Singular Degenerate Prarabolic Equations with Applications to the p-Laplace Diffusion Equation

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Singular Degenerate Prarabolic Equations with Applications to the p-Laplace Diffusion Equation

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Abstract. We consider singular degenerate parabolic equations including the p-Laplace diffusion equation. We establish a comparison priciple which is a natural extension of the paper [12] by Ishii and Souganidis. Once we get a comparison priciple we can construct the unique global-in-time viscosity solution to the Cauchy problem for the p-Laplace diffusion equation. The solution is bounded, uniformly continuous in $[0,T) \times \mathbf{R}^N$ if the initial data is bounded, uninformly continuous on $\mathbf{R}^N$.

§1. Introduction

We consider a singular degenerate parabolic equation of the form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \text{ in } Q_T = (0,T) \times \Omega, \tag{1.1}$$

where Ω is a domain in $\mathbf{R}^N$ and $T > 0$. Here $u_t = \partial u/\partial t$, ∇u and $\nabla^2 u$ denote, respectively, the time derivative of u, the gradient of u and the Hessian of u in space variables. The function $F = F(q,X)$ needs not to be bounded arouned $q = 0$ even for fixed X and F needs not to be geometric in the sense of [2].

A typical example is the p-Laplace diffusion equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \text{ in } Q_T = (0,T) \times \Omega, \tag{1.2}$$

especially when $p < 2$. For this equation, $F = F(q,X)$ is given by

$$F(q,X) = -|q|^{p-2}\operatorname{trace}\left\{(I + (p-2)\frac{q \otimes q}{|q|^2})X\right\}, \tag{1.3}$$

where $\otimes$ denotes the tensor product.

Our major goal is to extend the theory of viscosity solutions for singular degenerate parabolic equations including this type of equation. We introduce a notion of viscosity solutions so that the comparison principle holds. This is considered as a natural extension of the work of [12], where they assumed F is geometric in the sense of [2].

We shall establish the comparison theorem and the existence theorem based on Perron's method for a large class of equations including all geometric equations and the p-Laplace diffusion equation for $1 < p < 2$. (Note if $p \geq 2$, the p-Laplace diffusion equation has no singularity at $\nabla u = 0$). The proof of comparison theorem parallels that of [12]. However we give a simple proof by restricting bounded domain Ω. The proof of existence theorem needs a new choice of sub- and supersolution. Since we do not assume that F is geometric, an extra effort is necessary.

We apply our theory to the Cauchy problem for the p-Laplace diffusion equation. We prove the unique existence of global-in-time solution with every initial data which is bounded, uniformly continuous in $\mathbf{R}^N$ provided that $1 < p \leq 2$. The solution is bounded, uniformly continuous on $[0,T] \times \mathbf{R}^N$ for every $T > 0$. This result has already been known by interpreting solutions as usual weak solutions. The existence of the unique global-in-time weak solution was proved in [13]. The continuity of such a weak solution is known. For details, see the book [5] by DiBenedetto. However, since the proof of continuity was done by using the Harnack inequality and many a priori estimates, we need many procedures to get such a solution. We believe our approach is simple to construct continuous solutions. Moreover, our theory does not require the divergence structure of the equation. For example our theory applies to

$$u_t - |\nabla u|^{p-2}\mathrm{trace}\left\{(I + (p'-2)\frac{\nabla u \otimes \nabla u}{|\nabla u|^2})\nabla^2 u\right\} = 0, \tag{1.4}$$

where $p' \geq 1$. Note that if $p' = p$ this is nothing but the p-Laplace diffusion equation. We note that a C^2 solution of

$$|\nabla u|^{2-p} u_t - |\nabla u|^{2-p}\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$$

needs not to be a viscosity solution of (1.2). To see this we consider a symmetric nonnegative separable function

$$u(t,x) = \begin{cases} (t_* - t)^{\frac{1}{2-p}} U(x) & \text{if } t < t_*,\ 0 < x < 1, \\ 0 & \text{if } t \geq t_*,\ 0 < x < 1, \end{cases}$$

with $u(t,0) = u(t,1) = 0$ for fixed $t_* > 0$, where U is a positive C^2 function. For fixed t_* there are infinitely many C^2 solution u of the one dimensional equation

$$u_t |u_x|^{2-p} = (p-1)u_{xx},\ (t,x) \in (0,T) \times (0,1)$$

with $u(t,0) = u(t,1) = 0$. However, only one of them is the viscosity solution of (1.2).

Finally, we establish a general stability result. This would be useful to prove that our solution agrees with usual weak solution when the equation is the p-Laplace diffusion equation. However, we do not pursue this problem in this paper.

In the theory of viscosity solutions, many degenerate parabolic equations have been studied. Here we focus on the singularity of those equations. For example the existence and uniqueness of solutions for a given initial data was established in [2] when $F = F(q,X)$ is singular on $q = 0$ assuming that F can be extended continuously at $(q,X) = (0,O)$. The result is applicable to the level set equation of the mean curvature flow equation. Further developments are obtained in [14], [8], [6], [11]. Moreover, the existence and uniqueness theorems are obtained in [7], [12] even if F can not be extended continuously at $(q,X) = (0,O)$. This paper is organized as follows. In section 2 we give a notion of viscosity solutions and some basic propositions. In section 3 we establish the comparison principle of the viscosity solutions when Ω is bounded. Also we remark the comparison theorem for a domain not necessarily bounded. In section 4 we prove the existence of the solution to the Cauchy problem using Perron's method. In section 5 we consider the separable solutions to the p-Laplace diffusion equation with one space dimension. We prove that one of them is a viscosity solution. In section 6 we give a general stability result.

§2. Viscosity Solutions

We list the assumptions of F which are important to consider the viscosity solutions of (1.1).

(F1) F is continuous in $(\mathbf{R}^N\backslash\{0\}) \times S^N$.

(F2) F is degenerate elliptic, *i.e.*, if $X \geq Y$, then $F(q,X) \leq F(q,Y)$ for all $q \in (\mathbf{R}^N\backslash\{0\})$.

Definition 2.1 *We denote by $\mathcal{F}(F)$ the set of function $f \in C^2[0,\infty)$ which satisfies*

$$f(0) = f'(0) = f''(0) = 0,\ f''(r) > 0 \text{ for all } r > 0, \tag{2.1}$$

and

$$\lim_{|x|\to 0, x\neq 0} F(\nabla f(|x|), \nabla^2 f(|x|)) = 0. \tag{2.2}$$

Remark 2.2 *For F of (1.3) with $1 < p \leq 2$ we note $f(r) = r^{1+\sigma}$ with $\sigma > \dfrac{1}{p-1} \geq 1$ is a function in $\mathcal{F}(F)$. For F of (1.3) with $p > 2$ we see $f(r) = r^2 \in \mathcal{F}(F)$. Our definition of $\mathcal{F}(F)$ is an extension of that in [12]. Actually, if F is geometric in the sense of [2], i.e.,*

$$F(\lambda q, \lambda X + \mu q \otimes q) = \lambda F(q,X) \text{ for all } \lambda > 0, \mu \in \mathbf{R}, q \in (\mathbf{R}^N\backslash\{0\}), X \in S^N,$$

the set $\mathcal{F}(F)$ is the same in [12]. The set $\mathcal{F}(F)$ in [12] is not empty provided F satisfies (F1) and (F2), but our $\mathcal{F}(F)$ could be empty. Indeed, we know $\mathcal{F}(F)$ is empty for F of (1.3) with $p \leq 1$.

Definition 2.3 *A function $\varphi \in C^2(Q_T)$ is admissible(i.e., $\varphi \in \mathcal{A}(F)$) if for any $\hat{z} = (\hat{t}, \hat{x}) \in Q_T$ with $\nabla\varphi(\hat{z}) = 0$, there are a constant $\delta > 0$,$f \in \mathcal{F}(F)$ and $\omega \in C[0,\infty)$ satisfying $\omega \geq 0$ and $\lim_{r\downarrow 0}\omega(r)/r = 0$ such that for all $z = (t, x)$, $|z - \hat{z}| < \delta$ satisfies*

$$|\varphi(z) - \varphi(\hat{z}) - \varphi_t(\hat{z})(t - \hat{t})| \leq f(|x - \hat{x}|) + \omega(|t - \hat{t}|). \tag{2.3}$$

Note that if φ is admissible, so is $-\varphi$.

Next we recall that the upper semicontinuous envelope u^* and the lower semicontinuous envelope u_* of a function $u : Q_T \to \mathbf{R} \cup \{\pm\infty\}$ are defined by

$$\begin{aligned} u^*(z) &= \limsup_{r\downarrow 0}\{u(\zeta); |\zeta - z| < r\}, \\ u_*(z) &= \liminf_{r\downarrow 0}\{u(\zeta); |\zeta - z| < r\}, \end{aligned}$$

respectively.

Definition 2.4 *Assume that (F1) and (F2) hold and that $\mathcal{F}(F) \neq \emptyset$.*

1. *A function $u : Q_T \to \mathbf{R} \cup \{-\infty\}$ is a viscosity subsolution of (1.1) in Q_T if $u^* < \infty$ on $\overline{Q_T}$ and for all $\varphi \in \mathcal{A}(F)$ and all local maximum points z of $u^* - \varphi$ in Q_T,*

$$\begin{cases} \varphi_t(z) + F(\nabla\varphi(z), \nabla^2\varphi(z)) \leq 0 & \text{if } \nabla\varphi(z) \neq 0, \\ \varphi_t(z) \leq 0 & \text{otherwise.} \end{cases}$$

2. *A function $u : Q_T \to \mathbf{R} \cup \{\infty\}$ is a viscosity supersolution of (1.1) in Q_T if $u_* > -\infty$ on $\overline{Q_T}$ and for all $\varphi \in \mathcal{A}(F)$ and all local minimum points z of $u_* - \varphi$ in Q_T,*

$$\begin{cases} \varphi_t(z) + F(\nabla\varphi(z), \nabla^2\varphi(z)) \geq 0 & \text{if } \nabla\varphi(z) \neq 0, \\ \varphi_t(z) \geq 0 & \text{otherwise.} \end{cases}$$

3. *A viscosity solution of (1.1) in Q_T is a function which is both a viscosity sub- and super- solution of (1.1) in Q_T.*

We often suppress the word "viscosity", except in statements of theorems. As usual we obtain basic properties of viscosity solutions. We state them without their proof.

Proposition 2.5 *Assume that (F1) and (F2) hold and $\mathcal{F}(F) \neq \emptyset$. Let $\mathcal{S}$ be a set of subsolutions of (1.1) in Q_T. Set*

$$u(z) := \sup\{v(z); v \in \mathcal{S}, z \in Q_T\}.$$

If $u^* < \infty$ in $\overline{Q_T}$, then u is a subsolution of (1.1) in Q_T. A similar assertion holds for supersolutions of (1.1) in Q_T.

Proposition 2.6 *Assume that (F1) and (F2) hold and $\mathcal{F}(F) \neq \emptyset$. Let $\mathcal{S}$ be a set of subsolutions of (1.1) in Q_T. Let ℓ and h be a subsolution and a supersolution of (1.1) in Q_T, respectively. Assume that ℓ and h are locally bounded in Q_T and $\ell \leq h$ holds. We define u as follows:*

$$u(z) := \sup\{v(z);\ v \in \mathcal{S},\ \ell \leq v \leq h \text{ in } Q_T,\ z \in Q_T\}.$$

Then u is a solution of (1.1) in Q_T.

§3. Comparison Theorem

Let Ω be a domain in $\mathbf{R}^N$ and $T > 0$. We consider a degenerate parabolic equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \text{ in } Q_T = (0,T) \times \Omega, \tag{3.1}$$

In section 4 we shall consider the initial value problem of (1.1) as a Cauchy problem. So we have to obtain the comparison theorem in the case $\Omega = \mathbf{R}^N$. The proof is almost same as in [12]. For readers we shall give a comparison theorem for a bounded domain Ω since we believe the proof is more simple than that of the $\Omega = \mathbf{R}^N$. After we proved the theorem, we shall give a lemma for the proof of the theorem in the case $\Omega = \mathbf{R}^N$. We need the lemma because our definition of $\mathcal{F}(F)$ is different from that in [12]. Here we give our main theorem of the comparison principle. For $Q_T = (0,T) \times \Omega$, we call

$$\partial_p Q_T = \{0\} \times \Omega \cup [0,T] \times \partial\Omega \tag{3.2}$$

the parabolic boundary of Q_T.

Theorem 3.1 (Comparison Theorem) *Let Ω be a bounded domain in $\mathbf{R}^N$ and $T > 0$. Suppose that F satisfies (F1) and (F2) and that $\mathcal{F}(F) \neq \emptyset$. Let u and v be a viscosity sub- and super- solution of (1.1) in Q_T, respectively. If $u^* \le v_*$ on $\partial_p Q_T$, then $u^* \le v_*$ in Q_T.*

We argue by contradiction. Let w be upper semicontinuous on $\overline{Q_T} \times \overline{Q_T}$, and $\overline{Q_T}$ be compact. Then there is a positive constant M satisfying

$$w(t,x,s,y) \le M \text{ for all } (t,x),\ (s,y) \in \overline{Q_T}. \tag{3.3}$$

Now, we set

$$\alpha = \sup\{w(t,x,t,x);\ x \in \overline{\Omega}, 0 \le t < T\}, \tag{3.4}$$

$$\Phi(t,x,s,y) = \frac{1}{\varepsilon}P(|x-y|) + \frac{1}{\varepsilon}(t-s)^2 + \frac{\gamma}{T-s} + \frac{\gamma}{T-t}, \tag{3.5}$$

where $\varepsilon, \gamma > 0$, with $P \in \mathcal{F}(F)$. We define $\Psi(t,x,s,y) := w(t,x,s,y) - \Phi(t,x,s,y)$.

Proposition 3.2 *Suppose that w is upper semicontinuous on $\overline{Q_T} \times \overline{Q_T}$ and $\alpha > 0$. Then there is a positive constant γ_0 such that*

$$\sup\{\Psi(t,x,s,y); (t,x),(s,y) \in \overline{Q_T}\} \ge \frac{\alpha}{2} \tag{3.6}$$

holds for all $0 < \gamma < \gamma_0$, $\varepsilon > 0$.

Proof. Since w is bounded on $\overline{Q_T} \times \overline{Q_T}$ and $\alpha > 0$, we see that there is a point $(t_0, x_0) \in \overline{Q_T}$ such that $w(t_0,x_0,t_0,x_0) > \frac{3}{4}\alpha$. Choose γ which is sufficiently small so that

$$\frac{2\gamma}{T-t_0} < \frac{\alpha}{4}. \tag{3.7}$$

Then we see

$$\sup\{\Psi(t,x,s,y); (t,x),(s,y) \in \overline{Q_T}\} \ge \Psi(t_0,x_0,t_0,x_0) \ge \frac{\alpha}{2}.$$

∎

Proposition 3.3 *Suppose that w be upper semicontinuous on $\overline{Q_T} \times \overline{Q_T}$ and $\alpha > 0$. Let γ_0 be as in Proposition 3.2. Let Ψ attain its maximum at $(\hat{t},\hat{x},\hat{s},\hat{y}) \in \overline{Q_T} \times \overline{Q_T}$ for all $0 < \gamma < \gamma_0$. Then $|\hat{x}-\hat{y}| \to 0$ and $|\hat{t}-\hat{s}| \to 0$ as $\varepsilon \to 0$; these convergences are uniform in $0 < \gamma < \gamma_0$.*

Proof. By Proposition 3.2, $\Psi(\hat{t},\hat{x},\hat{s},\hat{y}) > 0$. Therefore,

$$\frac{1}{\varepsilon}P(|\hat{x}-\hat{y}|) + \frac{1}{\varepsilon}(\hat{t}-\hat{s})^2 + \frac{\gamma}{T-\hat{s}} + \frac{\gamma}{T-\hat{t}} < w(\hat{t},\hat{x},\hat{s},\hat{y}) \le M.$$

Note that $P \in \mathcal{F}(F)$ is monotone increasing since P satisfies (2.1). Its inverse function denotes P^{-1}. Thus we obtain

$$|\hat{x}-\hat{y}| \le P^{-1}(\varepsilon M),\ |\hat{t}-\hat{s}| \le (\varepsilon M)^{1/2}. \tag{3.8}$$

∎

By those propositions, we now get:

Proposition 3.4 *Suppose that w is upper semicontinuous on $\overline{Q_T} \times \overline{Q_T}$ and $\alpha > 0$. Let γ_0 be as in Proposition 3.2. Set $U = Q_T \times Q_T$ and $\partial_p U = (\partial_p Q_T \times \overline{Q_T}) \cup (\overline{Q_T} \times \partial_p Q_T)$. Suppose that there is a modulus function m (i.e., $m : [0,\infty) \to [0,\infty)$ such that continuous, nondecreasing and $m(0) = 0$) which satisfies $w(t,x,s,y) \le m(|x-y| + |t-s|)$ for all $(t,x,s,y) \in \partial_p U$. Then there exists $\varepsilon_0 > 0$ such that Ψ attains its maximum over $\overline{U}$ at an interior point $(\hat{t},\hat{x},\hat{s},\hat{y}) \in U$ for all $0 < \varepsilon < \varepsilon_0$ and $0 < \gamma < \gamma_0$.*

Proof. Suppose the conclusion were false. By the properties of the function $\gamma/(T-s) + \gamma/(T-t)$ we see $\hat{t} < T$ and $\hat{s} < T$. There would exist sequences $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$ and $\{\gamma_j\} \subset (0,\gamma_0)$ such that $\partial_p U$ contains a maximum point $(\hat{t}_j, \hat{x}_j, \hat{s}_j, \hat{y}_j)$ of Ψ for the value $\varepsilon = \varepsilon_j$, $\gamma = \gamma_j$. By Proposition 3.2 and the assumption of Proposition 3.4, we see

$$\frac{\alpha}{2} \le \Psi(\hat{t}_j, \hat{x}_j, \hat{s}_j, \hat{y}_j) \le w(\hat{t}_j, \hat{x}_j, \hat{s}_j, \hat{y}_j) \le m(|\hat{x}_j - \hat{y}_j| + |\hat{t}_j - \hat{s}_j|). \tag{3.9}$$

Letting $\varepsilon_j \to 0$, and applying Proposition 3.3 yields $|\hat{x}_j - \hat{y}_j| \to 0$ and $|\hat{t}_j - \hat{s}_j| \to 0$, which leads a contradiction. ∎

We give basic properties on the relation between elliptic superjets $J^{2,+}$ and parabolic superjets $\mathcal{P}^{2,+}$. Moreover, we shall obtain a similar relation for $\overline{J^{2,+}}$ and $\overline{\mathcal{P}^{2,+}}$. Although we do not state explicitly, those relations hold for elliptic subjets $J^{2,-}$ and parabolic subjets $\mathcal{P}^{2,-}$. For the definitions of $J^{2,\pm}$, $\mathcal{P}^{2,\pm}$, $\overline{J^{2,\pm}}$ and $\overline{\mathcal{P}^{2,\pm}}$, see the review paper [4].

Lemma 3.5 *Suppose*

$$\left(\begin{pmatrix} \tau \\ q \end{pmatrix}, \begin{pmatrix} a & {}^t l \\ l & X \end{pmatrix} \right) \in J^{2,+} u(\hat{t}, \hat{x}),$$

where $(\tau, q) \in \mathbf{R} \times \mathbf{R}^N$, $a \in \mathbf{R}$, $l \in \mathbf{R}^N$, $X \in \mathcal{S}^N$. Then $(\tau, q, X) \in \mathcal{P}^{2,+} u(\hat{t}, \hat{x})$.

Proof. The assumption is equivalent to

$$\begin{aligned} u(t,x) \;\le\; & u(\hat{t},\hat{x}) + \tau(t-\hat{t}) + \langle q, x-\hat{x}\rangle + \frac{1}{2}a(t-\hat{t})^2 + (t-\hat{t})\langle l, x-\hat{x}\rangle \\ & + \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle + o(|t-\hat{t}|^2 + |x-\hat{x}|^2) \text{ as } t \to \hat{t}, x \to \hat{x}, \end{aligned}$$

where $\langle , \rangle$ denotes the Euclidean inner product. By Young's inequality we see

$$\begin{aligned} (t-\hat{t})\langle l, x-\hat{x}\rangle &\le |l||t-\hat{t}||x-\hat{x}| \\ &\le |l|(\frac{2}{3}|t-\hat{t}|^{3/2} + \frac{1}{3}|x-\hat{x}|^3) \\ &= o(|t-\hat{t}| + |x-\hat{x}|^2). \end{aligned}$$

Finally we obtain

$$u(t,x) \le u(\hat{t},\hat{x}) + \tau(t-\hat{t}) + \langle q, x-\hat{x}\rangle + \tfrac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle + o(|t-\hat{t}| + |x-\hat{x}|^2) \quad \text{as } t \to \hat{t},\ x \to \hat{x}.$$

∎

By the definitions of $\overline{J^{2,+}}$, $\overline{\mathcal{P}^{2,+}}$, we see:

Corollary 3.6 *Suppose*

$$\left(\begin{pmatrix} \tau \\ q \end{pmatrix}, \begin{pmatrix} a & {}^t l \\ l & X \end{pmatrix} \right) \in \overline{J^{2,+}} u(\hat{t}, \hat{x}),$$

where $(\tau, q) \in \mathbf{R} \times \mathbf{R}^N$, $a \in \mathbf{R}$, $l \in \mathbf{R}^N$, $X \in \mathcal{S}^N$. Then $(\tau, q, X) \in \overline{\mathcal{P}^{2,+}} u(\hat{t}, \hat{x})$.

We shall state a lemma on admissible test functions without its proof.

Lemma 3.7 *Assume that (F1) and (F2) hold and that $\mathcal{F}(F) \neq \emptyset$. Suppose that $\varphi : Q_T \to \mathbf{R}$ is admissible (i.e., $\varphi \in \mathcal{A}(F)$). If $\nabla\varphi(\hat{t},\hat{x}) = 0$ where $(\hat{t},\hat{x}) \in Q_T$ with $\hat{t} < T$, then $\varphi(t,x) + \frac{\gamma}{T-t}$ is also admissible for all $\gamma > 0$.*

Now we are position to the proof of Theorem 3.1.

Proof. We may assume that u and v are upper semicontinuous and lower semicontinuous, respectively, in $\overline{Q_T}$ so that

$$w(t,x,s,y) = u(t,x) - v(s,y)$$

is upper semicontinuous in $\overline{U} = \overline{Q_T} \times \overline{Q_T}$. Note that $u \leq v$ on $\partial_p Q_T$ implies the existence of modulus m satisfying

$$w(t,x,s,y) \leq m(|x-y| + |t-s|) \text{ on } \partial_p U = (\partial_p Q_T \times \overline{Q_T}) \cup (\overline{Q_T} \times \partial_p Q_T)$$

since $\partial_p Q_T$ is compact. We assume that

$$\alpha = \sup\{w(t,x,t,x);\, x \in \overline{\Omega}, 0 \leq t < T\} > 0,$$

then derive a contradiction. By the assumption $\alpha > 0$ we verify all conclusion of Proposition 3.2 - 3.4 would hold for $\Psi = w - \Phi$. We know Ψ attains its maximum over $\overline{U}$ at $(\hat{t},\hat{x},\hat{s},\hat{y}) \in U = Q_T \times Q_T$ for small ε, γ because Proposition 3.4 holds. Since

$$\Psi(t,x,\hat{s},\hat{y}) \leq \Psi(\hat{t},\hat{x},\hat{s},\hat{y}),$$

we obtain

$$\begin{aligned} &u(t,x) - v(\hat{s},\hat{y}) - \frac{1}{\varepsilon}P(|x-\hat{y}|) - \frac{1}{\varepsilon}(t-\hat{s})^2 - \frac{\gamma}{T-t} - \frac{\gamma}{T-\hat{s}} \\ &\leq u(\hat{t},\hat{x}) - v(\hat{s},\hat{y}) - \frac{1}{\varepsilon}P(|\hat{x}-\hat{y}|) - \frac{1}{\varepsilon}(\hat{t}-\hat{s})^2 - \frac{\gamma}{T-\hat{t}} - \frac{\gamma}{T-\hat{s}}. \end{aligned}$$

By Lemma 3.7,

$$\varphi^+(t,x) := \frac{1}{\varepsilon}P(|x-\hat{y}|) + \frac{1}{\varepsilon}(t-\hat{s})^2 + \frac{\gamma}{T-t}$$

is an admissible test function which satisfies

$$\max_{(t,x)\in\overline{Q_T}}(u-\varphi^+) = (u-\varphi^+)(\hat{t},\hat{x}).$$

In the same way since $\Psi(\hat{t},\hat{x},s,y) \leq \Psi(\hat{t},\hat{x},\hat{s},\hat{y})$ holds,

$$\begin{aligned} &u(\hat{t},\hat{x}) - v(s,y) - \frac{1}{\varepsilon}P(|\hat{x}-y|) - \frac{1}{\varepsilon}(\hat{t}-s)^2 - \frac{\gamma}{T-\hat{t}} - \frac{\gamma}{T-s} \\ &\leq u(\hat{t},\hat{x}) - v(\hat{s},\hat{y}) - \frac{1}{\varepsilon}P(|\hat{x}-\hat{y}|) - \frac{1}{\varepsilon}(\hat{t}-\hat{s})^2 - \frac{\gamma}{T-\hat{t}} - \frac{\gamma}{T-\hat{s}}. \end{aligned}$$

Hence,

$$\varphi^-(s,y) := -\frac{1}{\varepsilon}P(|\hat{x}-y|) - \frac{1}{\varepsilon}(\hat{t}-s)^2 - \frac{\gamma}{T-s}$$

is an admissible test function which satisfies

$$\min_{(s,y)\in\overline{Q_T}}(v-\varphi^-) = (v-\varphi^-)(\hat{s},\hat{y}).$$

1. In the case $\hat{x} = \hat{y}$.
   Since $\nabla\varphi^+(\hat{t},\hat{x}) = \nabla\varphi^-(\hat{s},\hat{y}) = 0$, the definitions of viscosity subsolution and supersolution yield the following inequalities;

$$\varphi_t^+(\hat{t},\hat{x}) = \frac{2}{\varepsilon}(\hat{t}-\hat{s}) + \frac{\gamma}{(T-\hat{t})^2} \leq 0, \tag{3.10}$$

$$\varphi_s^-(\hat{s},\hat{y}) = \frac{2}{\varepsilon}(\hat{t}-\hat{s}) - \frac{\gamma}{(T-\hat{s})^2} \geq 0. \tag{3.11}$$

Subtracting (3.10) from (3.11), we obtain

$$\frac{\gamma}{(T-\hat{t})^2} + \frac{\gamma}{(T-\hat{s})^2} \leq 0. \tag{3.12}$$

Thus we get a contradiction.

2. In the case $\hat{x} \neq \hat{y}$.

We set $\xi = (t,x)$, $\eta = (s,y)$. Since $w(\xi,\eta) - \Phi(\xi,\eta) \leq w(\hat{\xi},\hat{\eta}) - \Phi(\hat{\xi},\hat{\eta})$,

$$\left(\begin{pmatrix} \hat{\Phi}_\xi \\ \hat{\Phi}_\eta \end{pmatrix}, A \right) \in J^{2,+}w(\hat{\xi},\hat{\eta}),$$

with $\hat{\xi} = (\hat{t},\hat{x})$, $\hat{\eta} = (\hat{s},\hat{y})$, and

$$A = \begin{pmatrix} \hat{\Phi}_{\xi\xi} & \hat{\Phi}_{\xi\eta} \\ \hat{\Phi}_{\eta\xi} & \hat{\Phi}_{\eta\eta} \end{pmatrix}.$$

where $\hat{\Phi}_\xi = \nabla_\xi \Phi(\hat{\xi},\hat{\eta})$, $\hat{\Phi}_{\xi\xi} = \nabla^2{}_{\xi\xi}\Phi(\hat{\xi},\hat{\eta})$ and so on. Applying the elliptic version of Crandall-Ishii's Lemma (c.f. [3], [4]), we see that

for all positive λ, there exist $X_1, Y_1 \in S^{N+1}$ such that

(i)

$$\left(\hat{\Phi}_\xi, X_1\right) \in \overline{J^{2,+}}u(\hat{t},\hat{x}),$$

$$\left(\hat{\Phi}_\eta, Y_1\right) \in \overline{J^{2,+}}(-v(\hat{s},\hat{y})), \textit{i.e.}, \ \left(-\hat{\Phi}_\eta, -Y_1\right) \in \overline{J^{2,-}}(v(\hat{s},\hat{y})).$$

(ii)

$$-(\frac{1}{\lambda} + ||A||)I_{2N+2} \leq \begin{pmatrix} X_1 & O \\ O & Y_1 \end{pmatrix} \leq A + \lambda A^2. \tag{3.13}$$

Here, $||A||$ denotes the operator norm of A. Since $X_1, Y_1 \in S^{N+1}$, we can represent that

$$X_1 = \begin{pmatrix} a & {}^t l \\ l & X_2 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} b & {}^t m \\ m & Y_2 \end{pmatrix},$$

for some $a, b \in \mathbf{R}$, $l, m \in \mathbf{R}^N$ and X_2, Y_2 in S^N. By Corollary 3.6, we see

$$(\hat{\Phi}_t, \hat{\Phi}_x, X_2) \in \overline{\mathcal{P}^{2,+}}u(\hat{t},\hat{x}), \quad (-\hat{\Phi}_s, -\hat{\Phi}_y, -Y_2) \in \overline{\mathcal{P}^{2,-}}v(\hat{s},\hat{y}).$$

Since u is a viscosity subsolution, we get

$$0 \geq \hat{\Phi}_t + F(\hat{\Phi}_x, X_2) = \frac{2(\hat{t}-\hat{s})}{\varepsilon} + \frac{\gamma}{(T-\hat{t})^2} + F(\hat{\Phi}_x, X_2). \tag{3.14}$$

In the same way, we get

$$0 \leq -\hat{\Phi}_s + F(-\hat{\Phi}_y, -Y_2) = \frac{2(\hat{t}-\hat{s})}{\varepsilon} - \frac{\gamma}{(T-\hat{s})^2} + F(\hat{\Phi}_x, -Y_2). \tag{3.15}$$

Subtracting (3.15) from (3.14), we get

$$\frac{\gamma}{T^2} + F(\hat{\Phi}_x, X_2) - F(\hat{\Phi}_x, -Y_2) \leq 0. \tag{3.16}$$

From matrix inequality (3.13), we observe that $X_2 + Y_2 \le O$. Since F is degenerate elliptic we observe that

$$\begin{aligned} 0 &\ge \frac{\gamma}{T^2} + F(\hat{\Phi}_x, X_2) - F(\hat{\Phi}_x, -Y_2) \\ &\ge \frac{\gamma}{T^2} + F(\hat{\Phi}_x, X_2) - F(\hat{\Phi}_x, X_2) \\ &= \frac{\gamma}{T^2}. \end{aligned} \tag{3.17}$$

We get a contradiction. Now we have completed the proof of Theorem 3.1.

∎

Lemma 3.8 *Let Ω be a domain in $\mathbf{R}^N$ not necessarily bounded and $T > 0$. Suppose that F satisfies (F1) and (F2) and that $\mathcal{F}(F) \neq \emptyset$. Let u and v be upper semicontinuous and lower semicontinuous on $[0,T) \times \overline{\Omega}$. Suppose that u and v are a viscosity subsolution and a supersolution of (1.1) in $Q_T = (0,T) \times \Omega$, respectively. Set*

$$u(T,x) := \lim_{r \to 0} \sup\{u(s,y); (s,y) \in Q_T, |y-x| + |s-T| \le r\},$$

$$v(T,x) := \lim_{r \to 0} \inf\{v(s,y); (s,y) \in Q_T, |y-x| + |s-T| \le r\}.$$

Then u and v are a viscosity subsolution and a supersolution of (1.1) in $(0,T] \times \Omega$, respectively.

Proof. We only prove the subsolution case. Let $\varphi \in \mathcal{A}(F)$ and let $\hat{z} = (T, \hat{y})$ satisfy

$$\max_{z \in (0,T] \times \Omega} (u - \varphi)(z) = (u - \varphi)(\hat{z}).$$

We may assume that $u - \varphi$ attains its strict maximum at $\hat{z}$. Then we shall show

$$\begin{cases} \varphi_t(z) + F(\nabla\varphi(z), \nabla^2\varphi(z)) \le 0 & \text{if } \nabla\varphi(z) \neq 0, \\ \varphi_t(z) \le 0 & \textit{otherwise}. \end{cases}$$

1. In the case $\nabla\varphi(\hat{z}) \neq 0$.

   For sufficiently large n, there exists $(t_n, y_n) \in Q_T$ such that

   $$\max_{z \in (0,T] \times \Omega} \left(u - \varphi - \frac{1}{n(T-t)} \right) = (u - \varphi)(t_n, y_n) - \frac{1}{n(T-t_n)}$$

   with $(t_n, y_n) \to \hat{z}$ as $n \to \infty$. Thanks to Lemma 3.7, $\varphi + \dfrac{1}{n(T-t)} \in \mathcal{A}(F)$. Since $\nabla\varphi$ is continuous on Q_T, $\nabla\varphi(t_n, y_n) \neq 0$ for sufficiently large n. Moreover, u is a subsolution of (1.1) in Q_T,

   $$\begin{aligned} 0 &\ge \varphi_t(t_n, y_n) + F(\nabla\varphi(t_n, y_n), \nabla^2\varphi(t_n, y_n)) + \frac{1}{n(T-t_n)^2} \\ &\ge \varphi_t(t_n, y_n) + F(\nabla\varphi(t_n, y_n), \nabla^2\varphi(t_n, y_n)). \end{aligned}$$

   Letting $n \to \infty$ yields

   $$0 \ge \varphi_t(\hat{z}) + F(\nabla\varphi(\hat{z}), \nabla^2\varphi(\hat{z})).$$

2. In the case $\nabla\varphi(\hat{z}) = 0$. Since $\varphi \in \mathcal{A}(F)$, there are $\delta > 0$, $f \in \mathcal{F}(F)$ and $\omega(r) \in C[0,\infty)$ satisfying $\omega(r) \ge 0$ and $\omega(r) = o(r)$ such that far all $z = (t,x) \in Q_T$ if $|z - \hat{z}| < \delta$ then

   $$|\varphi(z) - \varphi(\hat{z}) - \varphi_t(\hat{z})(t-T)| < f(|x - \hat{x}|) + \omega(T-t).$$

   Set $\psi(t,x) = \varphi_t(\hat{z})(t-T) + 2f(|x-\hat{y}|) + 2\omega(T-t)$. We may assume that $\omega(r) \in C^2[0,\infty)$, $\omega(0) = \omega'(0) = 0$ and $\omega(r) > 0$ for all $r > 0$. Note that $\psi \in \mathcal{A}(F)$ and

   $$\max_{z \in (0,T] \times \Omega} (u - \psi)(z) = (u - \psi)(\hat{z})$$

with $\hat{z} = (T, \hat{y})$. As in the former case, there exists $(t_n, y_n) \in Q_T$ such that

$$\max_{z \in (0,T] \times \Omega} \left(u - \psi - \frac{1}{n(T-t)} \right) = (u - \psi)(t_n, y_n) - \frac{1}{n(T-t_n)}$$

with $(t_n, y_n) \to \hat{z}$ as $n \to \infty$. Set $\psi_n(t,x) = \psi(t,x) + \dfrac{1}{n(T-t)}$, then $\psi_n \in \mathcal{A}(F)$. We shall show $\varphi_t(\hat{z}) \leq 0$. We consider the following two cases.

(a) In the case $\nabla \psi_n(t_n, y_n) \neq 0$. Since u is a subsolution in Q_T, we observe

$$\begin{aligned} 0 &\geq (\psi_n)_t(t_n, y_n) + F(\nabla \psi_n(t_n, y_n), \nabla^2 \psi_n(t_n, y_n)) \\ &= \psi_t(t_n, y_n) + \frac{1}{n(T-t_n)^2} + F(\nabla \psi(t_n, y_n), \nabla^2 \psi(t_n, y_n)) \\ &\geq \varphi_t(\hat{z}) + 2\omega'(T - t_n) + F(2\nabla f(|y_n - \hat{y}|), 2\nabla^2 f(|y_n - \hat{y}|)). \end{aligned}$$

Since $f \in \mathcal{F}(F)$, letting $n \to \infty$ yields $0 \geq \varphi_t(\hat{z})$.

(b) In the case $\nabla \psi_n(t_n, y_n) = 0$. Since u is a subsolution in Q_T, we observe

$$\begin{aligned} 0 &\geq (\psi_n)_t(t_n, y_n) \\ &= \psi_t(t_n, y_n) + \frac{1}{n(T-t_n)^2} \\ &\geq \varphi_t(\hat{z}) + 2\omega'(T - t_n). \end{aligned}$$

Letting $n \to \infty$ yields $0 \geq \varphi_t(\hat{z})$.

∎

Now we state the comparison principle for (1.1) when the domain Ω is not necessarily bounded.

Theorem 3.9 *Let Ω be a domain in $\mathbf{R}^N$ not necessarily bounded and $T > 0$. Suppose that F satisfies (F1) and (F2) and that $\mathcal{F}(F) \neq \emptyset$. Let u and v be upper semicontinuous and lower semicontinuous on $[0,T) \times \overline{\Omega}$, respectively. Let u and v be a viscosity sub- and super- solution of (1.1) in Q_T, respectively. Assume that*

$$\lim_{r \to 0} \sup\{u(z) - v(\zeta) | (z, \zeta) \in (\partial_p Q_T \times ([0,T) \times \overline{\Omega})) \cup (([0,T) \times \overline{\Omega}) \times \partial_p Q_T),\ |z - \zeta| \leq r\} \leq 0. \quad (3.18)$$

Then

$$\lim_{r \to 0} \sup\{u(z) - v(\zeta) | z, \zeta \in [0,T) \times \overline{\Omega},\ |z - \zeta| \leq r\} \leq 0.$$

Moreover, $u \leq v$ in $[0,T) \times \overline{\Omega}$.

§4. Existence of Solutions.

We shall construct a viscosity solution to the Cauchy problem of (1.1). We shall use Perron's method for this purpose. To apply Perron's method, we shall construct a subsolution and a supersolution of (1.1) for a given initial data. In this section, we only prove the supersolution case, since the subsolution case can be proved similarly. From the degenerate elliptic condition (F2), we have a sufficient condition that a C^2 function to be a supersolution or a subsolution.

Lemma 4.1 *Assume that F satisfies (F1) and (F2). Suppose that $\mathcal{F}(F) \neq \emptyset$. If $u \in C^2(Q_T)$ satisfies*

$$\begin{cases} u_t(z) + F(\nabla u(z), \nabla^2 u(z)) \geq 0 & \text{if } \nabla u(z) \neq 0, \\ u_t(z) \geq 0 & \text{if } \nabla u(z) = 0, \end{cases}$$

$$\left(\text{resp. } \begin{cases} u_t(z) + F(\nabla u(z), \nabla^2 u(z)) \leq 0 & \text{if } \nabla u(z) \neq 0, \\ u_t(z) \leq 0 & \text{if } \nabla u(z) = 0, \end{cases} \right)$$

then u is a viscosity supersolution (resp. subsolution) of (1.1) in Q_T.

Proof. Let $\varphi \in \mathcal{A}(F)$ and let $\hat{z}$ satisfies $\min_{Q_T}(u-\varphi) = (u-\varphi)(\hat{z})$. Then we have $u_t = \varphi_t, \nabla u = \nabla\varphi, \nabla^2 u \geq \nabla^2\varphi$ at $z = \hat{z}$.

In the case $\nabla\varphi(\hat{z}) \neq 0$, *i.e.*, $\nabla u(\hat{z}) \neq 0$. By the assumptions and (F2), we obtain

$$0 \leq u_t(\hat{z}) + F(\nabla u(\hat{z}), \nabla^2 u(\hat{z})) \leq \varphi_t(\hat{z}) + F(\nabla\varphi(\hat{z}), \nabla^2\varphi(\hat{z})).$$

In the case $\nabla\varphi(\hat{z}) = 0$, *i.e.*, $\nabla u(\hat{z}) = 0$. We know $0 \leq u_t(\hat{z}) = \varphi(\hat{z})$. Thus u is a viscosity supersolutions of (1.1) in Q_T. ∎

We construct a viscosity supersolution and subsolution of (1.1) satisfying the assumption of Lemma 4.1. We introduce a set of function $\mathcal{G}$;

$$\mathcal{G} := \{g(r) \in C^2[0,\infty), g(0) = g'(0) = 0, g'(r) > 0 \text{ if } r > 0, \lim_{r\to\infty} g(r) = \infty\}.$$

Note that $g(|x|) \in C^2(\mathbf{R}^N)$ if $g \in \mathcal{G}$. A direct calculation yields

$$\nabla^2 g(|x|) = \frac{g'(|x|)}{|x|}I + \left(g''(|x|) - \frac{g'(|x|)}{|x|}\right)\left(\frac{x}{|x|} \otimes \frac{x}{|x|}\right).$$

Although $\nabla^2 g(|x|)$ does not appear to be continuous at $x = 0$, it is continuous. Indeed, $\nabla^2 g(0) = g''(0)I$ holds since $\lim_{r\to 0} \frac{g'(r)}{r} = g''(0)$ by the definition of $\mathcal{G}$.

Our goal is to construct a viscosity solution for a given initial data a. By Perron's method it suffices to construct a sub- and a supersolution for a given initial data. Since the constructions of a sub- and a supersolution are similar procedures, we only explain the latter. To do this, we show in Lemma 4.2 that $u_+(t,x) = B(\varepsilon)t + A(\varepsilon)g(|x|)$ with $g \in \mathcal{G}$ could be a supersolution when we choose $A, B > 0$ in a suitable way. Since the equation is invariant under the translation and additions of constants to u_+,

$$u_{+,\xi}(t,x;\varepsilon) = a(\xi) + Bt + Ag(|x-\xi|) + \varepsilon$$

is also a supersolution, where $\xi \in \mathbf{R}^N$ and $\varepsilon, A, B > 0$. We take A and B in a way so that

$$\inf_{0<\varepsilon<1, \xi\in\mathbf{R}^N} u_{+,\xi}(0,x;\varepsilon) = a(x).$$

By Proposition 2.5 the infimum of a family of supersolutions is also a supersolution;

$$U_+(t,x) = \inf_{0<\varepsilon<1, \xi\in\mathbf{R}^N} u_{+,\xi}(t,x;\varepsilon)$$

is a desired supersolution. We shall also show that our solution is bounded and uniformly continuous in $[0,T) \times \mathbf{R}^N$ for each $T > 0$ provided that a is bounded and uniformly continuous on $\mathbf{R}^N$. For this purpose we have to work a little bit more. For example, we should check that the comparison theorem applies to our solution.

Lemma 4.2 *Assume that F satisfies (F1) and (F2) and that $\mathcal{F}(F) \neq \emptyset$. Suppose that $g \in \mathcal{G}$ and $A, B > 0$ satisfy*

$$F(\nabla(Ag(|x|)), \nabla^2(Ag(|x|))) \geq -B \text{ for all } x \in (\mathbf{R}^N \backslash \{0\}). \tag{4.1}$$

Then $u_+(t,x) := Bt + Ag(|x|)$ is a viscosity supersolution of (1.1) in Q_T.

Proof. Since $u_+(t,x) \in C^2(Q_T)$, we only have to check that $u_+(t,x)$ satisfies

$$\begin{cases} (u_+)_t(z) + F(\nabla(u_+)(z), \nabla^2(u_+)(z)) \geq 0 & \text{if } \nabla(u_+)(z) \neq 0, \\ (u_+)_t(z) \geq 0 & \text{if } \nabla(u_+)(z) = 0. \end{cases}$$

In the case $\nabla(u_+)(z) \neq 0$, *i.e.*, $\nabla(Ag(|x|)) \neq 0$, which is equivalent to $x \neq 0$.

Assumption yields

$$(u_+)_t(z) + F(\nabla(u_+)(z), \nabla^2(u_+)(z)) = B + F(\nabla(Ag(|x|)), \nabla^2(Ag(|x|))) \geq 0.$$

In the case $\nabla(u_+)(z) = 0$, *i.e.*, $\nabla(Ag(|x|)) = 0$.

We have $(u_+)_t(z) = B > 0$. Applying Lemma 4.1, we prove u_+ is a viscosity supersolution of (1.1) in Q_T. ∎

For the subsolution case, the condition (4.1) is replaced by

$$F(\nabla(-Ag(|x|)), \nabla^2(-Ag(|x|))) \leq B \text{ for all } x \in (\mathbf{R}^N \backslash \{0\}). \tag{4.2}$$

Remark 4.3 *For each $g \in \mathcal{G}$, there is $\tilde{g} \in \mathcal{G}$ such that*

1. $\tilde{g}(r) = g(r)$ *for* $0 \leq r \leq 1$.
2. $\inf_{r \geq 0} \tilde{g}'(r) > 0, \quad \sup_{r \geq 0} \tilde{g}'(r) < \infty.$

By this modification, $\tilde{g}$ satisfies (4.1)(resp. (4.2)) if g satisfies (4.1) (resp. (4.2)) with increasing the value of B if necessary.

We now consider the initial value problem;

$$u_t + F(\nabla u, \nabla^2 u) = 0 \qquad \text{in } Q_T = (0,T) \times \mathbf{R}^N, \tag{4.3}$$

$$u(0,x) = a(x) \qquad \text{on } \mathbf{R}^N, \tag{4.4}$$

where $a(x)$ is a given continuous function on $\mathbf{R}^N$. We suppose additional assumptions on F.

(F3)$_+$ There exists $g \in \mathcal{G}$ such that for all $A > 0$, there is $B > 0$ such that (4.1) holds.

(F3)$_-$ There exists $g \in \mathcal{G}$ such that for all $A > 0$, there is $B > 0$ such that (4.2) holds.

For $\varepsilon, A, B > 0$ and for $\xi \in \mathbf{R}^N$ we set $u_{\pm,\xi}(t,x;\varepsilon) := a(\xi) \pm Bt \pm Ag(|x-\xi|) \pm \varepsilon$ with $g \in \mathcal{G}$.

Lemma 4.4 *Suppose that $a(x)$ is a given bounded uniformly continuous function on $\mathbf{R}^N$; $a(x) \in BUC(\mathbf{R}^N)$. For all ε with $0 < \varepsilon < 1$ and for each $\xi \in \mathbf{R}^N$, there exist $A(\varepsilon) > 0$ and $B(\varepsilon) > 0$ such that*

$$u_{+,\xi}(0,x;\varepsilon) \geq a(x) \text{ for all } x \in \mathbf{R}^N, \tag{4.5}$$

and

$$\inf_{\xi \in \mathbf{R}^N} u_{+,\xi}(0,x;\varepsilon) \leq a(x) + \varepsilon \text{ for all } x \in \mathbf{R}^N. \tag{4.6}$$

Proof. It is easy to show (4.6). We put $x = \xi$ in the left side of (4.6) and observe that

$$\inf_{\xi \in \mathbf{R}^N} u_{+,\xi}(0,\xi;\varepsilon) = \inf_{\xi \in \mathbf{R}^N} a(\xi) + \varepsilon \leq a(x) + \varepsilon. \tag{4.7}$$

To prove the inequality (4.5), we have to show the existence of $A(\varepsilon)$ such that

$$|a(x) - a(\xi)| \leq A(\varepsilon) g(|x-\xi|) + \varepsilon. \tag{4.8}$$

Since $a \in BUC(\mathbf{R}^N)$, there exists a bounded modulus function m (*i.e.*, $m : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and $m(0) = 0$) such that

$$|a(x) - a(y)| \leq m(|x-y|) \text{ for all } x, y \in \mathbf{R}^N. \tag{4.9}$$

We set $\overline{r} := \min\{r \geq 0; m(r) = \varepsilon\}$. Since m is bounded, we can choose $A(\varepsilon)$ so that

$$Ag(\overline{r}) + \varepsilon \geq \sup_{r \geq 0} m(r). \tag{4.10}$$

Thus we obtain (4.8) and the inequality (4.5) is proved. ∎

Although we do not give a proof, we conclude a lemma such Lemma 4.4 for the subsolution case.

Lemma 4.5 *Suppose that $a(x) \in BUC(\mathbf{R}^N)$. For all ε with $0 < \varepsilon < 1$ and for each $\xi \in \mathbf{R}^N$, there exist $A(\varepsilon) > 0$ and $B(\varepsilon) > 0$ such that*

$$u_{-,\xi}(0,x;\varepsilon) \leq a(x) \text{ for all } x \in \mathbf{R}^N, \tag{4.11}$$

and

$$\sup_{\xi \in \mathbf{R}^N} u_{-,\xi}(0,x;\varepsilon) \geq a(x) - \varepsilon \text{ for all } x \in \mathbf{R}^N. \tag{4.12}$$

Combining Lemma 4.2, Lemma 4.4 and Lemma 4.5, we shall construct a supersolution and a subsolution satisfies the given initial data.

Lemma 4.6 *Assume that F satisfies (F1), (F2), and (F3)$_\pm$ and that $\mathcal{F}(F) \neq \emptyset$. Suppose that $a(x) \in BUC(\mathbf{R}^N)$. Then for all $T > 0$, there exist U_+, U_- : $[0,T) \times \mathbf{R}^N \to \mathbf{R}$ such that U_+ is a supersolution of (1.1), U_- is a subsolution of (1.1) and $(U_+)_*(0,x) = (U_-)^*(0,x) = a(x)$. Moreover, $U_+(t,x) \geq U_-(t,x)$ in $[0,T) \times \mathbf{R}^N$.*

Proof. We shall prove the supersolution case. Applying Lemma 4.2, we see that $u_{+,\xi}(t,x;\varepsilon)$ is a supersolution of (1.1). From Proposition 2.5

$$U_+(t,x) := \inf\{u_{+,\xi}(t,x;\varepsilon); 0 < \varepsilon < 1, \xi \in \mathbf{R}^N\}$$

is also a supersolution of (1.1). Applying Lemma 4.4 and (4.5), we observe that $(U_+)(0,x) \geq a(x)$ for all $x \in \mathbf{R}^N$. Moreover, we obtain $(U_+)_*(0,x) \geq a(x)$ for all $x \in \mathbf{R}^N$. By (4.5) and (4.6) we easily see $U_+(0,x) = a(x)$ for all $x \in \mathbf{R}^N$. Since generally we know $(U_+)_*(0,x) \leq U_+(0,x)$, we can prove that $(U_+)_*(0,x) = a(x)$. Set $U_-(t,x) := \sup\{u_{-,\xi}(t,x;\varepsilon); 0 < \varepsilon < 1, \xi \in \mathbf{R}^N\}$. In the same way we can prove the case of subsolutions. By the definitions of U_+ and U_-, we see $U_+(t,x) \geq U_+(0,x)$ and $U_-(0,x) \geq U_-(t,x)$. Since we know $U_+(0,x) = a(x) = U_-(0,x)$, we obtain $U_+(t,x) \geq U_-(t,x)$ in $[0,T) \times \mathbf{R}^N$. ∎

Lemma 4.7 *Assume that F satisfies (F1), (F2) and (F3)$_\pm$ and that $\mathcal{F}(F) \neq \emptyset$. Suppose that $a(x) \in BUC(\mathbf{R}^N)$. Let U_+ and U_- be as in Lemma 4.6. Then there is a modulus function ω such that*

$$U_+(t,x) - U_-(0,y) \leq \omega(|x-y| + t) \text{ for all } t \in [0,T), x, y \in \mathbf{R}^N, \tag{4.13}$$

and

$$U_+(0,x) - U_-(s,y) \leq \omega(|x-y| + s) \text{ for all } s \in [0,T), x, y \in \mathbf{R}^N. \tag{4.14}$$

Moreover,

$$U_+(t,x) < \infty, \quad U_-(t,x) > -\infty \text{ in } [0,T) \times \mathbf{R}^N.$$

Proof. We shall prove only (4.13). By Lemma 4.6 we see

$$a(y) = U_-(0,y) = U_+(0,y) = \inf\{u_{+,\xi}(0,y;\varepsilon); 0 < \varepsilon < 1, \xi \in \mathbf{R}^N\}.$$

This is equivalent to that for all $\delta > 0$ there are $\xi(\delta) \in \mathbf{R}^N$ and $\varepsilon(\delta)$ with $0 < \varepsilon(\delta) < 1$ such that

$$a(y) \geq u_{+,\xi(\delta)}(0,y;\varepsilon(\delta)) - \delta.$$

By Remark 4.3 g is global Lipschitz continuous. Then L denotes the Lipschitz constant. By the definition of U_+ we observe that

$$\begin{aligned} U_+(t,x) - a(y) &\leq U_+(t,x) - u_{+,\xi(\delta)}(0,y;\varepsilon(\delta)) + \delta \\ &\leq u_{+,\xi(\delta)}(t,x;\varepsilon(\delta)) - u_{+,\xi(\delta)}(0,y;\varepsilon(\delta)) + \delta \\ &= A(\varepsilon(\delta))t + A(\varepsilon(\delta))\left(g(|x-\xi(\delta)|) - g(|y-\xi(\delta)|)\right) + \delta \\ &\leq A(\varepsilon(\delta))t + LA(\varepsilon(\delta))|x-y| + \delta \\ &\leq \max\{1, L\}A(\varepsilon(\delta))(t + |x-y|) + \delta. \end{aligned}$$

For all $r > 0$, $\delta > 0$ and $\tilde{A} > 0$ we set

$$f_\delta(r) = \delta + \tilde{A}r$$

and

$$\omega(r) = \inf\{f_\delta(r); 0 < \delta < 1, r \geq 0\}.$$

Then ω with $\tilde{A} = \max\{1, L\}A(\varepsilon(\delta))$ is a desired modulus function. Indeed, ω is continuous on $[0,\infty)$, nondecreasing and $\omega(0) = 0$. In the same way we can prove (4.14). We shall prove the boundedness of $U_+(t,x)$. Putting $y = x$ in (4.13), we obtain $U_+(t,x) \leq a(x) + \omega(t)$. For all $T > 0$ we see

$$U_+(t,x) \leq \sup_{x\in\mathbf{R}^N} a(x) + \omega(T) < \infty \text{ in } [0,T) \times \mathbf{R}^N.$$

We can prove the boundedness of U_- similarly. ∎

Using Perron's method we obtain the existence of a viscosity solution of (4.3) and (4.4).

Theorem 4.8 *Suppose that F satisfies (F1), (F2) and (F3)$_\pm$ and that $\mathcal{F}(F) \neq \emptyset$. Assume that $a(x) \in BUC(\mathbf{R}^N)$. Then there exists a (unique) viscosity solution $u \in BUC([0,T) \times \mathbf{R}^N)$ of (4.3) and (4.4).*

Here we consider

$$u_t - |\nabla u|^{p-2}\text{trace}\left\{(I + (p'-2)\frac{\nabla u \otimes \nabla u}{|\nabla u|^2})\nabla^2 u\right\} = 0 \qquad \text{in } Q_T = (0,T) \times \mathbf{R}^N, \tag{4.15}$$

$$u(0,x) = a(x) \qquad \text{on } \mathbf{R}^N \tag{4.16}$$

with $1 < p \leq 2$, $p' \geq 1$. This equation (4.15) is given by

$$F(q,X) = -|q|^{p-2}\text{trace}\left\{(I + (p'-2)\frac{q \otimes q}{|q|^2})X\right\} \tag{4.17}$$

in (4.3). Note that if $p' = p$ this is nothing but the p-Laplace diffusion equation. We take $g(r) = \frac{p-1}{p} r^{\frac{p}{p-1}} \in \mathcal{G}$ to conclude the following.

Lemma 4.9 *Let F be defined by (4.17). Then F satisfies (F3)$_\pm$.*

Proof. Note that if $\nabla g(|x|) \neq 0$ (*i.e.*, $|x| \neq 0$), then

$$F(\nabla g(|x|), \nabla^2 g(|x|)) = -g'(|x|)^{p-2}\left\{(p'-1)g''(|x|) + \frac{N-1}{|x|}g'(|x|)\right\}.$$

Choose $B \geq \left(\frac{p'-1}{p-1} + N - 1\right) A^{p-1}$ we observe that (F3)$_\pm$ are fullfiled. ∎

For F of (4.17) we know $f(r) = r^{1+\sigma}$ with $\sigma > \dfrac{1}{p-1} \geq 1$ is a function of $\mathcal{F}(F)$. Now we conclude the following corollary.

Corollary 4.10 *Assume that $a(x) \in BUC(\mathbf{R}^N)$. Then there exists a (unique) viscosity solution $u \in BUC([0,T) \times \mathbf{R}^N)$ of (4.15) and (4.16).*

Also Theorem 4.8 is applicable to (4.15) and (4.16) with $p > 2$, $p' \geq 1$. For this purpose we only prove Lemma 4.9 for $p > 2$, $p' \geq 1$.

Lemma 4.11 *Let F be defined by (4.17) $p > 2$, $p' \geq 1$. Then F satisfies (F3)$_\pm$.*

Proof. We choose $g(r) = r - \arctan(r) \in \mathcal{G}$, where $\arctan(r)$ denotes arc tangent of r. Note that if $\nabla g(|x|) \neq 0$ (*i.e.*, $|x| \neq 0$), then

$$F(\nabla g(|x|), \nabla^2 g(|x|)) = -\left(\frac{|x|^2}{1+|x|^2}\right)^{p-2}\left\{(p'-1)\frac{2|x|}{(1+|x|^2)^2} + (N-1)\frac{|x|}{1+|x|^2}\right\}. \tag{4.18}$$

Since (4.18) is continuous,

$$\lim_{|x|\to 0} F(\nabla g(|x|), \nabla^2 g(|x|)) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} F(\nabla g(|x|), \nabla^2 g(|x|)) = 0,$$

we know $F(\nabla g(|x|), \nabla^2 g(|x|))$ is bounded on $\mathbf{R}^N$. Set

$$C(p, p', N) = \max_{x\in\mathbf{R}^N} F(\nabla g(|x|), \nabla^2 g(|x|)).$$

Choose $B \geq A^{p-1}C(p,p',N)$, we observe that $(\mathrm{F3})_\pm$ are fullfiled. ∎

Remark 4.12 *In the literature on the p-Laplace diffusion equation, many authors have studied and they dealt with the solutions defined in distribution sense. It is crucial to define them that the equation has divergence structure. But our definition of viscosity solutions does not depend on such like structure of the equation. Indeed, (4.15) does not depend on the divergence structure if $p' \neq p$.*

Remark 4.13 *If F satisfies (F1) and (F2) and F is geometric in the sense of [2], then the conclusion of Theorem 4.8 holds. Indeed, if F satisfies (F1) and (F2) and F is geometric, then $\mathcal{F}(F) \neq \emptyset$ (see [12]). By taking an $f \in \mathcal{F}(F)$ as a function in $\mathcal{G}$, then F satisfies $(F3)_\pm$.*

Remark 4.14 *If $p > 2$, then there is a standard definition of viscosity solutions in [4]. We wonder whether or not our definition of solutions agree with the usual one. For example it is not diffcult to prove that both defintions are equivalent although we do not present the proof here.*

§5. Separable Solution.

Throughout this section, we only consider the Dirichlet boundary problem of the p-Laplace diffusion equation (1.2) with one space variable. Rescaling the space variable to simplify the equation, we consider of the form

$$u_t = |u_x|^{p-2}u_{xx}, \qquad (t,x) \in (0,T)\times(0,1), \tag{5.1}$$
$$u(t,0) = u(t,1) = 0, \qquad t \in [0,T], \tag{5.2}$$

where $1 < p < 2$. We seek a non-negative separable solution of the form

$$u(t,x) = U(x)\cdot\Lambda(t), \tag{5.3}$$

where $U(x)$ and $\Lambda(t)$ are assumed to be a non-negative C^2 function and a C^1 function, respectively.

Thus, substituting (5.3) to (5.1), we get

$$\Lambda^{1-p}(t)\Lambda'(t) = U^{-1}(x)|U'(x)|^{p-2}U''(x) = \text{const}, \tag{5.4}$$

Since $U(0) = U(1) = 0$ and $U \geq 0$, we get $U''(x) \leq 0$. Moreover, the following equations for $U(x)$ and $\Lambda(t)$ hold;

$$\Lambda(t)^{-p+1}\cdot\Lambda'(t) = -c, \tag{5.5}$$
$$U''(x) = -cU(x)|U'(x)|^{2-p} \tag{5.6}$$

for some positive constant c. When $\tilde{U}(x) = \beta U(x)$ and $\tilde{\Lambda}(t) = \beta^{-1}\Lambda(t)$, where β is a positive constant, then $U(x)\Lambda(t) = \tilde{U}(x)\tilde{\Lambda}(t)$ and

$$\tilde{\Lambda}^{-p+1}(t)\tilde{\Lambda}'(t) = \tilde{U}^{-1}(x)|\tilde{U}'(x)|^{p-2}\tilde{U}''(x) = -\beta^{p-2}c.$$

By a suitable choice of β, we may assume $c = \dfrac{1}{2-p}$ in (5.5) and (5.6) without loss of generality. Integrating (5.5), we easily see that the separable solution is of the form

$$u(t,x) = (t_* - t)^{\frac{1}{2-p}}U(x), \ 0 \leq t \leq t_*,$$

where $t_* > 0$ is a positive constant (called the extinction time) and $U(x)$ is a solution of the following equation

$$U''(x) = -\tfrac{1}{2-p}U(x)|U'(x)|^{2-p},\ U(x) > 0, \quad 0 < x < 1, \tag{5.7}$$

$$U(0) = U(1) = 0. \tag{5.8}$$

We shall solve (5.7), (5.8).

Proposition 5.1 *Suppose $V(x) \geq 0$ solves*

$$V''(x) = -\frac{1}{2-p}V(x)(V'(x))^{2-p}, \qquad 0 \leq x \leq 1/2, \tag{5.9}$$

$$V(0) = 0, \tag{5.10}$$

$$V'(x) \geq 0 \text{ in a neighborhood of } x = 0. \tag{5.11}$$

1. *Then there exists a positive number x_* such that $V'(x) > 0$ for all $0 < x < x_*$, $V'(x_*) = 0$ and that $V(x)$ solves (5.9) in $0 \leq x \leq x_*$.*
2. *Furthermore, there exists x_* such that $x_* \leq 1/2$.*
3. *In the case $x_* \leq 1/2$,*

$$U(x) = \begin{cases} V(x), & 0 \leq x \leq x_*, \\ V(x_*), & x_* \leq x \leq 1 - x_*, \\ V(1-x), & 1 - x_* \leq x \leq 1 \end{cases} \tag{5.12}$$

is a symmetric solution of (5.7), (5.8).

Proof.

1. Multiply both sides of (5.9) by $(V'(x))^{p-1}$ and integrate them from 0 to x to get

$$V'(x) = (K^p - c_p V(x)^2)^{\frac{1}{p}},$$

where $c_p = \dfrac{p}{2(2-p)}$ and K denotes $V'(0)$. This equation is valid while x is sufficiently small. Integrating this yields

$$V(x) = K^{\frac{p}{2}} c_p^{-\frac{1}{2}} W_p^{-1}(K^{\frac{2-p}{2}} c_p^{\frac{1}{2}} x).$$

Here $W_p^{-1}(x)$ is the inverse function of a non-decreasing function W_p such that

$$W_p(y) := \int_0^y (1 - s^2)^{-\frac{1}{p}} ds,\ 0 \leq y \leq 1.$$

Note that the integral is convergent at $y = 1$ and we put $W_p(1) = M_p(< \infty)$. To say the least this representation of V is valid while x is small. We take the supremum of x at which the representation of $(K^p - c_p V(x)^2)^{\frac{1}{p}} > 0$ is valid, which is denoted by x_*. The value of x_* can be written explicitly as follows;

$$x_* = K^{\frac{p-2}{2}} c_p^{-\frac{1}{2}} M_p. \tag{5.13}$$

Then this x_* satisfies all desired properties.

2. By seeing (5.13), we see that $x_* \leq 1/2$ holds if we choose sufficiently large K.
3. It suffices to prove U given by (5.12) is C^2 across $x = x_*$, which holds since $V'(x_*) = V''(x_*) = 0$.

∎

Theorem 5.2 *Set*

$$u(t,x)=\begin{cases}(t_*-t)^{\frac{1}{2-p}}U(x), & 0\le t\le t_*,\\ 0, & t\ge t_*\end{cases}$$

with U *in (5.12). Then* u *is a viscosity solution of (5.1)-(5.2) if and only if* $V(x)$ *attains its unique maximum value (i.e.,* $x_*=1/2$*).*

Proof. Note that u_t is continuous at $t=t_*$ since $\dfrac{1}{2-p}>1$. We easily see that u satisfies the conditions to be a viscosity solution in the case $t\ge t_*$. So we consider the case $t\le t_*$. We see that if $x_*<1/2$, then $u(t,x)=U(x)\cdot\Lambda(t)$ is not a viscosity super solution. Indeed, we can choose an admissible test function $\varphi(t,x)$ such that $u-\varphi$ attains its local minimum in $(0,T)\times(x_*,1-x_*)$, where $u_t=\varphi_t<0$, which contradicts the definition of viscosity supersolution.

In the case $x_*=1/2$, it is obvious that the $u(t,x)=U(x)\cdot\Lambda(t)$ is a viscosity subsolution. To show that u is a viscosity supersolution, we are going to show that for any admissible test function φ, $u-\varphi$ never attains its local minimum on $x=1/2$, where $u_x=0$.

Suppose that $u-\varphi$ attains its local minimum at $(\hat{t},1/2)$. Then there exist $f\in\mathcal{F}(F)$ and $\delta>0$ such that

$$|\varphi(\hat{t},x)-\varphi(\hat{t},1/2)|<f(|x-1/2|) \tag{5.14}$$

for all x such that $|x-1/2|<\delta$. Note that in this case, $f\in\mathcal{F}(F)$ is equivalent to $f(0)=f'(0)=f''(0)=0$, $f''(r)>0$ for all $r>0$, and

$$\lim_{r\to0}(f'(r))^{p-2}\cdot f''(r)=0. \tag{5.15}$$

Since $u-\varphi$ attains its minimum at $(\hat{t},1/2)$ and (5.14) holds, we observe that

$$\begin{aligned}u(\hat{t},1/2)-u(\hat{t},x) &= \Lambda(\hat{t})\cdot(U(1/2)-U(x))\\ &\le \varphi(\hat{t},1/2)-\varphi(\hat{t},x)\\ &\le f(1/2-x)\end{aligned} \tag{5.16}$$

for $x\in[0,1/2]$.

Set $g(r):=\Lambda(\hat{t})\cdot(U(1/2)-U(1/2-r))$ for $r\in[0,1/2]$ Then (5.16) is equivalent to

$$g(r)\le f(r). \tag{5.17}$$

On the other hand, We consider an inequality obtained by replacing f by g in (5.15). Note that $g(0)=0$, $g'(r)=\Lambda(\hat{t})\cdot U'(1/2-r)$, $g''(r)=-\Lambda(\hat{t})\cdot U''(1/2-r))$. Utilizing (5.7), we get

$$\lim_{r\to0}(g'(r))^{p-2}\cdot g''(r)=\frac{1}{2-p}\cdot\Lambda(\hat{t})^{p-1}\cdot U(1/2)>0. \tag{5.18}$$

From (5.15) and (5.18) we see

$$(g'(r))^{p-2}\cdot g''(r)>(f'(r))^{p-2}\cdot f''(r) \tag{5.19}$$

for all r such that $0<r<\delta$ if a positive number δ is sufficiently small. Integrating both sides from 0 with respect to r, we get

$$\frac{1}{p-1}(g'(r))^{p-1}>\frac{1}{p-1}(f'(r))^{p-1}. \tag{5.20}$$

Hence $g'(r)>f'(r)$ holds for sufficiently small r, we get

$$g(r)>f(r), \tag{5.21}$$

which contradicts (5.17). ∎

Remark 5.3 *A C^2 class function u can be a supersolution even if it violates*

$$u_t(z) + F(\nabla u(z), \nabla^2 u(z)) \geq 0, \qquad \textit{if } \nabla u(z) \neq 0, \tag{5.22}$$
$$u_t(z) \geq 0, \qquad \textit{if } \nabla u(z) = 0. \tag{5.23}$$

Indeed, our u defined in Theorem 5.2 is a supersolution which does not satisfy (5.23). This is the reason why we do not apply Lemma 4.1 to show that u is a supersolution.

Remark 5.4 *The existence of the unique solution of (5.7) and (5.8) was proved by Ôtani [9], [10] in usual weak solution sense.*

§6. Stability of the Solution

In this section we are concerned with the stability of viscosity solutions; one of the basic properties of viscosity solutions. We have in mind the p-Laplace diffusion equation. For functions $u_n : Q_T \to \mathbf{R} \cup \{\pm\infty\}$ $(n \in \mathbf{N})$, where $Q_T = (0,T) \times \Omega$, $\Omega \subset \mathbf{R}^N$; not necessarily bounded, we define $\overline{u}, \underline{u} : Q_T \to \mathbf{R} \cup \{\pm\infty\}$ by

$$\overline{u}(z) = \lim_{r \to 0} \sup\{u_n; |\zeta - z| \leq r, n > \frac{1}{r}\}, \tag{6.1}$$
$$\underline{u}(z) = \lim_{r \to 0} \inf\{u_n; |\zeta - z| \leq r, n > \frac{1}{r}\}. \tag{6.2}$$

We give a stability theorem as in [12].

Theorem 6.1 *Assume that F and F_n $(n \in \mathbf{N})$ satisfy (F1) and (F2) and that $F_n \to F$ uniformly in any compact set of $(\mathbf{R}^N \backslash \{0\}) \times S^N$. Suppose that $\mathcal{F}(F) \neq \emptyset$ and $\mathcal{F}(F) \subset \mathcal{F}(F_n)$ for all $n \in \mathbf{N}$ and that for all $f \in \mathcal{F}(F)$,*

$$\lim_{|x| \to 0, n \to \infty} F_n(\nabla f(|x|), \nabla^2 f(|x|)) = 0. \tag{6.3}$$

Let u_n $(n \in \mathbf{N})$ be subsolutions (resp. supersolutions) of

$$\frac{\partial u_n}{\partial t} + F_n(\nabla u_n, \nabla^2 u_n) = 0 \textit{ in } Q_T. \tag{6.4}$$

Assume that $\overline{u}(z) < +\infty$ (resp. $\underline{u}(z) > -\infty$) for all $z \in Q_T$. Then $\overline{u}$ (resp. $\underline{u}$) is a subsolution (resp. a supersolution) of (1.1) in Q_T.

Proof. We only prove a subsolution case, since a supersolution case can be proved similarly.

We take $\varphi \in \mathcal{A}(F)$ and assume that $\max_{Q_T}(\overline{u} - \varphi) = (\overline{u} - \varphi)(\hat{z})$ with $\hat{z} = (\hat{t}, \hat{x})$. We may assume $\overline{u} - \varphi$ attains its strict maximum at $\hat{z}$. If $\nabla\varphi(\hat{z}) \neq 0$, standard argument yields the conclusion. We only have to check that if $\nabla\varphi(\hat{z}) = 0$, then $\varphi_t(\hat{z}) \leq 0$.

Since $\varphi \in \mathcal{A}(F)$, there exist $\delta > 0$, $f \in \mathcal{F}(F)$ and $\omega \in C[0,\infty)$ with $\omega(r) = o(r)$ as r goes to zero such that if $|x - \hat{x}| + |t - \hat{t}| < \delta$ then

$$|\varphi(z) - \varphi(\hat{z}) - \varphi_t(\hat{z})(t - \hat{t})| \leq f(|x - \hat{x}|) + \omega(|t - \hat{t}|). \tag{6.5}$$

We may assume that $\omega \in C^2[0,\infty)$, $\omega(0) = \omega'(0) = 0$ and $\omega(r) > 0$ for all $r > 0$. Moreover, we set

$$\tilde{\omega}(r) = \begin{cases} \omega(r), & r \geq 0, \\ \omega(-r), & r < 0. \end{cases}$$

Note that $\tilde{\omega} \in C^2(\mathbf{R})$ and that $\tilde{\omega}'(0) = 0$. We choose a sequence $\{\omega_n\}_n \subset C^2(\mathbf{R})$ such that $\omega_n(r)$ and $\omega_n'(r)$ uniformly converge to $\tilde{\omega}(r)$, $\tilde{\omega}'(r)$, respectively , on any compact set in $\mathbf{R}$ and that $\omega_n(r) = o(r)$ and $\omega_n(0) = \omega_n'(0) = 0$. We set

$$\psi(t,x) := \varphi_t(\hat{z})(t - \hat{t}) + 2f(|x - \hat{x}|) + 2\tilde{\omega}(t - \hat{t}), \tag{6.6}$$
$$\psi_n(t,x) := \varphi_t(\hat{z})(t - \hat{t}) + 2f(|x - \hat{x}|) + 2\omega_n(t - \hat{t}). \tag{6.7}$$

Note that $\overline{u} - \psi$ has its strict maximum at $z = \hat{z}$. Since $\omega_n(r) = o(r)$ and $\omega'_n(0) = 0$, we know $\psi_n \in \mathcal{A}(F)$. In view of Barles - Perthame's Lemma (Lemma A.3 in [1]) we may assume that $u_n^* - \psi_n$ attains its maximum at some point (t_n, x_n) with (t_n, x_n) goes to $\hat{z}$ as $n \to \infty$, since $\omega_n \to \tilde{\omega}$ uniformly on any compact set in $\mathbf{R}$. Since u_n is a subsolution, we have:

1. In the case $\nabla\psi_n(t_n, x_n) \neq 0$, *i.e.*, $x_n \neq \hat{x}$. We get

$$\varphi_t(\hat{z}) + 2\omega'_n(t_n - \hat{t}) + F_n(\nabla f(|x_n - \hat{x}|), \nabla^2 f(|x_n - \hat{x}|)) \leq 0.$$

   Letting $n \to \infty$ and assumption (6.3) yield $\varphi_t(\hat{z}) \leq 0$.

2. In the case $\nabla\psi_n(t_n, x_n) = 0$, *i.e.*, $x_n = \hat{x}$. We get

$$\varphi_t(\hat{z}) + 2\omega'_n(t_n - \hat{t}) \leq 0.$$

Letting $n \to \infty$, we obtain $\varphi_t(\hat{z}) \leq 0$. ∎

We want to check that Theorem 6.1 is applicable to the p-Laplace diffusion equation (1.2) with $1 < p \leq 2$. We give an approximate equation of (1.1) by

$$F_n(q, X) = -|q|_\varepsilon{}^{p-2}\mathrm{trace}\left\{(I + (p-2)\frac{q \otimes q}{|q|_\varepsilon^2})X\right\}$$

with $|q|_\varepsilon = (|q|^2 + \varepsilon^2)^{1/2}$, $\varepsilon = 1/n$.

Proposition 6.2 *Let F be as in (1.3) and let F_n be as above. Then $\mathcal{F}(F) \subset \mathcal{F}(F_n)$ for all $n \in \mathbf{N}$, and*

$$\lim_{|x|\to 0, n\to+\infty} F_n(\nabla f(|x|), \nabla^2 f(|x|)) = 0$$

for all $f \in \mathcal{F}(F)$.

Proof. Let $f \in \mathcal{F}(F)$, *i.e.*,

$$f \in C^2[0,\infty), \quad f(0) = f'(0) = f''(0) = 0, \quad \lim_{|x|\to 0} F(\nabla f(|x|), \nabla^2 f(|x|)) = 0$$

hold. Then we have

$$\frac{1}{r}f'(r)^{p-1} \to 0 \qquad \text{as } r \to 0, \tag{6.8}$$

$$f''(r)f'(r)^{p-2} \to 0 \qquad \text{as } r \to 0 \tag{6.9}$$

with $r = |x|$. For such $f \in \mathcal{F}(F)$, we have to show

$$\lim_{|x|\to 0} F_n(\nabla f(|x|), \nabla^2 f(|x|)) = 0 \text{ for all } n \in \mathbf{N}.$$

This is equivalent to

$$\frac{1}{r}f'(r)|f'(r)|_\varepsilon{}^{p-2} \to 0 \quad \text{as } r \to 0,$$

$$\left\{1 + (p-2)\frac{f'(r)^2}{|f'(r)|_\varepsilon{}^2}\right\} f''(r)|f'(r)|_\varepsilon{}^{p-2} \to 0 \quad \text{as } r \to 0.$$

Since

$$\frac{1}{r}f'(r)|f'(r)|_\varepsilon{}^{p-2} \leq \frac{1}{r}f'(r)^{p-1} \tag{6.10}$$

and since (6.8) holds, we obtain

$$\lim_{r\to 0}\frac{1}{r}f'(r)|f'(r)|_\varepsilon{}^{p-2} = 0.$$

By the definition of $|\cdot|_\varepsilon$ and since $1 < p \leq 2$, we observe that

$$\left|1 + (p-2)\frac{f'(r)^2}{|f'(r)|_\varepsilon^2}\right| \leq 1 + |p-2|\cdot 1, \tag{6.11}$$

$$f''(r)|f'(r)|_\varepsilon{}^{p-2} \le f''(r)f'(r)^{p-2}. \tag{6.12}$$

By (6.11), (6.12) and (6.9) we have

$$\lim_{r\to 0}\left\{1+(p-2)\frac{f'(r)^2}{|f'(r)|_\varepsilon^2}\right\} f''(r)|f'(r)|_\varepsilon{}^{p-2} = 0.$$

Thus we have proved $\mathcal{F}(F) \subset \mathcal{F}(F_n)$ for all $n \in \mathbf{N}$. Next we shall verify for all $f \in \mathcal{F}(F)$,

$$\lim_{|x|\to 0, n\to +\infty} F_n(\nabla f(|x|), \nabla^2 f(|x|)) = 0,$$

which is equivalent to

$$\lim_{r\to 0,\varepsilon\to 0} \frac{1}{r} f'(r)|f'(r)|_\varepsilon{}^{p-2} = 0, \tag{6.13}$$

$$\lim_{r\to 0,\varepsilon\to 0}\left\{1+(p-2)\frac{f'(r)^2}{|f'(r)|_\varepsilon^2}\right\} f''(r)|f'(r)|_\varepsilon{}^{p-2} = 0. \tag{6.14}$$

From (6.10) and (6.8) we conclude (6.13) holds. Since (6.11), (6.12) and (6.9) hold, we also obtain (6.14). ∎

Also Theorem 6.1 is applicable to the p-Laplace diffusion equation (1.2) with $p > 2$.

Proposition 6.3 *Let F be as in (1.3) and let F_n be as follows.*

$$F_n(q, X) = -|q|^{p-2}\mathrm{trace}\left\{(I + (p-2)\frac{q\otimes q}{|q|_\varepsilon^2})X\right\}.$$

Then the conclusion of Proposition 6.2 holds.

The proof parallels that of Proposition 6.2.

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