On strong maximum principle and large time behaviour of generalized mean curvature flow with the Neumann boundary condition

Y. Giga, M. Ohnuma and M.-H. Sato

Series #391. September 1997

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#366 A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, 40 pages. 1996.

#367 A. Arai, A class of representations of the *-algebra of the canonical commutation relations over a Hilbert space and instability of embedded eigenvalues in quantum field models, 12 pages. 1996.

#368 K. Ito, BV-solutions of a hyperbolic-elliptic system for a radiating gas, 33 pages. 1997.

#369 M. Nakamura and T. Ozawa, Nonlinear Schrödinger equations in the Sobolev space of critical order, 20 pages. 1997.

#370 N.H. Bingham and A. Inoue, An Abel-Tauber theorem for Hankel transforms, 8 pages. 1997.

#371 T. Nakazi and H. Sawada, The commutator ideal in Toeplitz algebras for uniform algebras and the analytic structure, 9 pages. 1997.

#372 M.-H. Giga and Y. Giga, Stability for evolving graphs by nonlocal weighted curvature, 70 pages. 1997.

#373 T. Nakazi, Brown-Halmos type theorems of weighted Toeplitz operators, 14 pages. 1997.

#374 J. Inoue and S.-E. Takahashi, On characterizations of the image of Gelfand transform of commutative Banach algebras, 30 pages. 1997.

#375 L. Solomon and H. Terao, The double coxeter arrangement, 21 pages. 1997.

#376 G. Ishikawa and T. Morimoto, Solution surfaces of Monge-Ampère equations, 15 pages. 1997.

#377 G. Ishikawa, A relative transversality theorem and its applications, 16 pages. 1997.

#378 J. Inoue and T. Nakazi, On the zeroes of solutions of an extremal problem in H^1, 14 pages. 1997.

#379 Y. Giga and K. Ito, On pinching of curves moved by surface diffusion, 12 pages. 1997.

#380 F. Hiroshima, Weak coupling limit removing an ultraviolet cut-off for a Hamiltonian of particles interacting with a scalar field, 39 pages. 1997.

#381 Y. Giga, S. Matsui and Y. Shimizu, On estimates in Hardy spaces for the Stokes flow in a half space, 13 pages. 1997.

#382 A. Arai, A new estimate for the ground state energy of Schrödinger operators, 12 pages. 1997.

#383 M. Nakamura and T. Ozawa, The Cauchy problem for nonlinear wave equations in the Sobolev space of critical order, 24 pages. 1997.

#384 K. Ito, Asymptotic stability of planar rarefaction wave for scalar viscous conservation law, 8 pages. 1997.

#385 A. Arai, Representation-theoretic aspects of two-dimensional quantum systems in singular vector potentials: canonical commutation relations, quantum algebras, and reduction to lattice quantum systems, 32 pages. 1997.

#386 P. Aviles and Y. Giga, On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg-Landau type energy for gradient fields, 21 pages. 1997.

#387 T. Nakazi and T. Yamamoto, Norms of some singular integral operators and their inverse operators, 28 pages. 1997.

#388 M.-H. Giga and Y. Giga, Remarks on convergence of evolving graphs by nonlocal curvature, 18 pages. 1997.

#389 T. Tsukada, Reticular Lagrangian singularities, 41 pages. 1997.

#390 M. Nakamura and T. Ozawa, The Cauchy problem for nonlinear wave equations in the homogeneous Sobolev space, 12 pages. 1997.

On strong maximum principle and large time behaviour of generalized mean curvature flow with the Neumann boundary condition

Yoshikazu Giga, Masaki Ohnuma
and
Moto-Hiko Sato

1. Introduction

We are concerned with the large time behaviour of solutions of the level set equation of the mean curvature flow equation with the (homogeneous) Neumann boundary condition in a cylindrical domain Ω of form

$$\Omega = \{x = (x', x_n) \in \mathbf{R}^n;\ x' \in \Omega',\ x_n \in \mathbf{R}\},$$

where Ω' is a smoothly bounded domain in $\mathbf{R}^{n-1}$ with $n \geq 2$. We consider

$$u_t - |\nabla u| \text{ div } (\nabla u / |\nabla u|) = 0 \quad \text{in } (0,\infty) \times \Omega, \tag{1.1a}$$

$$\partial u / \partial \nu = 0 \quad \text{on } (0,\infty) \times \partial\Omega, \tag{1.1b}$$

where ν denotes the outward unit normal of the boundary $\partial\Omega$. The system (1.1a)-(1.1b) is the level set equation of the mean curvature flow equation with the Neumann boundary condition since each level set of (1.1a)-(1.1b) moves by its mean curvature in Ω and it perpendicularly intersects $\partial\Omega$ at least formally. Even for such a surface evolution problem with boundary condition it is known that a level set method developed in [CGG], [ES] is applicable [GS1, GS2, S1, S2] (see also [G1], [G2]). Let us recall typical fundamental results. Consider (1.1a)-(1.1b) with the initial condition

$$u(0, x) = a(x). \tag{1.1c}$$

If a is continuous in $\overline{\Omega}$, i.e., $a \in C(\overline{\Omega})$ and equals a constant for large x_n, then there exists a unique global-in-time solution $u \in C([0,\infty) \times \overline{\Omega})$ of (1.1a)-(1.1b) with (1.1c) with the property that u is constant for large x_n. Moreover, c-level set (at time t)

$$\Gamma_t = \{x \in \overline{\Omega};\ u(t,x) = c\} \tag{1.2a}$$

is determined by the set

$$\Gamma_0 = \{x \in \overline{\Omega};\ a(x) = c\} \tag{1.2b}$$

Typeset by $\mathcal{AMS}$-TEX

and independent of the choice of a and c. The family $\{\Gamma_t\}_{t\geq 0}$ satisfying (1.2a)-(1.2b) is often called a *generalized interface evolution* by mean curvature with right angle boundary condition starting from Γ_0. Since for a given compact set Γ_0 in $\overline{\Omega}$ it is easy to find a (and c) satisfying (1.2b) and the requirement of global solvability of (1.1a)-(1.1c), there always exists a unique generalized interface evolution Γ_t (given by (1.2a)).

We are interested in the behaviour of Γ_t as time tends to infinity. If Γ_0 is the graph of a function on Ω', then there is a global-in-time graph-like smooth solution Γ_t of the mean curvature flow equation with right angle boundary condition starting from Γ_0. Moreover, the solution Γ_t converges to a hyperplane perpendicular to $\partial\Omega$ in C^∞ topology. These results are due to Huisken [H]. (Since it is not difficult to see that the smooth graph-like solution is a generalized interface evolution, we have used the symbol Γ_t to describe the solution obtained in [H].) It is interesting to study the large time behaviour of generalized interface evolution with a given initial (compact) hypersurface Γ_0 not necessarily a graph-like surface. It is too naive to guess that the limit of Γ_t as $t \to \infty$ is always a single hyperplane. Consider an initial hypersurface Γ_0 given by $r = r(x_n)$ where r is a distance from x_n-axis and Ω' is a ball in $\mathbf{R}^{n-1}$ centered at the origin. If $r = r(x_n)$ is an even convex function, we expect that Γ_t pinches in a finite time if $r(0)$ is very small so that Γ_0 has a thin neck near the origin of $\mathbf{R}^n$ provided that $n \geq 3$. Then it is natural to guess that Γ_t becomes two pieces and each piece converges to a different hyperplane. This suggests that the limit of Γ_t may consist of several hyperplanes perpendicular to $\partial\Omega$. As already pointed out in [ES] Γ_t may have interior even if Γ_0 has no interior; see also [G1], [G2] for the boundary value problems and references therein. This suggests that the limit of Γ_t may have interior. So the best we conjecture for general initial Γ_0 is that the limit of Γ_t as $t \to \infty$ is a closed set in D whose boundary consists of hyperplanes parallel to $\partial\Omega'$.

In this paper we prove a weaker version supporting the above conjecture. We list our main results.

Theorem 1.1(Convergence). *Assume that Ω' is a smoothly bounded convex domain in $\mathbf{R}^{n-1}$. Assume that $a \in C(\overline{\Omega})$ fulfills*

$$\begin{aligned} a(x', x_n) &= c_1 \quad \text{for} \quad x_n \geq m, \quad x' \in \overline{\Omega}', \\ a(x', x_n) &= c_2 \quad \text{for} \quad x_n \leq -m, \ x' \in \overline{\Omega}' \end{aligned} \tag{1.3}$$

with some constants $m > 0$, $c_1, c_2 \in \mathbf{R}$. Then the unique viscosity solution $u \in C([0,\infty) \times \overline{\Omega})$ of (1.1a)-(1.1c) satisfying (1.3) with the same m, c_1, c_2 at each time *converges uniformly on* $\overline{\Omega}$ *to a function* $v \in C(\overline{\Omega})$ *as* $t \to \infty$ *that satisfies the level set minimal surface equation with the Neumann condition*

$$-|\nabla v| \text{ div } (\nabla v/|\nabla v|) = 0 \quad \text{in } \Omega, \tag{1.4a}$$

$$\partial v/\partial \nu = 0 \quad \text{on } \partial\Omega \tag{1.4b}$$

in the viscosity sense. (If a *is Lipschitz continuous, so is* v*). Moreover,* v *fulfills (1.3).*

Remark.. The assertion is still valid for arbitrary smoothly bounded convex domain Ω not necessarily a cylinder in $\mathbf{R}^n$ except the statement related to (1.3).

Theorem 1.2 (Strong maximum principle). *Let* Ω' *be a smoothly bounded domain in* $\mathbf{R}^{n-1}$. *Assume that* $v \in C(\overline{\Omega})$ *is a viscosity solution of (1.4a)-(1.4b). If* $v(x', x_n)$ *is a constant for sufficiently large* x_n *(or* $-x_n$*), then* v *is independent of* x' *as a function in* $\overline{\Omega}$.

Combining Theorems 1.1 and 1.2 we have:

Theorem 1.3. *Under the same hypothesis of Theorem 1.1 the solution* $u(t,x)$ *converges to a function* $v = v(x_n)$ *(satisfying (1.3)) uniformly in* $\overline{\Omega}$ *as* $t \to \infty$. *In particular for each* $c \in \mathbf{R}$

$$\lim_{t \to \infty} \sup\{\text{dist } (x, \ \Gamma_\infty); \ x \in \Gamma_t\} = 0 \tag{1.5}$$

with

$$\Gamma_\infty = \{(x', x_n) \in \mathbf{R}^n; \ v(x_n) = c, \ x' \in \overline{\Omega}\},$$
$$\Gamma_t = \{(x', x_n) \in \mathbf{R}^n; \ u(t, x', x_n) = c\},$$

where $\text{dist } (x, A) = \inf \{|x - y|; \ y \in A\}$.

We conjecture that

$$\lim_{t \to \infty} \sup\{\text{dist } (y, \Gamma_t); \ y \in \Gamma_\infty\} = 0. \tag{1.6}$$

In the last remark of this paper we prove (1.6) when Γ_∞ consists of a finite collection of parallel hyperplanes (perpendicular to x_n-axis). If (1.6) is proved, combining (1.5) and (1.6) implies that Γ_t converges to Γ_∞ in the topology of the Hausdorff distance as $t \to \infty$.

To show our convergence theorem we consider an approximate equation

$$u_t - (\varepsilon^2 + |\nabla u|^2)^{1/2} \operatorname{div}(\nabla u/(\varepsilon^2 + |\nabla u|^2)^{1/2}) = 0, \tag{1.7a}$$

$$\partial u/\partial \nu = 0 \tag{1.7b}$$

with $\varepsilon > 0$ for smoother initial data a, say $a \in C^2(\overline{\Omega})$. (We may assume that Ω is bounded for our purpose.) Key ingredients to show the convergence are an a priori gradient bound for solution u^ε of (1.7a)-(1.7b):

$$\sup_{x,t} |\nabla u^\varepsilon|(t,x) \le C \tag{1.8}$$

with C independent of ε and the existence of a Lyapunov function which together with (1.8) yields

$$\int_\Omega \int_0^\infty |u_t^\varepsilon(t,x)|^2 dx dt \le C' \tag{1.9}$$

with C' independent of ε. To get (1.8) we are forced to assume that Ω is convex. In [H] a priori gradient bound for (1.7a)-(1.7b) with $\varepsilon = 1$ has been established for general (bounded) domain Ω. By rescaling the dependent variable his estimate provides a gradient estimate but it depends on ε if we write in the form of (1.8). As remarked in §2 it turns out that (1.8) may not hold for general bounded non-convex domain Ω with smooth boundary. Once we get convergence for C^2 initial data a, the extension to general a is obtained by comparison principle. Similar convergence result is known for the Dirichlet problem of (1.1a) when Ω is bounded, mean-convex domain [StZ], [IStZ]. Our general strategy is similar to theirs.

Our strong maximum principle (Theorem 1.2) asserts that all level set of the limit function of $u(t,\cdot)$ as $t \to \infty$ is perpendicular to x_n-axis. The assumption that $v(x', x_n)$ equals a constant for large x_n cannot be removed since $v(x) = x_1$ is always a viscosity solution of the boundary value problem with $\Omega' = (0,1)$ by definition [S1]. The proof of Theorem 1.2 based on propagation of maximum point in the tangential direction of level sets and the normal direction to the boundary $\partial\Omega$. Such propagation of maximum for degenerate nonlinear elliptic equations like (1.4a)-(1.4b) seems to be new and its proof is a natural generalization of that of the classical strong maximum principle explained for example in [PW], [GT]. For uniformly elliptic equations maximum point propagates to all directions and this property called strong maximum principle has been extended to viscosity solutions; see [KK] and references cited there. There are several strong maximum principles for minimal surfaces with singularities [I], [SoW] but their results are for rectifiable currents and may not be comparable to ours. They do not treat the boundary version.

There are several analysis on the large time behaviour of mean curvature flow in a cylinder with precribed contact angle as in [H], [AW1, 2], [Gu] but all results are for graph-like surface. It would be interesting to extend these results for general non graph-like surface.

2. Convergence

We consider the initial value problem for the approximate equation (1.7a)-(1.7b) (with $\varepsilon > 0$) in a smoothly bounded domain in $\mathbf{R}^n$:

$$u_t - (\varepsilon^2 + |\nabla u|^2)^{1/2} \operatorname{div} (\nabla u/(\varepsilon^2 + |\nabla u|^2)^{1/2}) = 0 \quad \text{in} \quad (0,\infty) \times D, \qquad (2.1a)$$

$$\partial u/\partial \nu = 0 \quad \text{on} \quad (0,\infty) \times \partial D, \qquad (2.1b)$$

$$u(0,x) = a(x) \quad \text{on} \quad \overline{D}. \qquad (2.1c)$$

If we set $w = u/\varepsilon$, then w solves

$$w_t - (1 + |\nabla w|^2)^{1/2} \operatorname{div} (\nabla w/(1 + |\nabla w|^2)^{1/2}) = 0 \quad \text{in} \quad (0,\infty) \times D, \qquad (2.2a)$$

$$\partial w/\partial \nu = 0 \quad \text{on} \quad (0,\infty) \times \partial D, \qquad (2.2b)$$

$$w(0,x) = a(x)/\varepsilon \quad \text{on} \quad \overline{D}. \qquad (2.2c)$$

The equations (2.2a)-(2.2b) describe motion of the graph of w moved by the mean curvature in a cylindrical domain $D' \times \mathbf{R}$ with right contact angle on $\partial(D' \times \mathbf{R})$. The existence of unique global smooth solution as well as its large time behaviour is established in [H] at least for $a \in C^\infty(\overline{D})$. Thus (2.1a)-(2.1c) admits a unique global smooth solution for each $\varepsilon > 0$. We shall derive several estimates for u^ε independent of ε. For vectors and tensors f we set $||f||_\infty^2 = \sup_D(\sum |f_j(x)|^2)$, where the sum is over all components f_j of f. By Δa and $\nabla\nabla a$ we mean the Laplacian and the Hessian of a, respectively.

Proposition 2.1(Gradient and time derivative bound). *Let u^ε be a smooth solution of (2.1a)-(2.1c) with $a \in C^2(\overline{D})$. Then*

$$\sup_{t>0} ||u_t^\varepsilon||_\infty(t) \le ||\Delta a||_\infty + ||\nabla\nabla a||_\infty.$$

If moreover D is convex, then

$$\sup_{t>0} ||\nabla u^\varepsilon||_\infty(t) \le ||\nabla a||_\infty.$$

Proof. Although the proof is well-known, (e.g. [Sp]) we give it for completeness. Differentiate in time in (2.1a) and set $v = u_t^\varepsilon$. Then v solves a linear parabolic differential equation with $\partial v/\partial\nu = 0$ on ∂D. Since u^ε is C^2 up to $t = 0$ (except on $\{0\} \times \partial D$), then

$$\begin{aligned} v(0,x) &= (\varepsilon^2 + |\nabla a|^2)^{1/2} \operatorname{div} (\nabla a/(\varepsilon^2 + |\nabla a|^2)^{1/2}) \\ &= \Delta a - \sum_{1\le i,j\le n} a_{ij}a_ia_j/(\varepsilon^2 + |\nabla a|^2), \end{aligned}$$

where $a_j = \partial a/\partial x_j$, $a_{ij} = \partial^2 a/\partial x_i \partial x_j$. By the maximum principle

$$||v||_\infty(t) \le ||v||_\infty(0) \le ||\Delta a||_\infty + ||\nabla\nabla a||_\infty.$$

By a standard argument we see that $W = |\nabla u|^2 (= \sum_{j=1}^n u_j^2)$ is a subsolution of a linear homogeneous parabolic differential equation with no zero-th order term. Since $\partial u/\partial\nu = 0$, we observe that

$$\begin{aligned} \partial W/\partial\nu &= \langle \nu, \nabla\, |\nabla u|^2\rangle \\ &= 2 \sum_{1\le j,k\le n} \left\{ \left(\frac{\partial}{\partial x_j}(\nu_k u_k) \right) u_j - \frac{\partial \nu_k}{\partial x_j} u_k u_j \right\} \\ &= -2 \sum_{1\le j,k\le n} \frac{\partial \nu_k}{\partial x_j} u_k u_j. \end{aligned}$$

Note that $\sum_{j=1}^n u_j\, \partial/\partial x_j$ is tangential differentiation, since $\partial u/\partial\nu = 0$, so the terms involving $\partial\nu_k/\partial x_j$ is well defined if we take the sum over j. Since D is convex and $\nu = (\nu_1, \cdots \nu_n)$ is an outward unit normal, the last term is nonpositive; see [Sp, Theorem 9.7]. Thus the maximum principle yields

$$||W||_\infty(t) \le ||W||_\infty(0) = ||\nabla a||_\infty^2. \quad \square$$

Proposition 2.2 (Integral bound). *Let u^ε be a smooth solution of (2.1a)-(2.1c) with $a \in C^2(\overline{D})$. Then*

$$\int_0^T \int_D \frac{(u_t^\varepsilon(t,x))^2}{(\varepsilon^2 + |\nabla u^\varepsilon(t,x)|^2)^{1/2}} dx dt \le \int_D (\varepsilon^2 + |\nabla a|^2)^{1/2} dx.$$

Proof. The proof is rather standard. We give it for completeness. We shall show that

$$J(t) = \int_D (\varepsilon^2 + |\nabla u^\varepsilon(t,x)|^2)^{1/2} dx$$

plays a role of Lyapunov function in our system as in [StZ]. Integrating by parts with (2.1b) yields

$$\begin{aligned} \frac{d}{dt} J(t) &= \int_D \frac{\langle \nabla u^\varepsilon, \nabla u_t^\varepsilon \rangle}{(\varepsilon^2 + |\nabla u^\varepsilon|^2)^{1/2}} dx \\ &= -\int_D \operatorname{div}\left(\frac{\nabla u^\varepsilon}{(\varepsilon^2 + |\nabla u^\varepsilon|^2)^{1/2}} \right) u_t^\varepsilon dx. \end{aligned}$$

It now follows from (2.1a) that

$$\frac{d}{dt} J(t) + \int_D \frac{(u_t^\varepsilon)^2}{(\varepsilon^2 + |\nabla u^\varepsilon|^2)^{1/2}} dx = 0.$$

Integrate over $(0,T)$ to get

$$\int_0^T \int_D \frac{(u_t^\varepsilon)^2}{(\varepsilon^2 + |\nabla u^\varepsilon|^2)^{1/2}} dx dt = J(0) - J(T) \le J(0). \quad \square$$

We apply Propositions 2.1 and 2.2 to get several properties of solutions of (1.1a)-(1.1c).

Proposition 2.3. *Let D be a smoothly bounded convex domain in $\mathbf{R}^n$. Let u^ε be a smooth solution of (2.1a)-(2.1c) with $a \in C^2(\overline{D})$.*

(i) u^ε converges to a unique viscosity solution $u \in C([0,\infty) \times \overline{D})$ of (1.1a)-(1.1c) with $\Omega = D$ locally uniformly on $[0,\infty) \times \overline{D}$ as $\varepsilon \to 0$.

(ii) Let Ω' be a smoothly bounded convex domain in $\mathbf{R}^{n-1}$. Assume that

$$D \cap \{|x_n| < m+1\} = \Omega' \times (-m-1, m+1)$$

and that $a \in C^2(\overline{\Omega})$ satisfies (1.3) with $\Omega = \Omega' \times \mathbf{R}$. Then the limit u of u^ε fulfills (1.3) with the same m, c_1, c_2 at each time. (so that u is extended to a unique viscosity solution $u \in C([0,\infty) \times \overline{\Omega})$ as a constant function outside $\overline{D}$.)

(iii) u fulfills

$$\int_0^\infty \int_D (u_t(t,x))^2 dx dt \le \|\nabla a\|_\infty^2 \int_D dx. \tag{2.3}$$

Proof. (i) By Proposition 2.1 and Ascoli-Arzela's theorem u^ε converges to a function u locally uniformly in $(0,\infty) \times \overline{D}$ as $\varepsilon \to 0$ by taking a subsequence. By the stability of viscosity solutions (e.g. [CGG]) u is a viscosity solution of

$$u_t - |\nabla u| \operatorname{div} (\nabla u / |\nabla u|) = 0 \quad \text{in} \quad (0,\infty) \times D, \tag{2.4a}$$

$$\partial u / \partial \nu = 0 \quad \text{on} \quad (0,\infty) \times \partial D \tag{2.4b}$$

with $u(0,x) = a(x)$ on $\overline{D}$. By the comparison theorem [GS1, GS2], the solution u is unique so that u^ε converges to u without taking a subsequence.

(ii) Since a satisfies (1.3) there are continuous functions a^+ and a^- depending only on x_n that satisfies $a^- \leq a \leq a^+$ in $\overline{D}$ and $a^- = a = a^+$ for $|x_n| \geq m$. Since a^- and a^+ are stationary solution of (2.4a)-(2.4b), we see that $a^- \leq u(t,\cdot) \leq a^+$ for all $t \geq 0$ by the comparison theorem [GS1, GS2]. This implies our $u \in C([0,\infty) \times \overline{D})$ satisfies the property (1.3) at each time. We extend u outside $\overline{D}$ as a constant function so that $u \in C([0,\infty) \times \overline{\Omega})$. Then u has all desired property of (ii).

(iii) Since $||\nabla u^\varepsilon||_\infty(t) \leq ||\nabla a||_\infty$ by Proposition 2.1, Proposition 2.2 yields

$$\int_0^T \int_D (u_t^\varepsilon(t,x))^2 dx dt \leq (||\nabla a||_\infty^2 + \varepsilon^2)^{1/2} \int_D (\varepsilon^2 + |\nabla a|^2)^{1/2} dx.$$

This implies u_t^ε converges the distributional derivative u_t weakly in $L^2((0,T) \times D)$ as $\varepsilon \to 0$ and

$$\int_0^T \int_D u_t^2 dx dt \leq ||\nabla a||_\infty^2 \int_D dx$$

by lower semicontinuity. Thus (2.3) follows. □

Proof of Theorem 1.1. The proof as well as that of Remark is essentially the same as in [IStZ] once (2.3) is established. We give the proof for completeness. We take a smoothly bounded convex domain D as in Proposition 2.3 (ii) and assume that $a \in C^2(\overline{D})$. Then there is a unique smooth solution u^ε of (2.1a)-(2.1c) [H]. As in Proposition 2.3 a unique solution u of (1.1a)-(1.1c) is given by the limit of u^ε as $\varepsilon \to 0$. We consider the sequence of function $\{u_k\}$ defined on $\overline{D}_1$ with

$$u_k(t,x) = u(k+t,x), \quad k = 1,2,\cdots$$

where $D_1 = (0,1) \times D$. By Proposition 2.1 and Ascoli-Arzela's theorem there is a subsequence $\{u_{k_j}\}_{j=1}^\infty$ converges to a function $v \in C(\overline{D}_1)$ uniformly as $j \to \infty$. By the stability v is still a viscosity solution of

$$u_t - |\nabla u| \operatorname{div} (\nabla u/|\nabla u|) = 0 \quad \text{in} \quad D_1, \tag{2.5a}$$

$$\partial u/\partial \nu = 0 \quad \text{on} \quad (0,1) \times \partial D. \tag{2.5b}$$

It remains to argue that v is independent of t and $u(t,\cdot)$ converges to v uniformly along the full sequence of times. The uniform convergence in particular implies that $(u_{k_j})_t$ converges to the distributional derivative v_t in distribution sense. By Proposition 2.3 (iii) we see

$$\lim_{j\to\infty} \int_0^1 \int_D ((u_{k_j})_t)^2 dx dt = 0.$$

Thus $(u_{k_j})_t \to 0$ weakly in $L^2(D_1)$ so that $v_t = 0$ in the distribution sense, i.e.,

$$\int_D \int_0^1 v \frac{\partial \varphi}{\partial t} dx dt = 0$$

for all φ of the form $\varphi = \psi\xi$, $\psi \in C_0^\infty(D)$, $\xi \in C_0^\infty(0,1)$. Since φ is arbitrary, this implies

$$\int_0^1 v(t,x) \frac{\partial \xi}{\partial t}(t) dt = 0$$

for each $x \in D$ and $\xi \in C_0^\infty(0,1)$. Thus the distributional time derivative of $v(\cdot, x)$ equals zero, so that $v(t,x)$ is constant in t for each x. Since v is continuous, v is independent of t. Thus, v is a solution of (1.4a)-(1.4b). Since u_{k_j} converges uniformly to v, for each $\varepsilon > 0$ there is sufficiently large j that satisfies

$$|u_{k_j}(t,x) - v(x)| < \varepsilon$$

for all $(t,x) \in \overline{D}_1$. Setting $t = 0$ implies

$$v(x) - \varepsilon < u_{k_j}(0,x) < v(x) + \varepsilon$$

for all $x \in \overline{D}$. Since $v \pm \varepsilon$ is a time-independent viscosity solution of (2.5a)-(2.5b), by the comparison

$$v(x) - \varepsilon \leq u_{k_j}(t,x) = u(t + k_j, x) \leq v(x) + \varepsilon$$

for $t \in [0, \infty), x \in \overline{D}$. Thus shows the uniform convergence of $u(t,x)$ to $v(x)$ as $t \to \infty$ in $\overline{D}$.

We shall prove the convergence for general $a \in C(\overline{\Omega})$. Since (1.1a)-(1.1b) is invariant under addition of constants, the comparison implies the contraction

$$||u^1 - u^2||_\infty(t) \leq ||a^1 - a^2||_\infty \tag{2.6}$$

for all $t \geq 0$ (see e.g. [ESS, 2.14]), where u^i is the solution of (1.1a)-(1.1b) with initial data $a^i \in C(\overline{\Omega})$ (satisfying (1.3)). If a is only continuous on $\overline{\Omega}$ (satisfying (1.3)), we approximate a by $a_\varepsilon^\pm \in C^2(\overline{D})$ (satisfying (1.3)) such that $a_\varepsilon^- < a < a_\varepsilon^+$ and $||a_\varepsilon^\pm - a||_\infty \leq \varepsilon$. Let $u_\varepsilon^\pm$ be the solution of (1.1a)-(1.1b) with initial data $a_\varepsilon^\pm$ and let u be the solution of (1.1a)-(1.1c). By the comparison $u_\varepsilon^- \leq u \leq u_\varepsilon^+$. Since $u_\varepsilon^\pm$ converges to a function $v_\varepsilon^\pm \in C(\overline{\Omega})$ uniformly in $\overline{\Omega}$,

$$(\limsup_{t\to\infty}{}^* u)(x) = \lim_{t\to\infty} \sup\{u(\tau, y);\ \tau \geq t,\ |x - y| \leq 1/t\} \leq v_\varepsilon^+(x),$$
$$(\limsup_{t\to\infty}{}^*(-u))(x) \leq -v_\varepsilon^-(x).$$

By the contraction (2.6) we see $||v_\varepsilon^- - v_\varepsilon^+||_\infty \le ||a_\varepsilon^- - a_\varepsilon^+||_\infty \le 2\varepsilon$. Since ε is arbitrary this now implies

$$\limsup_{t\to\infty}{}^* u + \limsup_{t\to\infty}{}^*(-u) = 0,$$

which yields the uniform converges of u as $t \to \infty$. The bound $||\nabla u||_\infty(t) \le ||\nabla a||_\infty$ follows from Proposition 2.1. □

Remark (Gradient estimate). It is not difficult to see that the solution u of (2.4a)-(2.4b) fulfills

$$\sup_{0\le t\le T} ||\nabla u||_\infty(t) = C_T < \infty \quad \text{for} \quad T < \infty \tag{2.7}$$

provided that the initial data a is Lipschitz. However, there is a non-convex (but mean-convex for $n \ge 3$) domain D (with smooth boundary) and a smooth initial data such that

$$\lim_{T\to\infty} C_T = \infty. \tag{2.8}$$

In particular the uniform gradient estimate (1.8) may not hold in general. Here is an example of a domain. For $\delta > 0$ let D_δ be an axisymmetric domain of form

$$D_\delta = \{(x', x_n) \in \mathbf{R}^n;\ x_n^2 + |x'|^2 < 1 \quad \text{or} \quad |x'| < \delta \quad \text{for} \quad |x_n| < 2\}.$$

Mollifying nonsmooth part and taking $\delta > 0$ small we obtain a non-convex domain (but mean-convex for $n \ge 3$) of form

$$D = \{(x', x_n) \in \mathbf{R}^n;\ |x'| < r(x_n),\ |x_n| < 2\}$$

with the property that

(i) $r(x_n) = r(-x_n)$,
(ii) $r'(x_n) = 0$ for $x_n \in [\alpha, \beta] \subset (0, 2)$,
(iii) $r'(x_n) < 0$ for $x_n \notin [\alpha, \beta],\ 0 < x_n < 2$,

where (α, β) is a neighborhood of $x_n = 1$. We take initial data $a(x) = \max(x_n, 0)$. The level set $x_n = 0$ is an unstable stationary solution of the mean curvature flow equation with right angle boundary condition while $x_n = \alpha$ is a stationary solution attracting surfaces located in its left neighborhood. The level set $x_n = q$ with $0 < q < \alpha$ is not stationary since it does not fulfill the boundary condition. As time develop all q-level set is attracted to $x_n = \alpha$. This yields

$$\lim_{t\to\infty} u(t,x) = \begin{cases} 0 & x_n < \alpha \\ \alpha & x_n = \alpha \end{cases}$$

for the solution u of (2.4a)-(2.4b). This evidently implies (2.8). The point of this example of a is that a has an attracting stationary solution as a level set.

The proof of (2.7) is standard. We give a formal proof for the reader's convenience. We take a nonnegative $\varphi \in C^2(\overline{D})$ satisfying $\partial\varphi/\partial\nu \geq 1$ on ∂D. We set $w = |\nabla u|^2 e^{-M\varphi}$ with $M = 2||\varphi||_{C^2(\overline{D})}$, where u solves (2.4a)-(2.4b). By the choice of M we see $\partial w/\partial\nu \leq 0$ on ∂D. In D w is a subsolution of a linear homogeneous parabolic equation with bounded coefficients, so the maximum principle yields

$$w(t,x) \leq e^{CT} \sup_x |w(0,x)|, \ (t,x) \in (0,T) \times \overline{D}$$

with some C depending only on φ through M. This yields a bound for C_T. To justify this calculation we should carry out the same procedure for solution u^ε of the approximate system (2.1a)-(2.1b). Recently, a time global gradient bound is provided for a little bit different equation in [ESS]. A typical example contained in [ESS] is

$$u_t - |\nabla u|(\mathrm{div}(\nabla u/|\nabla u|) - \mu - u) = 0,$$
$$\partial u/\partial\nu = 0,$$

where μ is a constant. They proved $\overline{\lim}_{T\to\infty} C_T < \infty$ for their solution for arbitrary domain. The term $-u$ plays an important role since if this term is missing we may have $\lim_{T\to\infty} C_T = \infty$ even for $\mu = 0$.

3. Strong maximum principle for the level set minimal surface equation

We prepare two lemmas on propagation of maximum points for the level set minimal surface equations to prove Theorem 1.2.

Lemma 3.1 (Propagation of maximum, interior version). *Let D' be a domain in $\mathbf{R}^{n-1}$ and let $D = D' \times (\alpha,\beta)$ with $\alpha,\beta \in \mathbf{R}$. Let w be an upper semicontinuous viscosity subsolution of*

$$-|\nabla w|\mathrm{div}\ (\nabla w/|\nabla w|) = 0 \quad \text{in } D.$$

Assume that w attain its maximum K in D.

Let $M \in \mathbf{R}$ be of form

$$M = \sup\{x_n \in (\alpha,\beta);\ w(x',x_n) = K \quad \text{for some } x' \in D'\}$$

If $M < \beta$ and $w(\cdot, M)$ attains its maximum K at some (interior) point $\xi' \in D'$, then $w(x',M) = K$ for all $x' \in D'$.

Lemma 3.2 (Boundary version). *Let D and D' be as in Lemma 3.1. Assume that $\partial D'$ is C^2. Let w be an upper semicontinuous viscosity subsolution of*

$$-|\nabla w| \operatorname{div}(\nabla w/|\nabla w|) = 0 \quad in \quad D,$$
$$\partial w/\partial \nu = 0 \quad on \quad \partial D' \times (\alpha, \beta).$$

Assume that w attains its maximum K in $\overline{D}$. Let $M \in \mathbf{R}$ be of form

$$M = \sup\{x_n \in (\alpha, \beta);\ w(x', x_n) = K \quad \text{for some} \quad x' \in \overline{D}'\}.$$

If $M < \beta$ and $w(\cdot, M)$ attains its maximum K at some point $\xi' \in \partial D'$, then $w(x', M) = K$ for all $x' \in \overline{D}'$.

Proof of Theorem 1.2 admitting Lemmas 3.1 and 3.2. We may assume that $v = v(x', x_n)$ is a constant c_1 for sufficiently large x_n, say $x_n \geq m$. We set

$$A_\lambda^+ = \{x \in \overline{\Omega};\ v(x) \geq \lambda\},\ A_\lambda^- = \{x \in \overline{\Omega};\ v(x) \leq \lambda\}.$$

To show that v is independent of x', it suffices to prove that A_λ^+ and A_λ^- are perpendicular to x_n-axis for all $\lambda > c_1$ and $\lambda < c_1$, respectively. Here a set A in $\overline{\Omega}$ is called *perpendicular* to x_n-axis if $(x', x_n) \in A$ for some $x' \in \overline{\Omega}'$ implies $(z, x_n) \in A$ for all $z \in \overline{\Omega}'$.

We shall only give a proof that A_λ^+ is perpendicular to x_n-axis for all $\lambda > c_1$ since the proof for A_λ^- is symmetric by taking $-v$ instead of v. We may assume that $\lambda = 0, c_1 < 0$ by replacing v by $v - \lambda$. We may assume that $v \leq 0$ on $\overline{\Omega}$ by replacing v by min $(v, 0)$ since (1.1a)-(1.1b) is geometric so that min $(v, 0)$ is still a viscosity solution of (1.1a)-(1.1b) [CGG, S1]. By these reduction it suffices to prove that

$$A_0^+ = \{x \in \overline{\Omega};\ v(x) = 0\}$$

is perpendicular to x_n-axis, when $v \leq 0$ on $\overline{\Omega}$ and $v = c_1 < 0$ for $x_n \geq m$. We may assume that A_0^+ is nonempty.

Let Σ be the projection of A_0^+ on x_n-axis, i.e.,

$$\Sigma = \{x_n \in \mathbf{R};\ (x', x_n) \in A_0^+\}.$$

Since $\overline{\Omega}'$ is compact and A_0^+ is closed by continuity of v, it is easy to see that Σ is a closed set in $\mathbf{R}$. Since $v = c_1 < 0$ for $x_n \geq m$, Σ is bounded from above. Since Σ

is closed, there exist at most countably many disjoint open intervals $\{I_j\}_{j=1}^r$, $1 \leq r \leq \infty$ with the property

$$\mathbf{R}\backslash\Sigma = \bigcup_{j=1}^{r} I_j, \ I_1 \supset [m, \infty).$$

We shall prove for each j that

(i) if $\hat{x}_n \in \partial I_j$, then $v(x', \hat{x}_n) = 0$ for all $x' \in \overline{\Omega}'$, i.e., $(x', \hat{x}_n) \in A_0^+$ for all $x' \in \overline{\Omega}'$;
(ii) if $\hat{x}_n \in$ int Σ, i.e., $\hat{x}_n$ is an interior point of Σ, then $(x', \hat{x}_n) \in A_0^+$ for all $x' \in \overline{\Omega}'$.

Clearly, (i) and (ii) imply that A_0^+ is perpendicular to x_n-axis.

Proof of (i). If $\hat{x}_n$ is the left end point of I_j, we take $\alpha < \beta$ so that $\alpha <$ inf $I_j = \hat{x}_n < \beta < \sup I_j$ and set

$$M = \sup\{x_n \in (\alpha, \beta); \ v(x', x_n) = 0 \quad \text{for some } x' \in \overline{\Omega}'\}.$$

By the choice of α and β it is clear that $\alpha < M = \hat{x}_n < \beta$. We now apply Lemma 3.1 and 3.2 by taking $D' = \Omega'$ to get

$$v(x', \hat{x}_n) = 0 \quad \text{for all } x' \in \overline{\Omega}'.$$

If $\hat{x}_n$ is the right end point of I_j, we consider $w(x', x_n) = v(x', -x_n)$ and reduce the argument to the case of left end points. Note that so far we have only use the fact that v is a subsolution of (1.1a)-(1.1b).

Proof of (ii). We argue by contradiction. Suppose that there were a point $(\hat{x}', \hat{x}_n) \in \overline{\Omega}$ with $\hat{x}_n \in$ int Σ and $v(\hat{x}', \hat{x}_n) = -\lambda_0 < 0$. Let A be the maximal closed interval that satisfies $A \subset \Sigma$ and $\hat{x}_n \in A$. Since $\hat{x}_n$ is an interior point of Σ, A is not a singleton. Since Σ is bounded from above, $q = \sup A$ is finite. Since $q \notin$ int Σ by maximality, we see that q belongs to the closure of $\cup_{j=1}^r \partial I_j$. (Indeed, for any ε-open neighborhood U_ε of q, there is I_j intersecting U_ε. Since U_ε is connected and $q \notin I_j$, U_ε intersects ∂I_j.) By (i) and the continuity of v we conclude that

$$v(x', q) = 0 \quad \text{for all } x' \in \overline{\Omega}'.$$

Since v is continuous and $v(\hat{x}', \hat{x}_n) = \lambda_0$ with $\hat{x}_n \in A$, this vanishing property of v on $x_n = q$ implies that $\sup S < q$ for

$$S = \{x_n \in A \subset \Sigma; \ v(x', x_n) \leq -\lambda_0 \quad \text{for some} \quad x' \in \overline{\Omega}'\}$$

since v is continuous and vanishes on $x_n = q$. Since (1.1a)-(1.1b) is geometric, $\tilde{v} = \max(v, -\lambda_0) + \lambda_0$ still solves (1.1a)-(1.1b) by invariance [CGG, S1]. We now invoke the assumption that v is a supersolution of (1.1a)-(1.1b) so that $w = -\tilde{v}$ a (nonpositive) subsolution of (1.1a)-(1.1b). Since $M = \sup S$ with

$$M := \sup\{x_n \in A;\ w(x', x_n) = 0 \quad \text{for some} \quad x' \in \overline{\Omega}'\}$$

and $M < q$, we apply Lemmas 3.1 and 3.2 to w with $D' = \Omega'$, $\alpha < M < \beta = q$ to observe that

$$w(x', M) = 0 \quad \text{for all } x' \in \overline{\Omega}'.$$

In other words, $v(x', M) \leq -\lambda_0$ for all $x' \in \overline{\Omega}'$. This contradicts $M \in \Sigma$. The proof is now complete.

Remark. (i) If we examine the proof, it is not difficult to see that we may relax the continuity assumption to semicontinuity assumption in Theorem 1.2. For example instead of continuity we only need to assume the upper semicontinuity on $\overline{\Omega}$ with the property that $v = (v_*)^*$ to conclude that v depends only on x_n. Here v^* and v_* denotes the upper and lower semicontinuous envelope of v, respectively (see e.g. [CGG], [S1] for definition).
(ii) The assumption that v is constant for large x_n is essentially invoked to prove that $v(x', \hat{x}_n)$ is a constant as a function of x' for $\hat{x}_n \in \operatorname{int} \Sigma$.

Proof of Lemma 3.1. We may assume that $K = 0$ since w plus a constant is still a subsolution when w is a subsolution. We may also assume that $M = 0$ by a translation.

We argue by contradiction. Suppose that there would exist $\zeta' \in D'$ such that $w(\zeta', 0) < 0 = K$. The basic strategy for the proof is to find a domain E in D and a test function $\varphi \in C^2(E)$ that satisfies

$$\max_E(w - \varphi) = (w - \varphi)(\hat{x}', \hat{x}_n), \tag{3.1}$$

$$-|\nabla\varphi| \operatorname{div}(\nabla\varphi/|\nabla\varphi|) > 0 \quad \text{at} \quad (\hat{x}', \hat{x}_n) \tag{3.2}$$

for some $\hat{x} = (\hat{x}', \hat{x}_n) \in E$. This evidently contradicts the assumption that w is a subsolution in D. Our construction of φ and E reflects the proof of the classical strong maximum principle in [PW], [GT].

1. Choice of a test function. Let w_0 be a function on D' of form

$$w_0(x') = w(x', 0).$$

Since w_0 is upper semicontinuous, there is an open ball B_0 with $\overline{B}_0 \subset D'$ that satisfies

$$w_0 < 0 \quad \text{in} \quad B_0 \quad \text{and}$$
$$w_0(y') = 0 \quad \text{for some} \quad y' \in \partial B_0.$$

This is standard; see e.g. [PW]. (Indeed, we take a curve γ starting from ζ' to ξ' and denote by η' the first point attaing $w_0 = 0$ on γ starting from ζ'. Then there exists a point ζ_1' on the arc $\zeta'\eta'$ such that

$$\zeta_1' \in B(\eta', d/2) \subset D',$$

where

$$d = \text{dist}(\gamma, \partial D')$$

and $B(\eta', \sigma)$ denotes the open ball in $\mathbf{R}^{n-1}$ of radius σ centered at η'. We set

$$r_0 = \sup\{r;\ w_0(x') < 0 \quad \text{for all} \quad x' \in B(\zeta_1', r) \subset D'\}$$

so that

$$r_0 < |\zeta_1' - \eta| < d/2.$$

If we set $B_0 = B(\zeta_1', r_0)$, then B_0 satisfies all desired properties.)

Let B_1 be a little bit smaller open ball in B_0 such that $\partial B_0 \cap \partial B_1 = \{y'\}$. Let a be the center of B_1 and $r_1 (< r_0)$ be the radius of B_1. We take

$$\varphi(x', x_n) = -\varepsilon_1\, z(x') - \varepsilon_2\, x_n,$$
$$z(x') = e^{-\gamma|x'-a|^2} - e^{-\gamma r_1^2}$$

with positive parameters $\varepsilon_1, \varepsilon_2$ and γ to be determined later. By definition one observe that

$$\begin{aligned} 0 < z(x') < 1 &\quad \text{in} \quad B_1 = B(a, r_1), \\ z(x') = 0 &\quad \text{on} \quad \partial B_1, \\ -1 < z(x') < 0 &\quad \text{outside} \quad \overline{B}_1. \end{aligned} \tag{3.3}$$

2. Choice of γ. For each $\mu = \varepsilon_2/\varepsilon_1$ there is $\gamma_0 = \gamma_0(\mu)$ such that for $\gamma \geq \gamma_0$ it holds

$$-|\nabla\varphi|\ \text{div}(\nabla\varphi/|\nabla\varphi|) > 0 \quad \text{at all} \quad (x', x_n) \tag{3.4}$$

with

$$\frac{r_1}{2} \le |x' - a| \le \frac{3r_1}{2},\ x_n \in \mathbf{R}.$$

Since

$$-|\nabla\varphi|\ \mathrm{div}(\nabla\varphi/|\nabla\varphi|) = \varepsilon_1(|\nabla' z(x')|^2 + \mu^2)^{1/2}\ H(z)$$

with $H(z) = \mathrm{div}'\{\nabla' z(x')/(\mu^2 + |\nabla' z(x')|^2)^{1/2}\}$, it suffices to prove that $H(z)(x') > 0$ for x' with $r_1 \le 2|x' - a| \le 3r_1$ when γ is sufficiently large. Here ∇' denotes the gradient in x' and div' denotes the divergence in x'.

Since $z(x')$ is radial, i.e.,

$$z(x') = g(|x' - a|) \quad \text{with} \quad g(\rho) = e^{-\gamma\rho^2} - e^{-\gamma r_1^2},$$
$$H(z) = \left(\frac{g'}{((g')^2 + \mu^2)^{1/2}}\right)' + \frac{n-2}{\rho}\,\frac{g'}{((g')^2 + \mu^2)^{1/2}}\Bigg|_{\rho = |x' - a|}.$$

Since $g'(\rho) = -2\,\gamma\rho e^{-\gamma\rho^2}$, $g''(\rho) = -2\gamma e^{-\gamma\rho^2} + 4\gamma^2\rho^2 e^{-\gamma\rho^2}$, we obtain

$$H(z) = \frac{\{4\mu^2\gamma^2\rho^2 - 2(n-1)\mu^2\gamma - 8(n-2)\gamma^3\rho^2 e^{-2\gamma\rho^2}\}\ e^{-\gamma\rho^2}}{(4\gamma^2\rho^2 e^{-2\gamma\rho^2} + \mu^2)((g')^2 + \mu^2)^{1/2}}$$

with $\rho = |x' - a|$. The quantity in $\{\ \}$ is uniformly positive for ρ, $r_1 \le 2\rho \le 3r_1$ provided that γ is sufficiently large say $\gamma > \gamma_0(\mu)$.

3. Choice of the domain E, $\varepsilon_1, \varepsilon_2$. Let y' be the point as in Step 1. By definition

$$w_0 < 0 \quad \text{in} \quad \overline{B}_1 \backslash \{y'\} \quad \text{and} \quad w_0(y') = 0.$$

We set $B_2 = B(y', r_1/2)$. Since $r_1 < r_0 < d/2$, B_2 is contained in D'. We take $\delta > 0$ so small that

$$\partial(B(a, r_1 + \delta)) \cap \partial B_2 \subset B_0.$$

We then divide the boudnary of B_2 into two pieces:

$$C_2' = \partial B_2 \cap \overline{B(a, r_1 + \delta)},\ C_2'' = \partial B_2 \backslash \overline{B(a, r_1 + \delta)};$$

clearly ∂B_2 is a disjoint union of C_2' and C_2''. Since $w_0 < 0$ on a compact set C_2', there exists a costant $\ell > 0$ that satisfies $w_0 \le -\ell$ on C_2' by upper semicontinuity of w_0. Since w is uppersemicontinuous,

$$w \le -\ell/2 \quad \text{on} \quad C_2' \times [\alpha', \beta'],\ [\alpha', \beta'] \subset (\alpha, \beta)$$

for $\alpha' < 0 < \beta'$ sufficiently close to zero. We first fix $\alpha' < 0$ since $|z(x')|$ on $\overline{B}_2$ is bounded by 1 by (3.3), we take $\mu > (-\alpha')^{-1}$ so that

$$\sup\{z(x');\ x' \in B_2\}(-\alpha')^{-1} < \mu \tag{3.5}$$

for all $\gamma > 0$. We fix γ with $\gamma > \gamma_0(\mu)$ so that (3.4) holds. We then take β' smaller so that

$$-\sup\{z(x');\ x' \in C_2''\}/\beta' > \mu. \tag{3.6}$$

We set

$$\sigma_1 = \sup\{w(x', x_n);\ x' \in C_2',\ \alpha' < x_n < \beta'\},$$
$$\sigma_2 = \sup\{w(x', \beta');\ x' \in \overline{B}_2\}.$$

By definition of C_2' and $M = 0$ we see that $\sigma_1 \le -\ell/2$, $\sigma_2 < 0$. Choose $\varepsilon_1, \varepsilon_2$ sufficiently small so that

$$\max\{\sigma_1, \sigma_2\} + \varepsilon_1 + \varepsilon_2\beta' < 0 \tag{3.7}$$

keeping $\mu = \varepsilon_2/\varepsilon_1$. We take $E = B_2 \times (\alpha', \beta')$ and fix $\alpha', \mu, \gamma, \beta'$, $\varepsilon_1, \varepsilon_2$ satisfying (3.5)-(3.7) with $\gamma > \gamma_0(\mu)$.

4. Completion of the proof. To show (3.1) it suffices to prove

$$\max_{\partial E}(w - \varphi) < 0 \tag{3.8}$$

since $(w - \varphi)(y', 0) = 0$ and $(y', 0) \in E$. We divide ∂E into four pieces

(a) $x' \in C_2'$ and $\alpha' < x_n < \beta'$,
(b) $x' \in C_2''$ and $\alpha' < x_n < \beta'$,
(c) $x' \in \overline{B}_2$ and $x_n = \alpha'$,
(d) $x' \in \overline{B}_2$ and $x_n = \beta'$.

On the part (a) because of a bound $w \le -\ell/2$ we conclude $w - \varphi$ is negative if $\varepsilon_1, \varepsilon_2$ is taken by (3.7); note that $|z|$ is bounded independent of γ by (3.3). On the part (b) by (3.3)

$$\sup\{z(x');\ x' \in C_2''\} < 0.$$

The negativity of $w - \varphi$ follows from (3.6). On the part (c) the negativity of $w - \varphi$ follows from (3.5). On the part (d) since $\sigma_2 < 0$, (3.7) implies the negativity of

$w - \varphi$. Thus we have proved (3.8), since (3.4) holds on $B_2 \times \mathbf{R}$, we get desired φ and E satisfying (3.1) and (3.2). □

Proof of Lemma 3.2. As in the proof of Lemma 3.1 we may assume that $K = 0$ and $M = 0$.

We argue by contradiction. Suppose that there would exist $\zeta' \in D'$ such that $w(\zeta', 0) < 0 = K$. The basic strategy is to find a domain E in D and a test function $\varphi \in C^2(E \cup \Lambda)$ with $\overline{\Lambda} = \partial D \cap \partial E$ that satisfies

$$\max_{E \cup \Lambda}(w - \varphi) = (w - \varphi)(\hat{x}', \hat{x}_n), \tag{3.9}$$

$$-|\nabla\varphi| \operatorname{div}(\nabla\varphi/|\nabla\varphi|) > 0 \quad \text{at} \quad (\hat{x}', \hat{x}_n), \tag{3.10}$$

$$\partial\varphi/\partial\nu > 0 \quad \text{at} \quad (\hat{x}', \hat{x}_n) \quad \text{if} \quad (\hat{x}', \hat{x}_n) \in \Lambda \tag{3.11}$$

for some $\hat{x} = (\hat{x}', \hat{x}_n) \in E \cup \Lambda$, where Λ is an open set in $\partial D'$. This evidently contradicts the assumption that w is a subsolution in $E \cup \Lambda$.

1. Choice of a test function. As before let w_0 be a function on D' of form $w_0(x') = w(x', 0)$. By Lemma 3.1 we may assume $w_0 < 0$ in D'. Since $\partial D'$ is C^2 (so that D' satisfies the interior sphere condition), there is an open ball $B_1 = B(a, r_1)$ in D' such that

$$\overline{B}_1 \cap \overline{D}' = \{\xi'\}, \ w_0 < 0 \quad \text{in} \quad \overline{B}_1 \backslash \{\xi'\} \quad \text{and} \quad w_0(\xi') = 0$$

We take φ and $\gamma_0 = \gamma_0(\mu)$ as in Steps 1 and 2 of the proof of Lemma 3.1.

2. Choice of the domain $E, \varepsilon_1, \varepsilon_2$. We set $B_2 = B(\xi', r_1/2)$ and take $\delta > 0$ so small that

$$\partial(B(a, r_1 + \delta)) \cap \partial B_2 \subset D'.$$

We then set

$$G = B_2 \cap B_3 \cap D' \quad \text{with} \quad B_3 = B(a, r_1 + \delta)$$

and divide the boundary ∂G into three pieces:

$$C_2' = \partial B_2 \cap \overline{B}_3, \ C_2'' = \partial B_3 \cap B_2 \cap D', \ Z = \partial D' \cap B_3.$$

By definition $w_0 < 0$ on C_2' and $z < 0$ on $\overline{C_2''}$. We take $\alpha', \mu, \gamma, \beta'$ as before and choose ε_1 and ε_2 small so that (3.7) holds with σ_2 replaced by

$$\sigma_2 = \sup\{w(x', \beta'); x' \in \overline{G}\}.$$

We then set

$$E = G \times (\alpha', \beta')$$

and observe that

$$\partial E \cap \partial D = \overline{\Lambda} \quad \text{with} \quad \Lambda = Z \times (\alpha', \beta').$$

3. Completion of the proof. By the construction of E it is not difficult to see that

$$\max_{\partial E \setminus \Lambda} (w - \varphi) < 0$$

(cf. Step 4 of the proof of Lemma 3.1). Since $w(\xi', 0) - \varphi(\xi', 0) = 0$, the maximum of $w - \varphi$ in $E \cup \Lambda$ is attained there. Since $(x', x_n) \in \overline{E}$ implies $r_1/2 \leq |x' - a| \leq 3r_1/2$, the inequality (3.10) holds everywhere in $E \cup \Lambda$. Since

$$\partial\varphi/\partial\nu = 2\varepsilon_1 \, \gamma \, e^{-\gamma|x'-a|^2} \, |x' - a| > 0$$

the inequality (3.11) holds everwhere on Λ. We thus obtain (3.9)-(3.11). □

Remark. From the proof the assertion of Lemma 3.2 can be localized. We shall state a version without detailed proof.

Lemma 3.2'. *Let D be a domain in $\mathbf{R}^n$. Let w be an upper semicontinuous viscosity subsolution of*

$$-|\nabla w| \; div(\nabla w / |\nabla w|) = 0 \quad in \quad D,$$
$$\partial w/\partial \nu = 0 \quad on \quad Z_0,$$

where Z_0 is an open neighborhood in ∂D of a point $\xi = (\xi', \xi_n) \in \partial D$ and is assumed to be C^2. Assume that w attains its maximum K in $D \cup Z_0$ and that $\nu(\xi)$ is orthogonal to x_n-axis. Assume that

$$\xi_n = \sup\{x_n; \; w(x', x_n) = K \quad for\ some \quad x' \quad with \quad (x', x_n) \in D \cup Z_0\}$$

and that $w(\xi', \xi_n) = K$. Then $w(x', \xi_n) = K$ for all x' with $(x', \xi_n) \in D \cup Z_0$.

Remark. Our Theorem 1.2 as well as Lemmas 3.1 and 3.2 applies more general equation than (1.4a). We may replace (1.4a) by

$$F(\nabla u, \; \nabla\nabla u) = 0 \tag{3.12}$$

with F satisfying

(i) $F : (\mathbf{R}^n\backslash\{0\}) \times \mathbf{S}^n \to \mathbf{R}$ is continuous and geometric in the sense of [CGG].
(ii) $F(p, O) = 0$ for all $p \in \mathbf{R}^n\backslash\{0\}$.
(iii) For each $\lambda_0 > 0$ there exists $N_0 > 0$ such that if $\lambda_{\max}(Q_{\overline{p}}(X)) \leq \lambda_0$ and $\lambda_{\min}(Q_{\overline{p}}(X)) \leq -N_0$ (resp. $\lambda_{\min} \geq -\lambda_0$, $\lambda_{\max} \geq N_0$) then $F(p, Q_{\overline{p}}(X)) > 0$ (resp. < 0) for all $X \in \mathbf{S}^n$ and $p \in \mathbf{R}^n\backslash\{0\}$, where $Q_{\overline{p}}(X) = (I - \overline{p} \otimes \overline{p})X(I - \overline{p} \otimes \overline{p})$ with $\overline{p} = p/|p|$.

Here $\mathbf{S}^n$ denotes the space of all real symmetric matrices and $\lambda_{\min}(Y)$ and $\lambda_{\max}(Y)$ are the smallest and the largest eigenvalues of $Y \in \mathbf{S}^n$, respectively. Even if (1.4a) is replaced by (3.12) the proof of Lemma 3.1 is the same except step 2 where we have to replace (3.4) by

$$F(\nabla\varphi, \nabla\nabla\varphi) > 0 \quad \text{at all} \quad (x', x_n) \tag{3.4'}$$

satisfying $r_1 \leq 2|x' - a| \leq 3r_1, x_n \in \mathbf{R}$. To prove (3.4') for large $\gamma \geq \gamma_0(\mu)$ the property (iii) is invoked. Although a direct but not short calculation shows (3.4'), we do not present it here. A similar remark applies Lemma 3.2. (Geometricity is not invoked for Lemmas 3.1 and 3.2.) To extend Theorem 1.2 for (3.12) we notice that properties (i)–(iii) are invariant under translation in space independent variables and order-preserving change of the dependent variable of (3.12); (i)-(iii) are invariant under multiplication with -1 to the dependent variable by taking $\tilde{F}(p, X) = F(-p, -X)$.

This extended theory applies level set equations of anisotropic mean curvature flow equations (see e.g. [G2]) provided that the Frank diagram of interfacial energy is strictly convex in the sense that its all (inward) principal curvatures are positive. We shall discuss this application elsewhere.

Remark. There are fundamental results [B] for propagation of maximum points for degenerate linear elliptic operators represented by a sum of square of vector fields. Also, for a linear degenerate elliptic operator with one fixed degenerate direction, the strong maximum principle has been established in [A]. However, these results do not apply to our situation, since the vector field ∇u representing the degenerate direction is not smooth and u is not differentiable.

Remark on convergence of Γ_t to Γ_∞. We shall show that the convergence (1.6) holds when Γ_∞ is a finite collection of parallel hyperplanes perpendicular to x_n-axis in the situation of Theorem 1.3.

Assume that (1.6) were false. Then there would exist a point $\hat{x}$ on $\Gamma_\infty \cap \Omega$ and neighborhood $B(\hat{x}, r)$ such that

$$\Gamma_{t_j} \cap B(\hat{x}, r) = \emptyset$$

for some sequence $t_j \to \infty$. By Theorem 1.3 and the assumption on Γ_∞ the set Γ_∞ is of form

$$\Gamma_\infty = \cup_{\ell=1}^m H_\ell, \ H_\ell = \{(x', x_n) \in \mathbf{R}^n;\ x_n = \sigma_\ell\ x' \in \overline{\Omega}'\}$$

with some m different constants $\sigma_\ell (\ell = 1, \cdots, m)$. We may assume that $\hat{x} \in H_1 \cap \Omega$ and that $B(\hat{x}, 2r)$ does not intersect H_ℓ for $\ell > 1$ by taking r smaller.

By the convergence (1.5) for each $\delta > 0$

$$\Gamma_t \subset \cup_{\ell=1}^m H_\ell(\delta)$$

for sufficiently large t, say $t \geq T(\delta)$ with

$$H_\ell(\delta) = \{(x', x_n) \in \mathbf{R}^n;\ |x_n - \sigma_\ell| < \delta,\ x' \in \overline{\Omega}'\}$$

which is the δ-neighborhood of H_ℓ. We take δ smaller than r and observe that

$$\Gamma_{t_j} \cap \partial H_1(\delta) = \emptyset,\ \Gamma_{t_j} \cap H_1(\delta) \subset H_1(\delta) \backslash B(\hat{x}, r)$$

for $t_j \geq T(\delta)$. Let u be as in Theorem 1.3. Since u is continuous and Γ_{t_j} is c-level set of $u(t_j, \cdot)$, for each j there is $\varepsilon_j > 0$ such that

$$W_{t_j} \cap \partial H_1(\delta) = \emptyset,\ W_{t_j} \cap H_1(\delta) \subset H_1(\delta) \backslash B(\hat{x}, r)$$

$$W_{t_j} = \{(x', x_n) \in \mathbf{R}^n;\ |u(t_j, x', x_n) - c| \leq \varepsilon_j,\ x' \in \overline{\Omega}\}.$$

For given $r > 0$ we shall construct, by taking δ small, an open set U in $\overline{\Omega}$ with the properties

(i) $\overline{H_1(\delta)} \backslash B(\hat{x}, r) \subset U$;
(ii) the generalized evolution $\{D(t)\}_{t \geq 0}$ [GS1, GS2] of the mean curvature flow equation with the right angle boundary condition with $D(0) = U$ extincts in a finite time.

Once such U is constructed, we fix t_j with $t_j \geq T(\delta)$ and observe that

$$H_1(\delta) \cap W_{t_j} \subset U.$$

By comparison of generalized evolutions (derived from comparison [GS1] [GS2] of the level set equation as in [AAG]) we see

$$H_1(\delta) \cap \{(x', x_n) \in \mathbf{R}^n;\ |u(t, x', x_n) - c| \le \varepsilon_j,\ x' \in \overline{\Omega}\} \subset D(t - t_j)$$

for all $t \ge t_j$. Since D extincts in a finite time, in $H_1(\delta)$

$$|u(t, x', x_n) - c| \ge \varepsilon_j$$

for sufficiently large time t, say $t \ge T'$. This contradicts the property that the limit v takes the value c on H_1.

It remains to construct a set U satisfying (i)-(iii). We may assume that $\hat{x} = 0$ and $\sigma_1 = 0$ by a translation. We consider an axisymmetric hypersurface in $\mathbf{R}^n$ of form

$$S_t = \{(x', x_n); R(t, x_n) = |x'|\}.$$

It is not difficult to see that S_t evolves by the mean curvature if and only if R solves

$$R_t - \frac{R_{zz}}{1 + R_z^2} + \frac{n-2}{R} = 0 \tag{3.13}$$

with $z = x_n$; see e.g.[AAG]. We shall construct a solution R of form $R(t, z) = \alpha t + \rho(z)$ with $\alpha > 0$. For a given $\alpha > 0$ and $\rho_0 > 0$ let ρ be a local solution of

$$\rho_z = (e^{2\alpha\rho}\rho^{2(n-2)}/e^{2\alpha\rho_0}\rho_0^{2(n-2)} - 1)^{1/2},\ \rho(0) = \rho_0 \tag{3.14}$$

near $z = 0$. By the definition ρ is even in z and $\rho_z > 0$ for $z > 0$ as far as $\rho(z)$ is defined. Moreover $\rho(z) \nearrow \infty$ as $z \nearrow z_0$ for some $z_0 \in \mathbf{R}$. Rearranging (3.14) yields

$$\frac{1 + \rho_z^2}{\rho^{2(n-2)}} = e^{2\alpha\rho}c,\ c = e^{-2\alpha\rho_0}\rho_0^{-2(n-2)}.$$

Taking logarithm of both sides yields

$$\frac{1}{2}\log(1 + \rho_z^2) = \log \rho^{n-2} + \alpha\rho + \frac{1}{2}\log c.$$

Differentiating in z to get

$$\frac{\rho_{zz}\rho_z}{1 + \rho_z^2} = \frac{(n-2)\rho_z}{\rho} + \alpha\rho_z.$$

Since $\rho_z(z) \ne 0$ for $z \ne 0$, we see

$$R(t, z) = \alpha t + \rho(z) \tag{3.15}$$

solves (3.13) so that S_t evolves by the mean curvature. For a given $B(\hat{x},r)(\hat{x}=0)$ we take $\rho_0 = r/2$ and set

$$U = \{(x', x_n) \in \overline{\Omega}; |x'| > R(0, x_n)\ x' \in \overline{\Omega}',\ x_n \in \text{dom } \rho\},$$

where R is given by (3.15) and dom ρ denotes the domain of definition of ρ, which is a finite interval of form $(-z_0, z_0)$ as we observed from (3.14). Clearly this U satisfies the property (i) by taking δ small. Let $D(t)$ be the generalized evolution of the mean curvature flow equation with the right angle boundary condition in Ω with $D(0) = U$. Then, by comparison

$$D(t) \subset \{(x', x_n) \in \overline{\Omega}; |x'| > R(t, x_n)\ x' \in \overline{\Omega}',\ x_n \in \text{dom } \rho\},$$

since the right hand side is a supersolution although S_t does not satisfy the boundary condition. The right hand side becomes empty in a finite time by the definition of R in (3.15) and boundedness of Ω'. Thus U fulfills the property (ii). This completes the proof.

Acknowledgement.

The first author was partly supported by the NISSAN SCIENCE FOUNDATION and Japanese Ministry of Education, Science, Sports and Culture through grant no. 08874005, 09242201 for scientific research. The work of the second author was completed when he was a JSPS Research Fellow.

References

[A] S. J. Altschuler, *A geometric heat flow for one-forms on three dimensional manifolds*, Illinois J. of Math. **39** (1995), 98-118.

[AAG] S. J. Altschuler, S. Angenent and Y. Giga, *Mean curvature flow through singularities for surfaces of rotation*, J. Geom. Anal. **5** (1995), 293-357.

[AW1] S. J. Altschuler and L. F. Wu, *Convergence to translating solutions for a class of quasilinear parabolic equations with fixed angle of contact to a boudnary*, Math. Ann. **295** (1993), 761-765.

[AW2] S. J. Altschuler and L. F. Wu, *Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle*, Calc. Var. **2** (1994), 101-111.

[B] J.-M. Bony, *Principle du Maximum, Inégalité de Harnack et Unicité du Problème du Cauchy pour les Opérateurs Elliptic Dégénérés*, Ann. Inst. Fourier Grenoble **19** (1969), 277-304.

[CGG] Y. G. Chen, Y. Giga and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geometry **33** (1991), 749-786.

[ESS] C. M. Elliott, R. Schätzle and B. Stoth, *Viscosity solutions of a degenerate parabolic elliptic system arising in the mean field theory of superconductivity*, preprint (1997).

[ES] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature, I*, J. Differential Geometry **33** (1991), 635-681.

[G1] Y. Giga, *Evolving curves with boundary conditions*, Proc. of "Curvature flow and related topics" (Levico, 1994)(eds. A. Damlamian et al.) GAKUTO Internat. Ser. Math. Sci. Appl. vol. 5, Gakkotosho, Tokyo (1995), pp. 99-109.

[G2] Y. Giga, *A level set method for surface evolution equations*, Sugaku **47**(1995), 321-340. English translation, Sugaku Expositions, to appear.

[GS1] Y. Giga and M.-H. Sato, *Generalized interface evolution with the Neumann boundary condition*, Proc. Japan Acad. **67A** (1991), 263-266.

[GS2] Y. Giga and M.-H. Sato, *Neumann problem for singular degenerate parabolic equations*, Differential and Integral Equations **6** (1993), 1217-1230.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Math. Wiss., vol.224, Springer-Verlag, New York 1983.

[Gu] B. Guan, *Mean curvature motion of nonparametric hypersurfaces with contact angle condition*, In: Elliptic and Parabolic Methods in Geometry, (eds. B. Chow, R. Gulliver, S. Levy and J. Sullivan), pp. 47-56 A K Peters, Wellesley, Massachusetts, USA.

[H] G. Huisken, *Non-parametric mean curvature evolution with boundary conditions*, J. Differential Equations **77** (1989), 369-378.

[I] T. Ilmanen, *A strong maximum principle for singular minimal hypersurfaces*, Calc. Var. **4** (1996), 443-467.

[IStZ] T. Ilmanen, P. Sternberg and W. Ziemer, *Equilibrium solutions to generalized motion by mean curvature*, J. Geom. Anal., to appear.

[KK] B. Kawohl and N. Kutev, *Strong maximum principle for semicontinuous viscosity solutions of nonlinear partial differential equations*, preprint(1997).

[PW] M. H. Protter and H. Weinberger, *Maximum principles in differential equations*, Prentice-Hall 1967.

[S1] M.-H. Sato, *Interface evolution with Neumann boundary condition*, Adv. Math. Sci. Appl. **4** (1994), 249-264.

[S2] M.-H. Sato, *Capillary problem for singular degenerate parabolic equations on a half space*, Differential and Integral Equations **9** (1996), 1213-1224.

[SoW] B. Solomon and B. White, *A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals*, Indiana Univ. Math. J. **38** (1989), 683-691.

[Sp] R. P. Sperb, *Maximum principles and their applications*, Academic Press, New York 1981.

[StZ] P. Sternberg and W. P. Ziemer, *Generalized motion by curvature with a Dirichlet condition*, J. Differential Equations **114** (1994), 580-600.

Department of Mathematics
Hokkaido University
Sapporo 060, Japan

Department of Mathematics
Tokyo Metropolitan University
Hachioji, Tokyo 192-03
Japan

Muroran Institute of Technology
27-1 Mizumoto, Muroran 050
Japan