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FIRST VARIATION OF HOLOMORPHIC FORMS AND SOME APPLICATIONS

BAHMAN KHANEDANI AND TATSUO SUWA

ABSTRACT. We study various local invariants associated with a singular holomorphic foliation on a complex surface admitting a possibly singular invariant curve. We establish the relation among them and prove/reprove formulas relating the total sum of these invariants to some global invariants of the foliation and the invariant curve.

For a holomorphic vector field v on a complex surface leaving a non-singular curve C invariant, C. Camacho and P. Sad [CS] introduced the index of v relative to C and proved an index formula, which says that the total sum of the indices is equal to the Chern number of the normal bundle of C. After the work of a number of authors, the theory has been generalized to the case of singular invariant curves in [S], and further, to the higher dimensional case in [LS]. In [S], the index formula was proved by taking desingularization of the curve and reducing to the case of non-singular invariant curves, while the proof in [LS] involves the Chern-Weil theory, the vanishing theorem and so forth. In this article, we first give a direct proof of the index theorem for a singular foliation $\mathcal{F}$ on a complex surface leaving a (possibly singular) compact curve C invariant by explicitly computing the Chern class of the normal bundle of C (Theorem 1.2).

We then consider "exponent forms" for holomorphic 1-forms defining the foliation $\mathcal{F}$ and define the "variation" of $\mathcal{F}$ relative to C at a singular point as the residue of an exponent form along the link of the singularity in C. This turns out to be a localized class of the (co)normal bundle of the foliation (Theorem 2.2). We extend the notion of the "multiplicity" of a vector field v along a (locally) irreducible invariant curve [CLS] to the case of possibly reducible curves so that it coincides with the "Schwartz index" [SS] of the restriction of v to the curve. After establishing the relation among these invariants in Lemma 2.3, we give a formula for the total sum of the (Schwartz) indices in Theorem 2.6, which is the "Poincaré-Hopf theorem" for a singular foliation, with possibly non-trivial tangent bundle, on a singular curve.

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In the final section, we discuss the geometric meaning of the variation and give an alternative proof of the fact that the index of $\mathcal{F}$ relative to C represents the first order term of the holonomy along the link of the singularity in C, which was shown earlier in [S].

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1. The index formula

We generally use the notation and the definitions in [S]. First we consider everything in a neighborhood of the origin 0 in $\mathbb{C}^2 = \{(x, y)\}$. Let v be a germ of holomorphic vector field at 0 with (at most) an isolated singularity at 0 and ω a germ of holomorphic 1-form with an isolated singularity at 0 which annihilates v. More explicitly, if $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ with a and b germs of holomorphic functions at 0, we may set $\omega = b\,dx - a\,dy$. Also, let C be a germ of reduced curve with defining function f. We quote Lemma (1.1) in [S]:

Lemma 1.1. *The vector field v leaves C invariant if and only if there exist germs of holomorphic functions g and h and a germ of holomorphic 1-form η such that h and f are relatively prime and that*

$$g\omega = hdf + f\eta. \tag{1.1}$$

The lemma is proved in [Li] when f is irreducible. Note that if ω is non-singular at 0, C is also non-singular at 0 and, by a suitable choice of f, we may set $\eta = 0$. Denoting by $\mathcal{F}$ the foliation defined by v (or ω), we define the index of $\mathcal{F}$ relative to C at 0 by

$$\mathrm{Ind}_0(\mathcal{F}; C) = \frac{\sqrt{-1}}{2\pi}\int_L \frac{\eta}{h},$$

where L denotes the link of the singularity 0 in C with natural orientation. When f is irreducible, this coincides with the one defined in [Li]. See [S] Proposition (1.4) for their relation in the general case.

Now let X be a (non-singular) complex surface. Recall that a (co)dimension one (singular) foliation $\mathcal{F}$ on X is defined by a system $\{(U_\lambda, \omega_\lambda, \varphi_{\lambda\mu})\}$, where
(i) $\{U_\lambda\}$ is an open covering of X,
(ii) for each λ, ω_λ is a (not identically zero) holomorphic 1-form on U_λ and
(iii) for each pair (λ, μ), $\varphi_{\lambda\mu}$ is a non-vanishing holomorphic function on $U_\lambda \cap U_\mu$ with $\omega_\mu = \varphi_{\lambda\mu}\omega_\lambda$.

The singular set $S(\mathcal{F})$ of $\mathcal{F}$ is defined to be the union of the singular sets of the ω_λ's. We assume that $S(\mathcal{F})$ consists of isolated points hereafter.

Theorem 1.2. *For a (co)dimension one foliation $\mathcal{F}$ on X and a compact reduced curve C in X which is invariant by $\mathcal{F}$, we have*

$$\sum_{p\in S} \mathrm{Ind}_p(\mathcal{F}; C) = C\cdot C,$$

where S denotes the set of singular points of $\mathcal{F}$ on C and $C\cdot C$ the self-intersection number of C.

This is proved in [S] Theorem (2.1) and the higher dimensional case is in [LS]. Here we give a simple direct proof.

Proof. We let $S=\{p_1,\dots,p_r\}$ and take a system $\{(U_\lambda,\omega_\lambda,\varphi_{\lambda\mu})\}$ as above so that it further satisfies:
(iv) C is defined by f_λ on U_λ,
(v) for each p_i, there is only one U_{λ_i} with $p_i\in U_{\lambda_i}$ and $U_{\lambda_i}\cap U_{\lambda_j}=\varnothing$, if $i\neq j$.

If we set $f_{\lambda\mu}=\frac{f_\lambda}{f_\mu}$ on $U_\lambda\cap U_\mu$, then the cocycle $\{f_{\lambda\mu}\}$ defines the line bundle L_C on X associated with the divisor C. We compute $c_1(L_C)\frown [C]=\int_C c_1(L_C)$ in two ways. First, since $c_1(L_C)$ is the Poincaré dual to the homology class [C], we see that it is equal to the self-intersection number $C\cdot C$. Next we compute it directly. If we let $\{\rho_\lambda\}$ be a partition of unity subordinate to $\{U_\lambda\}$, we have

$$c_1(L_C)|_{U_\lambda}=\frac{\sqrt{-1}}{2\pi}\sum_\mu d(\rho_\mu\, d\log f_{\mu\lambda}).$$

On each U_λ, we have a decomposition

$$g_\lambda\omega_\lambda=h_\lambda\, df_\lambda+f_\lambda\eta_\lambda \tag{1.1_λ}$$

as (1.1). We may assume that $\eta_\lambda=0$ for $\lambda\neq\lambda_i$. Evaluation of the both sides of the identity (1.1_λ) at each point of $U_\lambda\cap C$ gives

$$g_\lambda\omega_\lambda=h_\lambda\, df_\lambda. \tag{1.2_λ}$$

Also, from $dg_\lambda\wedge\omega_\lambda+g_\lambda d\omega_\lambda=(dh_\lambda-\eta_\lambda)\wedge df_\lambda+f_\lambda\, d\eta_\lambda$ and (1.2_λ), we have, at each point of $U_\lambda\cap C$,

$$d\omega_\lambda=\left(-\frac{\eta_\lambda}{h_\lambda}+d\log\frac{h_\lambda}{g_\lambda}\right)\wedge\omega_\lambda. \tag{1.3_λ}$$

From (1.2_λ) and (1.2_μ), we have, in $U_\lambda\cap U_\mu\cap C$,

$$\frac{h_\mu}{g_\mu}=f_{\lambda\mu}\varphi_{\lambda\mu}\frac{h_\lambda}{g_\lambda}. \tag{1.4}$$

Also, from (1.3_λ) and (1.3_μ), we have, in $U_\lambda \cap U_\mu \cap C$,

$$d\log \varphi_{\lambda\mu} = \frac{\eta_\lambda}{h_\lambda} - \frac{\eta_\mu}{h_\mu} + d\log \frac{h_\mu}{g_\mu} - d\log \frac{h_\lambda}{g_\lambda}. \tag{1.5}$$

Hence from (1.4) and (1.5), we have, at each point of $U_\lambda \cap U_\mu \cap C$,

$$d\log f_{\mu\lambda} = \frac{\eta_\lambda}{h_\lambda} - \frac{\eta_\mu}{h_\mu}. \tag{1.6}$$

Let $C' = C - \mathrm{Sing}(C)$ be the set of regular points of C (note that $\mathrm{Sing}(C) \subset S$). Then, from (1.6), we have

$$c_1(L_C)|_{U_\lambda \cap C'} = \frac{\sqrt{-1}}{2\pi} \sum_\mu d\rho_\mu \wedge \left(\frac{\eta_\lambda}{h_\lambda} - \frac{\eta_\mu}{h_\mu}\right) = -\frac{\sqrt{-1}}{2\pi} \sum_\mu d\rho_\mu \wedge \frac{\eta_\mu}{h_\mu}.$$

Since $\eta_\lambda = 0$ for $\lambda \neq \lambda_i$, we have

$$\int_C c_1(L_C) = \int_{C'} c_1(L_C) = \sum_{i=1}^r \int_{U_{\lambda_i} \cap C'} c_1(L_C).$$

We denote by D_{λ_i} a disk in U_{λ_i} with center p_i such that $\rho_{\lambda_i} \equiv 1$ on D_{λ_i}. Note that $\partial D_{\lambda_i} \cap C = L_{\lambda_i}$, the link of C at p_i. Then we have

$$\begin{aligned}
\int_{U_{\lambda_i} \cap C'} c_1(L_C) &= -\frac{\sqrt{-1}}{2\pi} \int_{U_{\lambda_i} \cap C'} d\rho_{\lambda_i} \wedge \frac{\eta_{\lambda_i}}{h_{\lambda_i}} \\
&= -\frac{\sqrt{-1}}{2\pi} \int_{(U_{\lambda_i} - D_{\lambda_i}) \cap C'} d\rho_{\lambda_i} \wedge \frac{\eta_{\lambda_i}}{h_{\lambda_i}} \\
&= -\frac{\sqrt{-1}}{2\pi} \int_{(U_{\lambda_i} - D_{\lambda_i}) \cap C'} d\left(\rho_{\lambda_i} \frac{\eta_{\lambda_i}}{h_{\lambda_i}}\right) \\
&= \frac{\sqrt{-1}}{2\pi} \int_{L_{\lambda_i}} \rho_{\lambda_i} \frac{\eta_{\lambda_i}}{h_{\lambda_i}} \\
&= \frac{\sqrt{-1}}{2\pi} \int_{L_{\lambda_i}} \frac{\eta_{\lambda_i}}{h_{\lambda_i}} = \mathrm{Ind}_{p_i}(\mathcal{F}; C). \qquad \square
\end{aligned}$$

2. Exponent forms

Suppose $\mathcal{F}$ is a germ of foliation at 0 in $\mathbb{C}^2$ with defining 1-form ω (or vector field v) and C a germ of reduced curve with defining function f which is invariant by $\mathcal{F}$. In a neighborhood of a non-singular point, there exists a holomorphic 1-form

α such that $d\omega = \alpha \wedge \omega$. If α' is another such 1-form, we have $\alpha' \equiv \alpha$ on every leaf. Thus in a neighborhood of 0 (away from 0) there exists a holomophic multi-valued 1-form α such that $d\omega = \alpha \wedge \omega$ and that its restriction to each leaf is single-valued. We call α an *exponent form* for ω. We consider the residue of α along C;

$$\mathrm{Res}_0(\alpha|_C) = \frac{1}{2\pi\sqrt{-1}} \int_L \alpha,$$

where L is the link of 0 in C as before.

Lemma 2.1. *The residue* $\mathrm{Res}_0(\alpha|_C)$ *is an invariant of the foliation.*

Proof. Suppose $\omega' = \varphi\omega$ with φ a non-vanishing holomorphic function. We have

$$d\omega' = d\varphi \wedge \omega + \varphi d\omega = d\varphi \wedge \omega + \varphi\alpha \wedge \omega = (\alpha + d\log\varphi) \wedge \omega'.$$

Since φ is non-vanishing, we obtain $\int_L (\alpha + d\log\varphi) = \int_L \alpha$. $\square$

In view of the above lemma, we set

$$\mathrm{Var}_0(\mathcal{F}; C) = \mathrm{Res}_0(\alpha|_C)$$

and call it the variation of $\mathcal{F}$ relative to C at 0. Note that if $C = \cup_{i=1}^r C_i$ is the irreducible decomposition of C at 0, $\mathcal{F}$ leaves each component C_i invariant and we have

$$\mathrm{Var}_0(\mathcal{F}; C) = \sum_{i=1}^{r} \mathrm{Var}_0(\mathcal{F}; C_i). \tag{2.1}$$

Now we go back to the global situation as in Theorem 1.2 and suppose the foliation $\mathcal{F}$ is defined on a complex surface X by a system $\{(U_\lambda, \omega_\lambda, \varphi_{\lambda\mu})\}$. Let T^*X denote the (holomorphic) cotangent bundle of X and F the line bundle defined by the cocycle $\{\varphi_{\lambda\mu}\}$. Then we have a bundle map on X;

$$F \xrightarrow{\omega} T^*X,$$

which is injective on $X - S(\mathcal{F})$. We call F the conormal bundle of the foliation $\mathcal{F}$.

Theorem 2.2. *In the above situation, if C is a compact curve in X invariant by $\mathcal{F}$, we have*

$$\sum_{p \in S} \mathrm{Var}_p(\mathcal{F}; C) = -c_1(F) \frown [C].$$

Proof. Take a system $\{(U_\lambda, \omega_\lambda, \varphi_{\lambda\mu})\}$ defining $\mathcal{F}$ so that it satiesfies also (iv) and (v) in the proof of Theorem 1.5. Let α_λ be an exponent form for ω_λ. For $\lambda \neq \lambda_i$, we may set $\alpha_\lambda = 0$, since we may choose a closed form as ω_λ. As in Theorem 1.2, we have

$$c_1(F)|_{U_\lambda} = \frac{\sqrt{-1}}{2\pi}\sum_\mu d(\rho_\mu\, d\log\varphi_{\mu\lambda}).$$

In $U_\lambda \cap U_\mu \cap C$, we have

$$d\log\varphi_{\lambda\mu} = \alpha_\lambda - \alpha_\mu$$

and the rest is done similarly as for Theorem 1.2. $\square$

Let C be a germ of reduced curve at 0 in $\mathbb{C}^2$ invariant by a foliation $\mathcal{F}$ defined by v. If C is irreducible, according to [CLS], one defines the *multiplicity of v along C at* 0 to be the topological index of $v|_C$ at 0, where C is seen as being homeomorphic to a two dimensional disk. Since it is also an invariant of the foliation $\mathcal{F}$, we denote it by $\mathrm{Ind}_0(\mathcal{F}_C)$. In general, let $C = \cup_{i=1}^r C_i$ be the irreducible decomposition of C at 0. We define $\mathrm{Ind}_0(\mathcal{F}_C)$ by

$$\mathrm{Ind}_0(\mathcal{F}_C) = \sum_{i=1}^r \mathrm{Ind}_0(\mathcal{F}_{C_i}) - r + 1 \tag{2.2}$$

and call it the index of the restriction of $\mathcal{F}$ to C at 0. Note that it coincides with the "Schwartz index" of $v|_C$ at 0 in the sense of [SS]. Recall that the Milnor number $\mu_0(C)$ of C at 0 is given by $\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]_0$, the intersection number of the curves defined by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at 0.

Lemma 2.3. *We have*

$$\mathrm{Ind}_0(\mathcal{F}_C) = \mathrm{Var}_0(\mathcal{F}; C) - \mathrm{Ind}_0(\mathcal{F}; C) + \mu_0(C).$$

Proof. First we prove the lemma when C is irreducible. If we take a decomposition as in Lemma 1.1, at each point of C we have (see (1.3))

$$d\omega = \left(-\frac{\eta}{h} + d\log\frac{h}{g}\right)\wedge\omega.$$

Hence we get

$$\mathrm{Var}_0(\mathcal{F}; C) = \mathrm{Ind}_0(\mathcal{F}; C) + [h, f]_0 - [g, f]_0. \tag{2.3}$$

Now, by a suitable choice of coordinates (x, y) of $\mathbb{C}^2$, we may set $g = \frac{\partial f}{\partial y}$ and $h = -a$, when we write $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ (see the proof of Lemma (1.1) in [S]). By [CLS] Proposition 3, $\mathrm{Ind}_0(\mathcal{F}_C)$ is computed as follows. Let $\pi : (D, 0) \to (C, 0)$ be a Puiseux parametrization. Then the vector field V in $D = \{t\}$ with $\pi_* V = v|_C$ is given by $V = \frac{a}{\dot{x}}\frac{d}{dt}$, $\dot{x} = \frac{dx}{dt}$. Thus

$$\text{(2.4)} \qquad \mathrm{Ind}_0(\mathcal{F}_C) = [h, f]_0 - [x, f]_0 + 1.$$

On the other hand, we know from [Li] (8) that

$$\text{(2.5)} \qquad \mu_0(C) = \left[\frac{\partial f}{\partial y}, f\right]_0 - [x, f]_0 + 1.$$

and the formula follows from (2.3), (2.4) and (2.5). Next, in general, if $C = \cup_{i=1}^r C_i$ is the irreducible decomposition of C, we have ([S] (1.11))

$$\mathrm{Ind}_0(\mathcal{F}; C) - \mu_0(C) = \sum_{i=1}^{r} (\mathrm{Ind}_0(\mathcal{F}; C_i) - \mu_0(C_i)) + r - 1.$$

Hence the lemma follows from the formula for the irreducible case together with (2.1) and (2.2). □

Remark 2.4. Let $\mathcal{F}^\circ$ be the foliation defined by df. Then, since we may set $\alpha = 0$ we have $\mathrm{Var}_0(\mathcal{F}^\circ; C) = 0$. Also, since we may set $\eta = 0$ in (1.1), we have $\mathrm{Ind}_0(\mathcal{F}^\circ; C) = 0$ and $\mathrm{Ind}_0(\mathcal{F}^\circ; C_i) = -\sum_{j \neq i} (C_i \cdot C_j)_0$ ([S] Proposition (1.4). Note that $\mathrm{Ind}_0(\mathcal{F}^\circ; C, C_i) = 0$ in the notation used there). Thus, by Lemma 2.3, we have

$$\mathrm{Ind}_0(\mathcal{F}^\circ_C) = \mu_0(C) \quad \text{and} \quad \mathrm{Ind}_0(\mathcal{F}^\circ_{C_i}) = \mu_0(C_i) + \sum_{j \neq i} (C_i \cdot C_j)_0.$$

The first equality also follows from the fact that the vector field defining $\mathcal{F}^\circ$ is tangent to the nearby Milnor fibers of f and has no singularities on the fiber ([SS] Proposition 5.3). The second equality shows that $\mathrm{Ind}_0(\mathcal{F}^\circ_{C_i})$ coincides with $c_0(C, C_i)$ in [S] (1.8). If we set $c_0(C) = \sum_{i=1}^r c_0(C, C_i)$, it is related to the Milnor number by $c_0(C) = \mu_0(C) + r - 1$ ([S] (1.9)).

The above remark may be used to prove the "adjunction formula" as follows, although we should note that the argument is essentially equivalent to the one in [K]. Let C be a compact (reduced) curve in a surface X. We take a covering $\{U_\lambda\}$ of X by coordinate neighborhoods with coordinates (x_λ, y_λ) so that C is defined by $f_\lambda = 0$ in U_λ. Let $\mathcal{F}^\circ_\lambda$ be the foliation on U_λ defined by df_λ. Then it is defined by the vector field $v_\lambda = \frac{\partial f_\lambda}{\partial y_\lambda}\frac{\partial}{\partial x_\lambda} - \frac{\partial f_\lambda}{\partial x_\lambda}\frac{\partial}{\partial y_\lambda}$. By computation, we see that, in $U_\lambda \cap U_\mu \cap C$,

$$v_\lambda = f_{\lambda\mu} \kappa_{\lambda\mu} v_\mu,$$

where $\kappa_{\lambda\mu} = \det \frac{\partial(x_\mu, y_\mu)}{\partial(x_\lambda, y_\lambda)}$, the Jacobian of (x_μ, y_μ) with respect to (x_λ, y_λ). Thus, if we let $\pi : \tilde{C} \to C \subset X$ be a resolution of C, the collection $\{v_\lambda|_C\}$ determines a section of the line bundle $\pi^*(L_C \otimes K_X) \otimes T\tilde{C}$, where K_X denotes the canonical bundle of X and $T\tilde{C}$ the tangent bundle of $\tilde{C}$. Hence from the second equality in Remark 2.4, we have the adjunction formula

$$\chi(\tilde{C}) = -K_X \cdot C - C \cdot C + \sum_{p \in S} c_p(C),$$

where $\chi(\tilde{C})$ denotes the Euler number of $\tilde{C}$ and $K_X \cdot C = c_1(K_X) \frown [C]$. Since the Euler number $\chi(C)$ of C is given by $\chi(C) = \chi(\tilde{C}) - \sum_{p \in S}(r_p - 1)$ with r_p the number of local branches of C at p, we have

$$\chi(C) = -K_X \cdot C - C \cdot C + \sum_{p \in S} \mu_p(C), \tag{2.6}$$

which is a special case of the formula in [SS] Theorem 5.5.

From Theorem 1.2 and (2.6), we have the following formula, which is a modified form of the one in [S] Theorem (2.5).

Theorem 2.5. *Let X, $\mathcal{F}$ and C be as in Theorem 1.2. We have*

$$\sum_{p \in S} (\mathrm{Ind}_p(\mathcal{F}; C) - \mu_p(C)) = -K_X \cdot C - \chi(C).$$

Now we recall that a foliation $\mathcal{F}$ on a complex surface X is also defined by a system $\{(U_\lambda, v_\lambda, \varepsilon_{\lambda\mu})\}$, where
(i) $\{U_\lambda\}$ is an open covering of X,
(ii)$'$ for each λ, v_λ is a (not identically zero) holomorphic vector field on U_λ and
(iii)$'$ for each pair (λ, μ), $\varepsilon_{\lambda\mu}$ is a non-vanishing holomorphic function on $U_\lambda \cap U_\mu$ with $v_\mu = \varepsilon_{\lambda\mu} v_\lambda$.

A system $\{(U_\lambda, \omega_\lambda, \varphi_{\lambda\mu})\}$ of 1-forms and a system $\{(U_\lambda, v_\lambda, \varepsilon_{\lambda\mu})\}$ of vector fields define the same foliation $\mathcal{F}$ if, for each λ, ω_λ and v_λ have isolated singularities and they annihilate each other. Suppose this is the case. Then the singular set $S(\mathcal{F})$ of $\mathcal{F}$ coincides with the union of the singular sets of the v_λ's. Let TX denote the tangent bundle of X and E the line bundle defined by the the cocycle $\{\varepsilon_{\lambda\mu}\}$. Then we have a bundle map on X;

$$E \xrightarrow{v} TX,$$

which is injective on $X - S(\mathcal{F})$. We call E the tangent bundle of the foliation $\mathcal{F}$. By a straightforward computation using the explicit relation between the forms and the vector fields defining $\mathcal{F}$, we have

$$F = E \otimes K_X.$$

Therefore, from Lemma 2.3 and Theorems 2.2 and 2.5, we have

Theorem 2.6. *For a foliation $\mathcal{F}$ on a complex surface X leaving a compact curve C invariant, we have*

$$\sum_{p\in S} \mathrm{Ind}_0(\mathcal{F}_C) = \chi(C) - c_1(E) \frown [C].$$

In particular, if $\mathcal{F}$ is defined by a global vector field, then, since E becomes trivial,

$$\sum_{p\in S} \mathrm{Ind}_0(\mathcal{F}_C) = \chi(C).$$

The second formula above is a special case of the Poincaré-Hopf theorem for singular varieties ([SS] Theorem 5.4). Also, when C is non-singular, the right hand side of the first formula above is equal to the Chern number of the normal sheaf of the foliation induced from $\mathcal{F}$ on C (cf. [BB]).

We finish this section by giving a remark on the topological invariance of some invariants associated with holomorphic foliations. Recall that the Milnor number is a topological invariant [Lê] and that the local intersection number of two analytic curves is also a topological invariant [GH]. We say that two foliations are topologically equivalent if there is a homeomorphism between the ambient spaces preserving the singular sets and the leaves. Let $\mathcal{F}$ be a foliation on a surface leaving a curve C invariant. If C is irreducible at a point p, it is shown that $\mathrm{Ind}_p(\mathcal{F}_C)$ is a topological invariant of holomorphic foliations [CLS]. Hence, by (2.2), it is a topological invariant in general. Thus, from Theorems 1.2, 2.2 and 2.6 and Lemma 2.3, we have;

Proposition 2.7. *For a foliation $\mathcal{F}$ on a surface X admitting a compact invariant curve C, $c_1(F) \frown [C]$ and $c_1(E) \frown [C]$ are topological invariants.*

Note that, in [GSV], it is already shown that $c_1(E)$ is a topological invariant of a dimension one foliation.

3. Relation with holonomy

Let $\mathcal{F}$ be a foliation on a complex surface and γ a loop in a leaf of $\mathcal{F}$. Suppose for the moment that $\mathcal{F}$ is defined by a *closed* multi-valued 1-form ω in a neighborhood of γ. Fixing a point p_0 on γ, let ω_0 be the restriction of a branch of ω to a neighborhood of p_0 and let ω_1 be the branch obtained after one revolution around γ. Then there exists a holomorphic function φ defined in a neighborhood of x_0 so that $\varphi\omega_1 = \omega_0$. Recall that the multiplier of $\mathcal{F}$ relative to γ is the derivative of the holonomy mapping at its basepoint.

Lemma 3.1. *In the above situation, the multiplier is given by $\varphi(p_0)$.*

Proof. Let p be a point in γ. Since ω is assumed to be closed, there is a biholomorphic map ζ_p, by the Frobenius theorem (or simply by 'straightening out'), from an open neighborhood U_p of p onto a neighborhood of 0 in $\mathbb{C}^2 = \{(x,y)\}$, $\zeta_p(p) = 0$, such that $\zeta_p^* dy = \omega|_{U_p}$. By compactness of γ, there is a finite set of charts $\{(U_i, \zeta_i)\}$, $i = 0, \cdots, n$, with $p_0 \in U_0 \cap U_n$, $U_i \cap U_{i+1} \neq \varnothing$, $\zeta_0^* dy = \omega_0$, and $\zeta_i^* dy$ equal to the restriction of the branch of ω to U_i obtained by analytic continuation along γ. We have $\zeta_i^* dy = \zeta_{i+1}^* dy$ in the common domain, from which we deduce that the second coordinate of $(\zeta_{i+1} \circ \zeta_i^{-1})(x,y)$ is y. Now $\zeta_0^* dy = \omega_0 = \varphi\omega_1 = \varphi\zeta_n^* dy$, and writing $\zeta_0 \circ \zeta_n^{-1} = (x', y')$, we see that $\varphi \circ \zeta_n^{-1}$ is equal to $\frac{\partial y'}{\partial y}$ and $\frac{\partial y'}{\partial x} = 0$. □

Suppose $\mathcal{F}$ is defined by a holomorphic 1-form ω in a neighborhood of γ. Then one can write $d\omega = \alpha \wedge \omega$, where α is a multi-valued 1-form in a neighborhood of γ, and the restriction of α to every leaf is single-valued.

Theorem 3.2. *The multiplier of $\mathcal{F}$ relative to γ is given by* $\exp\left(\int_\gamma \alpha\right)$.

Proof. We have $d\omega = \alpha \wedge \omega$ as above. Let Γ be a local transversal at a point p_0 of γ. Denote by h the backward projection on Γ along the leaves, defined in a neighborhood of γ. For p in a neighborhood of γ, define:

$$g(p) = \exp\left(-\int_{h(p)}^{p} \alpha\right),$$

where integration is performed along a curve from $h(p)$ to p on the leaf going through p which defines the holonomy. Since any two such curves are homotopic, the integration is well-defined. We have

$$d(g\omega) = dg \wedge \omega + g\, d\omega = -g \cdot d\left(\int_{h(p)}^{p} \alpha\right) \wedge \omega + g\alpha \wedge \omega.$$

Now we take a biholomorphic map ζ from a neighborhood of p_0 onto a neighborhood of 0 in $\mathbb{C}^2 = \{(x,y)\}$ such that $\zeta^* dy$ defines the foliation $\mathcal{F}$ in a neighborhood of p_0. Writing $\alpha = \zeta^*(k_1 dx + k_2 dy)$, we have, for p in a neighborhood of p_0, $\int_{h(p)}^{p} \alpha = \int_0^{x(p)} k_1 dx$ so that:

$$d\left(\int_{h(p)}^{p} \alpha\right) = \zeta^* d\left(\int_0^{x(p)} k_1 dx\right) = \zeta^*\left(k_1 dx + \left(\int_0^{x(p)} \frac{\partial k_1}{\partial y} dx\right) dy\right).$$

Therefore using analytic continuation we obtain:

$$d\left(\int_{h(p)}^{p} \alpha\right) \wedge \omega = \alpha \wedge \omega.$$

Then

$$d(g\omega) = -g\alpha \wedge \omega + g\alpha \wedge \omega = 0.$$

Applying Lemma 3.1 to the closed multi-valued 1-form $g\omega$, we obtain that the multiplier is $g(p_0)^{-1} = \exp(\int_\gamma \alpha)$, as desired. $\square$

Now let $\mathcal{F}$ be a germ of foliation at 0 in $\mathbb{C}^2$ and C a germ of reduced and irreducible curve which is invariant by $\mathcal{F}$. Since $\mathrm{Ind}_0(\mathcal{F}_C)$ and $\mu_0(C)$ are integers, from Lemma 2.3 we obtain the following result, which is proved in [S] Proposition (3.1) by different approach.

Corollary 3.3. *The quantity* $\exp\left(2\pi\sqrt{-1}\,\mathrm{Ind}_0(\mathcal{F}, C)\right)$ *gives the multiplier of* $\mathcal{F}$ *relative to the link of the singularity* 0 *in* C.

Note: After the preparation of the manuscript, the recent preprint of M. Brunella [B] was brought to our attention. Theorem 2.2 above together with Theorem 1.2 and Lemma 2.3 implies the first formula in [B] Lemme 3 and Theorem 2.6 is equivalent to the second formula there. We note that the formulas in [B] are given under the assumption that the ambient surface be compact, which is not necessary in this article.

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