\|=1} \{\lambda \|Ay\|^2 + |1 - \lambda| \cdot |(Ay, y)|\} \leq w_\rho(A)$$

and if $w_\rho(A) \geq 1$, then

$$\sup_{\|y\|=1} \{\lambda\|Ay\|^2 + |1-\lambda| \cdot |(Ay,y)|\}^{1/2} \leq w_\rho(A).$$

$w_\rho(A)$ is calculated using $|(Ay, y)|$ and $\|Ay\|$ (see (1) of Theorem 2). This shows the following general inequality (see Corollary 3). For any $w_\rho(A)$,

$$w_\rho(A) \leq \sup_{\|y\|=1} \{\sqrt{\lambda}\|Ay\| + |1-\lambda| \cdot |(Ay,y)|\}.$$

§2. ρ-dilation for $0 < \rho \leq 2$ and generalized numerical radius

In this section, we are interested in operators with $v_S(A) \leq 1$ when S is a special positive operator. If $S = |A|$ and $v_S(A) \leq 1$, then A is normal (*cf.* [2],[8]). We consider A when $S = \lambda I + \mu|A|^2, \lambda + \mu = \pm 1$ and $v_S(A) \leq 1$ where $\lambda \geq 0$ and μ are constants.

For $0 \leq \mu < \infty$

$$\mathbf{w}_\mu(A) = \sup\{\mu\|Ay\|^2 + |1-\mu| \cdot (Ay,y)| \; ; \; \|y\| = 1\}.$$

Then $\mathbf{w}_0(A) = w(A)$ and $\mathbf{w}_1(A) = \|A\|^2$.

Theorem 1. *Suppose* $0 < \rho \leq 2$ *and* $\rho \neq 1$. *Then* A *is of class* C_ρ *if and only if* $S \geq 0$ *and* $v_S(A) \leq 1$ *where* $S = (\rho I + (\rho - 2)|A|^2)/2|\rho - 1|$.

Proof. It is known that A is of class C_ρ if and only if

$$\|y\|^2 + \left(1 - \frac{2}{\rho}\right) |\zeta|^2\|Ay\|^2 - 2\left(1 - \frac{1}{\rho}\right) Re\zeta(Ay,y) \geq 0$$

for $\zeta \in D$ and $y \in H$ where $D = \{z \in \mathbf{C} : |z| < 1\}$. This inequality is equivalent to

$$Re\zeta(Ay,y) \leq \frac{\rho}{2|\rho-1|} \|y\|^2 + \frac{\rho-2}{2|\rho-1|} |\zeta|^2\|Ay\|^2$$

for $\zeta \in D$ and $y \in H$, and

$$|\zeta|(Ay,y)| \leq \frac{\rho}{2|\rho-1|} \|y\|^2 + \frac{\rho-2}{2|\rho-1|} |\zeta|^2\|Ay\|^2$$

for $\zeta \in D$ and $y \in H$. Since $0 < \rho \leq 2$, the last inequality is equivalent to

$$|(Ay,y)| \leq \frac{\rho}{2|\rho-1|}\|y\|^2 + \frac{\rho-2}{2|\rho-1|}\|Ay\|^2$$

for $y \in H$.

Corollary 1. *Suppose* $0 < \rho < 2$ *and* $\rho \neq 1$. *A is of class* C_ρ *if and only if A admits a factorization :* $A = S^{1/2}BS^{1/2}$ *where* $S = (\rho I + (\rho-2)|A|^2)/2|\rho - 1|$ *and* $w(B) \leq 1$.

Proof. Suppose A is of class C_ρ. If there exists $y \in H$ such that $Sy = 0$, then $|A|^2 y = \rho y/(2-\rho)$. Since $\rho w_\rho(A) \geq \|A\|$ [1],

$$1 \geq w_\rho(A) \geq \frac{\|A\|}{\rho} \geq \frac{1}{\sqrt{\rho(2-\rho)}}.$$

and hence $\rho = 1$. This contradiction implies that $\ker S = \{0\}$. Since $\ker S = \{0\}, v_S(A) \leq 1$ if and only if

$$|(S^{-1/2}AS^{-1/2}x, x)| \leq (x, x) \quad (x \in H).$$

Let $B = S^{-1/2}AS^{-1/2}$, then this inequality is equivalent to that

$$A = S^{1/2}BS^{1/2} \text{ and } w(B) \leq 1.$$

Now Theorem 1 implies the corollary.

Corollary 2. *Let A be a bounded linear operator on H and* $0 < \rho = \dfrac{2}{\mu+1} \leq 2$. *Then the following conditions are mutually equivalent :*

(1) $w_\rho(A) \leq 1$.

(2) $\mathbf{w}_\mu(A) \leq 1$

and

(3) $w(\mu|A|^2 + |1-\mu|e^{i\theta}A) \leq 1$

for any $\theta \in R$.

Proof. By Theorem 1,$w_\rho(A) \leq 1$ if and only if

$$\frac{2-\rho}{2|\rho-1|}\|Ay\|^2 + |(Ay, y)| \leq \frac{\rho}{2|\rho-1|}\|y\|^2$$

for $y \in H$. The last inequality is equivalent to

$$\mu\|Ay\|^2 + |1-\mu| \cdot |(Ay, y)| \leq \|y\|^2$$

where $\mu = (2-\rho)/\rho$. This implies the equivalence of (1) and (2). The equivalence of (2) and (3) is trivial.

(1),(2) of Corollary 2 implies that if $w_\rho(A) \leq 1$ then $\mathbf{w}_\mu(A) \leq w_\rho(A)$ and if $w_\rho(A) \geq 1$ then $\mathbf{w}_\mu(A)^{1/2} \leq w_\rho(A)$. By (1),(3) of Corollary 2, if $\mu\|A\|^2 + |1-\mu|w(A) \leq 1$

then $w_\rho(A) \le 1$. However the converse is not true. For if $A = \begin{bmatrix} 0 & \rho \\ 0 & 0 \end{bmatrix}$, then $\|A\| = \rho, w(A) = \rho/2$ and $w_\rho(A) = 1$. Hence $\mu = (2-\rho)/\rho$ and

$$\mu\|A\|^2 + |1-\mu|w(A) = \frac{2-\rho}{\rho} \times \rho^2 + \frac{2|\rho-1|}{\rho} \times \frac{\rho}{2} > 1.$$

§3. Operator radii for $0 < \rho \le 2$

In this section, we give two exact formulae and useful estimates for $w_\rho(A)$ when $0 < \rho \le 2$ and $\rho \ne 1$. Put

$$D = D(A, \rho, y) = |(Ay, y)|^2 - \frac{\rho(\rho-2)}{(\rho-1)^2}\|Ay\|^2$$

for a bounded linear operator $A, \rho > 0$ and y in H.

Theorem 2. *Let A be a bounded linear operator on H.*
(1) If $0 < \rho \le 2$ and $\rho \ne 1$, then

$$w_\rho(A) = \frac{|\rho-1|}{\rho} \sup_{\|y\|=1} \{|(Ay, y)| + \sqrt{D}\}.$$

(2) If $0 < \rho \le 2$, and $\rho \ne 1$ then

$$w_\rho(A) = \frac{2}{\rho} \sup_{\|y\|=1} \sup_{0 \le t \le 1} \{\sqrt{\rho(2-\rho)}\|Ay\|\sqrt{t(1-t)} + |\rho - 1| \cdot |(Ay, y)|t\}.$$

Proof. (1) By Theorem 1, if $t \ge w_\rho(A)$ then $\lambda\|y\|^2 t^2 - |(Ay, y)|t + (1-\lambda)\|Ay\|^2 \ge 0$ for $y \in H$ where $\lambda = \rho/2|\rho-1|$. Hence

$$w_\rho(A) \le \frac{|(Ay, y)| - \sqrt{D}}{2\lambda\|y\|^2}$$

or

$$w_\rho(A) \ge \frac{|(Ay, y)| + \sqrt{D}}{2\lambda\|y\|^2}.$$

Put

$$t_0 = \sup_{y \ne 0} \frac{|(Ay, y)| + \sqrt{D}}{2\lambda\|y\|^2}.$$

If $t \geq t_0$, then

$$\lambda\|y\|^2t^2 - |(Ay,y)|t + (1-\lambda)\|Ay\|^2 \geq 0.$$

for $y \in H$ and so by Theorem 1 $w_\rho(A) \leq t_0$. When $0 < \rho \leq 2, |(Ay,y)| - \sqrt{D} \leq 0$ and so $w_\rho(A) \geq t_0$. Thus $w_\rho(A) = t_0$.

(2) Put $g(t,y) = \sqrt{\rho(2-\rho)}\|Ay\|\sqrt{t(1-t)} + |\rho - 1| \cdot |(Ay,y)|t$ for each y with $\|y\| = 1$. Then

$$\begin{aligned} &2\left(\frac{d}{dt}g(t,y)\right)\sqrt{t(1-t)} \\ = &\sqrt{\rho(2-\rho)}\|Ay\|(1-2t) + 2|\rho-1| \cdot |(Ay,y)|\sqrt{t(1-t)}. \end{aligned}$$

Hence $\frac{d}{dt}g(t,y)|_{t=t_0} = 0$ and $0 \leq t_0 \leq 1$ if and only if

$$t_0 = \frac{1}{2} + \frac{|(Ay,y)|}{2\sqrt{D}}$$

and so

$$\sqrt{t_0(1-t_0)} = \frac{\sqrt{\rho(2-\rho)}\|Ay\|}{2|\rho-1|\sqrt{D}}.$$

Therefore

$$\begin{aligned} &\frac{2}{\rho}\sup_{\|y\|=1}\sup_{0\leq t\leq 1}\{\sqrt{\rho(2-\rho)}\|Ay\|\sqrt{t(1-t)} + |\rho-1| \cdot |(Ay,y)|t\} \\ &= \frac{2}{\rho}\sup_{\|y\|=1}\left\{\sqrt{\rho(2-\rho)}\|Ay\| \times \frac{\sqrt{\rho(2-\rho)}\|Ay\|}{2|\rho-1| \cdot \sqrt{D}} + |\rho-1| \cdot |(Ay,y)| \times \left(\frac{1}{2} + \frac{|(Ay,y)|}{2\sqrt{D}}\right)\right\} \\ &= \frac{|\rho-1|}{\rho}\sup_{\|y\|=1}\{|(Ay,y)| + \sqrt{D}\}. \quad \square \end{aligned}$$

Corollary 3. *If* $0 < \rho \leq 2$, *then*

$$w_\rho(A) \leq \sup_{\|y\|=1}\{\sqrt{\frac{2-\rho}{\rho}}\|Ay\| + 2\left|1 - \frac{1}{\rho}\right| \cdot |(Ay,y)|\}.$$

Proof. Since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}, (a,b > 0)$ for $\rho \neq 1$

$$\sqrt{D} \leq |(Ay,y)| + \frac{\sqrt{\rho(2-\rho)}}{|\rho-1|}\|Ay\|.$$

This inequality and (1) of Theorem 2 imply the corollary.

Corollary 4. *If* $0 < \rho \leq 2$, *then*

$$\max\left\{2\left|1-\frac{1}{\rho}\right|w(A), \sqrt{\frac{2-\rho}{\rho}}\|A\|\right\} \leq w_\rho(A) \leq 2\left|1-\frac{1}{\rho}\right|w(A) + \sqrt{\frac{2-\rho}{\rho}}\|A\|$$

Proof. Since $-\rho(\rho-2)\|Ay\|^2/(\rho-1)^2 \geq 0$. by (1) of Theorem 2

$$w_\rho(A) \geq 2\frac{|\rho-1|}{\rho}|(Ay,y)|, \ \frac{\sqrt{\rho(2-\rho)}}{\rho}\|Ay\|$$

for $y \in H$. Hence

$$w_\rho(A) \geq \max\left\{2\left|1-\frac{1}{\rho}\right|w(A), \sqrt{\frac{2-\rho}{\rho}}\|A\|\right\}.$$

We can get the upper estimate of $w_\rho(A)$ using Corollary 3.

Corollary 5. *If* $0 < \rho \leq 2$, *and* $\rho \neq 1$ *then*

$$\begin{aligned} w_\rho(A) &\leq \frac{2}{\rho}\sup_{0\leq t\leq 1}\{\sqrt{\rho(2-\rho)}\|A\|\sqrt{t(1-t)} + |\rho-1|\cdot w(A)t\} \\ &= \frac{|\rho-1|}{\rho}\left\{w(A) + \sqrt{w(A)^2 - \frac{\rho(\rho-2)}{(\rho-1)^2}\|A\|^2}\right\} \end{aligned}$$

and for any $\|y\| = 1$ and $0 \leq t \leq 1$

$$w_\rho(A) \geq \frac{2}{\rho}\{\sqrt{\rho(2-\rho)}\|Ay\|\sqrt{t(1-t)} + |\rho-1|\cdot|(Ay,y)|t\}.$$

Corollary 6. [5] *If* $0 < \rho \leq 2$, *then*

$$\rho w_\rho(A) = 2w\left(\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)} & A \\ 0 & (1-\rho) & A \end{bmatrix}\right).$$

Proof. For $0 < \rho \leq 2$,

$$\begin{aligned} &2w\left(\begin{bmatrix} 0 & \sqrt{\rho(2-\rho)} & A \\ 0 & (1-\rho) & A \end{bmatrix}\right) \\ &= 2\sup_{\|x\|^2+\|z\|^2=1}|\sqrt{\rho(2-\rho)}(Az,x) + (1-\rho)(Az,z)| \\ &= 2\sup_{\|x\|^2+\|z\|^2=1}\{\sqrt{\rho(2-\rho)}|(Az,x)| + |1-\rho|\cdot|(Az,z)|\} \\ &= 2\sup_{0\leq\lambda\leq 1}\sup_{\|z\|=\sqrt{\lambda}}\{\sqrt{\rho(2-\rho)}\|Az\|\sqrt{1-\lambda} + |1-\rho|\cdot|(Az,z)|\} \\ &= 2\sup_{0\leq\lambda\leq 1}\sup_{\|y\|=1}\{\sqrt{\rho(2-\rho)}\|Ay\|\sqrt{\lambda(1-\lambda)} + |1-\rho|\cdot|(Ay,y)|\sqrt{\lambda}\} \\ &= \rho\, w_\rho(A). \end{aligned}$$

We used (2) of Theorem 2 to show the last eqnality. □

§4. The case of $2 < \rho < \infty$

In this section, we consider Theorems 1 and 2 for $2 < \rho < \infty$. Unfortunately the results for $2 < \rho < \infty$ are more complicated than those for $0 < \rho \leq 2$.

For $0 \leq \lambda \leq 1$, put

$$\mathrm{w}'_\lambda(A) = \sup\{|(Ay, y)| \ ; \ \lambda\|y\|^2 + (1-\lambda)\|Ay\|^2 \leq 1\}.$$

Then $\mathrm{w}'_1(A) = w(A)$, and $\mathrm{w}'_0(A) = w(A^{-1})$ if A is invertible.

Proposition 3. *Suppose $2 < \rho < \infty$ and for $0 < t \leq 1$*

$$S_t = \frac{1}{t}\,\frac{\rho}{2|\rho-1|}I + t\,\frac{\rho-2}{|\rho-1|}\,|A|^2.$$

A is of class C_ρ if and only if $S_t \geq 0$ and $v_{S_t}(A) \leq 1$ for $0 < t \leq 1$.

Proof is almost same to that of Theorem 1.

Using Proposition 3 we can show a version of Corollary 1 for $2 < \rho < \infty$. For $0 < \lambda < 1$ and arbitrary bounded operator A on H,

$$\mathrm{w}'_\lambda(A) \leq 1/\sqrt{\lambda(1-\lambda)}.$$

In fact, for any constant $t > 0$

$$|(Ay, y)| \leq (\sqrt{t}\|y\|)(\frac{1}{\sqrt{t}}\|Ay\|) \leq 2t\|y\|^2 + \frac{1}{2t}\|Ay\|^2.$$

Assuming $\lambda \neq 0, 1$, if $k \geq 1/\sqrt{\lambda(1-\lambda)}$ then $k\lambda \geq 1/k(1-\lambda)$. Hence if $k\lambda = 2t$ then $k(1-\lambda) \geq 1/2t$. Therefore if $k \geq 1/\sqrt{\lambda(1-\lambda)}$, then

$$|(Ay, y)| \leq k(\lambda|y\|^2 + (1-\lambda)\|Ay\|^2)$$

and so $\mathrm{w}'_\lambda \leq 1/\sqrt{\lambda(1-\lambda)}$.

$\mathrm{w}'_0(A) < \infty$ if and only if there exists a bounded operator B such that $BA = P$ where P is an orthogonal projection to $(\ker A)^\perp$. For if $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is the matrix of A with respect to the decomposition $H = (\ker A)^{\perp} \oplus \ker A$, then $A_{12} = A_{22} = 0$. If $k = \mathrm{w}_0'(A) < \infty$, then for $z = (x, y) \in (\ker A)^{\perp} \oplus \ker A$

$$|(A_{11}x, x) + (A_{21}x, y)| \leq k(\|A_{11}x\|^2 + \|A_{21}x\|^2).$$

This implies $A_{21} = 0, A_{11}$ is one to one on $(\ker A)^{\perp}$ and $|(A_{11}x, x)| \leq k\|A_{11}x\|^2$. If $B_{11}A_{11}x = x$, then by the inequality above B_{11} is bounded and so $B = B_{11} \oplus O$ satisfies $BA = P$. The proof is reversible.

Corollary 7. *Let A be a bounded linear operator and let $0 \leq \lambda \leq 1$.*

(1) Suppose $\frac{1}{2} < \lambda \leq 1$ and $\rho = 2\lambda/(2\lambda - 1) \geq 2$. Then $\mathrm{w}_\lambda'(tA) \leq 1$ for any $0 < t \leq 1$ and if and only if $w_\rho(A) \leq 1$

(2) Suppose $0 \leq \lambda < \frac{1}{2}, \rho = 2(\lambda - 1)/(2\lambda - 1) \geq 2$ and A is invertible. Then $\mathrm{w}_\lambda'(tA) \leq 1$ for any $t \geq 1$ if and only if $w_\rho(A^{-1}) \leq 1$.

Proof. It is clear by Proposition 3. □

Proposition 4. *Let A be a bounded linear operator on H*

(1) If $2 < \rho < \infty$, then

$$w_\rho(A) = \frac{\rho - 1}{\rho} \sup_{\|y\|=1} \{|(Ay, y)| + \sqrt{D}\ ;\ D \geq 0\}.$$

(2) If $2 < \rho < \infty$, then

$$w_\rho(A) = \frac{2}{\rho} \sup_{\|y\|=1} \inf_{t \geq 1} \{-\sqrt{\rho(\rho-2)}\|Ay\|\sqrt{t(t-1)} + (\rho - 1)|(Ay, y)|t\ ;\ D \geq 0\}.$$

Proof. (1) The proof is almost same to that of (1) of Theorem 2.

(2) Put $f(t, y) = -\sqrt{\rho(\rho-2)}\|Ay\|\sqrt{t(t-1)} + (\rho - 1)|(Ay, y)|t$ for each y with $\|y\| = 1$. Then

$$\begin{aligned} &2\left(\frac{d}{dt}f(t, y)\right)\sqrt{t(t-1)} \\ &\quad = -\sqrt{\rho(\rho-2)}\|Ay\|(2t - 1) + 2(\rho - 1)|(Ay, y)|\sqrt{t(t-1)}. \end{aligned}$$

Hence $\frac{d}{dt}f(t, y)|_{t=t_0} = 0$ and $t_0 \geq 1$ if and only if

$$t_0 = \frac{1}{2} + \frac{|(Ay, y)|}{2\sqrt{D}}$$

and so

$$\sqrt{t_0(t_0-1)} = \frac{\sqrt{\rho(\rho-2)}\|Ay\|}{2(\rho-1)\sqrt{D}}$$

Therefore

$$\frac{2}{\rho} \sup_{\substack{\|y\|=1 \\ D\geq 0}} \inf_{t\geq 1}\{-\sqrt{\rho(\rho-2)}\|Ay\|\sqrt{t(t-1)} + (\rho-1)|(Ay,y)|t\}$$

$$= \frac{2}{\rho} \sup_{\substack{\|y\|=1 \\ D\geq 0}} \left\{-\sqrt{\rho(\rho-2)}\|Ay\| \times \frac{\sqrt{\rho(\rho-2)}\|Ay\|}{2(\rho-1)\sqrt{D}} + (\rho-1)|(Ay,y)| \times \left(\frac{1}{2} + \frac{|(Ay,y)|}{2\sqrt{D}}\right)\right\}$$

$$= \frac{\rho-1}{\rho} \sup_{\substack{\|y\|=1 \\ D\geq 0}} \{|(Ay,y)| + \sqrt{D}\}$$

Corollary 8. *Suppose* $\rho \geq 2$ *and* $w(A) \geq \dfrac{\sqrt{\rho(\rho-2)}}{\rho-1}\|A\|$. *Then*

$$\begin{aligned} & 2(1-\frac{1}{\rho})w(A) - \sqrt{\frac{\rho-2}{\rho}}\|A\| \\ \leq\ & (1-\frac{1}{\rho})\left\{w(A) + \left(w(A)^2 - \frac{\rho(\rho-2)}{(\rho-1)^2}\|A\|^2\right)^{1/2}\right\} \\ \leq\ & w_\rho(A) \\ \leq\ & (1-\frac{1}{\rho})\left\{w(A) + \left(w(A)^2 + \frac{\rho(\rho-2)}{(\rho-1)^2}\|A\|^2\right)^{1/2}\right\} \\ \leq\ & 2(1-\frac{1}{\rho})w(A) + \sqrt{\frac{\rho-2}{\rho}}\|A\|. \end{aligned}$$

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