NONEXISTENCE OF STABLE TURING PATTERNS WITH SMOOTH LIMITING INTERFACIAL CONFIGURATIONS IN HIGHER DIMENSIONAL SPACES

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NONEXISTENCE OF STABLE TURING PATTERNS WITH SMOOTH LIMITING INTERFACIAL CONFIGURATIONS IN HIGHER DIMENSIONAL SPACES

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abstract

When the thickness of the interface (denoted by ε) tends to zero, any *stable* stationary internal layered solutions to a class of reaction diffusion systems cannot have a smooth limiting interfacial configuration. This means that if the limiting configuration of the interface has a smooth limit, it must become unstable for small ε, which makes a sharp contrast with one-dimensional case as in [5]. This suggests that stable layered patterns must become very fine and complicated in this singular limit. In fact we can formally derive that the rate of shrinking of stable patterns is of order $\varepsilon^{1/3}$ as well as the rescaled reduced equation which determines the morphology of magnified patterns. A variational characterization of the critical eigenvalue combined with the matched asymptotic expansion method is a key ingredient for the proof, although the original system is not of gradient type.

1. Introduction

Dynamics of interfacial patterns attracts much interests in many fields such as population dynamics, combustion, chemical reaction, solidification and so on. One of the pioneering works in pattern formation problem can be traced back to Turing [9] who found that spatially inhomogeneous patterns can be formed by diffusion-driven instability if the inhibitor diffuses faster than the activator. A typical model system is of the form

$$(1.1) \qquad \begin{cases} u_t = \varepsilon^2 \Delta u + f(u,v), & (x,t) \in \Omega \times (0,\infty), \\ v_t = D\Delta v + g(u,v), & \\ \dfrac{\partial u}{\partial n} = 0 = \dfrac{\partial v}{\partial n} & (x,t) \in \partial\Omega \times (0,\infty). \end{cases}$$

where u is the activator, v is the inhibitor, Ω is a smooth bounded domain in $\mathbf{R}^N$, and ε is a small positive parameter. The nonlinearity f has at least two stable branches for a fixed v and the signs of g are different on these branches, typically $(f,g) = (u(1-u)(u-a) - v, u - \gamma v)$, where $0 < a < 1$, $\gamma > 0$. More precise assumptions for (f,g) are displayed at the end of this section. Although (1.1) exhibits a variety of patterns depending on diffusion and/or reaction rates, we focus on the stationary ones in higher space dimensions, especially we are interested in layered solutions which have internal transition layers from one stable branch to the other one. The basic issue is that *"Does* (1.1) *has nonconstant stable stationary solutions up to $\varepsilon = 0$? And, if it does, what are the asymptotic configurations of them as $\varepsilon \downarrow 0$?"* As we shall see, this is closely related to finding the location of free boundary called the *interface* separating two different states.

Numerically as well as experimentally, for a *fixed* $\varepsilon > 0$, a variety of stationary patterns have been observed such as spots, stripes, and labyrinthine patterns for (1.1) (see for instance [1] and the references therein). Hence naively one can expect that (1.1) has a lot of stable stationary solutions for small ε up to $\varepsilon = 0$.

In fact, for one-dimensional case, it is proved rigorously (see [5]) that many stable layered solutions coexist up to $\varepsilon = 0$. It should be noted that each layer position has a definite limit and the distance between layer positions remains finite as $\varepsilon \downarrow 0$.

On the other hand, for higher dimensional case, we know very little about the limiting configuration of *stable* stationary solutions to (1.1) when ε tends to zero. For instance, the planar front does not persist as a stable one (see [7]) and more complicated patterns take over it for small ε. We rephrase our basic question in the following way: *Does* (1.1) *has an ε-family of stable stationary layered solutions $(U^\varepsilon, V^\varepsilon)$ with smooth interface Γ^ε up to $\varepsilon = 0$?* Here Γ^ε is defined by

$$\Gamma^\varepsilon \equiv \{x \in \Omega | U^\varepsilon(x) = \alpha^*\},$$

where α^* is an intermediate value between two stable branches and, for instance, is equal to 1/2 for the above example. Note that "smooth up to $\varepsilon = 0$ " means that there exist a $(N-1)$-dimensional smooth compact connected manifold Γ^0 without boundary embedded in $\mathbf{R}^N$ such that Γ^ε converges to Γ^0 smoothly as $\varepsilon \downarrow 0$.

The goal in this paper is to give a *negative* answer to this question under the assumption that it has a matched asymptotic expansion(see Section 2 for details). Namely we have

MAIN THEOREM *Suppose that* (1.1) *has an ε-family of stationary matched asymptotic solutions whose interface is smooth up to $\varepsilon = 0$. Then it must become unstable for small ε.*

This instability result leads to the following natural question, i.e., how about the fate of stable ones when $\varepsilon \downarrow 0$? The above theorem strongly suggests that stable patterns somehow must become very fine and/or complicated when $\varepsilon \downarrow 0$, and if it happens, can we characterize the domain size of those patterns and their morphologies ? We shall discuss on these issues in Section 4 (see also [5] and [8]).

We prove the above theorem under the following assumptions.

(A.0) Γ^ε is a $(N-1)$-dimensional, smooth, compact, connected manifold without boundary inside of Ω, and the domain surrounded by Γ^ε is simply-connected.

(A.1) f and g are smooth functions of u and v defined on some open set $\mathcal{O}(\supset \Omega)$ in $\mathbf{R}^2$ and the partial desrivatives f_v(resp. g_u) is a negative (resp. positive) constant function.

(A.2) (a) The nullcline of f is sigmoidal and consists of three smooth curves $u = h_-(v)$, $h_0(v)$ and $h_+(v)$ defined on the intervals I_-, I_0, and I_+, respectively. Let $\min I_- = \underline{v}$ and $\max I_+ = \overline{v}$, then the inequality $h_-(v) < h_0(v) < h_+(v)$ holds for $v \in I^* \equiv (\underline{v}, \overline{v})$ and $h_+(v)$ (resp. $h_-(v)$) coincides with $h_0(v)$ at only one point $v = \overline{v}$ (resp. $\underline{v}$) respectively.

(b) The nullcline of g intersects with that of f at one or three points transversally as in Fig.1.1. The critical point on $u = h_-(v)$ (resp. $h_+(v)$ or $h_0(v)$), if exists, is denoted by $P = (u_-, v_-) = (h_-(v_-), v_-)$ (resp. $Q = (u_+, v_+) = (h_+(v_+), v_+)$ or $R = (u_0, v_0) = (h_0(v_0), v_0))$.

(A.3) $J(v)$ has an isolated zero at $v = v^* \in I^*$ such that $dJ/dv < 0$ at $v = v^*$, where $J(v) = \int_{h_-(v)}^{h_+(v)} f(s,v)ds$. Moreover we assume that $v_- < v^* < v_+$.

(A.4) $f_u < 0$ on $\mathcal{H}_+ \cup \mathcal{H}_-$, where $\mathcal{H}_-$ (resp. $\mathcal{H}_+$) denotes the part of the curve $u = h_-(v)$ (resp. $h_+(v)$) defined by $\mathcal{H}_-$ (resp. $\mathcal{H}_+$) $= \{(u,v)|u = h_-(v)($ resp. $h_+(v))$ for $v_- \leq v < v^*(v^* < v \leq v_+)\}$, respectively. Note that $v_- \leq$ (resp. $\leq v_+$) is replaced by $\underline{v} <$ (resp. $< \overline{v}$) when there are no critical points on the branch $u = h_-(v)$ (resp. $h_+(v)$). See thick solid part of $f = 0$ in Figure 1.1.

$(A.5)$ (a)
$$g|_{\mathcal{H}_-} < 0 < g|_{\mathcal{H}_+}$$

(b)
$$\det\left(\frac{\partial(f,g)}{\partial(u,v)}\right)\Bigg|_{\mathcal{H}_+\cup\mathcal{H}_-} > 0.$$

$(A.6)$
$$g_v|_{\mathcal{H}_+\cup\mathcal{H}_-} < 0.$$

REMARK 1.1. *The assumption for the partial derivatives in (A.1) is necessary for the technical reason in Section 3.*

REMARK 1.2. *Under the above assumptions. it is natural to assume that the internal layered solutions have matched asymptotic expansions as described in Section 2.*

2. Matched asymptotic expansion of singularly perturbed stationary solution

In this section, we summarize the necessary conditions for the existence of the ε-family of matched asymptotic solutions with internal transition layers of the following stationary problem:

$$(2.1)\qquad \begin{cases} 0 = \varepsilon^2 \Delta u + f(u,v), \\ 0 = D\Delta v + g(u,v), \end{cases} \qquad \text{in } \Omega,$$

$$(2.2)\qquad \frac{\partial u}{\partial n} = 0 = \frac{\partial v}{\partial n} \qquad \text{on } \partial\Omega,$$

Before presenting the precise form of matched asymptotic expansion, we need to do a change of variables near the interface. Let us assume that there exist an ε-family of smooth solutions $(U^\varepsilon(x), V^\varepsilon(x))$ to (2.1) and (2.2) with interior transition layers such that the interface defined by

$$(2.3)\qquad \Gamma^\varepsilon \equiv \{x \in \Omega | U^\varepsilon(x) = \alpha^* \equiv \frac{1}{2}\}$$

is a compact smooth manifold of dimension $N-1$ embedded in $\mathbf{R}^N$ and have a definite limit Γ^0 with the same properties as $\varepsilon \downarrow 0$. Let (X_ϕ, ϕ) be a local chart on Γ^0, with $\phi(X_\phi)$ an open subset of $\mathbf{R}^{N-1}$. For $x_0 \in X_\phi$, $\phi(x_0) = s = (s^1, \cdots, s^{N-1})$ and we denote the inverse of ϕ by

$$x_0 = (x_0^1(s), \cdots, x_0^N(s)).$$

In some tubular neighborhood $U_d(\Gamma^0) = \{x \in \mathbf{R}^N || y(x)| \le d\}$ of Γ^0, local coordinate system $(s,y) = (s^1, \cdots, s^{N-1}, y)$ is defined and for $x \in U_d(\Gamma^0)$,

$$(2.4)\qquad x = X(s,y) \equiv x_0(s^1, \cdots, s^{N-1}) + y\nu(s^1, \cdots, s^{N-1})$$

holds, where $\nu(s^1, \cdots, s^{N-1})$ is the outward unit normal vector at $s = (s^1, \cdots, s^{N-1})$ to Γ^0. Then, X becomes a diffeomorphism from $[-d,d] \times \Gamma^0$ to $U_d(\Gamma^0)$ if d is strictly smaller than the infimum of the radii of curvature of Γ^0. Its inverse is denoted by $(S(x), Y(x))$. Then Γ^ε can be represented by

$$\Gamma^\varepsilon = \{x_0(s) + \gamma(s,\varepsilon)\nu(s) | s \in \Gamma^0\}$$

where

$$\gamma(s,\varepsilon) = \sum_{k=1}^{m} \varepsilon^k \gamma_k(s) + \varepsilon^m \hat{\gamma}_{m+1}(s,\varepsilon).$$

Here we introduce local shift variable τ by the following relation:

$$(2.5)\qquad y = \tau + \omega(\frac{\tau}{d})\gamma(s,\varepsilon),$$

where $\omega(\tau) \in C^\infty(\mathbf{R})$ is a cut off function such that

$$\omega(\tau) = 1 \ \text{ for } \ |\tau| \le \frac{1}{2}, \ \omega(\tau) = 0 \ \text{ for } \ |\tau| \ge 1,$$

$$0 \le \omega(\tau) \le 1, \ \ |\omega'| \le 3.$$

Then, by the implicit function theorem, $\tau = \mathcal{T}(s, y, \varepsilon)$ satisfying (2.5) is defined for sufficiently small ε. In place of x, we use a new independent variable $\hat{x}$, defined by

$$\hat{x} = \hat{X}(x, \varepsilon) = \begin{cases} x, & x \in \Omega \backslash U_d(\Gamma^0), \\ X(S(x), \mathcal{T}(S(x), Y(x), \varepsilon)), & x \in U_d(\Gamma^0). \end{cases}$$

Let Ω_ε^+ (resp. Ω_0^+) be the region surrounded by Γ^ε (resp. Γ^0) and $\Omega_\varepsilon^- \equiv \Omega \backslash \overline{\Omega_\varepsilon^+}$ (resp. $\Omega_0^- \equiv \Omega \backslash \overline{\Omega_0^+}$). Then, note that $\hat{x} = \hat{X}(x, \varepsilon)$ maps Γ^ε to Γ^0, and $\Omega_\varepsilon^\pm$ to $\Omega_0^\pm$, respectively, namely the free boundary Γ^ε becomes a fixed boundary Γ^0 in the new coodinate. Throughout the paper, we shall use the following notation

$$u(x) = u(s, y), \quad \hat{u}(\hat{x}) = \hat{u}(s, \tau).$$

Using the above transformation, stationary problem (2.1) with (2.2) can be rewritten as

$$\begin{cases} \varepsilon^2 M^\varepsilon \hat{u} + f(\hat{u}, \hat{v}) = 0, \\ DM^\varepsilon \hat{v} + g(\hat{u}, \hat{v}) = 0, \end{cases} \quad \text{in } \Omega, \tag{2.6}$$

$$\frac{\partial \hat{u}}{\partial n} = 0 = \frac{\partial \hat{v}}{\partial n} \quad \text{on } \partial\Omega, \tag{2.7}$$

where $\hat{u} = \hat{u}(\hat{x})$, $\hat{v} = \hat{v}(\hat{x})$ and M^ε is the representation of Laplacian Δ_x in $\hat{x}$. In $\Omega \backslash U_d(\Gamma^0)$, M^ε is equal to $\Delta_{\hat{x}}$. On the other hand, in the neighborhood $U_d(\Gamma^0)$, M^ε is defined in the following way: For the local coordinate system (s, y) defined by (2.4) in $\mathbf{R}^N$, let g^{ij} be the contravariant metric tensor and $g = \det(g^{ij})$. Then for $u(x) = u(s, y)$, Laplacian Δ_x is expressed by

$$\begin{aligned} (\Delta_x u)(x) &= (\Delta_{(s,y)} u)(s, y) \\ &\equiv \frac{\partial^2}{\partial y^2} u(s, y) + (N-1) H(s, y) \frac{\partial}{\partial y} u(s, y) \\ &\quad + \frac{1}{\sqrt{g}} \sum_{i=1}^{N-1} \frac{\partial}{\partial s^i} \left(\sqrt{g} \sum_{j=1}^{N-1} g^{ij} \frac{\partial}{\partial s^j} u(s, y) \right), \end{aligned} \tag{2.8}$$

where $H = H(s, y)$ is the mean curvature of the hypersurface $\Gamma(y) = \{x_0(s) + y\nu(s) \mid s \in \Gamma^0\}$ at (s, y). Using this representation, for $\hat{u}(\hat{x}) = \hat{u}(s, \tau)$, M^ε is defined by

$$(M^\varepsilon \hat{u})(\hat{x}) \equiv \Delta_{(s,y)} \hat{u}(s, \mathcal{T}(s, y, \varepsilon)).$$

It follows from this definition that M^ε can be expanded as $M^\varepsilon = \sum_{k\geq 0} \varepsilon^k M_k$, where

$$M_0 \equiv \Delta_{\hat{x}}, \qquad \hat{x} \in \Omega,$$

and for $k \geq 1$,

$$M_k = \begin{cases} 0, & \hat{x} \in \Omega \backslash U_d(\Gamma^0), \\ \text{at most the second order differential operator in } s^i \ (i = 1, \cdots, N-1) \text{ and } \tau, & \hat{x} \in U_d(\Gamma^0). \end{cases}$$

In the following, we consider only (2.6) and (2.7), so we omit symbol hat ˆ. A family of solutions $(U^\varepsilon, V^\varepsilon)$ of (2.6) and (2.7) is called an *ε-family of matched asymptotic solutions* when it has the following expansions (2.9)-(2.12). Roughly speaking, $(U^\varepsilon, V^\varepsilon)$ is expanded separately in two regions $\Omega_0^\pm$ divided by the interface Γ^0, and they are matched smoothly at Γ^0. It should be recalled that the boundary condition (2.3) is also satisfied at Γ^0. More precisely we have

(2.9)
$$U^\varepsilon(x) \approx \begin{cases} U_+^\varepsilon(x) \equiv U_m^+(x,\varepsilon) + \Phi_m^+(x,\varepsilon), & x \in \Omega_0^+, \\ U_-^\varepsilon(x) \equiv U_m^-(x,\varepsilon) + \Phi_m^-(x,\varepsilon), & x \in \Omega_0^-, \end{cases}$$

$$V^\varepsilon(x) \approx \begin{cases} V_+^\varepsilon(x) \equiv V_m^+(x,\varepsilon) + \varepsilon^2 \Psi_m^+(x,\varepsilon), & x \in \Omega_0^+, \\ V_-^\varepsilon(x) \equiv V_m^-(x,\varepsilon) + \varepsilon^2 \Psi_m^-(x,\varepsilon), & x \in \Omega_0^-, \end{cases}$$

where

(2.10)
$$U_m^\pm(x,\varepsilon) = \sum_{k=0}^{m} u_k^\pm(x)\varepsilon^k, \quad V_m^\pm(x,\varepsilon) = \sum_{k=0}^{m} v_k^\pm(x)\varepsilon^k,$$

(2.11)
$$\Phi_m^\pm(x,\varepsilon) = \begin{cases} \omega(\frac{Y(x)}{d}) \sum_{k=0}^{m} \phi_k^\pm(S(x), \frac{Y(x)}{\varepsilon})\varepsilon^k, & x \in U_d(\Gamma^0) \cap \Omega_0^\pm, \\ 0, & x \in \Omega_0^\pm \backslash U_d(\Gamma^0), \end{cases}$$

(2.12)
$$\Psi_m^\pm(x,\varepsilon) = \begin{cases} \omega(\frac{Y(x)}{d}) \sum_{k=0}^{m} \psi_k^\pm(S(x), \frac{Y(x)}{\varepsilon})\varepsilon^k, & x \in U_d(\Gamma^0) \cap \Omega_0^\pm, \\ 0, & x \in \Omega_0^\pm \backslash U_d(\Gamma^0), \end{cases}$$

$\phi_k^\pm$ and $\psi_k^\pm$ are functions of s and ξ, and ξ is *stretched variable* $\xi \equiv \tau/\varepsilon$ (recall that $Y(\hat{x}) = \tau$). The coefficients $u_k^\pm$, $v_k^\pm$, $\phi_k^\pm$, and $\psi_k^\pm$ satisfy the equations listed below in appropriate function spaces, which can be obtained by making *outer* and *inner* expansions and equating the same powers of ε^k. The inner and outer solutions are not independent in the sense that they must satisfy the boundary conditions as well as C^1-*matching conditions* between $(U_+^\varepsilon, V_+^\varepsilon)$ and $(U_-^\varepsilon, V_-^\varepsilon)$ on Γ^0. Let $\beta^\varepsilon(s) = v^* + \sum_{k=1}^m \beta_k(s)\varepsilon^k + \varepsilon^m \beta_{m+1}(s,\varepsilon)$ be the expansion of the value of V^ε on Γ^0. Note that the $0-$ *thorder* should be v^* from (A.2), since $(U^\varepsilon, V^\varepsilon)$ is a stationary solution.

We briefly explain the algorithm of matched asymptotic expansion method and display the equations and relations up to order $O(\varepsilon^m)$. For more detailed aruguments, see [2] and [6].

First we divide (2.6) into two problems as follows:

$$(2.13)_+ \qquad \begin{cases} \varepsilon^2 M^\varepsilon u^+ + f(u^+, v^+) = 0, & \text{in } \Omega_0^+, \\ DM^\varepsilon v^+ + g(u^+, v^+) = 0, & \\ u^+ = \alpha^*, \quad v^+ = \beta^\varepsilon & \text{on } \Gamma^0, \end{cases}$$

$$(2.13)_- \qquad \begin{cases} \varepsilon^2 M^\varepsilon u^- + f(u^-, v^-) = 0, & \text{in } \Omega_0^-, \\ DM^\varepsilon v^- + g(u^-, v^-) = 0, & \\ u^- = \alpha^*, \quad v^- = \beta^\varepsilon & \text{on } \Gamma^0, \\ \dfrac{\partial u^-}{\partial n} = 0 = \dfrac{\partial v^-}{\partial n} & \text{on } \partial\Omega. \end{cases}$$

Then the interface is regarded as the boundary layer at Γ^0.

OUTER EXPANSION

Let

$$(2.14) \qquad u^\pm = \sum_{k=0}^m u_k^\pm(x)\varepsilon^k, \quad v^\pm = \sum_{k=0}^m v_k^\pm(x)\varepsilon^k,$$

where both $u_k^\pm(x)$ and $v_k^\pm(x)$ belong to $C^\infty(\overline{\Omega_0^\pm})$. Substituting (2.14) into $(2.13)_\pm$ and equating like power of ε^k, we have the following problem for $(u_k^\pm(x), v_k^\pm(x))$:

$k = 0$

$$(2.15) \qquad \begin{cases} f(u_0^\pm, v_0^\pm) = 0, & \text{in } \Omega_0^\pm, \\ DM_0 v_0^\pm + g(u_0^\pm, v_0^\pm) = 0, & \\ \dfrac{\partial v_0^-}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

$k \geq 1$

$$
(2.16) \quad \begin{cases} f_u^{0\pm} u_k^{\pm} + f_v^{0\pm} v_k^{\pm} = - \sum_{i+j=k-2} M_i u_j^{\pm} + P_{k-1}^{\pm}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega_0^{\pm}, \\ DM_0 v_k^{\pm} + g_u^{0\pm} u_k^{\pm} + g_v^{0\pm} v_k^{\pm} = -D \sum_{i+j=k, i\geq 1} M_i v_j^{\pm} + Q_{k-1}^{\pm}, \\ \dfrac{\partial v_k^-}{\partial n} = 0 \qquad \text{on } \partial\Omega, \end{cases}
$$

where $f_u^{0\pm} \equiv \frac{\partial}{\partial u} f(u_0^{\pm}, v_0^{\pm})$, $f_v^{0\pm} \equiv \frac{\partial}{\partial v} f(u_0^{\pm}, v_0^{\pm})$, and so on. $P_{k-1}^{\pm}$ and $Q_{k-1}^{\pm}$ are functions determined only by $u_0^{\pm}, v_0^{\pm}, \cdots, u_{k-1}^{\pm}, v_{k-1}^{\pm}$. This expansion is insufficient because the layer part is not represented. For example, u_0^+ and u_0^- are discontinuous on Γ^0. So we need new variable $\xi = \tau/\varepsilon$ that rescale a neighborhood of interface. Also we note that the boundary conditions of $v_k^{\pm}$ are determined by matching conditions.

INNER EXPANSION

We introduce the stretched variable $\xi = \tau/\varepsilon$ and let

$$
(2.17) \quad \begin{aligned} u^{\pm} &= U_m^{\pm}(x, \varepsilon) + \sum_{k=0}^{m} \phi_k^{\pm}(S(x), \frac{Y(x)}{\varepsilon}) \varepsilon^k, \\ v^{\pm} &= V_m^{\pm}(x, \varepsilon) + \varepsilon^2 \sum_{k=0}^{m} \psi_k^{\pm}(S(x), \frac{Y(x)}{\varepsilon}) \varepsilon^k, \end{aligned}
$$

where $\phi_k^{\pm} = \phi_k^{\pm}(s, \xi)$ and $\psi_k^{\pm} = \psi_k^{\pm}(s, \xi)$. Since the definition domain of $\phi_k^{\pm}$ and $\psi_k^{\pm}$ is semi-infinite, these functions and the inhomogeneous terms of their equations listed below must have some decaying property for solvability. An appropriate function space for this purpose is the following.

DEFINITION *Let $\mathcal{E}^{\pm}$ be the set of functions $E^{\pm}(s, \xi, \varepsilon)$ defined on $\Gamma_0 \times I^{\pm} \times [0, \varepsilon_0)$ with the property that for each C^{∞} linear differential operator D of any order in the variables s and ξ, there exist positive constants $C_{\pm}$ and K (possibly depending on D and $E^{\pm}$, but not on s, ξ, and ε) with $|DE^{\pm}| \leq Ke^{-C_{\pm}|\xi|}$. Here $I^- \equiv (-\infty, 0)$ and $I^+ \equiv (0, \infty)$.*

Substituting (2.17) into $(2.13)_{\pm}$ and equating like power of ε^k, we obtain the following equations:

$k = 0$

$$
\begin{cases} \ddot{\phi}_0^{\pm} + f(h_{\pm}(v^*) + \phi_0^{\pm}, v^*) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \xi \in I^{\mp}, s \in \Gamma^0, \\ D\ddot{\psi}_0^{\pm} = g(h_{\pm}(v^*), v^*) - g(h_{\pm}(v^*) + \phi_0^{\pm}, v^*), \\ \phi_0^{\pm}(s, \mp\infty) = 0, \quad \psi_0^{\pm}(s, \mp\infty) = 0 = \dot{\psi}_0^{\pm}(s, \mp\infty), \end{cases}
$$

$k = 1$

$$\begin{cases} \ddot{\phi}_1^{\pm} + \tilde{f}_u^{0\pm}\phi_1^{\pm} = -\tilde{M}_1\phi_0^{\pm} - \tilde{f}_u^{0\pm}\{u_1^{\pm}(s,0) + u_{0\tau}^{\pm}(s,0)\xi\} \\ \qquad -\tilde{f}_v^{0\pm}\{v_1^{\pm}(s,0) + v_{0\tau}^{\pm}(s,0)\xi\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \xi \in I^{\mp}, s \in \Gamma^0, \\ D\ddot{\psi}_1^{\pm} = -D\tilde{M}_1\psi_0^{\pm} + \tilde{Q}_0^{\pm}, \\ \phi_1^{\pm}(s,\mp\infty) = 0, \quad \psi_1^{\pm}(s,\mp\infty) = 0 = \dot{\psi}_1^{\pm}(s,\mp\infty), \end{cases}$$

$k \geq 2$

$$\begin{cases} \ddot{\phi}_k^{\pm} + \tilde{f}_u^{0\pm}\phi_k^{\pm} = -\sum_{i+j=k, i\geq 1} \tilde{M}_i\phi_j^{\pm} + \tilde{P}_{k-1}^{\pm}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \xi \in I^{\mp}, s \in \Gamma^0, \\ D\ddot{\psi}_k^{\pm} = -D\sum_{i+j=k, i\geq 1} \tilde{M}_i\psi_j^{\pm} + \tilde{Q}_{k-1}^{\pm}, \\ \phi_k^{\pm}(s,\mp\infty) = 0, \quad \psi_k^{\pm}(s,\mp\infty) = 0 = \dot{\psi}_k^{\pm}(s,\mp\infty), \end{cases}$$

where $\cdot = \frac{\partial}{\partial\xi}$, $\tilde{f}_u^{0\pm} \equiv \frac{\partial}{\partial u} f(h_\pm(v^*) + \phi_0^{\pm}, v^*)$, and so on. $\tilde{P}_{k-1}^{\pm}$ depends on $u_0^{\pm}, v_0^{\pm}, \cdots, u_k^{\pm}, v_k^{\pm}$, $\phi_0^{\pm}, \psi_0^{\pm}, \cdots, \phi_{k-1}^{\pm}, \psi_{k-2}^{\pm}$ and $\tilde{Q}_{k-1}^{\pm}$ does moreover on $\psi_{k-1}^{\pm}$. We define the right-hand sides of the first two equation of (2.16) are equal to zero when $k = 1$ and $\psi_{-1} \equiv 0$. The solvability of the above equations in the space $\mathcal{E}^{\pm}$ can be shown in a similar way as in [2], so we leave the details to the reader. $\tilde{M}^{\varepsilon}$ is the representation of M^{ε} in variables s and ξ, and expanded as

$$\tilde{M}^{\varepsilon} \equiv \frac{1}{\varepsilon^2}\sum_{k\geq 0} \varepsilon^k \tilde{M}_k.$$

Here $\tilde{M}_k$ $(k \geq 0)$ are at most second order differential operator in s and ξ. The precise forms of $\tilde{M}_k$ is presented at the end of this section.

BOUNDARY CONDITIONS AND C^1-MATCHING CONDITIONS

Now we describe the boundary conditions of $v_k^{\pm}$ and $\phi_k^{\pm}$ on Γ^0. Then $u_k^{\pm}$, $v_k^{\pm}$, $\phi_k^{\pm}$, and $\psi_k^{\pm}$ are determined recursively. These conditions are given by

$$\alpha^* = \sum_{k=0}^{m} u_k^{\pm}(s,0)\varepsilon^k + \sum_{k=0}^{m} \phi_k^{\pm}(s,0)\varepsilon^k,$$

$$v^* + \sum_{k=1}^{m} \beta_k(s)\varepsilon^k = \sum_{k=0}^{m} v_k^{\pm}(s,0)\varepsilon^k + \varepsilon^2 \sum_{k=0}^{m-2} \psi_k^{\pm}(s,0)\varepsilon^k.$$

Equating like power of ε^k, we have the following boundary conditions:

$k = 0$

(2.18) $$\phi_0^{\pm}(s,0) = \alpha^* - u_0^{\pm}(s,0), \quad v_0^{\pm} = v^* \text{ on } \Gamma^0,$$

$k \geq 1$

$$\phi_k^{\pm}(s,0) = -u_k^{\pm}(s,0), \quad v_k^{\pm} = \beta_k(s) - \psi_{k-2}^{\pm}(s,0) \text{ on } \Gamma^0.$$

In this way, we obtain the formal asymptotic solution of $(2.13)_{\pm}$. In order that $(U^{\varepsilon}, V^{\varepsilon})$ become a formal stationary solution of (2.6), $(U^{\varepsilon}_{\pm}, V^{\varepsilon}_{\pm})$ must satisfy the C^1-matching conditions, that is,

$$\varepsilon\frac{\partial U^{\varepsilon}_{+}}{\partial\nu} = \varepsilon\frac{\partial U^{\varepsilon}_{-}}{\partial\nu}, \quad \varepsilon\frac{\partial V^{\varepsilon}_{+}}{\partial\nu} = \varepsilon\frac{\partial V^{\varepsilon}_{-}}{\partial\nu} \quad \text{on } \Gamma^0.$$

After some computation, we have

$k = 0$

$$\frac{\partial v_0^{+}}{\partial\nu}(s,0) = \frac{\partial v_0^{-}}{\partial\nu}(s,0), \quad \dot{\phi}_0^{+}(s,0) = \dot{\phi}_0^{-}(s,0) \quad \text{on } \Gamma^0. \tag{2.19}$$

$k \geq 1$

$$\left.\begin{aligned} &\frac{\partial v_k^{+}}{\partial\nu}(s,0) + \dot{\psi}_{k-1}^{+}(s,0) = \frac{\partial v_k^{-}}{\partial\nu}(s,0) + \dot{\psi}_{k-1}^{-}(s,0), \\ &\dot{\phi}_k^{+}(s,0) + \frac{\partial u_{k-1}^{+}}{\partial\nu}(s,0) = \dot{\phi}_k^{-}(s,0) + \frac{\partial u_{k-1}^{-}}{\partial\nu}(s,0). \end{aligned}\right\} \quad \text{on } \Gamma^0 \tag{2.20}$$

The second equation of (2.15) with the boundary and C^1-matching conditions (see (2.18) and (2.19)) is called the *reduced problem*, namely

$$\begin{cases} D\Delta v_0^{\pm} + g(h_{\pm}(v_0^{\pm}), v_0^{\pm}) = 0 & \text{in } \Omega_0^{\pm}, \\ v_0^{\pm} = v^*, \quad \dfrac{\partial v_0^{+}}{\partial\nu} = \dfrac{\partial v_0^{-}}{\partial\nu} \quad \text{on } \Gamma^0, \\ \dfrac{\partial v_0^{-}}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.21}$$

This is a free boundary problem for Γ^0 which determines the asymptotic configuration of stationary interfacial patterns.

We close this section by presenting a lemma on the representations of $\tilde{M}_k$, which will become useful in the next section. The proof is delegated to [6].

LEMMA 2.1. *$\tilde{M}_0$, $\tilde{M}_1$, and $\tilde{M}_2$ have the following form:*

$$\tilde{M}_0 \equiv \frac{\partial^2}{\partial\xi^2}, \quad \tilde{M}_1 \equiv (N-1)H_0(s)\frac{\partial}{\partial\xi},$$

$$\begin{aligned} \tilde{M}_2 \equiv \Delta^{\Gamma^0} - (P_1(s) + P_2(s))\frac{\partial}{\partial\xi} + P_3(s)\frac{\partial^2}{\partial\xi^2} \\ -D_s\frac{\partial}{\partial\xi} - H_1(s)(\xi + \gamma_1(s))\frac{\partial}{\partial\xi}, \end{aligned} \tag{2.22}$$

where

$$P_1(s) = \frac{1}{2G} \sum_{i=1}^{N-1} G_{s^i} \sum_{j=1}^{N-1} G^{ij} \partial_{s^j} \gamma_1,$$

$$P_2(s) = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} [G^{ij}_{s^i} \partial_{s^j} \gamma_1 + G^{ij} \partial_{s^i s^j} \gamma_1],$$

$$P_3(s) = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} G^{ij} \partial_{s^i} \gamma_1 \partial_{s^j} \gamma_1 > 0,$$

$$D_s = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} G^{ij} \left(\partial_{s^i} \gamma_1 \frac{\partial}{\partial s^j} + \partial_{s^j} \gamma_1 \frac{\partial}{\partial s^i} \right),$$

$$H_1(s) \equiv \sum_{i=1}^{N-1} \kappa_i(s)^2.$$

$H_0(s)$ (*resp.* $\kappa_i(s)$) *are the mean (resp. principal) curvature of* Γ^0 *at* $s \in \Gamma^0$, G^{ij} *is the contravariant metric tensor for the manifold* Γ^0 *of dimension* $N-1$, $G = \det(G^{ij})$, *and* Δ^{Γ^0} *Laplace-Beltrami's operator defined on* Γ^0. *Particularly coefficients of* $\frac{\partial}{\partial s^j}$ *in* D_s *are independent of* ξ.

3. Instability result for stationary patterns as $\varepsilon \downarrow 0$

In this section we prove that the internal layered patterns in the previous section must become unstable when ε tends to zero. For this purpose, we show that the following linearized eigenvalue problem around $(u^\varepsilon, v^\varepsilon)$:

$$
(3.1) \qquad \begin{cases} \lambda w = \varepsilon^2 M^\varepsilon w + f_u^\varepsilon w + f_v^\varepsilon z, \\ \lambda z = DM^\varepsilon z + g_u^\varepsilon w + g_v^\varepsilon z, \end{cases}
$$

has an unstable eigenvalue where λ is the eigenvalue parameter, $f_u^\varepsilon = \frac{\partial}{\partial u} f(u^\varepsilon, v^\varepsilon)$ and so on. Our main result is the following.

THEOREM 3.1. *Suppose that* (1.1) *has an ε-family of stationary matched asymptotic solutions whose interface is smooth up to $\varepsilon = 0$. Then,* (3.1) *has a positive (i.e., unstable) eigenvalue of $O(\varepsilon)$ for small ε.*

Apparently (3.1) is not a self-adjoint problem, since $f_v^\varepsilon \neq g_u^\varepsilon$. We first consider the following auxiliary problem.

$$
(3.2) \qquad \begin{cases} \lambda w = \varepsilon^2 M^\varepsilon w + f_u^\varepsilon w + f_v^\varepsilon z, \\ \eta z = DM^\varepsilon z + g_u^\varepsilon w + g_v^\varepsilon z, \end{cases}
$$

where η is an auxiliary parameter. For the proof of the above theorem, we show two lemmas related to (3.2) : the first one deals with the case $\eta = 0$ where, by solving the second equation with respect to z (see assumption (A.6)), and substituting it into the first equation, we have a self-adjoint problem of w; the second one shows the existence of positive eigenvalue of $O(\varepsilon)$ of (3.2) for each $\eta \neq 0$. The proof of Theorem 3.1 is an immediate consequence of these two lemmas. Note that linearized instability implies a nonlinear one for the class of evolutional systems like (1.1).

Step 1 $\quad \eta = 0$ case.

Solving the second equation of (3.2) with $\eta = 0$ with respect to z as

$$
(3.3) \qquad z = (N^\varepsilon)^{-1} g_u^\varepsilon w,
$$

where $(N^\varepsilon)^{-1} \equiv (-DM^\varepsilon - g_v^\varepsilon)^{-1}$, which is well-defined from (A.6), then substituting (3.3) into the first equation of (3.2), we obtain a scalar problem for w:

$$
(3.4) \qquad \lambda w = \varepsilon^2 M^\varepsilon w + f_u^\varepsilon w + f_v^\varepsilon (N^\varepsilon)^{-1} g_u^\varepsilon w.
$$

From (A.1), (3.4) becomes a self-adjoint problem.

LEMMA 3.2. (3.4) *has a positive (i.e., unstable) eigenvalue of $O(\varepsilon)$ for small ε.*

PROOF.

In what follows, for simplicity of notation, we can assume that $f_v \equiv -1$ and $g_u \equiv 1$ without loss of generality. Since the linearized operator (3.4) is self-adjoint, we shall prove that the largest eigenvalue λ_0^ε of (3.4) becomes positive for small ε, which is characterized by

$$\lambda_0^\varepsilon = \sup_{w\in H^1(\Omega)} \frac{\int_\Omega \{-\varepsilon^2|\nabla_{M^\varepsilon} w|^2 + f_u^\varepsilon w^2 - |(N^\varepsilon)^{-1/2} w|^2\}dx}{\int_\Omega w^2 dx}, \tag{3.5}$$

where ∇_{M^ε} is the representation of ∇ with respect to the coordinate $\hat{x}$ (see §2). Recall that we write x instead of $\hat{x}$ and hence $Y(x) = \tau$ and $\xi \equiv \tau/\varepsilon$. Now we construct an suitable test function for our purpose. Let

$$Q(\xi) = \begin{cases} \omega\left(\dfrac{\varepsilon\xi}{d}\right)\dot{U}(\xi) & \text{for} \quad |\xi| \le \dfrac{d}{\varepsilon}, \\ 0 & \text{for} \quad |\xi| \ge \dfrac{d}{\varepsilon}, \end{cases}$$

where U is a solution of

$$\begin{cases} \ddot{U} + f(U, v^*) = 0, \\ U(\pm\infty) = h_\mp(v^*), \quad U(0) = \alpha^*. \end{cases} \tag{3.6}$$

We define $w(x)$ by the following product with $\Theta \in L^2(\Gamma^0)$ and $\| \Theta \|_{L^2(\Gamma^0)} = 1$

$$w(x) = \begin{cases} Q\left(\dfrac{Y(x)}{\varepsilon}\right)\Theta(S(x)), & x \in U_d(\Gamma^0) \\ 0, & x \in \Omega \backslash U_d(\Gamma^0). \end{cases}$$

For this $w(x)$, $\varepsilon^2|\nabla_{M^\varepsilon} w|^2$ is computed as

$$\varepsilon^2|\nabla_{M^\varepsilon} w|^2 = \varepsilon^2 \sum_{k=1}^{N-1} G_{kk}\{\sum_{i=1}^{N} G^{ik}(\dot{U}\partial_{s^i}\Theta - \partial_{s^i}\gamma_1\Theta\ddot{U})\}^2 + \ddot{U}^2\Theta^2 + O(\varepsilon^3) \tag{3.7}$$

in $U_d(\Gamma^0)$, where G_{ij} is the covariant metric tensor for Γ^0. The remainder term $O(\varepsilon^3)$ depends only on Γ^ε and the L^2-norm of Θ. Here we used the fact that

$$\nabla_{M^\varepsilon}\hat{u} \equiv \nabla_{(s,y)}\hat{u}(s, \mathcal{T}(s, y, \varepsilon))$$

for $\hat{u} = \hat{u}(s, \tau)$, where

$$(\nabla u)_k \equiv \sum_{i=1}^{N} g^{ik}\sqrt{g_{kk}}\frac{\partial u}{\partial s^i} \quad (k = 1, \cdots, N-1), \quad (\nabla u)_N \equiv \frac{\partial u}{\partial y},$$

and

$$|\nabla_{(s,y)} u|^2 = \sum_{k=1}^{N-1}\left(\sqrt{g_{kk}}\sum_{i=1}^{N} g^{ik}\frac{\partial u}{\partial s^i}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

for $u = u(s,y)$. Hence, for a function $\varphi = \varphi(s, \tau(s,y,\varepsilon)/\varepsilon)$, we have $\frac{\partial \varphi}{\partial s^i} = \varphi_{s^i} + \frac{1}{\varepsilon}\varphi_\xi \tau_{s^i}$, $\frac{\partial \varphi}{\partial y} = \frac{1}{\varepsilon}\varphi_\xi \tau_y$, $\tau_{s^i} = -\varepsilon \partial_{s^i}\gamma_1 + O(\varepsilon^2)$, and $\tau_y \equiv 1$ in a neighborhood of $\tau = 0$ (see (2.5)). Integrating (3.7) over Ω, the first term of numerator of (3.5) becomes

$$(3.8)\quad \int_\Omega \varepsilon^2 |\nabla_{M^\varepsilon} w|^2 dx = \varepsilon \int_{\Gamma^0}\int_{|\xi|\le\frac{d}{\varepsilon}} \{\varepsilon^2 \dot{U}^2 |\nabla^{\Gamma^0}\Theta|^2 + \varepsilon^2 \ddot{U}^2 \hat{P}_1(s)\Theta^2 + \ddot{U}^2\Theta^2\} d\xi dS + O(\varepsilon^4),$$

where

$$|\nabla^{\Gamma^0}\Theta|^2 = \sum_{k=1}^{N-1} G_{kk}\left(\sum_{i=1}^{N} G^{ik}\partial_{s^i}\Theta\right)^2$$

and

$$\hat{P}_1(s) = \sum_{k=1}^{N-1} G_{kk}\left(\sum_{i=1}^{N} G^{ik}\partial_{s^i}\gamma_1\right)^2 > 0.$$

In order to compute the second term of numerator of (3.5), first note that

$$(3.9)\qquad f_u^\varepsilon = \begin{cases} \tilde{F}_u^{0+} + \varepsilon \tilde{F}_u^{1+} + \varepsilon^2 \tilde{F}_u^{2+} + O(\varepsilon^3) & \text{in } \Omega_0^+ \cap U_d(\Gamma^0) \\ \tilde{F}_u^{0-} + \varepsilon \tilde{F}_u^{1-} + \varepsilon^2 \tilde{F}_u^{2-} + O(\varepsilon^3) & \text{in } \Omega_0^- \cap U_d(\Gamma^0) \end{cases}$$

where

$$\tilde{F}_u^{0\pm} \equiv f_u(h_\pm(v^*) + \phi_0^\pm, v^*),$$

$$\tilde{F}_u^{1\pm} \equiv \tilde{f}_{uu}^{0\pm}\{\xi(u_0^\mp)_\tau(s,0) + u_1^\mp(s,0) + \phi_1^\pm\} + \tilde{f}_{uv}^{0\pm}\{\xi(v_0^\mp)_\tau(s,0) + v_1^\mp(s,0)\},$$

$(\cdot)_\tau = \frac{\partial}{\partial\tau}$, and the remainder term $O(\varepsilon^3)$ depends only on the stationary pattern $(U^\varepsilon, V^\varepsilon)$. The $O(1)$ term of (3.9) multiplied by w^2 combined with the third term of (3.8) vanishies, which is easily seen by differentiating (3.6) with respect to ξ. Hence we only focus on the contribution of (3.9) coming from the $O(\varepsilon)$-term and higher. The next quantity is a key ingredient for the proof.

$$(3.10)\qquad \int_{-\infty}^0 \tilde{F}_u^{1+}\dot{U}^2 d\xi + \int_0^\infty \tilde{F}_u^{1-}\dot{U}^2 d\xi = \left.\frac{\partial v_0}{\partial \nu}\right|_{\Gamma^0} \frac{d}{dv} J(v^*) > 0,$$

where $J(v) = \int_{h_-(v)}^{h_+(v)} f(t,v)dt$. In order to show (3.10), we note that $\dot{U} = \dot{\phi}_0^\pm$, $\ddot{U} = \ddot{\phi}_0^\pm$ for $\xi \in I^\mp$ (so we omit the superscript $\pm$ of $\dot{\phi}_0$ and $\ddot{\phi}_0$), and $p^\pm \equiv \phi_1^\pm$ satisfy the next equation (see §2):

$$(3.11)\qquad \ddot{p}^\pm + \tilde{F}_u^{0\pm} p^\pm = \Omega^\pm,$$

where

$$\Omega^\pm(s,\xi) \equiv H^\pm(s,\xi) - \{(u_0^\mp)_\tau(s,0)\tilde{f}_u^{0\pm} + (v_0^\mp)_\tau(s,0)\tilde{f}_v^{0\pm}\}$$

and

$$(3.12)\qquad H^\pm \equiv -(N-1)H_0\ddot{\phi}_0 - \tilde{F}_u^{1\pm}\dot{\phi}_0.$$

Multiplying $\dot{\phi}_0$ on both sides of (3.11) and using the following relations

$$\int_{\mp\infty}^{0} \tilde{f}_u^{0\pm}\dot{\phi}_0(z)dz = -\ddot{\phi}_0(0), \qquad \int_{\mp\infty}^{0} \tilde{f}_v^{0\pm}\dot{\phi}_0(z)dz = \int_{h_\pm(v^*)}^{\alpha} f_v(u,v^*)du,$$

we obtain by integration by parts

$$\begin{aligned}\int_{\mp\infty}^{0} H^\pm(s,z)\dot{\phi}_0(z)dz &= \ddot{\phi}_1^\pm(s,0)\dot{\phi}_0(0) - \ddot{\phi}_0(0)\{\dot{\phi}_1^\pm(s,0) + (u_0^\pm)_\tau(s,0)\} \\ &+(v_0^\pm)_\tau(s,0)\int_{h_\pm(v^*)}^{\alpha} f_v(u,v^*)du.\end{aligned} \tag{3.13}$$

On the other hand, multiplying $\dot{\phi}_0$ on both sides of (3.12) and using (3.13), we have

$$\int_{-\infty}^{0} \tilde{F}_u^{1+}\dot{U}^2 d\xi + \int_{0}^{\infty} \tilde{F}_u^{1-}\dot{U}^2 d\xi = \int_{-\infty}^{0} \tilde{F}_u^{1+}\dot{\phi}_0^2 d\xi + \int_{0}^{\infty} \tilde{F}_u^{1-}\dot{\phi}_0^2 d\xi$$

$$= -(N-1)H_0\int_{-\infty}^{0} \ddot{\phi}_0\dot{\phi}_0 d\xi - (N-1)H_0\int_{0}^{\infty} \ddot{\phi}_0\dot{\phi}_0 d\xi$$

$$-\ddot{\phi}_1^+(s,0)\dot{\phi}_0(0) + \ddot{\phi}_0\{\dot{\phi}_1^+(s,0) + (u_0^+)_\tau(s,0)\} - (v_0^+)_\tau(s,0)\int_{h_+(v^*)}^{\alpha^*} f_v(u,v^*)du$$

$$+\ddot{\phi}_1^-(s,0)\dot{\phi}_0(0) - \ddot{\phi}_0\{\dot{\phi}_1^-(s,0) + (u_0^-)_\tau(s,0)\} + (v_0^-)_\tau(s,0)\int_{h_-(v^*)}^{\alpha^*} f_v(u,v^*)du$$

$$= \left.\frac{\partial v_0}{\partial \nu}\right|_{\Gamma^0}\int_{h_-(v^*)}^{h_+(v^*)} f_v(u,v^*)du,$$

which is the required result (3.10). Here we used the fact that $\ddot{\phi}_1^+(s,0) = \ddot{\phi}_1^-(s,0)$ and the C^1-matching condition of $\phi_1^\pm$ (see (2.20)). Using (3.8), (3.9), and (3.10) and the Hopf boundary Lemma for v^0 on Γ^0 (see (2.15)), we obtain

$$\begin{aligned}\lambda_0^\varepsilon \geq C\Bigg[&\varepsilon\frac{m_*}{K_1}\frac{d}{dv}J(v^*) \\ &+\int_\Gamma \varepsilon^2\left\{-|\nabla^{\Gamma^0}\Theta|^2 - \frac{1}{K_1}(K_2\hat{P}_1(s) - \hat{P}_2(s))\Theta^2\right\}dS \\ &-\frac{1}{K_1\varepsilon}\int_\Omega |(N^\varepsilon)^{-1/2}w|^2 dx\Bigg] + O(\varepsilon^3),\end{aligned} \tag{3.14}$$

where

$$m_* \equiv \min_{\Gamma^0}\frac{\partial v_0}{\partial \nu} < 0, \qquad K_1 \equiv \int_{-\infty}^{\infty} \dot{U}^2 d\xi, \qquad K_2 \equiv \int_{-\infty}^{\infty} \ddot{U}^2 d\xi,$$

$$\hat{P}_2(s) \equiv \int_{-\infty}^{0} \tilde{F}_u^{2+}\dot{U}^2 d\xi + \int_{0}^{\infty} \tilde{F}_u^{2-}\dot{U}^2 d\xi,$$

and C is a positive constant. The objective is to choose an appropriate test function in order to make the first term of $[\cdot]$ of (3.14) dominant, which is positive and $O(\varepsilon)$. First we choose Θ as the k-th eigenfunction Θ_k of the following eigenvalue problem

$$\Delta^\Gamma \Theta_k - \frac{1}{K_1}(K_2\hat{P}_1(s) - \hat{P}_2(s))\Theta_k = \mu_k\Theta_k \qquad \text{on } \Gamma^0.$$

Then the second term of (3.14) is equal to $\varepsilon^2\mu_k$. Note also that Θ_k converges to 0 as $k \to \infty$ in weak $L^2(\Gamma^0)$-sense. As for the third term of (3.14), which comes from the nonlocal part, we first note that when ε tends to zero,

$$\text{(3.15)} \qquad \frac{Q}{\varepsilon} = \frac{1}{\varepsilon}\dot{U}(\frac{\tau}{\varepsilon})\omega(\frac{\tau}{d}) \longrightarrow c_0\delta(\tau) \qquad \text{in } H^{-1}((-d,d))\text{-sense,}$$

where $\delta(\tau)$ is a Dirac's δ-function at 0 and c_0 a positive constant. Let K_k^ε be

$$K_k^\varepsilon \equiv \int_\Omega |(N^\varepsilon)^{-1/2}\left(\frac{w_k}{\varepsilon}\right)|^2 dx = \int_\Omega \left[(N^\varepsilon)^{-1}\left(\frac{w_k}{\varepsilon}\right)\right]\left(\frac{w_k}{\varepsilon}\right) dx.$$

In view of (3.15) and that $(N^\varepsilon)^{-1}$ is a uniformly bounded operator mapping from $H^{-1}(\Omega)$ to $H^1(\Omega)$ with respect to ε, we see that K_k^ε is uniformly bounded with respect to ε and k, and that $\int_{-d}^{d}(Q(\tau/\varepsilon)/\varepsilon)\bullet d\tau$ converges to the trace operator on Γ^0 from $H^1(\Omega)$ to $H^{1/2}(\Gamma^0)$ in operator norm sense. Therefore, by using the fact that Θ_k converges weakly to 0 as $k \to \infty$, we see that for any given small $c^* > 0$, there exists an ε_0 and k_0 such that

$$\text{(3.16)} \qquad K_k^\varepsilon < c^* \text{ for } 0 < \varepsilon \le \varepsilon_0,\ k \ge k_0.$$

Substituting $\Theta = \Theta_k$ and $w = w_k$ into (3.14), we have

$$\lambda_0^\varepsilon \ge C\varepsilon\left[\frac{m_*}{K_1}\frac{d}{dv}J(v^*) - \frac{1}{K_1}K_k^\varepsilon + \varepsilon\mu_k\right] + O(\varepsilon^3),$$

where C is a positive constant. Using (3.16) and taking ε smaller, if necessary, we see that

$$\frac{m_*}{K_1}\frac{d}{dv}J(v^*) - \frac{1}{K_1}K_k^\varepsilon > 0.$$

Therefore the right hand side of (3.5) becomes positive for sufficiently small $\varepsilon > 0$, which is greater or equal to $O(\varepsilon)$ quantity. In the rest of the proof, we show that the upper bound of (3.5) is also order ε. Since the first and the third terms of the numerator of (3.5) is nonpositive, it holds obviously that

$$\lambda_0^\varepsilon \le \sup_{w\in H^1(\Omega)} \frac{\int_\Omega f_u^\varepsilon w^2 dx}{\int_\Omega w^2 dx}.$$

In view of the expansion (3.9) and the assumptions (A.2) and (A.4), f_u^ε has a positive sign only in the ε-neighborhood along the normal direction of Γ^0. Therefore we have the estimate

$$\lambda_0^\varepsilon \le C\varepsilon|\Gamma^0|,$$

where C is a positive constant and $|\cdot|$ denotes the area, which completes the proof.

Step 2 $\eta \neq 0$ case

Rewriting (3.2) as

$$\begin{cases} \lambda w = \varepsilon^2 M^\varepsilon w + f_u^\varepsilon w + f_v^\varepsilon z, \\ 0 = DM^\varepsilon z + g_u^\varepsilon w + (g_v^\varepsilon - \eta) z, \end{cases} \tag{3.17}$$

and noting that $g_v^\varepsilon - \eta < 0$ for $\eta \geq 0$ from (A.6), we see that all the computation in Step 1 is also valid for (3.17) with $\eta \geq 0$. Therefore we have the following lemma.

LEMMA 3.3. *(3.17) has a positive eigenvalue $\lambda = \lambda^\varepsilon(\eta)$ for $\eta \geq 0$. Moreover, there exist positive constants C_0 and C_1 $(C_0 < C_1)$ which are independent of η and ε such that*

$$C_0\varepsilon < \lambda^\varepsilon(\eta) < C_1\varepsilon \tag{3.18}$$

holds for $\eta \geq 0$.

PROOF OF THEOREM 3.1. It follows from Lemma 3.3 that $\lambda^\varepsilon(\eta)$ is a continuous function of η for $\eta \geq 0$. Since $\lambda^\varepsilon(\eta)$ has lower and upper bounds like (3.18), we see that $\eta = \lambda^\varepsilon(\eta)$ holds at least at one point $\eta = \eta^*$ $(0 < \eta^*)$ by the intermediate value theorem. This η^* is a requiered unstable eigenvalue for (3.1).

4. Concluding remarks

As was mentioned in §1, Main Theorem strongly suggests that stable patterns becomes very fine and/or complicated in the limit of $\varepsilon \downarrow 0$ in higher dimensional spaces. What we discuss here is to find an appropriate scaling in space and time by which the resulting singular limit dynamics could have stable patterns of *finite* size. Such patterns are usually maintained by the balance of two competing forces as described below. In the course of the following formal analysis, it turns out clear intuitively why the stable patterns of (1.1) must become fine in the original scale.

Suppose there is a sharp transition layer (interface) Γ connecting two stable bulk states, there are two forces that drives the interface: one is the bulk force causing the translation of interface with certain speed $W(v|_\Gamma)$ which depends on the value of v at Γ, the other is a geometric force, i.e., mean-curvature effect.

In one word, the characteristic size of stable patterns is determined by the *balance between the above two forces*, and turns out to be proportional to $\varepsilon^{1/3}$. It should be noted that the scale $\varepsilon^{1/3}$ coincides with the fastest growing wavelength of the planar front of (1.1)(see [7]). In what follows we consider a smooth subdomain $\tilde{\Omega}_\varepsilon(t)$ ($\subset \Omega$), and assume that both u and v satisfy the Neumann boundary conditions on $\partial\tilde{\Omega}_\varepsilon(t)$ and the diameter of $\tilde{\Omega}_\varepsilon(t)$ shrinks to zero as $\varepsilon \downarrow 0$ with order ε^α. Here α $(0 < \alpha < 1)$ is an unknown exponent. Typically $\tilde{\Omega}_\varepsilon(t)$ is a unit cell of some periodic structure in $\mathbf{R}^N$.

Applying a change of variable with unknown exponent α

$$\mathbf{y} = \frac{\mathbf{x}}{\varepsilon^\alpha} \qquad (0 < \alpha < 1)$$

to (1.1) ($D = 1$ for simplicity), we have

$$(4.1) \qquad \begin{cases} u_t = \varepsilon^{2(1-\alpha)}\Delta_{\mathbf{y}} u + f(u,v) \\ v_t = \varepsilon^{-2\alpha}\Delta_{\mathbf{y}} v + g(u,v) \end{cases} \quad \text{in } \hat{\Omega}_\varepsilon(t)$$

where $\Delta_{\mathbf{y}}$ stands for the Laplacian in $\mathbf{y}$-variable and $\hat{\Omega}_\varepsilon(t)$ is the stretched domain of $\tilde{\Omega}_\varepsilon(t)$. It is more convenient to rewrite (4.1) in the following form.

$$(4.2) \qquad \begin{cases} \varepsilon^{-(1-\alpha)} u_t = \varepsilon^{1-\alpha}\Delta_{\mathbf{y}} u + \varepsilon^{-(1-\alpha)} f(u,v), \\ \varepsilon^{2\alpha} v_t = \Delta_{\mathbf{y}} v + \varepsilon^{2\alpha} g(u,v). \end{cases}$$

Suppose $\hat{\Omega}_\varepsilon(t)$ has a smooth limit $\hat{\Omega}(t)$ as $\varepsilon \downarrow 0$, and taking a limiting procedure similar to [3], we obtain the following interfacial dynamics.

$$(4.3) \qquad \begin{cases} \varepsilon^{-(1-\alpha)}\Gamma_t = \{W(v|_\Gamma) - \varepsilon^{1-\alpha}\kappa\}\mathbf{N} & \text{on } \Gamma(t), \\ \varepsilon^{2\alpha} v_t^\pm = \Delta_{\mathbf{y}} v^\pm + \varepsilon^{2\alpha} g(h^\pm(v^\pm), v^\pm) & \text{in } \hat{\Omega}^\pm(t), \end{cases}$$

where $\Gamma(t)$ stands for the limiting configuration of the interface, κ denotes the mean curvature of $\Gamma(t)$, $\mathbf{N}$ is the unit normal vector at Γ pointing from $\hat{\Omega}^+$ to $\hat{\Omega}^-$, $W(\cdot)$ is the travelling velocity of the first equation of (1.1) with $\varepsilon = 1$ for a fixed v and typically a monotone decreasing function of v, the domain $\hat{\Omega}(t)$ is divided into two parts $\hat{\Omega}^{\pm}(t)$ by $\Gamma(t)$ where $u = h^{\pm}(v)$ on each subdomain, respectively, and v is matched in C^1-sense at $\Gamma(t)$. In view of the second equation of (4.3), $v^{\pm}$ may be expanded as

$$v^{\pm} = v_0^{\pm}(\mathbf{y}, t) + \varepsilon^{2\alpha} v_1^{\pm}(\mathbf{y}, t) + O(\varepsilon^{4\alpha}). \tag{4.4}$$

Substituting (4.4) into (4.3), and equating like the powers of ε, we easily see that $v_0^{\pm} \equiv v^*$, where v^* is the equal area level of $f(u, v)$ (see (A.3)) with $W(v^*) = 0$. Expanding $W(v|_\Gamma)$ into Taylor series, the principal part of the next order of (4.3) becomes

$$\begin{cases} \varepsilon^{-(1-\alpha)}\Gamma_t = \{\varepsilon^{2\alpha} W'(v^*) v_1|_\Gamma - \varepsilon^{1-\alpha}\kappa\}\mathbf{N} & \text{on } \Gamma(t), \\ 0 = \Delta_{\mathbf{y}} v_1^{\pm} + g(h^{\pm}(v^*), v^*) & \text{in } \hat{\Omega}^{\pm}(t). \end{cases} \tag{4.5}$$

The first term of the right-hand side of (4.5) is the bulk force and the second one is the mean curvature effect. In order to make these two terms comparable, namely, in order that the bulk force is balanced with the curvature effect, the exponent α must be taken as $\alpha = \frac{1}{3}$. Suppose $\alpha \neq \frac{1}{3}$, then either the bulk force or the curvature effect becomes dominant as $\varepsilon \downarrow 0$, hence there is no chance to have nontrivial stationary patterns of finite size in such a ε^{α}-rescaled domain. Employing this exponent $\alpha = \frac{1}{3}$ and introducing a new time scale $\tau \equiv \varepsilon^{4/3} t$, the rescaled interfacial dynamics is given by

$$\begin{cases} \Gamma_\tau = \{W'(v^*) v_1|_\Gamma - \kappa\}\mathbf{N} & \text{on } \Gamma(t), \\ 0 = \Delta_{\mathbf{y}} v_1^{\pm} + g(h^{\pm}(v^*), v^*) & \text{in } \hat{\Omega}^{\pm}(t). \end{cases} \tag{4.6}$$

Suppose $\tilde{\Omega}_\varepsilon$ is unit cell of a periodic structure such as hexagonal lattice and that $\tilde{\Omega}_\varepsilon / \varepsilon^{1/3}$ has a definite limit $\hat{\Omega}$ as $\varepsilon \downarrow 0$, then the stationary problem of (4.6)

$$\begin{cases} 0 = \{W'(v^*) v_1|_\Gamma - \kappa\}\mathbf{N} & \text{on } \Gamma, \\ 0 = \Delta_{\mathbf{y}} v_1^{\pm} + g(h^{\pm}(v^*), v^*) & \text{in } \hat{\Omega}^{\pm}. \\ v_1^{\pm} \text{ are matched in } C^1\text{-sense at } \Gamma, \end{cases} \tag{4.7}$$

is expected to give a stable morphology of unit cell. We call (4.7) the *morphology equations* of (1.1). Note that (4.7) is exactly the same as (2.19) in [6] where he used the matched asymptotic method to obtain it. However little is known about the richness of the solution set of (4.7) as well as their geometric profiles.

There is another observation due to [4] for a related system to (1.1) from a different point of view, which claims that the global minimizer of the following functional must oscillate rapidly with frequency being proportional to $\varepsilon^{1/3}$. The functional is given by

$$\int_\Omega \{\varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u) + \frac{1}{\varepsilon}|(-\Delta + \gamma I)^{-1/2} u|^2\} dx, \tag{4.8}$$

where $W(u)$ is a double-well potential like $u^4/4 - u^2/2$. This is related to our problem in the following sense. Suppose the relaxation time of v is much shorter than u (i.e., the quasi-static assumption for v is valid), then (1.1) can be replaced by

$$(4.9)\qquad \begin{cases} u_t = \varepsilon\Delta u + \frac{1}{\varepsilon}f_0(u) - \frac{1}{\varepsilon}v, & \\ & (x,t) \in \Omega \times (0,\infty), \\ 0 = D\Delta v + u - \gamma v, & \\ \dfrac{\partial u}{\partial n} = 0 = \dfrac{\partial v}{\partial n} & (x,t) \in \partial\Omega \times (0,\infty), \end{cases}$$

where $f_0(u) = u - u^3$. Solving the second equation with respect to v and substituting it to the first equation, we have a scalar equation for u with nonlocal term

$$u_t = \varepsilon\Delta u + \frac{1}{\varepsilon}f_0(u) - \frac{1}{\varepsilon}(-\Delta + \gamma I)^{-1}u,$$

which is the L^2-gradient equation of the functional (4.8). Suppose that $\Omega = Q = (0,1)^{\mathrm{N}}$ (N-dimensional cube) with periodic boundary conditions, we see by employing the arguments of [4] that the global minimizer u_ε of (4.8) has to satisfy the following inequality:

$$(4.10)\qquad C_1\varepsilon^{-1/3} \leq \frac{\displaystyle\int_Q |\nabla H(u_\varepsilon)|dx}{\displaystyle\int_Q |u_\varepsilon|dx} \leq C_2\varepsilon^{-1/3},$$

where $H(z) = \int_0^z W^{1/2}(s)ds$ and C_1,C_2 are positive constants independent of ε. Roughly speaking, the middle term of (4.10) counts the number of interface, and hence, (4.10) means that the global minimizer has to take a fine structure, although we do not know whether u_ε is spacially periodic or not. Finally it should be noted that the estimate (4.10) is valid only for the global minimizer and not for the other local minimizers.

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