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Series #401. January 1998

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Global solutions in the critical Sobolev space for the wave equations with nonlinearity of exponential growth

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1 Introduction

We consider the existence of global solutions for nonlinear wave equations of the form

$$\begin{aligned} &\partial_t^2 u(t,x) - \Delta u(t,x) = f(u(t,x)) \\ &u(0,\cdot) = \phi \in \dot{H}^{n/2}(\mathbf{R}^n), \quad \partial_t u(0,\cdot) = \psi \in \dot{H}^{n/2-1}(\mathbf{R}^n), \end{aligned} \tag{1.1}$$

where u is a complex-valued function of $(t,x) \in \mathbf{R} \times \mathbf{R}^n$, $\partial_t = \partial/\partial t$, Δ denotes the Laplacian of spatial variables, f is a complex function, and

$$\dot{H}^\mu(\mathbf{R}^n) \equiv (-\Delta)^{-\mu/2} L^2(\mathbf{R}^n)$$

is the homogeneous Sobolev space of order μ. We prove the global solvability of (1.1) in the critical Sobolev space $\dot{H}^{n/2}$ with nonlinearity f of exponential type, a typical example of which is given by $u^2 e^{\lambda|u|^2}$ with $\lambda > 0$.

There is a large literature on the Cauchy problem for the equation (1.1), see for instance [2,5,7,9,10,12,13,16,17,21,22]. It is well-known that the Cauchy problem (1.1) is locally well-posed in the usual Sobolev space $H^\mu(\mathbf{R}^n)$ if $\mu > n/2$ and f is any smooth function with $f(0) = 0$ [16], or if $1/2 \le \mu < n/2$ and f is given as a single power nonlinearity $\lambda|u|^{p-1}u$ with $\lambda \in \mathbf{C}$ and $p \le 1 + 4/(n-2\mu)$ [9, 13, 17, 22]. Moreover if $p = 1 + 4/(n-2\mu)$ and $1/2 \le \mu < n/2$, then we have global $\dot{H}^\mu$-solutions with the Cauchy data sufficiently small [13, 17]. The same situation happens for the nonlinear Schrödinger equations as well [3, 4, 11, 15, 23]. The critical power $p = 1+4/(n-2\mu)$ at the level of $\dot{H}^\mu$ naturally

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arises by the standard scaling argument and is consistent with the Sobolev embedding $\dot{H}^\mu \hookrightarrow L^{n(p-1)/2}$ for $\mu = n/2 - 2/(p-1)$. The last embedding, however, breaks down for $\mu = n/2$. In view of the circumstantial evidence above, it seems natural to call $n/2$ another critical exponent and accordingly, in this paper we are interested in the existence of solutions of (1.1) when $\mu = n/2$ and especially in the admissible class of nonlinearities in that theory. $\dot{H}^{n/2}$ solutions are of particular interest as finite energy and strong solutions when $n = 2$ and 4, respectively.

To state our theorem, first we clarify the class of nonlinear terms.

Definition Let $n \geq 2$. We define a class of functions G as follows. We say that $f \in G$ if f satisfies $f \in C^{[n/2]}(\mathbf{R}^2, \mathbf{C})$ and $f(0) = 0$, and there exists some positive number λ such that the following estimates hold for all $z \in \mathbf{C}$:

$$
\begin{aligned}
&|f'(z)| \leq \begin{cases} C|z|^4 e^{\lambda|z|^2} & \text{for } n = 2; \\ C|z|^2 e^{\lambda|z|^2} & \text{for } n = 3; \end{cases} \\
&|f'(z)| \leq C|z|^2 e^{\lambda|z|^2}, \quad |f''(z)| \leq C|z| e^{\lambda|z|^2} \quad \text{for } n = 4; \qquad (1.2) \\
&|f'(z)| \leq C|z| e^{\lambda|z|^2}, \quad \sup_{2 \leq k \leq [n/2]} |f^{(k)}(z)| \leq C e^{\lambda|z|^2} \quad \text{for } n \geq 5;
\end{aligned}
$$

where f is regarded as a function of z and $\bar{z}$, $f'(z) = (\partial f/\partial z, \partial f/\partial \bar{z})$, for $k \geq 2$ $f^{(k)}$ denotes any of the derivatives of f of k-th order with respect to z and $\bar{z}$, and $[s]$ denotes the integer part of $s \in \mathbf{R}$.

To show the existence of the global solutions of (1.1) with f in G, we use the standard fixed point argument on the corresponding integral equation. Let $1/q_0 = (n-1)/2(n+1)$. We introduce the following complete metric space

$$
\begin{aligned}
& X(R) \\
\equiv\ & \{u \in L^\infty(\mathbf{R}, \dot{H}^{n/2}) \cap L^{q_0}(\mathbf{R}, B^{(n-1)/2}_{q_0} \cap \dot{B}^0_{q_0}) \mid \qquad (1.3) \\
& \max\{\|u; L^\infty(\mathbf{R}, \dot{H}^{n/2})\|, \|u; L^{q_0}(\mathbf{R}, B^{(n-1)/2}_{q_0})\|, \|u; L^{q_0}(\mathbf{R}, \dot{B}^0_{q_0})\|\} \leq R\},
\end{aligned}
$$

endowed with the metric

$$d(u, v) \equiv \|u - v; L^{q_0}(\mathbf{R}, \dot{B}^0_{q_0})\|, \qquad (1.4)$$

where B^s_q [resp.$\dot{B}^s_q$] denotes the [resp.homogeneous] Besov space and we made abbreviation such as $B^s_q = B^s_{q,2}(\mathbf{R}^n)$, $\dot{B}^s_q = \dot{B}^s_{q,2}(\mathbf{R}^n)$, $H^{s,q} = H^{s,q}(\mathbf{R}^n)$, $\dot{H}^{s,q} = \dot{H}^{s,q}(\mathbf{R}^n)$. For the definitions and its properties of Sobolev and Besov spaces, we refer to [1, 6, 27]. For convenience of description, we define the following space of the Cauchy data by

$$
\begin{aligned}
Y &\equiv (\dot{H}^{n/2} \times \dot{H}^{n/2-1}) \cap (\dot{H}^{1/2} \times \dot{H}^{-1/2}) \qquad (1.5) \\
\|(\phi, \psi)\|_Y &\equiv \max\{\|\phi; \dot{H}^{n/2}\|, \|\phi; \dot{H}^{1/2}\|, \|\psi; \dot{H}^{n/2-1}\|, \|\psi; \dot{H}^{-1/2}\|\} \qquad (1.6)
\end{aligned}
$$

Now we state our theorem. In the following, we use the operators

$$K(t) \equiv (\sin t\sqrt{-\Delta})/\sqrt{-\Delta}, \quad \dot{K}(t) \equiv \cos t\sqrt{-\Delta}.$$

Theorem 1. 1 *Let $n \geq 2$. Let $f \in G$. Let $(\phi, \psi) \in Y$ be sufficiently small. Then there exists $R > 0$ such that (1.1) has a unique global solution in $X(R)$. Moreover the solution u belongs to $(C \cap L^\infty)(\mathbf{R}, \dot{H}^{n/2} \cap \dot{H}^{1/2})$ and depends on the initial data continuously. In addition, there exists a pair $(\phi_+, \psi_+) \in Y$ such that*

$$\|(u(t) - \dot{K}(t)\phi_+ - K(t)\psi_+, \partial_t(u(t) - \dot{K}(t)\phi_+ - K(t)\psi_+))\|_Y \longrightarrow 0 \quad \text{as} \quad t \to \infty.$$

Conversely, for sufficiently small $(\phi_-, \psi_-) \in Y$, there exists a solution u_- of (1.1) in $X(R)$ such that

$$\|(u_-(t) - \dot{K}(t)\phi_- - K(t)\psi_-, \partial_t(u_-(t) - \dot{K}(t)\phi_- - K(t)\psi_-))\|_Y \longrightarrow 0 \quad \text{as} \quad t \to -\infty.$$

Moreover the scattering operator $(\phi_-, \psi_-) \mapsto (\phi_+, \psi_+)$ is continuous with respect to the $\dot{H}^{1/2} \times \dot{H}^{-1/2}$-norm.

Remark 1. We now comment on the relation between the dimension and the vanishing order of the functions of (1.2) at the origin. As we mentioned above, for the existence of the global solutions it is natural to require the vanishing order at least the conformal power $1+4/(n-1)$ which is identical to the requirement in (1.2) when $n = 2, 3, 5$. As we see below, we need to differentiate the nonlinearity f at least $(n-1)/2$-times and as far as f is supposed to behave as a power u^p at the origin, we must impose $p \geq (n-1)/2$ except p is an integer. Therefore, if p is not an integer, then we are restricted to the case $n \leq 5$. We keep ourselves away from the resulting technical difficulty and assume sufficient smoothness of f at the origin.
Remark 2. An analogous result has been proved by the same authors for the nonlinear Schrödinger equations in the critical Sobolev space $H^{n/2}$ [18].
Remark 3. In (1.4), the exponent q_0 of the auxiliary function space is that of the Strichartz space-time estimate in the diagonal case [25]. In Theorem 1. 1 , as for the local well-posedness, the smallness assumption on the Cauchy data may be removed if f satisfies

$$\sup_{0 \leq k \leq [n/2]} |f^{(k)}(z)| \leq C|z|^m e^{\lambda|z|^{2-\epsilon}}, \quad 0 < \epsilon \leq 2.$$

for m sufficiently large (see [16] for details).

We prove Theorem 1. 1 in the next section. The proof depends on the Strichartz estimate [8, 13, 25] and on the estimates on the nonlinear terms in the form of the power series expansion given by the RHS of (1.2). To ensure the convergence of the corresponding series of norms on L^q or $\dot{B}^0_q$ with $q \to \infty$, we use the sharp Gagliardo-Nirenberg inequalities [18, 19]. Those inequalities are closely related to Trudinger's inequality [14, 19, 20, 24, 26, 28] and in this sense the power 2 in the exponential functions on the RHS of (1.2) also seems critical.

2 Proof of Theorem 1.1

In this section we prove Theorem 1. 1 . For $(\phi, \psi) \in Y$ and $u \in X(R)$ with R to be determined later, we define the operator Φ by

$$\Phi(u)(t) \equiv \dot{K}(t)\phi + K(t)\psi + \int_0^t K(t-\tau)f(u(\tau))d\tau. \tag{2.7}$$

For the existence of the solutions of (1.1), it suffices to show that Φ is a contraction map on $X(R)$ for some R. By the Strichartz estimate and the standard duality argument, we have the following linear estimates

$$\begin{aligned} & \max\{\|\Phi(u); L^\infty(\mathbf{R}, \dot{H}^{n/2})\|, \|\Phi(u); L^{q_0}(\mathbf{R}, \dot{B}^{(n-1)/2}_{q_0})\|\} \\ \leq \;& C(\|\phi; \dot{H}^{n/2}\| + \|\psi; \dot{H}^{n/2-1}\| + \|f(u); L^{q_0'}(\mathbf{R}, \dot{B}^{(n-1)/2}_{q_0'})\|), \quad (2.8) \\ & \|\Phi(u); L^{q_0}(\mathbf{R}, \dot{B}^0_{q_0})\| \\ \leq \;& C(\|\phi; \dot{H}^{1/2}\| + \|\psi; \dot{H}^{-1/2}\| + \|f(u); L^{q_0'}(\mathbf{R}, \dot{B}^0_{q_0'})\|), \quad (2.9) \end{aligned}$$

where $1/q_0' \equiv 1 - 1/q_0$, and C is independent of ϕ, ψ, f and u (see also [8]). In (2.9), we may replace $\dot{B}^0_{q_0}$ and $\dot{B}^0_{q_0'}$ with L^{q_0} and $L^{q_0'}$ respectively using the embedding $\dot{B}^0_r \hookrightarrow L^r$ for $2 \leq r < \infty$. Regarding the norms of $f(u)$ on RHS in (2.8) and (2.9), we have the following lemma.

Lemma 2. 1 *Let $f \in G$. Then there exists a monotone increasing function ρ on $\mathbf{R}$ such that for any u and v in $X(R)$*

$$\|f(u); L^{q_0'}(\mathbf{R}, L^{q_0'})\| \leq C\rho(R)R^2, \tag{2.10}$$

$$\|f(u); L^{q_0'}(\mathbf{R}, \dot{B}^{(n-1)/2}_{q_0'})\| \leq C\rho(R)R^2, \tag{2.11}$$

$$\|f(u) - f(v); L^{q_0'}(\mathbf{R}, L^{q_0'})\| \leq C\rho(R)R\|u - v; L^{q_0}(\mathbf{R}, L^{q_0})\|, \tag{2.12}$$

where C is independent of u and v.

Proof of Lemma 2. 1

We prove Lemma 2. 1 for $n \geq 5$, since the lemma for $n = 2, 3, 4$ could be shown quite analogously. First we prove (2.10). We recall the inequalities

$$\|u; L^q\| \leq Cq^{1/2+(r-2)/2q}\|u; \dot{H}^{n/2}\|^{1-r/q}\|u; L^r\|^{r/q}, \tag{2.13}$$

$$\|u; \dot{B}^0_q\| \leq Cq^{1/2+(r-2)/2q}\|u; \dot{H}^{n/2}\|^{1-r/q}\|u; \dot{B}^0_r\|^{r/q}, \tag{2.14}$$

for any $1 < r \leq q < \infty$, where C is independent of q and u (see [18, 19]). By expanding the RHS of (1.2) and estimating the resulting power series by the Hölder inequality in space and (2.13), we have

$$\begin{aligned} &\|f(u); L^{q_0'}\| \\ \leq\ & C\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\|u; L^{r^*}\|^{2j+1}\|u; L^{q_0}\| \qquad (2.15)\\ \leq\ & C\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}r^{*(1/2+(q_0-2)/2r^*)(2j+1)}\|u; \dot{H}^{n/2}\|^{(1-q_0/r^*)(2j+1)} \\ & \qquad \cdot\|u; L^{q_0}\|^{1+(2j+1)q_0/r^*}, \qquad (2.16) \end{aligned}$$

where $r^* = (n+1)(2j+1)/2$ and C is independent of u. Therefore we have by the Hölder inequality in time

$$\|f(u); L^{q_0'}(\mathbf{R}, L^{q_0'})\| \leq C\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}r^{*(1/2+(q_0-2)/2r^*)(2j+1)}R^{2j+2}. \qquad (2.17)$$

Since the series in (2.17) converges for sufficiently small R, we shall regard it as $\rho(R)R^2$. This completes the proof of (2.10). The inequality (2.12) follows analogously if we use the equality

$$f(u) - f(v) = \int_0^1 f'(v + \theta(u-v))(u-v)d\theta.$$

Next we show (2.11) for $n \geq 5$ odd. By the embedding $\dot{H}^{(n-1)/2}_{q_0'} \hookrightarrow \dot{B}^{(n-1)/2}_{q_0'}$, we have

$$\|f(u); \dot{B}^{(n-1)/2}_{q_0'}\| \leq C\sum_{k=1}^{(n-1)/2}\sum_{|\alpha|=(n-1)/2}\sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ |\beta_i|\geq 1}}\|f^{(k)}(u)\prod_{i=1}^{k}\partial^{\beta_i}u; L^{q_0'}\|. \qquad (2.18)$$

For $k \geq 2$ and $1 \leq i \leq k$, let $1/r^*$ and $1/r_i^*$ be

$$1/r^* \equiv 2/(n+1)(2j+k-1), \quad 1/r_i^* \equiv (1-2|\beta_i|/(n-1))/r^* + 2|\beta_i|/(n-1)q.$$

Then we have by the interpolation of the Besov space

$$\|\partial^{\beta_i}u; L^{r_i^*}\| \leq C\|u; \dot{B}^0_{r^*}\|^{1-2|\beta_i|/(n-1)}\|u; \dot{B}^{(n-1)/2}_{q_0}\|^{2|\beta_i|/(n-1)} \quad \text{for } 1 \leq i \leq k, \qquad (2.19)$$

where C is independent of j and u (see [6, Lemma A.1]). Therefore we have for $k \geq 2$

$$\|f^{(k)}(u)\prod_{i=1}^{k}\partial^{\beta_i}u; L^{q_0'}\| \leq C\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\|u; \dot{B}^0_{r^*}\|^{2j+k-1}\|u; \dot{B}^{(n-1)/2}_{q_0}\|, \qquad (2.20)$$

which corresponds to (2.15), and by the same argument as above the inequality (2.11) follows, where the estimate on the terms with $k=1$ is similar and simpler.

For $n \geq 6$ even, we use the following equivalent norm on the homogeneous Besov space

$$\|u;\dot{B}_q^{(n-1)/2}\| \simeq \sum_{|\alpha|=(n-2)/2} \{\int_0^\infty t^{-2} \sup_{|y|<t} \|\partial^\alpha u - \partial^\alpha \tau_y u; L^q\|^2 dt\}^{1/2}, \quad (2.21)$$

where τ_y is the shift function by $y \in \mathbf{R}^n$. By (2.21), we have

$$\begin{aligned} &\|f(u);\dot{B}_{q_0'}^{(n-1)/2}\| \\ \leq\ & C\sum_{k=1}^{(n-2)/2} \sum_{|\alpha|=(n-1)/2} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ |\beta_i|\geq 1}} \{\int_0^\infty t^{-2} \sup_{|y|<t} \|f^{(k)}(u)\prod_{i=1}^k \partial^{\beta_i} u \\ &\qquad -f^{(k)}(\tau_y u)\prod_{i=1}^k \partial^{\beta_i}\tau_y u; L^{q_0'}\|^2 dt\}^{1/2}. \end{aligned} \quad (2.22)$$

Here we estimate (2.22) as follows. Let $1/r^*$, $1/\hat{r}$, $1/r_i^*$ be

$$\begin{aligned} &1/r^* \equiv 2/(n+1)(2j+k), \quad 1/\hat{r} \equiv (1-1/(n-1))/r^* + 1/(n-1)q_0, \\ &1/r_i^* \equiv (1-2|\beta_i|/(n-1))/r^* + 2|\beta_i|/(n-1)q_0 \ \text{ for } \ 1\leq i\leq k. \end{aligned}$$

Then we have by the Hölder inequality and an estimate similar to (2.19)

$$\begin{aligned} &\|(f^{(k)}(u)-f^{(k)}(\tau_y u))\prod_{i=1}^k \partial^{\beta_i} u; L^{q_0'}\| \\ \leq\ & C\sum_{j=0}^\infty \frac{\lambda^j}{j!} \||u|^{2j}(u-\tau_y u)\prod_{i=1}^k \partial^{\beta_i} u; L^{q_0'}\| \\ \leq\ & C\sum_{j=0}^\infty \frac{\lambda^j}{j!} \|u;\dot{B}_{r^*}^0\|^{2j+k-(n-2)/(n-1)} \|u;\dot{B}_{q_0}^{(n-1)/2}\|^{(n-2)/(n-1)} \\ &\qquad \cdot\|u-\tau_y u; L^{\hat{r}}\|. \end{aligned} \quad (2.23)$$

Therefore we have

$$\begin{aligned} &\{\int_0^\infty t^{-2} \sup_{|y|<t} \|(f^{(k)}(u)-f^{(k)}(\tau_y u))\prod_{i=1}^k \partial^{\beta_i} u; L^{q_0'}\|^2 dt\}^{1/2} \\ \leq\ & C\sum_{j=0}^\infty \frac{\lambda^j}{j!} \|u;\dot{B}_{r^*}^0\|^{2j+k} \|u;\dot{B}_{q_0}^{(n-1)/2}\|. \end{aligned} \quad (2.24)$$

For $k \geq 2$, let $1/r^* \equiv 2/(n+1)(2j+k-1)$ and let

$$1/\hat{r} \equiv (1-(2|\beta_1|+1)/(n-1))/r^* + (2|\beta_1|+1)/(n-1)q_0,$$
$$1/r_i^* \equiv (1-2|\beta_i|/(n-1))/r^* + 2|\beta_i|/(n-1)q_0 \quad \text{for} \quad 2 \leq i \leq k.$$

Then we have by an estimate similar to (2.23)

$$\begin{aligned} &\|f^{(k)}(u)(\partial^{\beta_1}u - \partial^{\beta_1}\tau_y u)\prod_{i=2}^{k}\partial^{\beta_i}u; L^{q_0'}\| \\ \leq\ & C\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\|u; \dot{B}^0_{r^*}\|^{2j+k-1-(n-2-2|\beta_1|)/(n-1)} \\ &\cdot\|u; \dot{B}^{(n-1)/2}_{q_0}\|^{(n-2-2|\beta_1|)/(n-1)}\|\partial^{\beta_1}u - \partial^{\beta_1}\tau_y u; L^{\hat{r}}\|. \end{aligned} \tag{2.25}$$

Therefore we have

$$\begin{aligned} &\{\int_0^{\infty} t^{-2}\sup\|f^{(k)}(u)(\partial^{\beta_1}u - \partial^{\beta_1}\tau_y u)\prod_{i=2}^{k}\partial^{\beta_i}u; L^{q_0'}\|^2 dt\}^{1/2} \\ \leq\ & C\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\|u; \dot{B}^0_{r^*}\|^{2j+k-1}\|u; \dot{B}^{(n-1)/2}_{q_0}\|. \end{aligned} \tag{2.26}$$

The inequalities (2.24) and (2.26) correspond to (2.15), so that we have the required inequality (2.11), where the estimate on the terms with $k=1$ is again similar and simpler. This completes the proof of lemma 2. 1 . □

We now turn to the proof of Theorem 1. 1 . By (2.8), (2.9) and Lemma 2. 1 , we have

$$\max\{\|\Phi(u); L^\infty(\mathbf{R}, \dot{H}^{n/2})\|, \|\Phi(u); L^q(\mathbf{R}, B_q^{(n-1)/2})\|, \|\Phi(u); L^q(\mathbf{R}, \dot{B}_q^0)\|\}$$
$$\leq C\|(\phi,\psi)\|_Y + C\rho(R)R^2, \tag{2.27}$$
$$d(\Phi(u),\Phi(v)) \leq C\rho(R)Rd(u,v), \tag{2.28}$$

for $u, v \in X(R)$, where C is independent of u and v. Therefore Φ becomes a contraction map on $X(R)$ if $\|(\phi,\psi)\|_Y$ and R are sufficiently small. The existence of asymptotic states and the continuity of the scattering operator follow by the standard argument (see [17]). □

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