FREE BOUNDARY PROBLEM FOR QUASILINEAR PARABOLIC EQUATION WITH FIXED ANGLE OF CONTACT TO A BOUNDARY

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Series #404. February 1998

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FREE BOUNDARY PROBLEM FOR QUASILINEAR PARABOLIC EQUATION WITH FIXED ANGLE OF CONTACT TO A BOUNDARY

YOSHIHITO KOHSAKA

1. Introduction

We consider the following free boundary problem of form;

$$u_t = (a(u_x))_x, \qquad s(t) < x < 0,\ t > 0, \tag{1.1}$$

$$u_x(s(t), t) = \tan\theta_0, \quad t \geq 0, \tag{1.2}$$

$$u_x(0, t) = \tan\theta_1, \qquad t \geq 0, \tag{1.3}$$

$$u(s(t), t) = 0, \qquad t \geq 0, \tag{1.4}$$

$$u(x, 0) = u_0(x), \qquad s(0) := s_0 \leq x \leq 0, \tag{1.5}$$

where $a \in C^2(\mathbb{R})$ and $a'(\sigma) > 0$ for $\sigma \in \mathbb{R}$ ($' = \frac{d}{d\sigma}$), and s_0 is a given negative number, and $u_0 \in C^2[s_0, 0]$. We also assume a compatibility condition $u_{0x}(s_0) = \tan\theta_0, u_{0x}(0) = \tan\theta_1, u_0(s_0) = 0$, and assume $u_0(x) > 0$ for $x \in (s_0, 0]$. The angles $\theta_i \in (0, \frac{\pi}{2})$ for $i = 0, 1$ will be measured counter-clockwise from the x-axis.

If we set $a(\sigma) = \arctan\sigma$, the equation (1.1) is the curvature flow equation for the graph of u separating two phase. The curvature flow equation is one of the typical evolution equations which describe the motion of the phase boundary. In this case, this problem is the curvature flow problem with prescribed angle on the boundary of the second quadrant. (cf. Figure 1.1)

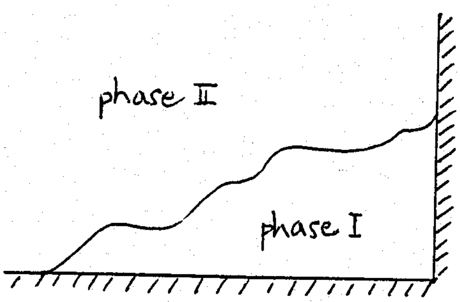

Figure 1.1

If we set $a(\sigma) = \sigma$, the equation (1.1) is the heat equation. In this case, this problem appears in the combustion theory.

In this note, we consider the convergence of the solution of (1.1)-(1.5) as t $\to \infty$ in the case $\theta_0 < \theta_1$. Our main goal of this paper is to show that the solution of (1.1)-(1.5) converges as $t \to \infty$ to the unique self-similar solution in the case $\theta_0 < \theta_1$.

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Main Theorem. *Assume that* $\theta_0 < \theta_1$.

(I) There exists an expanding self-similar solution S_t *corresponding to the problem (1.1)-(1.5) which is unique up to translation of time.*

(II) Let Γ_t *be a solution of (1.1)-(1.5). Then, for each* $0 < \delta < 1/2$, *there is a constant* C_δ *such that*

$$d_H(\Gamma_t, S_t) \leq C_\delta t^{-\delta} \quad \text{for } t \geq 1$$

where d_H *denotes the Hausdorff distance.*

To prove this theorem, we employ what is called similarity change of variables;

$$u(x,t) = \sqrt{2t+1}\, U(\xi, \tau), \ s(t) = \sqrt{2t+1}\, p(\tau), \tag{1.6}$$

where

$$\xi = \frac{x}{\sqrt{2t+1}}, \ \tau = \frac{1}{2}\log(2t+1). \tag{1.7}$$

Then, the equation (1.1) becomes

$$U_\tau = (a(U_\xi))_\xi + \xi U_\xi - U. \tag{1.8}$$

A stationary solution to (1.8) is called a self-similar solution.

We show in Section 2 that the self-similar solution corresponding to the problem (1.1)-(1.5) exists uniquely. We consider the following ordinary differential equation of form (P);

$$(a(U_\xi))_\xi + \lambda \xi U_\xi - \lambda U = 0, \quad \xi \in (q, 0), \tag{1.9}$$

$$U_\xi(q) = \tan\ \theta_0, \tag{1.10}$$

$$U_\xi(0) = \tan\ \theta_1, \tag{1.11}$$

$$U(q) = 0. \tag{1.12}$$

This is the stationary problem of (1.8) with the boundary conditions for $\lambda = 1$. For the proof, we shall employ the shooting method. That is, we first consider the set of the parameter λ (denoted J) as the solution of the initial value problem (1.9),(1.10),(1.12)(denoted $(P)_\lambda$) exists. Here, We define the map $\Phi : J \ni \lambda \mapsto U_\xi(0; \lambda)$. We prove that the map Φ is strictly monotone so that it is injective. Moreover, we prove that Φ is surjective from its domain of the definition to $[\tan\theta_0, \infty)$. Consequently, we obtain a unique solution of the problem (P).

We show in Section 3 that the solution of (1.1)-(1.5) converge as $t \to \infty$ to the self-similar solution. For the proof, we construct the subsolution and the supersolution converging to the stationary solution to (1.8) with the boundary conditions.

In this note, we do not prove the existence of the solution for the problem (1.1)-(1.5). As for the existence, we refer to A. Fasano and M. Primicerio [1]. They proved the local time existence and uniqueness of the equation $u_t = A(u)u_{xx}$ with the boundary conditions $u_x(s(t), t) = P/(1 - s(t))$ (P : a constant), $u_x(0,t) = h(t), u(s(t), t) = 0$, and the initial data $u(x, 0) = u_0(x)$, where $A \in C^2(\mathbb{R})$, $A(u) > 0$, and h, u_0 were given. We explain their idea of the proof. For given s, they find the solution of the problem excluding the condition $u(s(t),t) = 0$. From this u, one finds $\hat{s}$ as a solution of $ds(t)/dt = -(1 - s(t))u_t(s(t),t)/P$, which is obtained from $u(s(t),t) = 0$. They proved that the mapping $s \mapsto \hat{s}$ is contraction in some topology provided that the time interval is small. The fixed point is the desired free boundary. We note that one proves the local time existence of the solution for our problem by the similar method. For the local time existence and uniqueness of the free boundary problem for quasilinear parabolic equation, there is a paper by D. Andreucci and R. Gianni [2]. In a little bit different setting they studied the two phase problem with a jump condition for $|Du|$ across the free boundary.

There are several references studying the asymptotic analysis. We only refer these papers dealing with the problem with boundary conditions directly related to ours. We refer to S. J. Altschuler and L. F. Wu [3],[4], N. Ishimura [5], D. Hilhost and J. Hulshof [6], V. A. Galaktionov, J. Hulshof, and J. L. Vazques [7]. In [3] they studied the asymptotic behavior of the solution for the equation $u_t = (a(u_x))_x$ with the boundary conditions $u_x(j,t) = \tan\alpha_j (j = 0, d)$ (cf. Figure 1.2). They proved that this solution converges as $t \to \infty$ to a solution moving by translation with speed $(a(\tan\alpha_d) - a(\tan\alpha_0))/d$. In [4] the same problem is considered in two space dimension. In [5] N. Ishimura studied the evolution of plane curves which are described by entire graphs with prescribed opening angle. That is, he considered the asymptotic behavior of the solution for the curvature flow equation for the graph of u with the boundary conditions $u_x \to K_1$ as $x \to \infty$, $u_x \to -K_2$ as $x \to \infty$ $(0 < K_1 \leq K_2 < \infty)$. He assumed that the initial data u_0 is convex, and proved that this solution converges as $t \to \infty$ to the convex self-similar solution corresponding to his problem. The authors of [6] considered one-dimensional free boundary problem arising in combustion theory. They studied the asymptotic behavior of the solution for the heat equation with the boundary conditions $u_x(0,t) = 0, u_x(\zeta(t),t) = 1, u(\zeta(t),t) = 0$. They proved that all solutions are asymptotically equal to a self-similar shrinking solution which vanishes in a finite time. In [7] they extended the result of [6] to the radial symmetric multi-dimensional case.

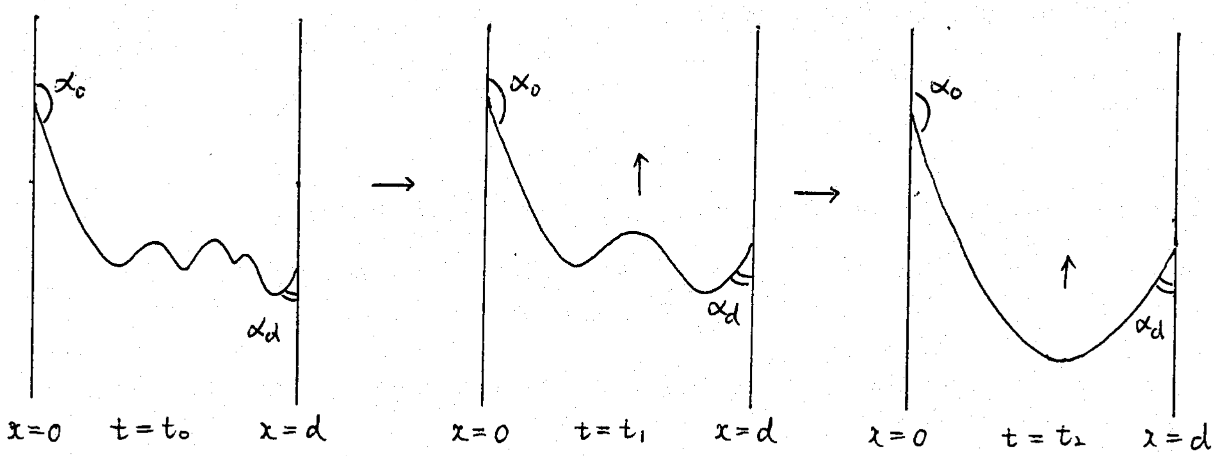

Figure 1.2

In this note, we proved with respect to the case $\theta_0 < \theta_1$. We shall discuss the case $\theta_0 = \theta_1$ and $\theta_0 > \theta_1$ in our forthcoming paper.

2. Structure of self-similar solutions

We consider the equation of form (P):

$$(a(U_\xi))_\xi + \lambda\xi U_\xi - \lambda U = 0, \ \xi \in (q, 0), \tag{2.1}$$

$$U_\xi(q) = \tan\ \theta_0, \tag{2.2}$$

$$U_\xi(0) = \tan\ \theta_1, \tag{2.3}$$

$$U(q) = 0, \tag{2.4}$$

where $a \in C^2(\mathbb{R})$ and $a'(\sigma) > 0$ for $\sigma \in \mathbb{R}({}' = \frac{d}{d\sigma})$, and q is negative constant. The angles $\theta_i \in (0, \frac{\pi}{2})$ for $i = 0, 1$ will be measured counter-clockwise from the x-axis. Here the function U and the number λ is unknown and we shall discuss the existence of solutions.

Theorem 2.1. *(Existence and uniqueness)* *Let q, θ_0, θ_1 be given constants. Assume that*

$$q < 0, \ 0 < \theta_0 \le \theta_1 < \frac{\pi}{2}.$$

Then there exists a unique solution $(\lambda, U) \in [0, \infty) \times C^2[q, 0]$ to (P). Moreover, $\lambda = 0$ is if and only if $\theta_0 = \theta_1$.

For given $\lambda \in [0, \infty)$, let $(P)_\lambda$ be the initial-value problem (2.1), (2.2), (2.4). We define the set J as

$$J := \{\lambda \in [0, \infty) \mid \text{there exists a } U \in C^2[q, 0] \text{ satisfying } (P)_\lambda \text{ for the interval } [q, 0]\}.$$

Remark 2.1. We first observe that $0 \in J$. Indeed, if $\lambda = 0$, (2.1) is $(a(U_\xi))_\xi = 0$ for $\xi \in [q, 0]$. Hence, $a'(U_\xi)U_{\xi\xi} = 0$ for $\xi \in [q, 0]$. Recalling $a' > 0$, we get

$$U_{\xi\xi} = 0 \quad \text{for} \quad \xi \in [q, 0]. \tag{2.5}$$

Then, by means of (2.2), (2.4), and (2.5),

$$U(\xi) = (\tan\theta_0)(\xi - q) \quad \text{for} \quad \xi \in [q, 0].$$

Since $U \in C^2[q, 0]$, J includes $\lambda = 0$. Thus, $J \neq \phi$.

For the proof, we use a shooting method. It consists of several steps.

Lemma 2.1. *(Openness of J)* *Assume that $\lambda_0 \in J$. Then there is a small $\hat{\delta} > 0$ so that the set $(\lambda_0 - \hat{\delta}, \ \lambda_0 + \hat{\delta}) \cap [0, \infty)$ is including in the set J.*

To prove this lemma we now rewrite the initial-value problem $(p)_\lambda$ by introducing a new dependent variable $y(\xi) < a(U_\xi(\xi))$. We set

$$a(U_\xi(\xi)) := y(\xi).$$

Since $a' > 0$, there exists a C^2 inverse function a^{-1} of the function a to get

$$U_\xi(\xi) = a^{-1}(y(\xi)).$$

The equation (2.1) becomes:

$$y_\xi + \lambda\xi a^{-1}(y) - \lambda U = 0.$$

It is easy to see that $(P)_\lambda$ is rewritten in the form of a system

$$\frac{d}{d\xi}\begin{pmatrix} y \\ U \end{pmatrix} = \begin{pmatrix} -\lambda\xi a^{-1}(y) + \lambda U \\ a^{-1}(y) \end{pmatrix}$$
$$\begin{pmatrix} y(q) \\ U(q) \end{pmatrix} = \begin{pmatrix} a(\tan\theta) \\ 0 \end{pmatrix}.$$

For later notation, we set

$$F(\xi,\ y,\ U,\ \lambda) = \begin{pmatrix} f_1(\xi,\ y,\ U,\ \lambda) \\ f_2(\xi,\ y,\ U,\ \lambda) \end{pmatrix} := \begin{pmatrix} -\lambda\xi a^{-1}(y) + \lambda U \\ a^{-1}(y) \end{pmatrix}$$

and denote the solution of $(P)_\lambda$ by $U(\xi;\lambda)$.

Proof of Lemma 2.1. For given $\lambda_0 \in J$, since $U(\cdot\ ;\lambda_0) \in C^2[q,0]$, there exists a constant $M > 0$ such that

$$|U_\xi(\xi;\lambda_0)| \le M,\ |U(\xi;\lambda_0)| \le M \text{ for } \xi \in [q,0].$$

Now, we define

$$D := \{(\xi,\ y,\ U,\ \lambda) \mid \xi \in [q,0],\ a(-K) < y < a(K),\ |U| < K,$$
$$\lambda \in (\lambda_0 - \mu,\ \lambda_0 + \mu) \cap [0,\infty)\}$$

where K, μ are constants with $K > 2M,\ \mu > 0$. Then, F is Lipschitz continuous or $\overline{D}$ with respect to $y,\ U$. Indeed,

$$\frac{\partial f_1}{\partial y} = -\lambda\xi\frac{\partial}{\partial y}a^{-1}(y) = -\frac{\lambda\xi}{a'(a^{-1}(y))},\ \frac{\partial f_1}{\partial U} = \lambda,$$
$$\frac{\partial f_2}{\partial y} = \frac{\partial}{\partial y}a^{-1}(y) = -\frac{1}{a'(a^{-1}(y))},\ \frac{\partial f_2}{\partial U} = 0.$$

Since the derivative of $a^{-1}(y)$ with respect to y is positive, we get

$$-K \le a^{-1}(y) \le K \quad \text{on} \quad \overline{D}.$$

Thus, recalling $a \in C^2(\mathbb{R})$ and $a' > 0$, there exist constants C_1, C_2 such that

$$0 < C_1 \le a'(a^{-1}(y)) \le C_2.$$

Hence, there exists a constant $C > 0$ (independent of a point of $\overline{D}$) that satisfies

$$\left|\frac{\partial f_1}{\partial y}\right| \le C,\ \left|\frac{\partial f_1}{\partial U}\right| \le C,\ \left|\frac{\partial f_2}{\partial y}\right| \le C \quad \text{on} \quad \overline{D}$$

where C depends on q, K, λ_0, μ. We thus observe that F is Lipschitz continuous on $\overline{D}$ with respect to y, U.

Then, from Chap.1, Sec.7 of E. A. Coddington and N. Levirson [8], there exists a constant $\hat{\delta} = \hat{\delta}(\lambda_0) \in (0, \mu)$ such that if $\lambda \in (\lambda_0 - \hat{\delta},\ \lambda_0 + \hat{\delta}) \cap [0, \infty)$, there exists a unique solution of $(P)_\lambda$ in $[q, 0]$. This concludes the proof of Lemma 2.1. □

Remark 2.2. (Differentiability on the parameter) We compute the first order partial derivatives of f_1, f_2 with respect to λ;

$$\frac{\partial f_1}{\partial \lambda} = -\xi a^{-1}(y) + U, \ \frac{\partial f_2}{\partial \lambda} = 0.$$

Then, these are continuous on $\overline{D}$. Thus, the solution of $(P)_\lambda$ in the proof of Lemma 2.1, which is given for $\lambda \in (\lambda_0 - \hat{\delta},\ \lambda_0 + \hat{\delta}) \cap [0, \infty)$, is

$$\begin{pmatrix} y(\xi\ ;\ \cdot) \\ U(\xi\ ;\ \cdot) \end{pmatrix} \in C^1((\lambda_0 - \hat{\delta},\ \lambda_0 + \hat{\delta}) \cap [0, \infty)) \quad \text{for} \quad \xi \in [q, 0].$$

(cf. [8])

Lemma 2.2. *(Connectedness) Assume that $\lambda_0,\ \lambda_1 \in J$ and $\lambda_0 < \lambda_1$. If $\lambda_0 \leq \lambda \leq \lambda_1$, then λ is included in the set J.*

In order to prove Lemma 2.2, we study qualitative properties of solution.

Lemma 2.3. *Assume that $\lambda \in [\alpha, \beta]$, with constants α, β satisfying $0 \leq \alpha < \beta$, and that $U \in C^2[q, \gamma]$ with constants q, γ satisfying $q < \gamma \leq 0$ fulfills*

$$\begin{aligned} &(a(U_\xi))_\xi + \lambda \xi U_\xi - \lambda U = 0, \ \xi \in [q, \gamma], \\ &U_\xi(q) = \tan\theta_0, \\ &U(q) = 0. \end{aligned} \tag{2.6}$$

Then, the following estimates are valid;
(i) $U_{\xi\xi}(\xi\ ;\ \lambda) > 0$ *for* $\xi \in [q, \gamma],\ \lambda > 0,$
(ii) $\dot{U}_\xi(\xi\ ;\ \lambda) > 0$ *for* $\xi \in [q, \gamma],\ \lambda \in [\alpha, \beta],$
(iii) $U_\xi(\xi\ ;\ \lambda) > 0$ *for* $\xi \in [q, \gamma],\ \lambda \geq 0,$
$\dot{U}(\xi\ ;\ \lambda) > 0$ *for* $\xi \in [q, \gamma],\ \lambda \in [\alpha, \beta],$
where $\cdot$ is the differential with respect to λ.

Proof of Lemma 2.3. (i). Since $U \in C^2[q, \gamma]$, $\xi \downarrow q$ in (2.6),

$$a'(U_\xi(q))U_{\xi\xi}(q) + \lambda q U_\xi(q) - \lambda U(q) = 0. \tag{2.7}$$

This equation (2.7) with $U_\xi(q) = \tan\theta_0$, $U(q) = 0$, and $\lambda > 0$ implies that

$$a'(U_\xi(q))U_{\xi\xi}(q) - \lambda q \tan\theta_0 > 0.$$

By $a' > 0$, we observe that $U_{\xi\xi}(q) > 0$. Thus, there is a some constant $\delta_1 > 0$ that satisfies

$$U_{\xi\xi}(\xi) > 0 \quad \text{for} \quad \xi \in [q,\ q + \delta_1). \tag{2.8}$$

Here, we define

$$\xi_1 := \sup\{\xi_2 \mid U_{\xi\xi}(\xi) > 0 \quad \text{for} \quad \xi \in [q, \xi_2)\}.$$

Then, by (2.8), we see $\xi_1 \geq q + \delta_1$.

Assume now that $\xi_1 \leq \gamma$. We derive contradiction. We first assume $\xi_1 < \gamma$. Then,

$$U_{\xi\xi}(\xi_1) = 0, \; U_{\xi\xi}(\xi) > 0 \quad \text{for} \quad \xi \in [q, \xi_1).$$

Thus, by $U_\xi(q) = \tan\theta_0 > 0$, we see

$$U_\xi(\xi) > 0 \quad \text{for} \quad \xi \in [q, \; \xi_1].$$

Moreover, since $U(q) = 0$, this implies

$$U(\xi) > 0 \quad \text{for} \quad \xi \in (q, \xi_1].$$

Setting $\xi = \xi_1$ in (2.6) now yields

$$a'(U_\xi(\xi_1))U_{\xi\xi}(\xi_1) + \lambda\xi_1 U_\xi(\xi_1) - \lambda U(\xi_1) = 0.$$

We recall $a' > 0$ and $\lambda > 0$ to get

$$U_{\xi\xi}(\xi_1) = \frac{\lambda(U(\xi_1) - \xi_1 U_\xi(\xi_1))}{a'(U_\xi(\xi_1))} > 0.$$

This contradicts $U_{\xi\xi}(\xi_1) = 0$.

Next, we assume $\xi_1 = \gamma$. Then,

$$U_{\xi\xi}(\gamma) = 0, \; U_{\xi\xi}(\xi) > 0 \quad \text{for} \quad \xi \in [q, \gamma).$$

Under the same discussion as in the case $\xi_1 < \gamma$, we get

$$U_\xi(\xi) > 0 \quad \text{for} \quad \xi \in [q, \gamma],$$

and

$$U(\xi) > 0 \quad \text{for} \quad \xi \in (q, \gamma].$$

Since $U \in C^2[q, \; \gamma]$, letting ξ to γ in (1.5) yields,

$$a'(U_\xi(\gamma))U_{\xi\xi}(\gamma) + \lambda\gamma U(\gamma) - \lambda U(\gamma) = 0.$$

Since $a' > 0$ and $\lambda > 0$, we now obtain

$$U_{\xi\xi}(\gamma) = \frac{\lambda(U(\gamma) - \gamma U(\gamma))}{a'(U_\xi(\gamma))} > 0.$$

This contradicts $U_{\xi\xi}(\gamma) = 0$.

Consequently, $\xi_1 > \gamma$ and consequently,

$$U_{\xi\xi}(\xi) > 0 \quad \text{for} \quad \xi \in [q,\ \gamma],\ \lambda > 0. \quad \square$$

Proof of Lemma 2.3. (ii). Since (i) holds and the graph of U is a straight line if $\lambda = 0$, $U_{\xi\xi}(\xi\ ;\ \lambda) \geq 0$ for $\xi \in [q,\ \gamma]$, $\lambda \geq 0$. Thus, by $U_\xi(q\ ;\ \lambda) = \tan\theta_0 > 0$.

$$U_\xi(\xi\ ;\ \lambda) > 0 \quad \text{for} \quad \xi \in [q, \gamma],\ \lambda \geq 0. \tag{2.9}$$

Moreover, by $U(q\ ;\ \lambda) = 0$,

$$U(\xi\ ;\ \lambda) > 0 \quad \text{for} \quad \xi \in (q, \gamma],\ \lambda \geq 0. \tag{2.10}$$

We next define

$$D_0 := \{(\xi,\ y,\ U,\ \lambda) \mid \xi \in [q,\ \gamma],\ a(-L) < y < a(L),\ |U| < L,\\ \lambda \in (\alpha - \hat{\mu},\ \beta + \hat{\mu}) \cap [0, \infty)\}$$

where $L, \hat{\mu}$ are constants and $L > 2M$, $\hat{\mu} > 0$.

Consider the initial-value problem $(P)_\lambda$ on D_0. Since the first order partial derivatives of f_1, f_2 with respect to y, U are continuous on $\overline{D}$. F is Lipschitz continuous on $\overline{D}_0$ with respect to y, U. Thus, from Chap.1, Sec.7 of E. A. Coddington and N. Levinson [8], there exists a constant $\delta_2 = \delta_2(\alpha, \beta) \in (0, \hat{\mu})$ such that there exists a unique solution of $(P)_\lambda$ in $[q, \gamma]$ for $\lambda \in (\alpha - \delta_2,\ \beta + \delta_2) \cap [0, \infty)$. Moreover, since the first order derivatives of f_1, f_2 with respect to λ are continuous on $\overline{D}_0$,

$$\begin{pmatrix} y(\xi\ ;\ \cdot) \\ U(\xi\ ;\ \cdot) \end{pmatrix} \in C^1((\alpha - \delta_2,\ \beta + \delta_2) \cap [0, \infty)) \quad \text{for} \quad \xi \in [q, \gamma].$$

Hence, we differentiable both sides of (2.6) with respect to λ to get

$$a'(U_\xi)\dot{U}_{\xi\xi} + a''(U_\xi)\dot{U}_\xi\ U_{\xi\xi} + \lambda\xi\dot{U}_\xi + \xi U_\xi - \lambda\dot{U} - U = 0. \tag{2.11}$$

Since $U_\xi(q\ ;\ \lambda) = \tan\theta_0$ and $U(q\ ;\ \lambda) = 0$,

$$\dot{U}_\xi(q\ ;\ \lambda) = 0,\ \dot{U}(q; \lambda) = 0.$$

Sending $\xi \downarrow q$ in (2.11),

$$a'(U_\xi(q\ ;\ \lambda))\dot{U}_{\xi\xi}(q\ ;\ \lambda) + qU_\xi(q\ ;\ \lambda) = 0.$$

Consequently,

$$a'(\tan\theta_0)\dot{U}_{\xi\xi}(q\ ;\ \lambda) = -q\tan\theta_0 > 0.$$

By $a' > 0$, this implies $\dot{U}_{\xi\xi}(q\ ;\ \lambda) > 0$. By continuity, there is a constant $\delta_3 > 0$ that satisfies

$$\dot{U}_{\xi\xi}(\xi\ ;\ \lambda) > 0 \quad \text{for} \quad \xi \in [q,\ q + \delta_3),\ \lambda \in [\alpha,\ \beta].$$

Since $\dot{U}_\xi(q\ ;\ \lambda) = 0$ and $\dot{U}(q\ ;\ \lambda) = 0$, this implies

$$\dot{U}_\xi(\xi\ ;\ \lambda) > 0 \quad \text{for} \quad \xi \in (q,\ q+\tilde{\delta}],\ \lambda \in [\alpha,\ \beta], \tag{2.12}$$

$$\dot{U}(\xi\ ;\ \lambda) > 0 \quad \text{for} \quad \xi \in (q,\ q+\tilde{\delta}],\ \lambda \in [\alpha,\ \beta]. \tag{2.13}$$

We now define

$$\xi_3 := \sup\{\xi_4 \mid \dot{U}_\xi(\xi\ ;\ \lambda) > 0 \quad \text{for} \quad \xi \in (q,\ \xi_4),\ \lambda \in [\alpha,\ \beta]\}.$$

Then, by (2.12), we see $\xi_3 > q + \delta_3$.

Now, let us assume $\xi_3 \leq \gamma (\leq 0)$ and show a contradiction. By the definition of ξ_3, there is $\overline{\lambda} \in [\alpha,\ \beta]$ that satisfies

$$\dot{U}_\xi(\xi_3\ ;\ \overline{\lambda}) = 0,\ \dot{U}_\xi(\xi\ ;\ \overline{\lambda}) > 0 \quad \text{for} \quad \xi \in (q,\ \xi_3). \tag{2.14}$$

We rewrite (2.11) if the form

$$(a'(U_\xi)\dot{U}_\xi)_\xi + \lambda\xi\dot{U}_\xi + \xi U_\xi - \lambda\dot{U} - U = 0. \tag{2.15}$$

Then, if $\lambda = \overline{\lambda}$, integrating both sides of (2.15) on $(q,\ \xi_3)$ with respect to ξ yield

$$a' \cdot \dot{U}_\xi(\xi_3\ ;\ \overline{\lambda}) - a' \cdot \dot{U}_\xi(q\ ;\ \overline{\lambda}) + \overline{\lambda} \int_q^{\xi_3} \xi\ \dot{U}_\xi\ d\xi + \int_q^{\xi_3} \xi U_\xi\ d\xi$$
$$- \overline{\lambda} \int_q^{\xi_3} \dot{U}\ d\xi - \int_q^{\xi_3} U\ d\xi = 0.$$

Integrating by parts with $\dot{U}_\xi(\xi_3\ ;\overline{\lambda}) = 0 = \dot{U}_\xi(q\ ;\ \overline{\lambda})$ yields

$$\xi_3(\overline{\lambda}\ \dot{U}(\xi_3\ ;\ \overline{\lambda}) + U(\xi_3\ ;\ \overline{\lambda})) = 2\left\{\overline{\lambda} \int_q^{\xi_3} \dot{U}\ d\xi + \int_q^{\xi_3} U\ d\xi\right\}. \tag{2.16}$$

By (2.13) and (2.14), we see

$$\dot{U}(\xi\ ;\ \overline{\lambda}) > 0 \quad \text{for} \quad \xi \in (q,\ \xi_3]. \tag{2.17}$$

Then, by (2.10), (2.17) and $q + \delta_3 < \xi_3 \leq \gamma \leq 0$, we see the left side of (2.16) is nonpositive while the right side of (2.16) is positive. This is a contradiction. Hence, $\xi_3 > \gamma$.

Consequently,

$$\dot{U}_\xi(\xi\ ;\ \lambda) > 0 \quad \text{for} \quad \xi \in (q,\ \gamma],\ \lambda \in [\alpha, \beta]. \quad \square$$

Proof of Lemma 2.3. (iii). By (2.9), $U_\xi(\xi\ ;\ \lambda) > 0$ for $\xi \in [q,\ \gamma]$, $\lambda \geq 0$. Moreover, from (ii) and $\dot{U}(q\ ;\ \lambda) = 0$, it follows that

$$\dot{U}(\xi\ ;\ \lambda) > 0 \quad \text{for} \quad \xi \in (q,\ \gamma],\ \lambda \in [\alpha,\ \beta]. \quad \square$$

Proof of Lemma 2.2. Let $[q,\ \iota(\lambda))$ denote the maximal existence interval of the solution of $(P)_\lambda$ for each λ. If $\iota(\lambda) > 0$ for any $\lambda \in [\lambda_0,\ \lambda_1]$, we get $\lambda \in J$. It suffices to prove that $\iota(\lambda) > 0$ for any $\lambda \in [\lambda_0,\ \lambda_1]$.

We first prove that $\iota(\lambda)$ is lower semi-continuous. Since the solution of $(P)_\lambda$ exists in $[q,\ \iota(\lambda))$, it exists in $[q,\ \iota(\lambda)-\varepsilon]$ for any $\varepsilon > 0$. Thus, from Chap.1, Sec.7 of E. A. Coddington and N. Levinson [8], there exists a constant $\delta_4 > 0$ such that there exists a unique solution of $(P)_{\hat{\lambda}}$ in $[q,\ \iota(\lambda)-\varepsilon]$ for $\hat{\lambda} \in (\lambda-\delta_4,\ \lambda+\delta_4)$. Then, by the definition of $\iota(\lambda)$,

$$\iota(\hat{\lambda}) > \iota(\lambda) - \varepsilon \quad \text{for} \quad \hat{\lambda} \in (\lambda-\delta_4,\ \lambda+\delta_4).$$

Consequently, $\iota(\lambda)$ is lower semi-continuous.

Here, since the lower semi-continuous function has a minimum value, we define

$$\iota_* := \min\{\iota(\lambda) \mid \lambda \in [\lambda_0,\ \lambda_1]\}.$$

It suffices to prove that $\iota_* > 0$.

We assume $\iota_* \leq 0$ and shall derive a contradiction. We take $\lambda_* \in [\lambda_0,\ \lambda_1]$ such that $\iota_* = \iota(\lambda_*)$, there exists a solution of $(P)_{\lambda_*}$ in $[q,\ \iota_*)$. Since $\lambda_1 \geq \lambda_*$, by Lemma 2.3 (ii),

$$U_\xi(\xi\ ;\ \lambda_1) \geq U_\xi(\xi\ ;\ \lambda_*) \quad \text{for} \quad \xi \in [q,\ \iota_*).$$

Thus,

$$\limsup_{\xi\uparrow\iota_*} U_\xi(\xi\ ;\ \lambda_1) \geq \limsup_{\xi\uparrow\iota_*} U_\xi(\xi\ ;\ \lambda_*). \tag{2.18}$$

Moreover, by Lemma 2.3 (iii),

$$U(\xi\ ;\ \lambda_1) \geq U(\xi\ ;\ \lambda_*) \quad \text{for} \quad \xi \in [q,\ \iota_*).$$

Hence

$$\limsup_{\xi\uparrow\iota_*} U_\xi(\xi\ ;\ \lambda_1) \geq \limsup_{\xi\uparrow\iota_*} U(\xi\ ;\ \lambda_*). \tag{2.19}$$

Here, by Lemma 2.3 (i), (iii), U_ξ and U are monotone increasing functions in $[q,\ \iota_*)$ with respect to ξ. Thus, by the definition of ι_*,

$$\limsup_{\xi\uparrow\iota_*} U_\xi(\xi\ ;\ \lambda_*) = \infty \quad \text{or} \quad \limsup_{\xi\uparrow\iota_*} U_\xi(\xi\ ;\ \lambda_*). \tag{2.20}$$

Then, (2.18), (2.19), (2.20) and $\iota_* \leq 0$ contradict $\lambda_1 \in J$. Consequently, $\iota_* > 0$ i.e. $\iota(\lambda) > 0$ for any $\lambda \in [\lambda_0,\ \lambda_1]$. □

By Lemma 2.1, J is the open set included in the interval $[0,\infty)$. Moreover, we define $\Lambda_0 \in (0,\infty]$ as the supremum of λ such that there exists a solution of $(P)_\lambda$ in $[q,0]$. Then, by Lemma 2.2, that J is an interval $[0,\Lambda_0)$.

We now define the mapping

$$\Phi : [0,\ \Lambda_0) \ni \lambda \mapsto U_\xi(0\ ;\ \lambda).$$

Then, Lemma 2.3 (ii) implies

$$\frac{\partial\Phi}{\partial\lambda} > 0.$$

Thus, Φ is a monotone increasing function, which is a bijection:

$$\Phi : [0,\ \Lambda_0) \to [\tan\theta,\ \lim_{\lambda\uparrow\Lambda_0} \Phi(\lambda)).$$

We shall prove that $\lim_{\lambda\uparrow\Lambda_0} \Phi(\lambda) = \infty$.

Lemma 2.4. *Assume that* $\Lambda_0 < \infty$. *Then* $\lim_{\lambda \uparrow \Lambda_0} \Phi(\lambda) = \infty$.

Proof of Lemma 2.4. Suppose that $\lim\limits_{\lambda \uparrow \Lambda_0} \Phi(\lambda) = L < \infty$. Since Φ is a monotone increasing function, we get

$$|U_\xi(0\ ;\ \lambda)| \leq L \quad \text{for} \quad \lambda \in [0,\ \Lambda_0). \tag{2.21}$$

While, by Lemma 2.3 (i),

$$U_{\xi\xi}(\xi\ ;\ \lambda) > 0 \quad \text{for} \quad \xi \in [q, 0],\ \lambda \in [0,\ \Lambda_0).$$

Thus, U_ξ is a monotone increasing function with respect to ξ. By $U_\xi(q\ ;\ \lambda) = \tan\theta_0$ and (2.21),

$$0 < \tan\theta_0 \leq U_\xi(\xi\ ;\ \lambda) \leq L \quad \text{for} \quad \xi \in [q, 0],\ \lambda \in [0,\ \Lambda_0). \tag{2.22}$$

Then, by (2.22),

$$(\tan\theta_0)(\xi - q) \leq U(\xi\ ;\ \lambda) \leq L(\xi - q) \quad \text{for} \quad \xi \in [q, 0],\ \lambda \in [0, \Lambda_0).$$

Thus,

$$0 \leq U(\xi\ ;\ \lambda) \leq -qL \quad \text{for} \quad \xi \in [q, 0],\ \lambda \in [0,\ \Lambda_0). \tag{2.23}$$

Moreover, by $a \in C^2(\mathbb{R})$, $a' > 0$ and (2.22), there exist constants C_1, C_2 such that

$$0 < C_1 \leq a'(U_\xi) \leq C_2. \tag{2.24}$$

Since $U_{\xi\xi} = (-\lambda\xi\ U_\xi + \lambda\ U)/a'(U_\xi)$, by (2.22), (2.23) and (2.24),

$$\sup\{|U_{\xi\xi}|\ |\xi \in [q, 0],\ \lambda \in [0,\ \Lambda_0)\} < \infty. \tag{2.25}$$

Now, we define

$$U^k := U(\xi\ ;\ \Lambda_0 - \frac{1}{k}),\ k \in \mathbb{N},\ k \in (\frac{1}{\Lambda_0},\ \infty).$$

Then, by (2.22), (2.23), and (2.25), there exists a constant $C_3 = C_3(L,\ \Lambda_0,\ q,\ \theta_0)$ such that

$$||U^k||_{C^2[q,0]} \leq C_3.$$

Here, since $C^2[q, 0]$ is compactly inbedded in $C^1[q, 0]$, there exist a subsequence $\{U^{k_j}\} \subset \{U^k\}$ and $\tilde{U} \in C^1[q, 0]$ such that

$$||U^{k_j} - \tilde{U}||_{C^1[q,0]} \to 0 \quad \text{as} \quad j \to \infty.$$

Now, we set

$$G(\xi,\ U,\ U_\xi,\ \lambda) = \begin{pmatrix} g_1(\xi,\ U,\ U_\xi,\ \lambda) \\ g_2(\xi,\ U,\ U_\xi,\ \lambda) \end{pmatrix} := \begin{pmatrix} U_\xi \\ (-\lambda\xi U_\xi + \lambda U)/a'(U_\xi) \end{pmatrix}.$$

Then, since U^{k_j} is a solution of $(P)_{\Lambda_0 - 1/k_j}$, we get

$$\begin{pmatrix} U^{k_j} \\ U^{k_j}_\xi \end{pmatrix} = \begin{pmatrix} 0 \\ \tan\theta_0 \end{pmatrix} + \int_q^\xi G(\eta,\ U^{k_j},\ U^{k_j}_\xi,\ \Lambda_0 - \frac{1}{k_j})d\eta. \tag{2.26}$$

Here, since G is continuous with respect to $U,\ U_\xi, \lambda$, we get

$$\begin{pmatrix} \tilde{U} \\ \tilde{U}_\xi \end{pmatrix} = \begin{pmatrix} 0 \\ \tan\theta_0 \end{pmatrix} + \int_q^\xi G(\eta,\ \tilde{U},\ \tilde{U}_\xi,\ \Lambda_0)\ d\eta \quad \text{as} \quad j \to \infty \quad \text{in (2.26)}.$$

Thus, $\tilde{U}$ satisfies $(P)_{\Lambda_0}$. Then, G is Lipschitz continuous on the set $\{(\xi,\ \tilde{U},\ \tilde{U}_\xi,\ \Lambda_0) \mid \xi \in [q,0],\ 0 \le \tilde{U} \le -qL,\ \tan\theta_0 \le \tilde{U}_\xi \le L,\ \Lambda_0 \in (0,\infty)\}$ with respect to $\tilde{U},\ \tilde{U}_\xi$. Because,

$$\frac{\partial g_1}{\partial \tilde{U}} = 0,\ \frac{\partial g_1}{\partial \tilde{U}_\xi} = 1,\ \frac{\partial g_2}{\partial \tilde{U}} = \frac{\Lambda_0}{a'(\tilde{U}_\xi)},$$
$$\frac{\partial g_2}{\partial \tilde{U}_\xi} = -\frac{\Lambda_0 \xi}{a'(\tilde{U}_\xi)} + \frac{a''(\tilde{U}_\xi)}{(a'(\tilde{U}_\xi))^2}\ (\Lambda_0 \xi \tilde{U}_\xi - \Lambda_0\ \tilde{U}).$$

Recalling $a \in C^2(\mathbb{R})$ and $\tan\theta_0 \le \tilde{U}_\xi \le L$, there exists a constant $C_4 > 0$ such that

$$|a''(\tilde{U}_\xi)| \le C_4. \tag{2.27}$$

Thus, by (2.24) and (2.27), there exists a constant $C_5 > 0$ such that

$$\left|\frac{\partial g_2}{\partial \tilde{U}}\right| \le C_5,\ \left|\frac{\partial g_2}{\partial \tilde{U}_\xi}\right| \le C_5$$

where C_5 depends on $q,\ L,\ \theta_0$. Then, we know that G is Lipschitz continuous on the set $\{(\xi,\ \tilde{U},\ \tilde{U}_\xi,\ \Lambda_0) \mid \xi \in [q,0],\ 0 \le \tilde{U} \le -qL,\ \tan\theta_0 \le \tilde{U}_\xi \le L,\ \Lambda_0 \in (0,\infty)\}$ with respect to $\tilde{U},\ \tilde{U}_\xi$.

Hence, since the solution of $(P)_{\Lambda_0}$ is unique, $\tilde{U} = U(\xi\ ;\ \Lambda_0)$. Consequently, $\Lambda_0 \in J$. Then, since J is an open set by Lemma 1.1 and $\Lambda_0 \in (0,\infty)$, there exist a constant $\delta_5 > 0$ such that $[\Lambda_0,\ \Lambda_0 + \delta_5) \subset J$. This contradicts that Λ_0 is the supremum of λ such that there exists a solution of $(P)_\lambda$ in $[q,0]$.

Consequently, $\lim_{\lambda \uparrow \Lambda_0} \Phi(\lambda) = \infty$. □

Lemma 2.5. *Assume that* $\Lambda_0 = \infty$. *Then* $\lim_{\lambda\uparrow\infty} \Phi(\lambda) = \infty$.

Proof of Lemma 2.5. Now, let us suppose $\lim_{\lambda\uparrow\infty} \Phi(\lambda) < \infty$. Then, setting $\lim_{\lambda\uparrow\infty} \Phi(\lambda) := L$, by $a \in C^2(\mathbb{R})$,

$$\lim_{\lambda\uparrow\infty} a(U_\xi(0\ ;\ \lambda)) = a(L) < \infty. \tag{2.28}$$

While, integrating both sides of $(a(U_\xi))_\xi + \lambda\xi U_\xi - \lambda U = 0$ on $[q,0]$ with respect to ξ and computing,

$$a(U_\xi(0\ ;\ \lambda)) = a(\tan\theta_0) + 2\lambda \int_q^0 U(\xi\ ;\ \lambda)d\xi.$$

Then, by Lemma 2.3 (iii),

$$a(U_\xi(0\ ;\ \lambda)) \geq a(\tan\theta_0) + 2\lambda \int_q^0 U(\xi\ ;\ 0)\ d\xi$$
$$= a(\tan\theta_0) + \lambda q^2 \tan\theta_0. \tag{2.29}$$

Since $q^2\ \tan\theta_0 > 0$, we get

$$\lim_{\lambda\uparrow\infty}\ a(U_\xi(0\ ;\ \lambda)) = \infty.$$

This contradicts (2.28).

Consequently, $\lim_{\lambda\uparrow\infty}\ \Phi(\lambda) = \infty$. □

Remark 2.3. (i) If a is bounded from the above, i.e. there exists a constant M such that $a(\sigma) < M$ for $\sigma \in \mathbb{R}$, then

$$\Lambda_0 \leq \frac{M - a(\tan\theta_0)}{q^2 \tan\theta_0} < \infty.$$

Indeed from (2.29), it follows that

$$a(U_\xi(0\ ;\ \lambda)) \geq a(\tan\theta_0) + \lambda\ q^2\ \tan\theta_0.$$

Thus, for $\lambda \in (0,\ \Lambda_0)$

$$a(\tan\theta_0) + \lambda q^2\ \tan\theta_0 \leq a(U_\xi(0\ ;\ \lambda)) < M.$$

Then, for any $\varepsilon > 0$

$$a(\tan\theta_0) + (\Lambda_0 - \varepsilon)q^2 \tan\theta_0 < M.$$

Hence,

$$\Lambda_0 < \frac{M - a(\tan\theta_0)}{q^2 \tan\theta_0} + \varepsilon.$$

Since ε is arbitrary,

$$\Lambda_0 \leq \frac{M - a(\tan\theta_0)}{q^2 \tan\theta_0}.$$

(ii) If the initial-value problem $(P)_\lambda$ is solvable for any $\lambda \in [0,\infty)$, i.e. $\sup_{\mathbb{R}} \frac{d}{dy}a^{-1}(y) < \infty$, $\Lambda_0 = \infty$. Because, if $\sup_{\mathbb{R}} \frac{d}{dy}a^{-1}(y) < \infty$, F is Lipschitz continuous with respect to y, U for any $\lambda \in [0,\infty)$.

Proof of Theorem 1.1. By Lemma 2.4 or Lemma 2.5,

$$\Phi([0,\ \Lambda_0)) = [\tan\theta_0,\ \infty).$$

Moreover, since $\partial\Phi/\partial\lambda > 0$ by Lemma 2.3 (ii), Φ is one-to-one. Thus, Φ is a bijection. Consequently, for any $\alpha := \tan\theta_1 \in [\tan\theta_0, \infty)$, there exist a unique $(\lambda, U) \in [0, \Lambda_0) \times C^2[q, 0]$ satisfying the initial-value problem $(P)_\lambda$ and $U_\xi(0) = \tan\theta_1$. □

Remark 2.4. (Relation between λ and q) We set $\lambda = \lambda(q)$. Then, $\lambda(\zeta q) = \lambda(q)/\zeta^2$ holds for $\zeta \in (0, \infty)$. Here, we set $q = -1$ and replace $-\zeta$ by q. Then,

$$\lambda(q) = \frac{\lambda(-1)}{q^2} \tag{2.30}$$

where $\lambda(-1)$ is a constant satisfying $\lambda(-1) = 0$ if $\theta_0 = \theta_1$ and $\lambda(-1) > 0$ if $\theta_0 < \theta_1$.

In Theorem 2.1, we determined (λ, U) by giving q, θ_0, θ_1. But since (2.30) holds, we can determine (q, U) by giving $\lambda, \theta_0, \theta_1$.

3. Convergence of a solution for $\theta_0 < \theta_1$

We consider the convergence of the solution of (1.1)-(1.5). Now, we employ the similarity change of variables (1.6)-(1.7). Then equations (1.1)-(1.5) become

$$U_\tau = (a(U_\xi))_\xi + \xi U_\xi - U, \quad p(\tau) < \xi < 0, \ \tau > 0, \tag{3.1}$$

$$U_\xi(p(\tau), \tau) = \tan\theta_0, \quad \tau \geq 0, \tag{3.2}$$

$$U_\xi(0, \tau) = \tan\theta_1, \quad \tau \geq 0, \tag{3.3}$$

$$U(p(\tau), \tau) = 0, \quad \tau \geq 0, \tag{3.4}$$

$$U(\xi, 0) = U_0(\xi), \quad p(0) = s_0 \leq \xi \leq 0. \tag{3.5}$$

Here, we shall discuss the convergence of a solution for problem (3.1)-(3.5).

Theorem 3.1. *Assume that $u_0 \in C^2[s_0, 0]$ satisfying $u_{0\xi}(s_0) = \tan\theta_0$, $u_{0\xi}(0) = \tan\theta_1$, $u_0(s_0) = 0$, and $u_0(\xi) > 0$ for $\xi \in (s_0, 0]$. Moreover, assume that $(U(\xi, \tau), p(\tau))$ is a smooth solution for problem (3.1)-(3.5). Then $(U(\xi, \tau), p(\tau))$ converge as $\tau \to \infty$ to $(U^*(\xi), p^*)$ satisfying*

$$(a(U^*_\xi))_\xi + \xi U^*_\xi - U^* = 0, \quad p^* < \xi < 0 \tag{3.6}$$

$$U^*_\xi(p^*) = \tan\theta_0 \tag{3.7}$$

$$U^*_\xi(0) = \tan\theta_1 \tag{3.8}$$

$$U^*(p^*) = 0 \tag{3.9}$$

Moreover, this convergence is exponential;

$$\sup_{\xi_0 \in [p^*, 0]} \ \sup_{\xi \in \mathcal{D}(\xi_0, \tau)} \left| \sqrt{\xi^2 + (U(\xi, \tau))^2} - \sqrt{\xi_0^2 + (U^*(\xi_0))^2} \ \right| \leq C e^{-\delta_0 \tau}$$

for some $\delta_0 \in (0, 2)$ and $\tau \geq 0$ where

$$\mathcal{D}(\xi_0, \tau) := \{\xi \mid \xi - \text{coordinate of intersection points of the straight line } \{(\xi, r) | U^*(\xi_0)\xi - \xi_0 r = 0\} \text{ and the graph } \{(\xi, r) | r = U(\xi, \tau), p(\tau) \leq \xi \leq 0\}\}$$

and C is a constant and is independent of τ.

For the proof of Theorem 3.1, we construct a subsolution and a supersolution for the problem (3.1)-(3.5), which converges as $\tau \to \infty$ to U^* satisfying (3.6)-(3.9), and use the strong maximum principle.

3.1 Structure of a subsolution

We first define $v_0(\xi)$ as the following. We set

$$K := \min\left\{\tan\theta_0, \inf_{\xi\in(s_0,0)}\left(\frac{u_0(\xi)}{\xi - s_0}\right)\right\}.$$

Here we choose a constant ℓ satisfying

$$\frac{s_0 K}{\tan\theta_1} < \ell < 0. \tag{3.10}$$

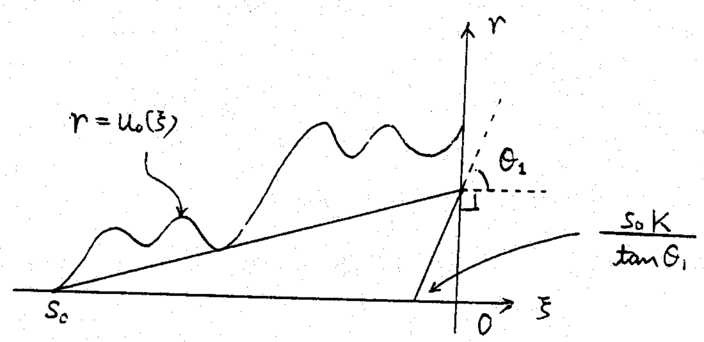

Figure 3.1

Then, by Theorem 2.1, there exist a unique $(\lambda_\ell, v_0) \in (0,\infty)\times C^2[\ell,0]$ satisfying

$$(a(v_{0\xi}))_\xi + \lambda_\ell\, \xi v_{0\xi} - \lambda_\ell\, v_0 = 0, \quad \ell < \xi < 0, \tag{3.11}$$

$$v_{0\xi}(\ell) = \tan\theta_0, \tag{3.12}$$

$$v_{0\xi}(0) = \tan\theta_1, \tag{3.13}$$

$$v_0(\ell) = 0. \tag{3.14}$$

By Remark 2.4, if necessary, we choose λ_ℓ such that $\lambda_\ell > 1$. Then, we get the following relation between u_0 and v_0.

Lemma 3.1. *Assume that v_0 satisfies (3.11)-(3.14). Then the following estimate is valid:*

$$u_0(\xi) > v_0(\xi) \quad \text{for} \quad \xi \in [\ell, 0]$$

Proof of Lemma 3.1. We set $\omega(\xi) := (\tan\theta_1)(\xi - \ell)$. By (3.10), we get $0 < -\ell\tan\theta_1 < -s_0 K$. Thus, by $K \le \tan\theta_0 < \tan\theta_1$,

$$\omega(\xi) < K(\xi - s_0) \quad \text{for} \quad \xi \in [\ell, 0].$$

Since $K(\xi - s_0) \le u_0(\xi)$ for $\xi \in [s_0, 0]$ by the definition of K,

$$\omega(\xi) < u_0(\xi) \quad \text{for} \quad \xi \in [\ell, 0]. \tag{3.15}$$

Next we compare ω with v_0. We set $\varphi(\xi) := \omega(\xi) - v_0(\xi)$. Since $\tan\theta_0 \le v_{0\xi}(\xi) \le \tan\theta_1$ for $\xi \in [\ell, 0]$ by Lemma 2.3 (i),

$$\frac{d}{d\xi}\varphi(\xi) = \tan\theta_1 - v_{0\xi}(\xi) \ge 0 \quad \text{for} \quad \xi \in [\ell, 0].$$

Thus, φ is a monotone increasing function. By $\varphi(\ell) = 0, \varphi(\xi) \ge 0$ for $\xi \in [\ell, 0]$. That is,

$$\omega(\xi) \ge v_0(\xi) \quad \text{for} \quad \xi \in [\ell, 0]. \tag{3.16}$$

By (3.15)-(3.16), we see $u_0(\xi) > v_0(\xi)$ for $\xi \in [\ell, 0]$. □

Moreover, we get the following relation between U^* and v_0.

Lemma 3.2. *Assume that U^* satisfies (3.6)-(3.9) and v_0 satisfies (3.11)- (3.14). Then U^* is represented by v_0 as the following;*

$$U^*(\xi) = \sqrt{\lambda_\ell}\, v_0\left(\frac{\xi}{\sqrt{\lambda_\ell}}\right).$$

Moreover, this representation is unique.

Proof of Lemma 3.2. We set $\tilde{U}^*(\xi) = \sqrt{\lambda_\ell}\, v_0(\rho)$ where $\rho = \xi/\sqrt{\lambda_\ell}$. Then, we get

$$\begin{aligned} &\left(a\left(\tilde{U}^*_\xi\right)\right)_\xi + \xi\tilde{U}^*_\xi - \tilde{U}^*_\xi \\ =&\frac{1}{\sqrt{\lambda_\ell}}\{a(v_{0\rho}))_\rho + \lambda_\ell\, \rho\, v_{0\rho} - \lambda_\ell\, v_0\} = 0. \end{aligned}$$

Since λ_ℓ is represented by ℓ, p^* as $\lambda_\ell = (p^*/\ell)^2$ (see Remark 2.4), we get

$$\begin{aligned} &\tilde{U}^*_\xi(p^*) = v_{0\rho}(p^*/\sqrt{\lambda_\ell}) = v_{0\rho}(\ell) = \tan\theta_0, \\ &\tilde{U}^*_\xi(0) = v_{0\rho}(0) = \tan\theta_1, \\ &\tilde{U}^*_\xi(p^*) = \sqrt{\lambda_\ell}\, v_0\ (p^*/\sqrt{\lambda_\ell}) = \sqrt{\lambda_\ell}\, v_0(\ell) = 0. \end{aligned}$$

By Theorem 2.1, the solution of (3.6)-(3.9) is unique. Consequently, we see $U^* = \tilde{U}^*$. That is,

$$U^*(\xi) = \sqrt{\lambda_\ell}\, v_0(\frac{\xi}{\sqrt{\lambda_\ell}}). \quad \square$$

Here we describe a subsolution for the problem (3.1)-(3.5).

Proposition 3.1. *For any $\delta_1 \in (0, 2]$, we define*

$$V(\eta, \tau) := \varphi(\tau)U^*\left(\frac{\eta}{\varphi(\tau)}\right) \tag{3.17}$$

where $\varphi(\tau) = 1 + \left(\frac{1}{\sqrt{\lambda_\ell}} - 1\right)e^{-\delta_1\tau}$. *Then* V *satisfies the following;*

$$\begin{aligned}
&(a(V_\eta))_\eta + \eta V_\eta - V - V_\tau > 0, && \varphi(\tau)p^* \le \eta \le 0,\ \tau \ge 0,\\
&V_\eta(\varphi(\tau)p^*, \tau) = \tan\theta_0, && \tau \ge 0,\\
&V_\eta(0,\tau) = \tan\theta_1, && \tau \ge 0,\\
&V(\varphi(\tau)p^*, \tau) = 0, && \tau \ge 0,\\
&V(\eta, 0) = v_0(\eta), && \ell \le \eta \le 0.
\end{aligned}$$

Proof of Proposition 3.1. By (3.17), it is a simple computation to show

$$\begin{aligned}
&(a(V_\eta))_\eta + \eta V_\eta - V - V_\tau\\
=&\frac{1}{\varphi(\tau)}(a(U^*_\xi))_\xi + \varphi(\tau)\xi\, U^*_\xi - \varphi(\tau)U^* - \dot\varphi(\tau)(U^* - \xi\, U^*_\xi)
\end{aligned}$$

where $\dot\varphi$ is the differential of φ with respect to τ. Since U^* satisfies equation (3.6), we get

$$\begin{aligned}
&(a(V_\eta))_\eta + \eta V_\eta - V - V_\tau\\
=&(U^* - \xi U^*_\xi)\left(\frac{1}{\varphi(\tau)} - \varphi(\tau) - \dot\varphi(\tau)\right).
\end{aligned}$$

Then, by Lemma 2.3 (i) and $a' > 0$, we see

$$U^* - \xi U^*_\xi = (a(U^*_\xi))_\xi = a'(U^*_\xi)U^*_{\xi\xi} > 0 \quad \text{for} \quad \xi \in [p^*, 0].$$

Moreover, by a simple computation,

$$\frac{1}{\varphi(\tau)} - \varphi(\tau) - \dot\varphi(\tau) = \left(\frac{1}{\sqrt{\lambda_\ell}} - 1\right)e^{-\delta_1\tau}\left\{\delta_1 - \frac{2 + (\frac{1}{\sqrt{\lambda_\ell}} - 1)e^{-\delta_1\tau}}{1 + (\frac{1}{\sqrt{\lambda_\ell}} - 1)e^{-\delta_1\tau}}\right\}.$$

Since $\lambda_\ell > 1$, we see $\frac{1}{\sqrt{\lambda_\ell}} - 1 < 0$. Moreover, by $\delta_1 > 0$,

$$2 < \frac{2 + (\frac{1}{\sqrt{\lambda_\ell}} - 1)e^{-\delta_1\tau}}{1 + (\frac{1}{\sqrt{\lambda_\ell}} - 1)e^{-\delta_1\tau}} \le \sqrt{\lambda_\ell} + 1 \quad \text{for} \quad \tau \ge 0.$$

Thus, from $\delta_1 \in (0, 2]$,

$$\delta_1 - \frac{2 + (\frac{1}{\sqrt{\lambda_\ell}} - 1)e^{-\delta_1\tau}}{1 + (\frac{1}{\sqrt{\lambda_\ell}} - 1)e^{-\delta_1\tau}} < 0 \quad \text{for} \quad \tau \ge 0.$$

Consequently, since $1/\varphi(\tau) - \varphi(\tau) - \dot\varphi(\tau) > 0$ for $\tau \ge 0$, we see

$$(a(V_\eta))_\eta + \eta V_\eta - V - V_\tau > 0 \quad \text{for} \quad \varphi(\tau)p^* \le \eta \le 0,\ \tau \ge 0.$$

Moreover, by the definition of V and (3.7)-(3.9),

$$V_\eta(\varphi(\tau)p^*,\tau) = U_\xi^*(p^*) = \tan\theta_0,$$
$$V_\eta(0,\tau) = U_\xi^*(0) = \tan\theta_1,$$
$$V(\varphi(\tau)p^*,\tau) = \varphi(\tau)U^*(p^*) = 0,$$

and by Lemma 3.2,

$$V(\eta,0) = \varphi(0)U^*\left(\frac{\eta}{\varphi(0)}\right) = \frac{1}{\sqrt{\lambda_\ell}}\,U^*(\sqrt{\lambda_\ell}\eta) = v_0(\eta).$$

Thus, the proof of Prop. 3.1 is completed. □

3.2 Structure of a supersolution

We first define $w_0(\xi)$ as the following. Now, we choose a constant L satisfying $L < s_0$ and

$$0 < \sup_{\xi\in[s_0,0]}\left(\frac{u_0(\xi)}{\xi - L}\right) \le \tan\theta_0. \tag{3.18}$$

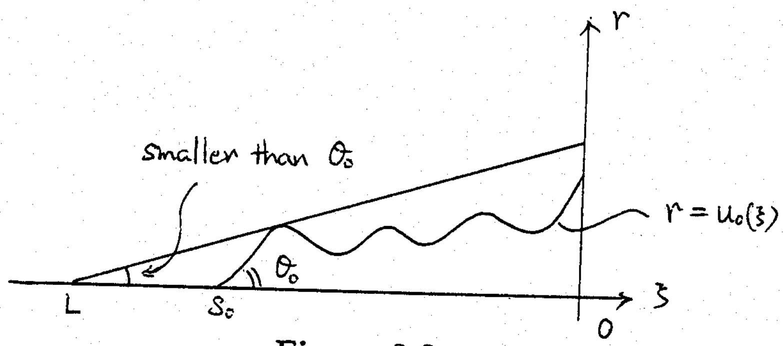

Figure 3.2

Then, by Theorem 2.1, there exist a unique $(\lambda_L, w_0) \in (0,\infty)\times C^2[L,0]$ satisfying

$$(a(w_{0\xi}))_\xi + \lambda_L\xi w_0 = 0,\ L < \xi < 0, \tag{3.19}$$
$$w_{0\xi}(L) = \tan\theta_0, \tag{3.20}$$
$$w_{0\xi}(0) = \tan\theta_1, \tag{3.21}$$
$$w_0(L) = 0. \tag{3.22}$$

By Remark 2.4, if necessary, we choose λ_L such that $0 < \lambda_L < 1$. Then, we get the following relation between u_0 and w_0.

Lemma 3.3. *Assume that w_0 satisfies (3.19)-(3.22). Then the following estimate is valid;*

$$u_0(\xi) < w_0(\xi) \quad \textit{for} \quad \xi\in[s_0,0].$$

Proof of Lemma 3.3. By Lemma 2.3 (i) and $w_{0\xi}(L) = \tan\theta_0$, we get $w_{0\xi}(\xi) > \tan\theta_0$ for $\xi\in(L,0]$. Since $w_0(L) = 0$,

$$(\tan\theta_0)(\xi - L) < w_0(\xi) \quad \text{for} \quad \xi\in[L,0]. \tag{3.23}$$

While, we see

$$u_0(\xi) \le \left\{ \sup_{\xi\in[s_0,0]} \left(\frac{u_0(\xi)}{\xi - L} \right) \right\} (\xi - L) \quad \text{for} \quad \xi \in [s_0, 0]. \tag{3.24}$$

Thus, by (3.18), (3.23) and (3.24),

$$u_0(\xi) < w_0(\xi) \quad \text{for} \quad \xi \in [s_0, 0]. \quad \square$$

Moreover, we get the following relation between U^* and w_0.

Lemma 3.4. *Assume that U^* satisfies (3.6)-(3.9) and w_0 satisfies (3.19)-(3.22). Then U^* is represented by w_0 as the following;*

$$U^*(\xi) = \sqrt{\lambda_L}\ w_0\left(\frac{\xi}{\sqrt{\lambda_L}}\right).$$

Moreover, this representation is unique.

The proof of Lemma 3.4 is the same as that of Lemma 3.2.
Here, we describe a supersolution for the problem (3.1)-(3.5).

Proposition 3.2. *For any $\delta_2 \in (0, \sqrt{\lambda_L} + 1)$, we define*

$$W(\rho,\tau) := \psi(\tau) U^*\left(\frac{\rho}{\psi(\tau)}\right) \tag{3.25}$$

where $\psi(\tau) = 1 + \left(\frac{1}{\sqrt{\lambda_L}} - 1\right)e^{-\delta_2\tau}$. Then W satisfies the following;

$$\begin{aligned}
&(a(W_\rho))_\rho + \rho W_\rho - W - W_\tau < 0, && \psi(\tau)p^* \le \rho \le 0,\ \tau \ge 0,\\
&W_\rho(\psi(\tau)p^*, \tau) = \tan\theta_0, && \tau \ge 0,\\
&W_\rho(0,\tau) = \tan\theta_1, && \tau \ge 0,\\
&W(\psi(\tau)p^*, \tau) = 0, && \tau \ge 0,\\
&W(\rho, 0) = w_0(\rho), && L \le \rho \le 0.
\end{aligned}$$

Proof of Proposition 3.2. Applying the same computation as the proof of Prop. 3.1 to (3.25), we get

$$\begin{aligned}
&(a(W_\rho))_\rho + \rho W_\rho - W - W_\tau\\
=&(U^* - \xi U^*_\xi)\left(\frac{1}{\psi(\tau)} - \psi(\tau) - \dot{\psi}(\tau)\right)
\end{aligned}$$

where $\dot{\psi}$ is the differential of ψ with respect to τ. Then, by a simple computation,

$$\frac{1}{\psi(\tau)} - \psi(\tau) - \dot{\psi}(\tau) = \left(\frac{1}{\sqrt{\lambda_L}} - 1\right)e^{-\delta_2\tau}\left\{\delta_2 - \frac{2 + (\frac{1}{\sqrt{\lambda_L}} - 1)e^{-\delta_2\tau}}{1 + (\frac{1}{\sqrt{\lambda_L}} - 1)e^{-\delta_2\tau}}\right\}.$$

Since $0 < \lambda_L < 1$, we see $\frac{1}{\sqrt{\lambda_L}} - 1 > 0$. Moreover, by $\delta_2 > 0$,

$$\sqrt{\lambda_L} + 1 \leq \frac{2 + (\frac{1}{\sqrt{\lambda_L}} - 1)e^{-\delta_2 \tau}}{1 + (\frac{1}{\sqrt{\lambda_L}} - 1)e^{-\delta_2 \tau}} < 2 \quad \text{for} \quad \tau \geq 0.$$

Thus, from $\delta_2 \in (0, \sqrt{\lambda_L} + 1)$,

$$\delta_2 - \frac{2 + (\frac{1}{\sqrt{\lambda_L}} - 1)e^{-\delta_2 \tau}}{1 + (\frac{1}{\sqrt{\lambda_L}} - 1)e^{-\delta_2 \tau}} < 0 \quad \text{for} \quad \tau \geq 0.$$

Consequently, since $1/\psi(\tau) - \psi(\tau) - \dot{\psi}(\tau) < 0$ for $\tau \geq 0$, recalling $U^* - \xi U^*_\xi > 0$ for $\xi \in [p^*, 0]$, we see

$$(a(W_\rho))_\rho + \rho W_\rho - W - W_\tau < 0 \quad \text{for} \quad \psi(\tau)p^* \leq \rho \leq 0,\ \tau \geq 0.$$

Moreover, by the definition W and (3.7)-(3.9),

$$\begin{aligned} W_\rho(\psi(\tau)p^*,\ \tau) &= U^*_\xi(p^*) = \tan\theta_0, \\ W_\rho(0, \tau) &= U^*_\xi(0) = \tan\theta_1, \\ W(\psi(\tau)p^*,\ \tau) &= \psi(\tau)U^*(p^*) = 0, \end{aligned}$$

and by Lemma 3.4,

$$W(\rho, 0) = \psi(0)U^*\left(\frac{\rho}{\psi(0)}\right) = \frac{1}{\sqrt{\lambda_L}}\, U^*(\sqrt{\lambda_L}\ \rho) = w_0(\rho).$$

Thus, the proof of Prop. 3.2 is completed. □

3.3 The gradient estimate

Lemma 3.5. *Assume that U satisfies (3.1)-(3.5). Then for $\tau \geq 0$*

$$|U_\xi(\xi, \tau)| \leq \sup_{\xi \in [s_0, 0]} |U_\xi(\xi, 0)|.$$

Proof of Lemma 3.5. Differentiating bath sides of (3.6),

$$U_{\tau\xi} = a'(U_\xi)U_{\xi\xi\xi} + a''(U_\xi)U^2_{\xi\xi} + \xi U_{\xi\xi}. \tag{3.26}$$

Here, we set

$$f(\xi, \tau) := e^{-\tau}(U_\xi(\xi, \tau) - \gamma)$$

where $\gamma = \sup_{\xi \in [s_0, 0]} U_\xi(\xi, 0)$. Then by (3.26), f satisfies

$$a'(fe^\tau + \gamma)f_{\xi\xi} + a''(fe^\tau + \gamma)e^\tau\ f^2_\xi - \xi f_\xi - f_\tau = f. \tag{3.27}$$

We prove that $f(\xi,\tau) \leq 0$ for $p(\tau) \leq \xi \leq 0$, $\tau \geq 0$. We set for any $T > 0$

$$Q_T := \{(\xi,\tau) \mid p(\tau) < \xi < 0,\ 0 < \tau < T\}.$$

Assume now that $\max_{\overline{Q}_T} f(\xi,\tau) = f(\xi_0,\tau_0) > 0$. Since

$$\begin{aligned} f(\xi,0) &= U_\xi(\xi,0) - \gamma \leq 0, \\ f(p(\tau),\tau) &= e^{-\tau}(U_\xi(p(\tau),\tau) - \gamma) = e^{-\tau}(\tan\theta_0 - \gamma) \\ &= e^{-\tau}(U_\xi(s_0,0) - \gamma) \leq 0, \\ f(0,\tau) &= e^{-\tau}(U_\xi(0,\tau) - \gamma) = e^{-\tau}(\tan\theta_1 - \gamma) \\ &= e^{-\tau}(U_\xi(0,0) - \gamma) \leq 0, \end{aligned}$$

we may consider that $(\xi_0,\tau_0) \in \{(\xi,\tau) \mid p(\tau) < \xi < 0,\ 0 < \tau \leq T\}$. Then, we see

$$f_{\xi\xi}(\xi_0,\tau_0) \leq 0,\ f(\xi)(\xi_0,\tau_0) = 0,\ f_\tau(\xi_0,\tau_0) \geq 0.$$

Since $a' > 0$, the left side of (3.27) is nonpositive at (ξ_0,τ_0). While, by the assumption, the right side of (3.27) is positive at (ξ_0,τ_0). This is a contradiction. Thus,

$$f(\xi,\tau) \leq 0 \quad \text{in} \quad \overline{Q}_T.$$

That is

$$U_\xi(\xi,\tau) \leq \gamma = \sup_{\xi\in[s_0,0]} U_\xi(\xi,0) \quad \text{in} \quad \overline{Q}_T.$$

By replacing U with $-U$, we obtain the same bound for $-U_\xi$. □

3.4. Comparison between a solution and a subsolution (supersolution)

We now set

$$d(\tau) := \inf\{[(\xi-\eta)^2 + (U(\xi,\tau) - V(\eta,\tau))^2]^{1/2} \\ |p(\tau) \leq \xi \leq 0,\ \varphi(\tau)p^* \leq \eta \leq 0\}.$$

Then, we get the following.

Lemma 3.6. *For $\tau \geq 0$, $d(\tau) > 0$.*

Proof of Lemma 3.6. Since $u_0(\xi) > v_0(\xi)$ for $\xi \in [\ell,0]$ by Lemma 3.1, we get $d(0) > 0$. Thus, there exists a constant $\mu_0 > 0$ such that $d(\tau) > 0$ for $0 \leq \tau < \mu_0$. We now set

$$\tau_0 := \sup\{\tau_1 \in (0,\infty) \mid d(\tau) > 0 \quad \text{for} \quad 0 \leq \tau < \tau_1\}.$$

Then we see $\tau_0 \geq \mu_0 > 0$.

Assume that there exists $\tau_0 \in [\mu_0,\infty)$. Then, by the definition τ_0, we get $d(\tau_0) = 0$. Here, the fact that $d(\tau_0) = 0$ is equivalent to the fact that there exists $\eta_0 \in [\varphi(\tau_0)p^*, 0]$ such that

$$\xi = \eta_0 \quad \text{and} \quad U(\xi,\tau_0) = V(\eta_0,\tau_0). \tag{3.28}$$

Thus, we consider three cases as the following and derive a contradiction for each case;

Case I: $\varphi(\tau_0)p^* < \eta_0 < 0$, (cf. Figure 3.3)
Case II: $\eta_0 = 0$, (cf. Figure 3.4)
Case III: $\eta_0 = \varphi(\tau_0)p^*$. (cf. Figure 3.5)

Here, we set

$$\hat{\Gamma}_\tau^{(1)} := \{(\xi,\tau) | r = U(\xi,\tau),\ p(\tau) \le \xi \le 0\},$$
$$\hat{\Gamma}_\tau^{(2)} := \{(\eta,\tau) | r = V(\eta,\tau),\ \varphi(\tau) \le \eta \le 0\}.$$

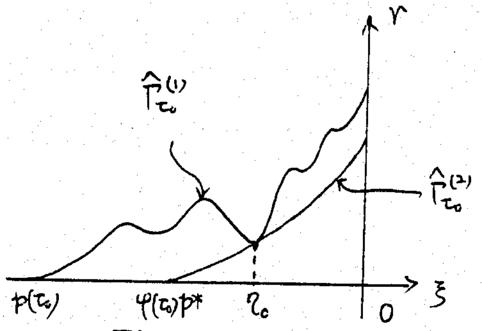

Figure 3.3

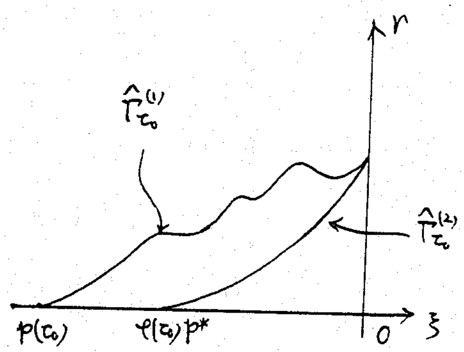

Figure 3.4

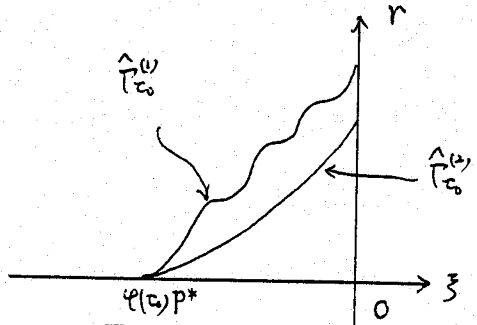

Figure 3.5

Case I: We now assume $\xi = \eta$ for $\varphi(\tau)p^* \le \xi \le 0$, $0 \le \tau \le \tau_0$ and set $\Psi(\eta,\tau) := U(\eta,\tau) - V(\eta,\tau)$. Then we see

$$\begin{aligned} &\Psi(\eta_0,\tau_0) = 0, \\ &\Psi(\eta,\tau) > 0 \quad \text{for} \quad \varphi(\tau)p^* \le \eta \le 0,\ 0 \le \tau < \tau_0. \end{aligned} \tag{3.29}$$

Moreover, Ψ satisfies

$$A(\eta,\tau)\Psi_{\eta\eta} + (A_\eta(\eta,\tau) + \eta)\Psi_\eta - \Psi - \Psi_\tau < 0,$$
$$\varphi(\tau)p^* < \eta < 0,\ 0 < \tau \le \tau_0$$

where

$$A(\eta,\tau) = \int_0^1 a'(\theta U_\eta + (1-\theta)V_\eta)d\theta > 0. \tag{3.30}$$

We set

$$\mathcal{L}[\Psi] := A(\eta,\tau)\Psi_{\eta\eta} + (A_\eta(\eta,\tau) + \eta)\Psi_\eta - \Psi - \Psi_\tau.$$

Since $V_{\eta\eta}$ is positive for $\varphi(\tau)p^* \leq \eta \leq 0$, $\tau \geq 0$ by the definition of V and Lemma 2.3 (i), we see

$$\tan\theta_0 \leq V_n \leq \tan\theta_1 \quad \text{for} \quad \varphi(\tau)p^* \leq \eta \leq 0,\ \tau \geq 0. \tag{3.31}$$

Thus, by Lemma 3.5 and (3.30),(3.31), there exist constants C_1, C_2 such that

$$0 < C_1 \leq A(\eta,\tau) \leq C_2 \quad \text{for} \quad \varphi(\tau)p^* \leq \eta \leq 0,\ \tau \geq 0. \tag{3.32}$$

Consequently, $\mathcal{L}$ is uniformly parabolic. We now set for $\varepsilon > 0$

$$Q^{\varepsilon}_{\tau_0} := \{(\eta,\tau) \mid \varphi(\tau)p^* < \eta < 0,\ \tau_0 - \varepsilon < \tau \leq \tau_0\}.$$

Since U is a smooth solution for problem (3.1)-(3.5), $|U_{\eta\eta}|$ is bounded in $\overline{Q}^{\varepsilon}_{\tau_0}$. Moreover, by Lemma 2.3 and $\frac{1}{\sqrt{\lambda_\ell}} \leq \varphi(\tau) < 1$, we get

$$0 < V_{\eta\eta} < C_3 = C_3(\lambda_\ell,\ U^*(0), p^*, \theta_1).$$

Moreover, by Lemma 3.5 and (3.31) and $a \in C^2(\mathbb{R})$, $|a''|$ is bounded in $\overline{Q}^{\varepsilon}_{\tau_0}$. Thus, $|A_\eta|$ is bounded in $\overline{Q}^{\varepsilon}_{\tau_0}$. Since $0 \leq |\eta| \leq |\varphi(\tau)|\ |p^*| < |p^*|$, we get

$$\sup\{|A_\eta(\eta,\tau) + \eta|\ |(\eta,\tau) \in \overline{Q}^{\varepsilon}_{\tau_0}\} < \infty. \tag{3.33}$$

Then, we set

$$m := \{(\eta,\tau) \in \overline{Q}^{\varepsilon}_{\tau_0} | (\eta,\tau) \text{ is connected with } (\eta_0,\tau_0) \text{ by a horizontal and a vertical line segment}\}$$

By (3.29), (3.32), (3.33) and $\mathcal{L}[\Psi] < 0$ in $Q^{\varepsilon}_{\tau_0}$, we can apply the strong minimum principle (cf. [9]). Consequently, we obtain

$$\Psi(\eta,\tau) = 0 \quad \text{for} \quad (\eta,\tau) \in m.$$

This contradicts the definition of τ_0. Thus, we can not have η_0 satisfying (3.28) in case I.

Case II: We set Ψ, $Q^{\varepsilon}_{\tau_0}$ as case I. Then we see

$$\Psi(0,\tau_0) = 0,\ \Psi(\eta,\tau) > 0 \quad \text{for} \quad (\eta,\tau) \in Q^{\varepsilon}_{\tau_0}.$$

Moreover, Ψ satisfies $\mathcal{L}[\Psi] < 0$ in $Q^{\varepsilon}_{\tau_0}$ with (3.32),(3.33). Applying the strong minimum principle, we get

$$\frac{\partial \Psi}{\partial \nu}(0,\tau_0) < 0 \tag{3.34}$$

where $\frac{\partial}{\partial \nu}$ is any outward directional derivative from $Q^{\varepsilon}_{\tau_0}$ at $(0,\tau_0)$.

But since $\Psi_\eta(0,\tau) = U_\eta(0,\tau) - V_\eta(0,\tau) = 0$ for $\tau \geq 0$, if we choose $\nu = (1,0)$, which is a outward vector from $Q^\varepsilon_{\tau_0}$, we get

$$\frac{\partial \Psi}{\partial \nu}(0,\tau_0) = \Psi_\eta(0,\eta_0) = 0.$$

This contradicts (3.34). Thus, $\eta_0 = 0$ does not satisfy (3.28).

Case III: Since $U_\xi(p(\tau),\tau) = \tan\theta_0 > 0$ for any $\tau \in [0,\tau_0]$. There exist positive constants ε_0 and $C_4 = C_4(\varepsilon_0)$ such that

$$U_\xi(\xi,\tau) \geq C_4 > 0 \quad \text{for} \quad p(\tau) \leq \xi \leq p(\tau) + \varepsilon_0,\ 0 \leq \tau \leq \tau_0. \tag{3.35}$$

Thus, U has a inverse function for $p(\tau) \leq \xi < p(\tau) + \mu_1$. We write it as $\sigma(r,\tau)$. Moreover, since $V_\eta \geq \tan\theta_0 > 0$ by Lemma 2.3 (i) and $V_\eta(\varphi(\tau)p^*,\tau) = \tan\theta_0$, V has a inverse function. We write it as $\hat{\sigma}(\hat{r},\tau)$. Then, σ satisfies

$$-\left(a\left(\frac{1}{\sigma_r}\right)\right)_r + r\sigma_r - \sigma - \sigma_\tau < 0$$
$$\text{for} \quad 0 < r < U(p(\tau)+\mu_1,\tau),\ 0 < \tau \leq \tau_0,$$

and $\hat{\sigma}$ satisfies

$$-\left(a\left(\frac{1}{\hat{\sigma}_{\hat{r}}}\right)\right)_{\hat{r}} + \hat{r}\hat{\sigma}_{\hat{r}} - \hat{\sigma} - \hat{\sigma}_\tau < 0$$
$$\text{for} \quad 0 < \hat{r} < V(0,\tau),\ 0 < \tau \leq \tau_0.$$

Moreover, by $U(p(\tau),\tau) = 0$ and $V(\varphi(\tau)p^*,\tau) = 0$, we get

$$\sigma(0,\tau) = p(\tau),\ \hat{\sigma}(0,\tau) = \varphi(\tau)p^*,$$

and by $U_\xi(p(\tau),\tau) = \tan\theta_0$ and $V_\eta(\varphi(\tau)p^*,\tau) = \tan\theta_0$, we get

$$\sigma_r(0,\tau) = \frac{1}{\tan\theta_0},\ \hat{\sigma}_{\hat{r}}(0,\tau) = \frac{1}{\tan\theta_0}.$$

We set $r_0(\tau) := \min\{U(p(\tau)+\varepsilon_0,\tau),\ V(0,\tau)\}$. Moreover, we assume $r = \hat{r}$ for $0 \leq \hat{r} \leq r_0(\tau)$, $0 \leq \tau \leq \tau_0$ and set $\tilde{\Psi}(r,\tau) := \sigma(r,\tau) - \hat{\sigma}(r,\tau)$. Then, we see

$$\begin{aligned} &\tilde{\Psi}(0,\tau_0) = 0, \\ &\tilde{\Psi}(r,\tau) > 0 \quad \text{for} \quad 0 < r < r_0(\tau),\ 0 \leq \tau \leq \tau_0. \end{aligned} \tag{3.36}$$

Moreover, $\tilde{\Psi}$ satisfies

$$\tilde{A}(r,\tau)\tilde{\Psi}_{rr} + (\tilde{A}_r(r,\tau) + r)\tilde{\Psi}_r - \tilde{\Psi} - \tilde{\Psi}_\tau < 0,$$
$$0 < r < r_0(\tau),\ 0 < \tau \leq \tau_0$$

where

$$\tilde{A}(r,\tau) = \frac{1}{\sigma_r\ \hat{\sigma}_r} \int_0^1 a'\left(\frac{\theta}{\hat{\sigma}_r} + \frac{1-\theta}{\sigma_r}\right) d\theta > 0. \tag{3.37}$$

We set

$$\tilde{\mathcal{L}}[\tilde{\Psi}] := \tilde{A}(r,\tau)\tilde{\Psi}_{rr} + (\tilde{A}_r(r,\tau) + r)\tilde{\Psi}_r - \tilde{\Psi} - \tilde{\Psi}_\tau,$$

and for $\varepsilon > 0$ (where ε is the same as case I)

$$P^\varepsilon_{\tau_0} := \{(r,\tau) \mid 0 < r < r_0(\tau),\ \tau_0 - \varepsilon < \tau \le \tau_0\}.$$

By Lemma 3.5 and (3.35), we get

$$0 < C_5 \le \sigma_r \le \frac{1}{C_4} \quad \text{in} \quad \overline{P}^\varepsilon_{\tau_0} \tag{3.38}$$

where C_5 is a constant depending only a $\sup_{\xi \in [s_0, 0]} |U_\xi(\xi, 0)|$. Moreover, by (3.31),

$$0 < \frac{1}{\tan\theta_1} \le \hat{\sigma}_r \le \frac{1}{\tan\theta_0} \quad \text{in} \quad \overline{P}^\varepsilon_{\tau_0}. \tag{3.39}$$

Thus, by (3.37)-(3.39) and $a \in C^2(\mathbb{R})$, there exist constants C_6, C_7 such that

$$0 < C_6 \le \tilde{A}(\eta,\tau) \le C_7 \quad \text{in} \quad \overline{P}^\varepsilon_{\tau_0}. \tag{3.40}$$

Consequently, $\tilde{\mathcal{L}}$ is uniformly parabolic in $P^\varepsilon_{\tau_0}$. Since we see

$$U_{\eta\eta} = -\frac{\sigma_{rr}}{\sigma_r^3},\ V_{\eta\eta} = -\frac{\hat{\sigma}_{rr}}{\hat{\sigma}_r^3},$$

and $|U_{\eta\eta}|$ and $|V_{\eta\eta}|$ and bounded in $\overline{Q}^\varepsilon_{\tau_0}$ (see case I), by (3.38),(3.39), we obtain that $|\sigma_{rr}|$ and $|\hat{\sigma}_{rr}|$ are bounded $\overline{P}^\varepsilon_{\tau_0}$. Moreover, by (3.38),(3.39) and $a \in C^2(\mathbb{R})$, $|a''|$ is bounded in $\overline{P}^\varepsilon_{\tau_0}$. Thus, $|\tilde{A}_r|$ is bounded in $\overline{P}^\varepsilon_{\tau_0}$. Since $0 \le r \le r_0(\tau) \le U^*(0)$, we get

$$\sup\{|\tilde{A}_r(r,\tau) + r| \mid (r,\tau) \in \overline{P}^\varepsilon_{\tau_0}\} < \infty. \tag{3.41}$$

Consequently, by (3.36), (3.40), (3.41) and $\tilde{\mathcal{L}}[\tilde{\Psi}] < 0$ in $P^\varepsilon_{\tau_0}$, we can apply the strong minimum principle. Then we obtain

$$\frac{\partial \tilde{\Psi}}{\partial \nu}(0,\tau_0) < 0 \tag{3.42}$$

where $\frac{\partial}{\partial \nu}$ is any outward directional derivative from $P^\varepsilon_{\tau_0}$ at $(0,\tau_0)$.

But since $\tilde{\Psi}_r(0,\tau) = \sigma_r(0,\tau) - \hat{\sigma}_r(0,\tau) = 0$ for $\tau \ge 0$, if we choose $\nu = (-1, 0)$, which is a outward vector from $P^\varepsilon_{\tau_0}$, we get

$$\frac{\partial \tilde{\Psi}}{\partial \nu}(0,\tau_0) = -\tilde{\Psi}_r(0,\tau_0) = 0.$$

This contradicts (3.42). Thus, $\eta_0 = \varphi(\tau_0)p^*$ does not satisfy (3.28).

Consequently, by case I, II, and III, we can not have $\eta_0 \in [\varphi(\tau_0)p^*, \tau_0]$ satisfying (3.28). This contradicts $d(\tau_0) = 0$. Thus, we can not have $\tau_0 \in [\mu_0, \infty)$. That is, $d(\tau) > 0$ for $\tau \geq 0$. □

Consequently, by Lemma 3.1 and Lemma 3.6, we get

$$p(\tau) < \varphi(\tau)p^*, \ U(\eta,\tau) > V(\eta,\tau) \quad \text{for} \quad \varphi(\tau)p^* \leq \eta \leq 0, \ \tau \geq 0.$$

In the same way, we get

$$\psi(\tau)p^* < p(\tau), \ W(\rho,\tau) > U(\rho,\tau) \quad \text{for} \quad p(\tau) \leq \rho \leq 0, \ \tau \geq 0.$$

3.5 Proof of Theorem 3.1

We now assume $\xi_0 \in [p^*, 0]$. Then, by the definition of V and W, the intersection points of the straight line $\{(\xi, r) \mid U^*(\xi_0)\xi - \xi_0\, r = 0\}$ and the graphs $\{(\xi, r) \mid r = V(\xi,\tau),\ \varphi(\tau)p^* \leq \xi \leq 0\}$, $\{(\xi, r) \mid r = W(\xi,\tau),\ \psi(\tau)p^* \leq \xi \leq 0\}$ are represented as the following;

$$(\varphi(\tau)\xi_0,\ \varphi(\tau)U^*(\xi_0)),\ (\psi(\tau)\xi_0,\ \psi(\tau)U^*(\xi_0)).$$

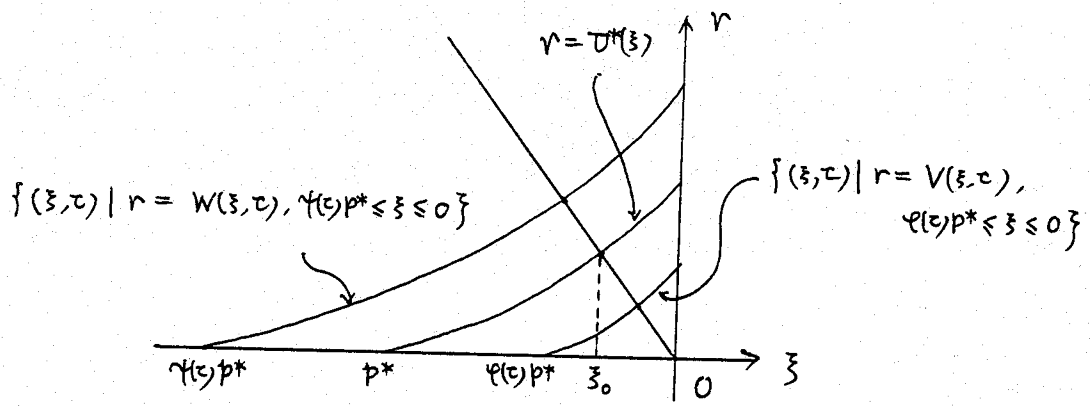

Figure 3.6

We set

$$\mathcal{D}(\xi_0, \tau) := \{\xi \mid \xi - \text{coordinate of intersection points of the straight line } \{(\xi, r) \mid U^*(\xi_0)\xi - \xi_0 r = 0\} \text{ and the graph } \{(\xi, r) \mid r = U(\xi,\tau),\ p(\tau) \leq \xi \leq 0\}.$$

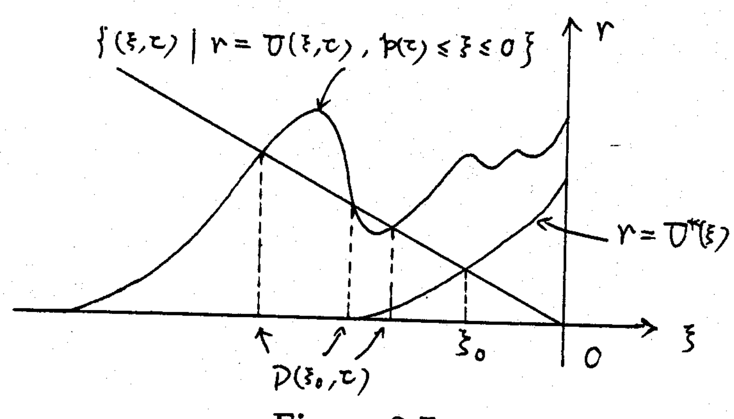

Figure 3.7

Since $U(\xi,\tau)$ is a smooth function in the set $\{(\xi,\tau)|\ p(\tau)\le\xi\le 0,\ \tau\ge 0\}$, we get $\mathcal{D}(\xi_0,\tau)\neq\emptyset$.

Then, from section 3.4, we obtain for $\xi\in\mathcal{D}(\xi_0,\tau)$

$$(\varphi(\tau)\xi_0)^2+(\varphi(\tau)U^*(\xi_0))^2\le\xi^2+(U(\xi,\tau))^2 \le(\psi(\tau)\xi_0)^2+(\psi(\tau)U^*(\xi_0))^2. \tag{3.43}$$

Here, we see

$$[(\varphi(\tau)\xi_0)^2+(\varphi(\tau)U^*(\xi_0))^2]^{1/2}-[\xi_0^2+(U^*(\xi_0))^2]^{1/2} =\left(\frac{1}{\sqrt{\lambda_\ell}}-1\right)e^{-\delta_1\tau}[\xi_0^2+(U^*(\xi_0))^2]^{1/2}, \tag{3.44}$$

$$[(\psi(\tau)\xi_0)^2+(\psi(\tau)U^*(\xi_0))^2]^{1/2}-[\xi_0^2+(U^*(\xi_0))^2]^{1/2} =\left(\frac{1}{\sqrt{\lambda_L}}-1\right)e^{-\delta_2\tau}[\xi_0^2+(U^*(\xi_0))^2]^{1/2}. \tag{3.45}$$

Thus, by (3.43)-(3.45) and $\lambda_\ell>1$ and $0<\lambda_L<1$, we get for $\xi\in\mathcal{D}(\xi_0,\tau)$

$$C\left(\frac{1}{\sqrt{\lambda_\ell}}-1\right)e^{-\delta_1\tau}<[\xi^2+U(\xi,\tau))^2]^{1/2}-[\xi_0^2+(U^*(\xi_0))^2]^{1/2} < C\left(\frac{1}{\sqrt{\lambda_L}}-1\right)e^{-\delta_2\tau}$$

where $C=\sup_{\xi_0\in[p^*,0]}[\xi_0^2+(U^*(\xi_0))^2]^{1/2}$.

Consequently, if we choose $\delta_0\in(0,\sqrt{\lambda_L}+1)$, we obtain for $\tau\ge 0$

$$\sup_{\xi_0\in[p^*,0]}\ \sup_{\xi\in\mathcal{D}(\xi_0,\tau)}\left|\,[\xi^2+(U(\xi,\tau))^2]^{1/2}-[\xi_0^2+(U^*(\xi_0))^2]^{1/2}\,\right|\le\hat{C}\,e^{-\delta_0\tau}$$

where $\hat{C}=\max\left\{-C\left(\frac{1}{\sqrt{\lambda_\ell}}-1\right),\ C\left(\frac{1}{\sqrt{\lambda_L}}-1\right)\right\}$. Thus, the proof of Theorem 3.1 is completed.

3.6 Proof of Main Theorem

We define

$$\hat{d}_H(\Gamma_t,S_t):=\sup_{X_0\in S_t}\ \sup_{Y\in\mathcal{Q}}\ |\,d(O,X_0)-d(O,Y)\,|$$

where

$\mathcal{Q}:=\{Y\in\Gamma_t\mid$ the intersection points between Γ_t and the straight line passing the origin O and $X_0(\in S_t)\}$.

Then, we note that $\hat{d}_H$ is equivalent to the Hausdorff distance d_H.

Consequently, if we choose $\hat{\delta}_0\in(1,2)$, by Theorem 3.1,

$$\hat{d}_H(\Gamma_t,S_t)\le C(2t+1)^{-(\hat{\delta}_0-1)/2}\le\tilde{C}t^{-(\hat{\delta}_0-1)/2}$$

Thus, the proof of Main Theorem is completed.

Acknowledgments. We wish to thank Y.Giga for useful discussion.

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