ON INSTANT EXTINCTION FOR VERY FAST DIFFUSION EQUATIONS

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ON INSTANT EXTINCTION FOR VERY FAST DIFFUSION EQUATIONS

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Dedicated to Professor Rentaro Agemi on the occasion of his sixtieth birthday

Abstract. In this paper we prove instant extinction of the solutions to Dirichlet and Neumann boundary value problem for some quasilinear parabolic equations whose diffusion coefficient is singular when the spatial gradient of unknown function is zero.

1. Introduction.

There are many works on quasilinear parabolic equations whose diffusion coefficient is singular when the gradient ∇u of unknown function $u = u(t,x)$ is zero. Examples include the level set equations of surface evolution equations ([2], [5], [11], [12]) and the p-Laplace diffusion equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{1.1}$$

with $1 < p < 2$ ([3], [14]). Another example is the equation

$$u_t - u_x^{-2}u_{xx} = 0, \tag{1.2}$$

for which the inverse function $x = x(t,u)$ of a solution $u(t,x)$ solves the heat equation $x_t = x_{uu}$. Except the last example (1.2), the initial value problem is globally well-posed for any bounded uniformly continuous initial data at least in the viscosity sense ([2], [5], [11], [12], [14]).

In this paper we study

$$u_t - |u_x|^{-\alpha}u_{xx} = 0 \tag{1.3}$$

for $\alpha > 0$ or its general form

$$u_t - a(u_x)u_{xx} = 0, \tag{1.4}$$

where $a > 0$ is C^1 except at the origin. If $\alpha = 2$, (1.3) is the same as (1.2) and if $\alpha < 1$, (1.3) is the same as (1.1) with $\alpha = 2 - p$ up to the dilation of u. We consider the homogeneous Dirichlet and Neumann boundary value problem for (1.4) on an interval Ω. We postulate that all reasonable solutions should be some limit of solutions of regularized equation of (1.4):

$$u_t - a^\varepsilon(u_x)u_{xx} = 0 \text{ in } Q = (0,T) \times \Omega \tag{1.5}$$

where $a^\varepsilon > 0$ is smooth everywhere and $a^\varepsilon \to a$ in some sense. Let u^ε be the solution of (1.5) with the zero boundary condition on $\partial\Omega$. Under very mild assumptions on a^ε and u_0 we show that if a is not integrable near zero, *i.e.*,

$$\int_{-1}^{1} a(p)dp = \infty, \tag{1.6}$$

then

$$u^\varepsilon(t,x) \to 0 \quad \text{for} \quad t > 0 \quad \text{as} \quad \varepsilon \to 0 \tag{1.7}$$

uniformly in $x \in \overline{\Omega}$ provided that $u^\varepsilon(0,x) = u_0(x)$. In other words any reasonable solution extincts instantaneously irrelevant to the initial data u_0. Let v^ε be the solution of (1.5) with the Neumann condition $v^\varepsilon_x = 0, x \in \partial\Omega$ and $v^\varepsilon(0,x) = u_0$. Then the structure condition (1.6) now implies (instead of (1.7))

$$v^\varepsilon(t,x) \to u_{0\#} \quad \text{for} \quad t > 0 \quad \text{as} \quad \varepsilon \to 0 \tag{1.8}$$

uniformly in $x \in \overline{\Omega}$, where $u_{0\#}$ is the average of u_0, *i.e.*,

$$u_{0\#} = \int_\Omega u_0 dx / (\text{length of } \Omega). \tag{1.9}$$

From the study of [14] it follows that if a fails to satisfy (1.6), then the set of admissible functions in [14] is not empty so that the theory of viscosity solutions applies to (1.4) and the instant extinction does not occur. The theory in [14] also applies to (1.1) so the instant extinction does not occur for (1.1) with $1 < p < 2$. So the condition (1.6) is optimal. Several parts of our argument are extendable to a multi-dimensional domain. We first discuss extinction problem in L^2 for equations of divergence type and then apply it for one dimensional problem. Our proof depends on integration by parts and is elementary. In [4] Evans studied (1.2) with the boundary condition $u(t,a) = -\infty$, $u(t,b) = \infty$ for $\Omega = (a,b)$. An approximate equation (1.5) with $a^\varepsilon(p) = 1/(p^2 + \varepsilon^2)$ is considered there. Let w^ε be the solution of (1.5) with $w^\varepsilon(0,x) = u_0(x)$. Then it is shown in [4] that

$$\lim_{t\to 0}\lim_{\varepsilon\to 0} w^\varepsilon(t,x) = (u_0)^*$$

where $(u_0)^*$ is the derivative of the convex hull of a primitive of u_0. Although the proof and the situation is different, this is interpreted as a similar phenomenon that there is no reasonable solution for (1.3) with $\alpha = 2$ continuous up to initial data.

For equations with very fast diffusion there are several results different from ours. The reader is referred to [16] and papers cited there. In our context the Cauchy problem for (1.3) is considered with a nondecreasing bounded initial data. A typical result says that the solution becomes unbounded (near $x = \infty$) instantaneously if and only if $\alpha \geq 2$. This also explains the effect of strong diffusion.

Recently, M.- H. Giga and the second author ([7], [8], [9]), constructed a continuous solution for (1.4) for any continuous (periodic) initial data even if a contains Dirac's delta function by extending the theory of viscosity solutions. Their results apply to the case when a is Dirac's delta function so that the instant extinction does not occur. Although the singularity of a is strong, it violates (1.6) so their theory is consistent with our theory. Such a type of singularity of a is typical in describing motion by crystalline curvature as introduced by Angenent and Gurtin [1] and independently by Taylor [15].

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2. Instant extinction.

Let Ω be a smoothly bounded domain in $\mathbf{R}^n$ and let $T > 0$. We consider

$$u_t - \operatorname{div}(\mathbf{A}^\varepsilon(\nabla u, t)) = 0 \quad \text{in} \quad Q = (0, T) \times \Omega \tag{2.1}$$

$$u(0, x) = u_0(x) \quad \text{at} \quad t = 0 \tag{2.2}$$

with either

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{2.3}$$

or

$$\mathbf{A}^\varepsilon(\nabla u, t) \cdot \nu = 0 \quad \text{on} \quad \partial\Omega \tag{2.4}$$

where ν is the outward normal on $\partial\Omega$. We assume for $0 < \varepsilon < 1$ that

(A1) $\mathbf{A}^\varepsilon = \mathbf{A}^\varepsilon(p, t) : \mathbf{R}^n \times [0, T] \to \mathbf{R}^n$ is continuous with its derivatives $\dfrac{\partial \mathbf{A}^\varepsilon}{\partial p_i}$, $\dfrac{\partial^2 \mathbf{A}^\varepsilon}{\partial p_i \partial p_j}$, $1 \le i,\ j \le n$,

(A2) for each $M > 0$ there is a constant $K_\varepsilon(M) > 0$ such that $\mathbf{A}^\varepsilon(p, t) \cdot p \ge K_\varepsilon(M)|p|^2$ for $|p| < M$, $0 \le t < T$, and $\lim_{\varepsilon \to 0} K_\varepsilon(M) = \infty$.

Our first observation is on the behavior of L^2 norm $||u^\varepsilon||_2(t) = (\int_\Omega |u^\varepsilon(x, t)|^2 dx)^{1/2}$.

Remark. Since the equation (2.1) is unchanged by replacing $\mathbf{A}^\varepsilon$ by $\mathbf{A}^\varepsilon + \mathbf{C}$ with $\mathbf{C}$ independent of p, the condition

$$\mathbf{A}^\varepsilon(p, t) \cdot p \ge K_\varepsilon(M)|p|^2$$

can be weakened as

$$(\mathbf{A}^\varepsilon(p, t) + \mathbf{C}) \cdot p \ge K_\varepsilon(M)|p|^2$$

with some vector $\mathbf{C}$ (independent of p).

It is not difficult to check that

$$\mathbf{A}^\varepsilon(p) = (|p|^2 + \varepsilon)^{(q-2)/2} p \qquad (q > 0)$$

satisfies (A2) or its modification remarked here if and only if $0 < q \le 1$ (cf. §3).

2.1. Lemma on L^2 estimates(Dirichlet problem).

Assume (A1)-(A2). Let $u^\varepsilon \in C(\overline{Q})$ be a solution of (2.1) and (2.2) with (2.3) such that u_t^ε, ∇u^ε, $\nabla^2 u^\varepsilon \in C(\overline{Q})$. (For brevity here and hereafter we shall simply say that u^ε is a $C^{1,2}$ solution.) Assume that ∇u^ε is bounded on Q, say $|\nabla u^\varepsilon| \leq M$ uniformly for $0 < \varepsilon < 1$ in Q. Then

$$||u^\varepsilon||_2^2(t) \leq \exp\big(-2\lambda_1 K_\varepsilon(M)t\big)||u_0||_2^2 \tag{2.5}$$

with $0 < \varepsilon < 1$, where λ_1 is the first eigenvalue of the minus Laplacian on Ω with the Dirichlet condition (2.3). In particular $||u^\varepsilon||_2^2(t) \to 0$ as $\varepsilon \to 0$ for $t > 0$.

Proof. Multiplying u^ε with (2.1) yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u^\varepsilon|^2 dx = -\int_\Omega \mathbf{A}^\varepsilon(\nabla u^\varepsilon, t)\cdot \nabla u^\varepsilon dx$$

by (A2) and the Poincaré inequality we have

$$\begin{aligned}\int_\Omega \mathbf{A}^\varepsilon(\nabla u^\varepsilon, t)\cdot\nabla u^\varepsilon dx &\geq K_\varepsilon(M)\int_\Omega |\nabla u^\varepsilon|^2 dx\\ &\geq K_\varepsilon(M)\lambda_1||u^\varepsilon||_2^2.\end{aligned}$$

This now yields

$$\frac{d}{dt}||u^\varepsilon||_2^2 \leq -2\lambda_1 K_\varepsilon(M)||u^\varepsilon||_2^2.$$

We thus obtain (2.5). The last assertion follows from (2.5) since $K_\varepsilon(M) \to \infty$ as $\varepsilon \to 0$.

2.2. Lemma on L^2 estimates(Neumann problem).

Assume (A1)-(A2). Let $v^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (2.1) and (2.2) with (2.4). Assume that $|\nabla v^\varepsilon| \leq M$ on Q with some $M > 0$ independent of ε, $0 < \varepsilon < 1$. Then

$$||v^\varepsilon - u_{0\#}||_2^2(t) \leq \exp\big(-2\mu K_\varepsilon(M)t\big)||u_0 - u_{0\#}||_2^2, \tag{2.6}$$

where $\mu > 0$ is the second eigenvalue of the (minus) Laplacian on Ω with the Neumann condition (2.4). In particular, $||v^\varepsilon - u_{0\#}||_2^2 \to 0$ as $\varepsilon \to 0$ for $t > 0$. Here $u_{0\#} = \int_\Omega u_0 dx/|\Omega|$ where $|\Omega|$ is the volume of Ω.

Proof. We set $u^\varepsilon = v^\varepsilon - u_{0\#}$ and observe that u^ε solves (2.1) and (2.4) with

$$u^\varepsilon(0,x) = u_0 - u_{0\#}.$$

Since u^ε solves (2.1) and (2.4), we see

$$\frac{d}{dt}\int_\Omega u^\varepsilon(t,x)dx = 0.$$

¿From $(u^\varepsilon(0,x))_\#$ it now follows that $(u^\varepsilon(0,\cdot))_\# = 0$. Multiplying u^ε with (2.1) and integrating by parts with (2.4) yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u^\varepsilon|^2 dx = -\int_\Omega \mathbf{A}^\varepsilon(\nabla u^\varepsilon, t)\cdot \nabla u^\varepsilon dx$$

Since $(u^\varepsilon(t,\cdot))_\# = 0$, the Poincaré inequality ([10], §7.8, (7.45)) with (A2) yields

$$\begin{aligned}\int_\Omega \mathbf{A}^\varepsilon(\nabla u^\varepsilon, t)\cdot\nabla u^\varepsilon dx &\geq K_\varepsilon(M)\int_\Omega |\nabla u^\varepsilon|^2 dx\\ &\geq K_\varepsilon(M)\mu||u^\varepsilon||_2^2\end{aligned}$$

As in the proof of Lemma 2.1 this now yields (2.6). The last assertion follows from (2.6) and $K_\varepsilon(M) \to \infty$ as $\varepsilon \to 0$.

2.3. Theorem on uniform convergence.

Assume (A1) - (A2).

(i) Let $u^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (2.1) and (2.2) with (2.3). Assume that $|\nabla u^\varepsilon| \le M$ on Q for $0 < \varepsilon < 1$. Then there is $C = C(n)$ such that

$$||u^\varepsilon||_\infty(t) \le CM^{1-\theta}\exp\Big(-\lambda_1 K_\varepsilon(M)\theta t\Big)||u_0||_2^\theta, \quad \theta = 2/(n+2).$$

In particular $||u^\varepsilon||_\infty(t) \to 0$ as $\varepsilon \to 0$ for $t > 0$.

(ii) Let $v^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (2.1) and (2.2) with (2.4). Assume that $|\nabla u^\varepsilon| \le M$ on Q for $0 < \varepsilon < 1$. Then there is $C = C(n)$ such that

$$||v^\varepsilon - u_{0\#}||_\infty(t) \le CM^{1-\theta}\exp\Big(-\mu K_\varepsilon(M)\theta t\Big)||u_0 - u_{0\#}||_2^\theta.$$

In particular $||v^\varepsilon - u_{0\#}||_\infty(t) \to 0$ as $\varepsilon \to 0$ for $t > 0$. Here $||f||_\infty = \sup\{|f(x)|; x \in \Omega\}$.

Proof. Our estimates follows from (2.5) and (2.6) with the Gagliardo-Nirenberg inequality

$$||f||_\infty \le C||f||_2^\theta \cdot ||\nabla f||_\infty^{1-\theta},\ \theta = \frac{2}{n+2}. \tag{2.7}$$

for f with $f = 0$ on $\partial\Omega$ or f with $f_\# = 0$. For $f \in C^1(\overline{\Omega})$ vanishing on $\partial\Omega$ the inequality (2.7) immediately follows from that for $f \in C_0^\infty(\mathbf{R}^n)$ which is standard ([6], Part 1, Theorem 9.3).

The weaker version of the inequality (2.7), *i.e.*,

$$||f||_\infty \le C_0||f||_2^\theta(||\nabla f||_\infty^{1-\theta} + ||f||_\infty^{1-\theta}) \tag{2.8}$$

holds for all $f \in C^\infty(\overline{\Omega})$ with C_0 independent of f. This inequality may be familiar ([6], Part1, Theorem 10.1). We note that (2.7) follows from (2.8) for f with $f_\# = 0$. Since (2.7) for f with $f_\# = 0$ is less familiar, we give the proof that (2.8) implies (2.7) with $f_\# = 0$ for convenience.

Suppose that (2.7) were false. Then there would exist $\{f_j\} \subset C^\infty(\overline{\Omega})$ with $||f_j||_\infty = 1$ and $f_\# = 0$ that satisfies

$$\frac{1}{j} = \frac{||f_j||_\infty}{j} > ||f_j||_2^\theta\,||\nabla f_j||_\infty^{1-\theta} \ge \frac{||f_j||_\infty}{C_0} - ||f_j||_2^\theta\,||f_j||_\infty^{1-\theta}; \tag{2.9}$$

the second inequality follows from (2.8). This yields either $||f_j||_2 \to 0$ or $||\nabla f_j||_\infty \to 0$ as $j \to \infty$. If $||f_j||_2 \to 0$, then by (2.9) it contradicts $||f_j||_\infty = 1$ for all j. If $||\nabla f_j||_\infty \to 0$, then the Ascoli-Arzela theorem says that $\{f_j\}$ converges subsequently to a constant function g in $C(\overline{\Omega})$. Since $f_{j\#} = 0$ implies $g = 0$, this contradicts the property $||f_j||_\infty = 1$ for all j.

3. One dimensional problem.

Unfortunately, in general the gradient bound may not be obtained in Q if $n > 1$ because of variety of domain, unless the equation is uniformly parabolic (see e.g. [13]). To avoid this extra difficulty we shall concentrate on one dimensional case. In this case we may assume $\Omega = (0, L)$. The Dirichlet problem can be reduced to the periodic boundary problem on $(-L, L)$ by extending u on $(-L, 0)$ by $-u(-x, t)$. The Neumann problem can be also reduced to the periodic boundary problem on $(-L, L)$ by extending u on $(-L, 0)$ by $u(-x, t)$ provided that $a^\varepsilon = \partial A^\varepsilon/\partial p$ is even in p, where a^ε is now scalar and positive. Since $w = u_x$ of (2.1) now solves

$$w_t - a^\varepsilon(u_x, t)w_{xx} - a_p^\varepsilon(u_x, t)w_x^2 = 0 \quad \text{on} \quad (0, T) \times \mathbf{R},$$

applying the maximum principle yields the bound of u_x for the bound of u_{0x}. Here $a_p^\varepsilon(p, t) = \dfrac{\partial^2 A^\varepsilon}{\partial p^2}$.

3.1. Lemma on bound for u_x.

Assume (A1) and $n = 1$.

(i) Let $u^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (2.1) and (2.2) with (2.3). Then

$$||u^\varepsilon{}_x||_\infty(t) \le ||u_{0x}||_\infty, \quad 0 \le t \le T, \quad 0 < \varepsilon < 1$$

(ii) Let $v^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (2.1) and (2.2) with (2.4). Then

$$||v^\varepsilon{}_x||_\infty(t) \le ||u_{0x}||_\infty, \quad 0 \le t \le T, \quad 0 < \varepsilon < 1$$

provided that $a^\varepsilon(p, t)$ is odd in p.

We now give a sufficient condition for (A2) when $n = 1$. We rewrite (2.1) as

$$u_t - a^\varepsilon(u_x, t)u_{xx} = 0, \tag{3.1}$$

where $a^\varepsilon(p, t) = \dfrac{\partial A^\varepsilon}{\partial p}$.

3.2. Proposition.

Assume that for $0 < \varepsilon < 1$,

(a1) $a^\varepsilon = a^\varepsilon(p, t) : \mathbf{R} \times [0, T] \to \mathbf{R}$ is continuous with its derivative $a_p^\varepsilon = \dfrac{\partial a^\varepsilon}{\partial p}$ and $a^\varepsilon > 0$,

(a2) $pa_p^\varepsilon \le 0$,

(a3) $a^\varepsilon(p, t)$ is even in p,

(a4) for each $M > 0$, there is $k_\varepsilon(M)$ such that

$$\int_0^M a^\varepsilon(p, t)dp \ge k_\varepsilon(M), \quad 0 \le t \le T$$

and

$$k_\varepsilon(M) \to \infty \quad \text{as} \quad \varepsilon \to 0.$$

Then $A^\varepsilon(p,t) = \int_0^p a^\varepsilon(q,t)dq$ satisfies (A1). Moreover $A^\varepsilon(p,t) = \beta^\varepsilon(p,t)p$ with continuous β^ε: $\mathbf{R} \times [0,T] \to \mathbf{R}$. (This condition guarantees that $u_x = 0$ implies $A^\varepsilon(u_x, t) = 0$).

Proof. From (a1) it follows (A1). By (a2) $A^\varepsilon(p,t)$ is concave in $p \in [0,\infty)$. Thus

$$A^\varepsilon(p,t) \geq \frac{A^\varepsilon(M,t)}{M} p \quad \text{for} \quad 0 \leq p \leq M.$$

By (a4) we now obtain

$$A^\varepsilon(p,t) \geq \frac{k_\varepsilon(M)}{M} p \quad \text{for} \quad 0 \leq p \leq M.$$

Since $A^\varepsilon(p,t)$ is odd by (a3), this yields

$$A^\varepsilon(p,t)p \geq \frac{k_\varepsilon(M)}{M} p^2 \quad \text{for} \quad |p| \leq M.$$

Setting $K_\varepsilon(M) = k_\varepsilon(M)/M$ leads to (A2) where the last property follows from definition of A^ε.

3.3. Theorem on instant extinction.

Assume (a1) - (a4).

(i) Let $u^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (3.1) in $Q = (0,T) \times \Omega$ with (2.2) and (2.3). If u_{0x} is bounded on Ω then there is a universal constant C that satisfies

$$||u^\varepsilon||_\infty(t) \leq C||u_{0x}||_\infty^{1-\theta} \exp\big(-\lambda_1 K_\varepsilon(||u_{0x}||_\infty)t\theta\big)||u_0||_2^\theta, \quad t > 0.$$

In particular

$$\lim_{\varepsilon \to 0} ||u^\varepsilon||_\infty(t) = 0 \quad \text{for} \quad t > 0.$$

(ii) Let $v^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (3.1) in Q with (2.2) and

$$u_x^\varepsilon = 0 \quad \text{on} \quad \partial\Omega. \tag{2.4'}$$

If u_{0x} is bounded on Ω then

$$||v^\varepsilon - u_{0\#}||_\infty(t) \leq C||u_{0x}||_\infty^{1-\theta} \exp\big(-\mu K_\varepsilon(||u_{0x}||_\infty)t\theta\big)||u_0 - u_{0\#}||_2^\theta, \quad t > 0.$$

In particular

$$\lim_{\varepsilon \to 0} ||v^\varepsilon - u_{0\#}||_\infty(t) = 0 \quad \text{for} \quad t > 0.$$

Proof. This follows from Theorem 2.3, Lemma 3.1 and Proposition 3.2.

3.4. Remark on the periodic boundary condition.

The second statement of Theorem 3.3 is still valid even if (2.4′) is replaced by the periodic boundary condition. Here $\mu > 0$ should be the second eigenvalue of the minus Laplacian on Ω with the periodic boundary condition. The proof is the same as the Neumann problem so is omitted.

3.5. Example.

We consider

$$u_t - a(u_x)u_{xx} = 0 \tag{3.2}$$

with $a \in C^1(\mathbf{R}\backslash\{0\})$, $a > 0$. We moreover assume that $a(p) = a(-p)$ and $a'(p) > 0$ for $p > 0$. Our main assumption on a is

$$\int_0^1 a(q)dq = \infty. \tag{3.3}$$

A typical example is

$$a(p) = |p|^{-\alpha} \quad \text{with} \quad \alpha > 1. \tag{3.4}$$

Let $a^\varepsilon(p) \to a(p)$ as $\varepsilon \to 0$ for a.e. p. Then the Fatou lemma with (3.3) implies

$$\varliminf_{\varepsilon\to 0} \int_0^M a^\varepsilon(p)dp = \infty$$

so such $a^\varepsilon(p)$ fulfills (a4). Of course there are many examples of such a sequence. For example, we may take

$$a^\varepsilon(p) = (|p|^2 + \varepsilon)^{\alpha/2} \quad \text{for} \quad a(p) = |p|^{-\alpha}.$$

Our Theorem 3.3 says if the solution of (3.2) and (2.2) with (2.3) or (2.4′) is obtained as a limit of solutions of regularized problem, then it must extinct instantaneously for the Dirichlet problem and it must equal the average of the initial data instantaneously for the Neumann problem. By Remark 3.4 the statement for the Neumann problem also holds for the periodic boundary value problem. In [9] a continuous solution for (3.2) is constructed for any continuous periodic initial data even if a contains Dirac's delta function.

It is unlikely that instant extinction occurs in their problem. However, their results do not contradict ours since in their paper the integral

$$\int_{-1}^1 adq$$

is assumed to be finite which excludes (3.3).

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