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# A simple proof for monotonicity of entropy in the quadratic family

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## Abstract

We give a simple proof for monotonicity of topological entropy in the quadratic family. We use a spectral property of a Ruelle operator.

## 1 Introduction

In this paper, we give a simple proof of the following theorem due to Sullivan, Douady-Hubbard and Milnor ([6]) for the dynamics of real quadratic maps  $Q_t(x) = t - x^2$ ,  $\mathbf{R} \rightarrow \mathbf{R}$ .

**Theorem 1** *The topological entropy  $h_{\text{top}}(Q_t)$  of  $Q_t$  is monotone increasing with respect to the parameter  $t$ .*

First we reduce theorem 1 to the following claim by the kneading theory.

**Theorem 2** *If the orbit of the critical point 0 is periodic with prime period  $n$  for  $Q_t$  (i.e.  $Q_t^n(0) = 0$  while  $Q_t^i(0) \neq 0$  for  $0 < i < n$ ), then*

$$\frac{\frac{\partial}{\partial s}(Q_s^n(0))|_{s=t}}{DQ_t^{n-1}(Q_t(0))} > 0$$

where  $D$  denote the derivative w.r.t. the space coordinate  $x$ .

Then we prove this claim by making use of a spectral property of a Ruelle operator. We also show a similar inequality for Collet-Eckmann case.

**Theorem 3** *If  $Q_t$  satisfies Collet-Eckmann condition, that is,*

$$\liminf_{n \rightarrow \infty} \sqrt[n]{|DQ_t^n(Q_t(0))|} > 1, \quad (1)$$

then we have

$$\lim_{n \rightarrow \infty} \frac{\frac{\partial}{\partial s}(Q_s^n(0))|_{s=t}}{DQ_t^{n-1}(Q_t(0))} > 0. \quad (2)$$

This, together with a version of Jakobson's theorem in [8], yields

**Corollary 4** If  $1/4 < t < 2$  is a parameter for which  $Q_t$  satisfies (1) and

$$\lim_{n \rightarrow \infty} n^{-1} \log |Q_t^n(0)| = 0, \quad (3)$$

then  $t$  is a density point of the set  $S$  of parameters  $s$  for which  $Q_s$  satisfies (1) and (3). Moreover,

$$\lim_{\epsilon \rightarrow +0} \frac{\log \text{Leb}([t - \epsilon, t + \epsilon] \setminus S)}{\log \epsilon} = 2$$

where  $\text{Leb}$  denotes the Lebesgue measure.

Another aim of this paper is to clarify the correspondance between monotonicity of quadratic family and that of the family of tent maps  $T_t(x) = 1 - t|x|$ . As we shall show in the last section, we can prove the monotonicity for the family of tent maps by quite analogous manner.

**Acknowledgement:** The author thanks Duncan Sands for his suggesting theorem 3.

## 2 Proof of theorem 1 and 2

Let us remember some definitions from the kneading theory[1, 6]. Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be a unimodal map with 0 its turning point, i.e., a continuous map that is strictly monotone increasing on the left of 0 and strictly monotone decreasing on the right of 0. The kneading invariant of  $f$  is an infinite sequence  $K(f) = (e_1, e_2, \dots)$  of symbols  $L, C$  and  $R$ , defined by

$$e_k = \begin{cases} L, & \text{if } f^k(0) < 0; \\ C, & \text{if } f^k(0) = 0; \\ R, & \text{if } f^k(0) > 0, \end{cases}$$

For sequences of symbols  $L, C$  and  $R$ , we define the so-called signed lexicographical order  $\prec$  in the following manner: for three sequences whose first  $n$ -entries are same,

$$\begin{aligned} I_L &= (e_1, e_2, \dots, e_n, L, \dots), \\ I_C &= (e_1, e_2, \dots, e_n, C, \dots), \\ I_R &= (e_1, e_2, \dots, e_n, R, \dots), \end{aligned}$$

we define  $I_L \prec I_C \prec I_R$  if the number of the symbol  $R$  in  $\{e_1, e_2, \dots, e_n\}$  is even, and define  $I_R \prec I_C \prec I_L$  otherwise. The topological entropy of a unimodal map depends only on its kneading invariant, and the dependence is monotone with respect to the order above. Thus theorem 1 follows from

**Theorem 5**  $K(Q_t)$  is monotone increasing with respect to  $t$ .

But this follows from theorem 2. In fact, let  $K_n(Q_t)$  be the truncated sequence of the kneading invariant  $K(Q_t)$  of length  $n$ . Then  $K_n(Q_t)$  is locally constant with respect to  $t$  unless  $Q_t^i(0) = 0$  for some  $0 < i \leq n$ . When  $Q_t^i(0) = 0$  for some  $0 < i \leq n$ , the derivative  $DQ_t^{i-1}(Q_t(0))$  is positive if and only if the number of the symbol  $R$  in  $K_{i-1}(Q_t)$  is even.

From the definition of the signed lexicographical order, theorem 2 implies that  $K_n(Q_t)$  is increasing at each of such parameters. Hence  $K_n(Q_t)$  is monotone increasing with respect to  $t$ . Since this holds for every  $n$ , the kneading invariant  $K(Q_t)$  is monotone increasing.

Now we prove theorem 2. Assume  $Q_t^n(0) = 0$  and  $Q_t^i(0) \neq 0$  for  $0 < i < n$ . Let  $Q : \mathbb{C} \rightarrow \mathbb{C}$  be the complex extension of  $Q_t$ . A Ruelle operator  $R$  is defined by

$$R(\psi)(z) = \sum_{Q(y)=z} \frac{\psi(y)}{DQ(y)^2}$$

for a function  $\psi$  on the complex plane. Let  $\chi_k(z) = \frac{1}{(z-Q^k(0))}$  and let  $S$  be the linear space of functions spanned by  $\chi_k$ ,  $k = 1, 2, \dots, n$ . By simple calculation, we can check

$$R(\chi_i) = \frac{\chi_{i+1}}{DQ(Q^i(0))} - \frac{\chi_1}{DQ(Q^i(0))}. \quad (4)$$

Hence the Ruelle operator  $R$  preserves  $S$ . The representation matrix for  $R|_S$  with respect to the basis  $\{\chi_i\}_{i=1}^n$  is of the form

$$\Phi = \begin{pmatrix} -\frac{1}{DQ(Q(0))} & -\frac{1}{DQ(Q^2(0))} & -\frac{1}{DQ(Q^3(0))} & -\frac{1}{DQ(Q^4(0))} & \cdots \\ \frac{1}{DQ(Q(0))} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{DQ(Q^2(0))} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{DQ(Q^3(0))} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (5)$$

We find

$$\begin{aligned} & \det(Id - \Phi) \\ &= \det \begin{pmatrix} 1 + \frac{1}{DQ(Q(0))} & \frac{1}{DQ(Q^2(0))} & \frac{1}{DQ(Q^3(0))} & \cdots \\ -\frac{1}{DQ(Q(0))} & 1 & 0 & \cdots \\ 0 & -\frac{1}{DQ(Q^2(0))} & 1 & \cdots \\ 0 & 0 & -\frac{1}{DQ(Q^3(0))} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\ &= \sum_{k=1}^n \prod_{j=1}^k \frac{1}{DQ(Q^j(0))} = \sum_{k=1}^n \frac{1}{DQ^k(Q(0))} = \frac{\frac{\partial}{\partial s}(Q_s^n(0))|_{s=t}}{DQ_t^n(Q_t(0))}. \end{aligned} \quad (6)$$

Since  $\Phi$  is a matrix with real entries, theorem 2 follows from

**Lemma 6** *The spectral radius of  $\Phi$  is smaller than 1.*

[proof] Let us consider the Perron-Frobenius operator

$$P(\psi)(x) = \sum_{Q(y)=x} \left| \frac{\psi(y)}{DQ(y)^2} \right|.$$

Obviously we have  $|R^k(\psi)(x)| \leq P^k(|\psi|)(x)$  for  $k > 0$ . Take a large disk  $D$  such that  $Q^{-1}(D) \subset D$ , and put  $A = D - f^{-1}(D)$ . Then

$$\|\psi\| = \int_A |\psi| |dz|^2$$

is a norm on the finite dimensional vector space  $S$ . We have

$$\sum_{k=0}^{\infty} \int_A P^k(|\psi|) |dz|^2 = \sum_{k=0}^{\infty} \int_{f^{-k}(A)} |\psi| |dz|^2 \leq \int_D |\psi| |dz|^2 < \infty.$$

This implies  $\|R^k(\psi)\| \rightarrow 0$  and, hence, the lemma.

*Q.E.D.*

We finished the proof of theorem 1 and 2. Here we give a remark on the relation between the above proof and that in [6]. In the original proof in [6], they consider the so-called Thurston map which acts on the Teichmuller space of finitely-pointed sphere. The point of the proof is that Thurston map is a contraction with respect to Teichmuller metric. The space  $S$  in our proof above corresponds to the cotangent space of the Teichmuller space, and the operator  $R$  corresponds to the action of Thurston map on the cotangent space at the fixed point. (cf. [3], [7]) Hence our proof is a local version of that in [6] in a sense. We showed that the local contraction was enough.

The Ruelle operator  $R$  in our proof is interesting in itself. For example, Levin, Sodin and Yuditski studied, in [4] and [5], the properties of  $R$  in more general setting, and get some formulae similar to (6).

### 3 Proof of theorem 3

The proof of theorem 3 is similar to that for theorem 2 except for the point that the space of function we consider is infinite dimensional. Let us assume that  $Q_t$  satisfies Collet-Eckmann condition (1) and choose  $r > 1$  such that

$$\liminf_{n \rightarrow \infty} \sqrt[n]{|DQ_t^n(Q_t(0))|} > r.$$

Below we assume that the critical point is not pre-periodic. The case when the critical point is pre-periodic is treated at the end of this section. Let us consider a Banach space

$$S = \left\{ (a_k)_{k=1}^{\infty} \mid \sum_{k=1}^{\infty} |a_k| \cdot |DQ_t^k(Q_t(0))| \cdot r^{-k} < \infty \right\}$$

with a norm  $\| (a_k)_{k=1}^{\infty} \| = \sum_{k=1}^{\infty} |a_k| \cdot |DQ_t^k(Q_t(0))| \cdot r^{-k}$ . Let  $\chi_k(z) = 1/(z - Q_t^k(0))$ . For each  $(a_k)_{k=1}^{\infty} \in S$ , we consider a holomorphic function defined on  $\mathbb{C} \setminus \mathbb{R}$ ,

$$\Psi((a_k)_{k=1}^{\infty})(z) = \sum_{k=1}^{\infty} a_k \chi_k(z).$$

**Lemma 7**  $\Psi$  is a injective map from  $S$  to the set of holomorphic functions defined on  $\mathbb{C} \setminus \mathbb{R}$ .

[proof] Assume that  $(a_k)_{k=1}^{\infty} \in S$  is not null. Take  $a_n \neq 0$  and  $m$  such that  $\sum_{k>m} |a_k| < |a_n|$ . Then, for  $y \in \mathbb{R} - \{0\}$ ,

$$|\Psi((a_k)_{k=1}^{\infty})(Q_t^n(0) + y\sqrt{-1})| \geq \frac{|a_n|}{y} - \sum_{k>m} \frac{|a_k|}{y} - \sum_{k \leq m, k \neq n} \frac{|a_k|}{|Q_t^k(0) - Q_t^n(0)|}$$

Since the last term is bounded as  $t \rightarrow 0$ , we see that  $\Psi((a_k)_{k=1}^{\infty})$  is not constant 0. Q.E.D.

We consider the Ruelle operator  $R$  as in the last section. From (4) and the definition of the norm  $||| \cdot |||$ , it is easy to see that  $R$  preserve the image  $\Psi(S)$  and induce a continuous linear map  $\Phi : S \rightarrow S$  such that  $\Psi \circ \Phi = R \circ \Psi$ . Remark that  $\Phi$  is of the form (5).

**Lemma 8** The spectrum of  $\Phi$  contained in  $\{|z| > r^{-1}\}$  is discrete and the corresponding eigenspace is finite dimensional.

[proof] Let  $\pi_1 : S \rightarrow S$  be the projection to the first component:  $\pi_1((a_k)_{k=1}^{\infty}) = (a_1, 0, 0, \dots)$ . We have  $|||(id - \pi_1) \circ \Phi|| \leq r^{-1}$  from the definition of the norm. Since  $\pi_1 \circ \Phi$  has one dimensional range, the lemma follows from the general spectral theory. ([2], pp 709). Q.E.D.

As in the last section, the operator  $\Phi$  contract the norm induced from

$$||\psi|| = \int_{A-R} |\psi| |dz|^2.$$

on  $\Psi(S)$ . ( $A$  is the annulus taken in the last section.) Hence the discrete spectrum in the above lemma should be contained in the unit disk  $\{|z| < 1\}$ . Therefore we get

**Proposition 9** The spectral radius of  $\Phi$  is smaller than 1.

Let  $\pi_n : S \rightarrow S$  be the projection to the first  $n$  components. We approximate  $\Phi$  by finite dimensional operators  $\Phi_n = \pi_n \circ \Phi \circ \pi_n$ .

**Lemma 10** There exists  $N > 0$ ,  $M > 0$  and  $0 < \mu < 1$  such that  $|||\Phi_n^k||| < M\mu^k$  for any  $n > N$  and  $k > 0$ .

[proof] Let  $B_n = \Phi_n + (Id - \pi_n) \circ \Phi$ . Since  $\Phi_n^m = \pi_n \circ B_n^m$ , we have  $|||\Phi_n^m||| \leq |||B_n^m|||$ . For  $a = (a_k)_{k=1}^{\infty} \in S$ , we have

$$\begin{aligned} |||\Phi(a) - B_n(a)||| &= |||\pi_1 \circ \Phi \circ (Id - \pi_n)(a)||| \\ &\leq \sum_{k>n} |a_k DQ_t^k(Q_t(0)) \cdot r^{-k} \cdot \frac{r^k}{DQ_t^{k+1}(Q_t(0))}| \\ &\leq |||a||| \cdot \max_{k>n} \left\{ \left| \frac{r^k}{DQ_t^{k+1}(Q_t(0))} \right| \right\}. \end{aligned}$$

Hence  $|||\Phi - B_n||| \rightarrow 0$  as  $n \rightarrow \infty$ . Now the lemma follows from

$$\lim_{m \rightarrow \infty} \sqrt[m]{|||\Phi^m|||} < 1,$$

which is equivalent to the claim of proposition 9. ([2, pp.567]) *Q.E.D.*

By calculation, we see

$$\det(Id - \Phi_n) = \frac{\frac{\partial}{\partial s}(Q_s^n(0))|_{s=t}}{DQ_t^{n-1}(Q_t(0))}. \quad (7)$$

On the other hand, for sufficiently large  $n$ , we have

$$\det(Id - \Phi_n) = \exp(\text{Trace} \log(\Phi_n)) = \exp\left(-\sum_{k=1}^{\infty} k^{-1} \text{Trace}(\Phi_n^k)\right).$$

Notice that  $p$ -th diagonal element of the matrix  $\Phi_n^k$  is null for  $p > k$  because of the special form of  $\Phi$ . Hence we have

$$|\text{Trace}(\Phi_n^k)| \leq k \|\Phi_n^k\| < kM\mu^k$$

for  $n \geq N$ . Therefore we obtain

$$\lim_{n \rightarrow \infty} \frac{\frac{\partial}{\partial s}(Q_s^n(0))|_{s=t}}{DQ_t^{n-1}(Q_t(0))} \geq \exp\left(\sum_{k=1}^{\infty} -M\mu^k\right) > 0.$$

We finished the proof of theorem 3 in the case when the critical point is not pre-periodic. When the critical point is pre-periodic, the space  $S$  of functions spanned by  $\chi_k$ ,  $k = 1, 2, \dots$ , is finite dimensional. By simple calculation, we can check

$$\det(Id - R|_S) = \lim_{n \rightarrow \infty} \frac{\frac{\partial}{\partial s}(Q_s^n(0))|_{s=t}}{DQ_t^{n-1}(Q_t(0))}.$$

By arguing as in the last section, we see that the spectral radius of  $R|_S$  is smaller than 1 and obtain (2).

## 4 On the family of tent maps

Monotonicity of the entropy is well-known for the family of tent maps  $T_t(x) = 1 - t|x|$ , ( $1 < t \leq 2$ ).

**Theorem 11** *The kneading invariant  $K(T_t)$  is monotone increasing with respect to  $t$ . Hence, so is the topological entropy.*

Here we give a proof which is analogous to that of theorem 2. Assume that  $T_t^n(0) = 0$ . We write  $T = T_t$ . Notice that the interval  $[T^2(0), T(0)]$  is invariant for  $T$  and contains 0. We consider the Perron-Frobenius operator  $P$  for  $T$ :

$$P(\psi)(x) = \sum_{T(y)=x} \left| \frac{\psi(y)}{DT(y)} \right|.$$



Let  $\chi_k(x)$  be a function defined by

$$\chi_k(x) = \begin{cases} 1, & x > T^k(0); \\ 0, & x = T^k(0); \\ -1, & x < T^k(0). \end{cases}$$

Then we have the following formula, which is similar to (4):

$$P(\chi_k) = \frac{\chi_{k+1}}{DT(T^k(0))} - \frac{\chi_1}{DT(T^k(0))}.$$

Let  $S$  be a linear space of functions spanned by  $\{\chi_k(0)\}_{k=1}^n$ . Let  $S'$  be the set of functions  $\varphi \in S$  such that  $\varphi(x) = 0$  for  $x \notin [T^2(0), T(0)]$ . Obviously,  $P(S) = S'$ . Since  $P$  preserves the integrals of functions,  $P$  also preserve the subspace

$$S'' = \left\{ \varphi \in S' \mid \int_{\mathbf{R}} \varphi dx = 0 \right\}.$$

Spectral radius of  $P|_S$  is not greater than 1. Moreover, from the ergodic properties of  $T$ , we know that 1 is a simple eigenvalue for  $P|_S$  whose eigenvector corresponds to the unique absolutely continuous ergodic measure for  $T$ . Remark that  $P|_{S''}$  do not have 1 as its eigenvalue. Therefore we have  $\det(Id_S - P|_S) = 0$  and  $\det(Id_{S''} - P|_{S''}) > 0$ . Theorem 11 follows from

**Lemma 12**  $\det(Id_{S''} - P|_{S''}) = t \cdot \frac{\partial (T_s^{n-1}(0))|_{s=t}}{DT^n(1)}.$

[proof] Let us define  $\pi : S' \rightarrow S$  by

$$\pi(\psi) = \psi - \frac{\int_{\mathbf{R}} \psi dx}{2} \cdot \chi_1.$$

Obviously  $\pi \circ P|_{S''} = P|_{S''}$ . Since  $\int_{\mathbf{R}} P(\chi_1) dx = 2$ , we see  $P \circ \pi(S') \subset S''$  and  $(\pi \circ P)^2(S) \subset S''$ . Hence  $\det(Id_{S''} - P|_{S''}) = \det(Id_S - \pi \circ P|_S)$ . We can check

$$\pi \circ P(\chi_k) = \frac{\chi_{k+1}}{DT(T^k(0))} - \frac{1 - (T(0) - T^{k+1}(0))}{DT(T^k(0))} \chi_1.$$

Hence, the representation matrix of  $\pi \circ P|_S$  w.r.t. the basis  $\{\chi_k\}_{k=1}^n$  is

$$\Phi = \begin{pmatrix} -\frac{1-(T(0)-T^2(0))}{DT(T(0))} & -\frac{1-(T(0)-T^3(0))}{DT(T^2(0))} & -\frac{1-(T(0)-T^4(0))}{DT(T^3(0))} & \dots \\ \frac{1}{DT(T(0))} & 0 & 0 & \dots \\ 0 & \frac{1}{DT(T^2(0))} & 0 & \dots \\ 0 & 0 & \frac{1}{DT(T^3(0))} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since

$$\det(Id_S - P|_S) = \det \begin{pmatrix} 1 + \frac{1}{DT(T(0))} & \frac{1}{DT(T^2(0))} & \frac{1}{DT(T^3(0))} & \cdots \\ -\frac{1}{DT(T(0))} & 1 & 0 & \cdots \\ 0 & -\frac{1}{DT(T^2(0))} & 1 & \cdots \\ 0 & 0 & -\frac{1}{DT(T^3(0))} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

equals to 0, we get

$$\det(Id_S - \pi \circ P|_S) = \det \begin{pmatrix} -\frac{T(0) - T^2(0)}{DT(T(0))} & -\frac{T(0) - T^3(0)}{DT(T^2(0))} & -\frac{T(0) - T^4(0)}{DT(T^3(0))} & \cdots \\ -\frac{1}{DT(T(0))} & 1 & 0 & \cdots \\ 0 & -\frac{1}{DT(T^2(0))} & 1 & \cdots \\ 0 & 0 & -\frac{1}{DT(T^3(0))} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Calculating the left hand side, we obtain the lemma.

Q.E.D.

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