LECTURES ON NONLINEAR DISPERSIVE EQUATIONS I

Edited by
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Sapporo, 2004

Partially supported by JSPS

- Grant-in-Aid for formation of COE
  "Mathematics of Nonlinear Structure via Singularities"
- Grant-in-Aid for Scientific Research
  S(2) #16104002 (T. Ozawa)
  A(2) #15204008 (Y. Tsutsumi)

PREFACE

This volume, together with the next, is intended as the proceedings of expository lectures in Special Months "Nonlinear Dispersive Equations. "

Nonlinear dispersive equations, such as nonlinear Schrödinger equations, KdV equation, and Benjamin-Ono equation, are of mathematical and physical importance. Expository courses in August 2004 are intended to cover a broad spectrum of the issues, from mathematical and physical backgrounds to the latest developments.

We wish to express our sincere thanks to

- J. Bona H. Koch, F. Planchon, P. Raphaël, and N. Tzvetkov for excellent lectures.
- M. Ikawa and A. Ogino for effitient arrangements.

T. Ozawa and Y. Tsutsumi

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Program

COE Program Special Months

Lectures On Nonlinear Dispersive Equations I

Organizers: T. Ozawa (Hokkaido Univ.)
Y. Tsutsumi (Kyoto Univ.)

Period :

August 23 - 27, 2004

Venue :

Department of Mathematics, Hokkaido University
Science Building #8 Room 309

Program :

		10:30–12:00	13:30–15:00	15:30–17:00
Aug.	23 (Mon)	Bona (1)	Bona (2)	Planchon (1)
	24 (Tue)	Bona (3)	Bona (4)	Planchon (2)
	25 (Wed)	Bona (5)	Raphaël (1)	Planchon (3)
	26 (Thu)	Raphaël (2)	Raphaël (3)	Planchon (4)
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- J. Bona (University of Illinois at Chicago)
  Derivation and some fundamental properties of nonlinear dispersive waves equations
- F. Planchon (Université de Paris-Nord)
  Schrödinger equations with variable coefficients
- P. Raphaël (Université de Cergy-Pontoise)
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– – *COE Program Special Months Homepage* – –

http://coe.math.sci.hokudai.ac.jp/news/special/index02.html.en

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Derivation and some fundamental properties of nonlinear dispersive waves equations.

Jerry Bona (University of Illinois at Chicago)

Abstract

This series of lectures aims to introduce some of the principal aspects of nonlinear dispersive wave theory. We start with an appreciation of the early history, and then introduce, within the original fluid mechanics context, the paradigm Korteweg-de Vries equation. Some of the more important properties of this equation are then outlined. These properties motivate and give direction to the further study of this and other nonlinear dispersive wave equations.

Further issues to be addressed will be chosen from among the following topics.

1. Existence theory for solitary waves
2. Stability and instability of solitary waves
3. Singularity formation
4. Initial-value and initial-boundary-value problems
5. Incorporation of damping into nonlinear dispersive wave equations
6. Application of the theory to problems in mechanics

Lectures notes on Schrödinger equations with variable coefficients
COE Program Special Months : Nonlinear Dispersive Equations, Hokkaïdo University, Sapporo

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August 2004

Introduction

The aim of this series of lectures is to give an overview of dispersive estimates for the Schrödinger equation. These estimates are a key tool for various problems, both linear and non-linear, and we will give examples along the way. We focus on the case where the domain Ω is the whole space $\mathbb{R}^n$: other situations (torus $\mathbb{T}^n$, bounded domain with Dirichlet conditions) are significantly more intricate and the subject of active research; a good knowledge of the $\mathbb{R}^n$ case is anyway a prerequisite.

As the title of the notes suggests, we would like to deal with variable coefficients: namely, what happens if we replace the standard Laplacian by, say, a Laplace-Beltrami operator associated to a metric g_{ij} ? As we will see in the first lecture, dispersive estimates for the flat case are obtained through harmonic analysis methods; these in turn rely heavily on the Fourier transform and admit no easy generalization to curved space. We will deal with the admittedly easiest case, namely $n = 1$: while some of the techniques which we will use are somewhat 1D specific, the problems one might encounter in the general case are already present. Moreover, we present much sharper results than those available at present for $n \geq 2$. Finally, we will deal with an application of these results to the Benjamin-Ono family of equations.

There exists a huge literature on the subject of dispersive equations. We have tried to give as many references as possible, but being exhaustive is an impossible task, so these references represent a snapshot of the author's current knowledge rather than an accurate picture or historical account.

We have tried to make the notes as self-contained as possible, assuming basic knowledge of functional analysis, distributions and Fourier analysis. There will be, however, blackboxes which won't be detailed: interpolation theory, for which we refer to [7], [6] or [63] which already contains most of what we need. Another blackbox which we will only half-open is the zoo of functional spaces: we will merely use Besov spaces and refer to [72] for an exhaustive reference, [52] for building up intuition or [7] if one just needs a quick summary. For those who have an interest in harmonic analysis by itself, [63] and its companion [65] are classic if not up to date with current

trends. Recent books like [64] or [32] are closer to a modern days perspective.
Comments and suggestions are welcome, *fab@math.univ-paris13.fr*.

Contents

Chapter 1

Dispersive estimates for the flat Schrödinger equation

Introduction

We consider the following Schrödinger equation

$$\left\{ \begin{array}{rcl} i\partial_t \phi + \Delta \phi & = & 0, \\ \phi(x,0) & = & \phi_0(x) \end{array} \right. \tag{1.1}$$

in $\mathbb{R}^n$. The reader may take $\phi_0 \in \mathcal{S}$, in order to avoid splitting hair on the meaning of a solution and focus on obtaining estimates in terms of various norms. Through functional analysis arguments (Hille-Yosida) the solution exists for H^1 or even L^2 datum. At any rate, one can prove there exists a unique solution defined as a tempered distribution, which reads (in Fourier variables)

$$\hat{\phi}(\xi, t) = e^{-it|\xi|^2} \phi_0(\xi), \tag{1.2}$$

and (in space variables)

$$\phi(x,t) = \frac{1}{(-4i\pi t)^{\frac{n}{2}}} \int e^{i\frac{|x-y|^2}{4t}} \phi_0(y) dy. \tag{1.3}$$

Notice that this last quantity is in fact an oscillatory integral and should be seen appropriately as a limit or as a distribution bracket when $\phi_0 \in \mathcal{S}$.
We will denote $\phi = S(t)\phi_0$. From the Fourier variable formulation, one trivially obtains conservation of L^2 mass,

$$\|S(t)\phi_0\|_2 = \|\phi_0\|_2, \tag{1.4}$$

through Plancherel. Alternatively, one may compute the time derivative of $\|\phi\|_2^2$ and use the equation and its conjugate. In a similar fashion, one has conservation of the L^2 norm of the gradient,

$$\|\nabla S(t)\phi_0\|_2 = \|\nabla\phi_0\|_2, \tag{1.5}$$

and this can be obtained as well by using the equation and integration by parts.

Remark 1
We highlight the fact that these conservation laws can be obtained by multiplier methods: as such, they do not require an explicit representation of the solution, and are flexible enough to adapt to more complicated settings: adding a (real) potential term $V(x)\phi$, a non-linear term $|u|^{p-1}u$, allowing variable coefficients Laplacians, etc...

From the space variable formulation, we obtain easily what is usually referred to as the dispersion inequality

$$\|S(t)\phi_0\|_\infty \lesssim \frac{1}{t^{\frac{n}{2}}}\|\phi_0\|_1, \tag{1.6}$$

by ignoring the imaginary exponential factor. This inequality captures some of the information on the "spreading" of a solution: if one takes a Gaussian packet, for which we can explicitly compute the solution, one readily observes a decrease of its maximum.

Remark 2
The main drawback of (1.6) *is the norm on the right hand side: the L^1_x norm is not preserved by the flow.*

1.1 Dispersion and Strichartz estimates

We intend to address the issue raised by the previous remark. In order to go further, we state an interpolation result.

Theorem 1 (Riesz-Thorin)
Let $1 \leq p_0, p_1, q_0, q_1 \leq +\infty$, $p_0 \neq p_1$; T a linear operator, bounded from $L^{p_0} \to L^{q_0}$, and from $L^{p_1} \to L^{q_1}$, with

$$\|Tf\|_{q_0} \leq M_0\|f\|_{p_0}, \;\; \|Tf\|_{q_1} \leq M_1\|f\|_{p_1}.$$

Then, for all $\theta \in (0,1)$ *the operator* T *is bounded from* $L^{p_\theta} \to L^{q_\theta}$*, with* $\frac{1}{q_\theta} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}$ *and* $\frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$*,*

$$\|Tf\|_{q_\theta} \leq M_\theta \|f\|_{p_\theta}, \quad M_\theta \leq M_0^\theta M_1^{1-\theta}. \tag{1.7}$$

Now, we are in position to state

PROPOSITION 1
Let $\phi_0 \in L^{p'}$, $1 \leq p' \leq 2$, *then* $S(t)\phi_0 \in L^p$ *and*

$$\|S(t)\phi_0\|_p \lesssim \frac{1}{t^{\frac{n}{2}(\frac{1}{p'}-\frac{1}{p})}} \|\phi_0\|_{p'}. \tag{1.8}$$

Proof: We only need to apply Theorem 1 using (1.4) and (1.6). □

Note that this Proposition is merely a rephrasing of the classical example which follows: the Fourier transform $\mathcal{F}$ maps $L^{p'}$ to L^p for $p \geq 2$ (incidentally the author knows of no other proof of this fact than interpolation). It certainly does not address the issue raised in the remark: we still have a norm which is not preserved by the flow.

However, we can use these estimates to obtain something which, if not satisfactory, hints at the right quantities. let us consider the nonlinear equation, for $n = 2$:

$$i\partial_t u + \Delta u = |u|^2 u,$$

and its integral formulation,

$$u = S(t)u_0 + \int_0^t S(t-s)|u|^2 u \, ds.$$

Looking at the Duhamel term, a quick (formal) sequence of computation gives

$$\|u\|_4(t) \lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \||u|^2 u\|_{\frac{4}{3}}(s) \, ds$$

$$\sup_t t^{\frac{1}{4}} \|u\|_4(t) \lesssim \int_0^1 \frac{1}{(1-\theta)^{\frac{1}{2}} \theta^{\frac{3}{4}}} \, d\theta \ \sup_t (t^{\frac{1}{4}} \|u\|_4(t))^3.$$

As such, we could set up a fixed point, if u_0 is such that $\sup_t t^{\frac{1}{4}} \|S(t)u_0\|_4$ is small enough. However, figuring out what this condition means, say, in term of Sobolev spaces for the initial data, is unclear. Given that the nonlinear equation is invariant by the rescaling $\phi_\lambda(x,t) = \lambda\phi(\lambda x, \lambda^2 t)$, and that $\sup_t (t^{\frac{1}{4}} \|u\|_4(t))$ is invariant as well, one may look at ϕ_0 in such an invariant norm: given $n=2$, the L^2 norm fits. The following theorem tells us that indeed the L^2 norm controls a weaker version of the weird looking time-space norm, and even better: the "right" norm for our non-linear problem should be $L^4_{t,x}$ (recall $t^{-\frac{1}{4}}$ won't be in L^4_t, but in $L^{4,\infty}_t$).

Theorem 2 (Strichartz estimates, [66],[38])
Let (q,r) and $(\tilde q,\tilde r)$ be admissible pairs, i.e. such that $\frac{2}{q}+\frac{n}{r}=\frac{n}{2}$, $q\geq 2$ (or $q>2$ if $n=2$, $q\geq 4$ if $n=1$). Let $\phi_0(x)\in L^2$, $F(t,x)\in L^{\tilde{q}'}(-T,T);L^{\tilde r'}_x)$. There exists $C(n,q)$, $\tilde C(n,q,\tilde q)$ (uniform with respect to $0<T\leq+\infty$) such that, if $\phi(x,t)$ is a solution of

$$i\partial_t\phi+\Delta\phi=F,\quad \phi(0,x)=\phi_0(x),$$

then

$$\|S(t)\phi_0\|_{L^q_t(L^r_x)} \leq C(n,q)\|\phi_0\|_2, \tag{1.9}$$

$$\|\phi-S(t)\phi_0\|_{C_t(L^2_x)} \leq C(n,\tilde q)\|F(x,t)\|_{L^{\tilde q'}_t(L^{\tilde r'}_x)}, \tag{1.10}$$

$$\|\phi-S(t)\phi_0\|_{L^q_t(L^r_x)} \leq \tilde C(n,q,\tilde q)\|F(x,t)\|_{L^{\tilde q'}_t(L^{\tilde r'}_x)}, \tag{1.11}$$

where $C_t(L^2_x)=C([0,T],L^2)$ and $L^q_t(L^r_x)=L^q((0,T);L^r_x)$.

Remark 3
This type of estimates has a long history. They apply to a large class of equations, beyond dispersive models, and notably include wave equations (for $n\geq 2$). They go back to Segal in the 60's (1D Klein-Gordon), and acquired fame with Strichartz'paper [66]: in said paper, the connection between such estimates (with $q=r$) and restriction estimates in harmonic analysis (following questions raised by E. Stein in the 60's) is established and has driven the subject ever since. Subsequent generalizations (different pairs) followed quickly, e.g. [9] (Klein-Gordon), [30] and references therein. The abstract formulation which we follow is due to Ginibre and Velo, whose systematic treatment led to a series of seminal papers on the Cauchy problem for various semi-linear problems.

Proof: The proof proceeds through several steps: the main point is an abstract functional analysis argument, usually referred to as $TT^\star$ ("TT star").

Lemma 1
Let H be an Hilbert space, B and its dual B' Banach spaces, and a linear operator T. The following three properties are equivalent:

- *the operator T is bounded from H to B, $\|Tf\|_B \leq C\|f\|_H$.*
- *Its adjoint $T^\star$ is bounded from B' to H, $\|T^\star F\|_H \leq C\|F\|_{B'}$.*
- *The operator $TT^\star$ is bounded from B' to B, $\|TT^\star F\|_B \leq C^2\|F\|_{B'}$.*

Let us prove the lemma: the first two properties are equivalent, being dual of each other; recall that the adjoint $T^\star$ is defined through

$$\forall f \in H, \forall g \in B', \ < T^\star g, f >_{H,H} = < g, Tf >_{B',B} .$$

Now we prove the last two properties to be equivalent: clearly, combining the first two we obtain the third one. Let us prove the converse. Writing

$$\begin{aligned}
\|T^\star g\|_H^2 &= < T^\star g, T^\star g >_{H,H} \\
&= < g, TT^\star g >_{B',B} \\
&\leq \|g\|_{B'}\|TT^\star g\|_B \\
&\leq C^2\|g\|_{B'}^2,
\end{aligned}$$

we obtain the result. $\square$

Now, we proceed with the theorem. One has to chose H, B, B' and T.

- We set $H = L^2_x$, $B = L^q_t(L^r_x)$ and
$$T(\phi_0) = \phi(x,t) = S(t)\phi_0.$$
- Using the procedure above, we obtain $T^\star$,
$$T^\star(g(x,s) = \int_{s\in\mathbb{R}} S(-s)g(x,s)ds.$$
Certainly $T^\star$ is defined as a tempered distribution for $g \in \mathcal{S}(\mathbb{R}^{n+1})$, and can be extended to B' by density.
- Finally, we have to study
$$TT^\star g(x,t) = \int_{s\in\mathbb{R}} S(t-s)g(x,s)ds.$$

The benefit of this approach can be immediately identified: here, we need to prove continuity from $L^{\tilde{q}'}_t(L^{\tilde{r}'}_x)$ to $L^q_t(L^r_x)$; a particular case will be $\tilde{q} = q$ and at fixed $t-s$, we have our $L^{r'} - L^r$ dispersion inequality at hand. In order to conclude, we use the Hardy-Littlewood-Sobolev inequality. We state it in its generic form, even if we only need the one dimensional version.

Proposition 2 (Hardy-Littlewood-Sobolev)
Let $f \in L^\gamma(\mathbb{R}^m)$, $1 < \gamma < +\infty$, $0 < \alpha < n$, then the convolution operator by the function $|x|^{-\alpha}$ is bounded from L^γ to L^β where $\beta^{-1} - \gamma^{-1} = \alpha n^{-1} - 1$. In other words,

$$\left\| \frac{1}{|x|^\alpha} * f \right\|_\beta \leq C(n, \gamma, \alpha) \|f\|_\gamma, \ \frac{1}{\gamma} - \frac{1}{\beta} + \frac{\alpha}{n} = 1. \tag{1.12}$$

We won't prove this inequality (which can be obtained e.g. by real interpolation and is readily seen as a generalization of Young's inequality: $|x|^{-\alpha}$ is "almost" in $L^{\frac{n}{\alpha}}$).

Remark 4
This inequality can be seen as an appropriate version of Sobolev embedding theorem. In our particular setting, we need $m = 1$, $\gamma = q'$ and $\beta = q$, hence $\alpha = 2/q$; in order to obtain (1.12) all is left to prove is $\dot{H}^{\frac{1}{2}-\frac{1}{q}}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ (exercise !). Such an estimate can be proved by "elementary" computations (though somehow one has to perform an argument reminiscent of real interpolation by hands).

Remark 5
Another version of the H-L-S inequality reads as follows: $f \in L^\gamma$, $g \in L^\rho$ imply

$$| < |x|^{-\alpha} * f, g > | \lesssim \|f\|_\gamma \|g\|_\rho, \text{ avec } \frac{1}{\gamma} + \frac{1}{\rho} + \frac{\alpha}{n} = 2.$$

When the space dimension is $n = 1$, we can replace convolution by $|x|^{-\alpha}$ by $|x|^{-\alpha}\chi_{x>0}$ or $|x|^{-\alpha}\chi_{x<0}$: indeed, for positive functions,

$$\int_{x<y} \frac{f(x)g(y)}{|x-y|^\alpha} dxdy \leq \int_{\mathbb{R}^2} \frac{f(x)g(y)}{|x-y|^\alpha} dxdy,$$

and this fact will be of help later on.

Now we can proceed with the $TT^\star$ argument: using Proposition 1,

$$\begin{aligned}
\|TT^\star g\|_r(t) &\lesssim \int_{s\in\mathbb{R}} \|S(t-s)g(s)\|_r ds \\
&\lesssim \int_{s\in\mathbb{R}} \frac{1}{(t-s)^{\frac{2}{q}}} \|g(s)\|_{r'} ds \\
&\lesssim \frac{1}{(\cdot)^{\frac{2}{q}}} * (\|g(\cdot)\|_{r'})(t) \\
\|TT^\star g\|_{L^q_t(L^r_x)} &\lesssim \|g\|_{L^{q'}_t(L^{r'}_x)}.
\end{aligned}$$

Remark 6

Remark the constant in this last inequality is essentially the H-L-S constant, which is known to blow-up when q gets close to 2. Hence, our proof is restricted to $q > 2$. However, when $n \geq 3$, the extremal value $q = 2$ is an admissible one, but requires a much more complicated proof which we will provide later as an add-on ([38]).

We just proved (1.9). We are left with the two other estimates in the theorem. The next one, namely (1.10), is an immediate consequence of the first,

$$\begin{aligned}
\phi - S(t)\phi_0 &= \int_0^t S(t-s)F(s)ds \\
&= S(t)\int_{s\in\mathbb{R}} S(-s)\chi_{s\in(0,t)}F(s)ds \\
&= S(t)T^\star(\chi_{s\in(0,t)}F(s)).
\end{aligned}$$

Therefore, at fixed t,

$$\begin{aligned}
\|\phi - S(t)\phi_0\|_2 &= \|S(t)T^\star(\chi_{s\in(0,t)}F(s))\|_2 \\
&= \|T^\star(\chi_{s\in(0,t)}F(s))\|_2 \\
&\lesssim \|\chi_{s\in(0,t)}F(s))\|_{L_s^{q'}L_x^{r'}},
\end{aligned}$$

which is the desired result.

Finally, we prove (1.11). Two different problems arise: first, we would like to deal with $\int_{s<t} S(t-s)F(s)ds$ rather than $\int_{\mathbb{R}} S(t-s)F(s)ds$, and second, two different admissible pairs (q,r) should be allowed. The first problem is taken care of through remark 5 and the second problem will be disposed of by interpolation with the already obtained estimates, after obtaining the case $(\tilde{q},\tilde{r}) = (q,r)$. Let us start with it: we already proved

$$\|TT^\star F\|_{L_t^qL_x^r} = \|\int_{\mathbb{R}} S(t-s)F(s)ds\|_{L_t^qL_x^r} \lesssim \|F\|_{L_t^{q'}L^{r'}}.$$

Writing $\int_0^t S(t-s)F(s)ds = \int_{s<t} S(t-s)\tilde{F}(s)ds$ where $\tilde{F}(s) = F(s)\chi_{s>0}$, relabelling $\tilde{F}$ as F, we are left to prove

$$\|\int_{s<t} S(t-s)F(s)ds\|_{L_t^qL_x^r} \lesssim \|F\|_{L_t^{q'}L^{r'}}.$$

By duality,

$$\begin{aligned}\left|< \int_{s<t} S(t-s)F(s)ds, G(t) >_{\mathbb{R}^{n+1}}\right| &= \left|\int_{s<t} < S(t-s)F(s), G(s) >_{\mathbb{R}^n} dsdt\right| \\ &\lesssim \int_{s<t} \frac{1}{(t-s)^{\frac{2}{q}}} \|F(s)\|_{r'} \|G(s)\|_{r'} dsdt \\ &\lesssim \int_{\mathbb{R}^2} \frac{1}{(t-s)^{\frac{2}{q}}} \|F(s)\|_{r'} \|G(s)\|_{r'} dsdt \\ &\lesssim \|F(s)\|_{L_t^{q'} L_x^{r'}} \|G(s)\|_{L_t^{q'} L_x^{r'}},\end{aligned}$$

which is our result. In summary, we have proven that for an admissible pair (q, r),

$$\| \int_{s<t} S(t-s)F(s)ds\|_{L_t^\infty L_x^2} \lesssim \|F(s)\|_{L_t^{q'} L_x^{r'}}, \tag{1.13}$$

$$\| \int_{s<t} S(t-s)F(s)ds\|_{L_t^q L_x^r} \lesssim \|F(s)\|_{L_t^{q'} L_x^{r'}}. \tag{1.14}$$

Given that $\int_{t<s} S(t-s)F(s)ds = \int_{\mathbb{R}} S(t-s)F(s)ds - \int_{s<t} S(t-s)F(s)ds = S(t)T^\star F - \int_{s<t} S(t-s)F(s)ds$, the two inequalities hold for $\int_{t<s} S(t-s)F(s)ds$ as well. We interpolate between (1.13) and (1.14), using an appropriate generalization of the Riesz-Thorin interpolation theorem.

Proposition 3
Let T be a bounded operator from $L_x^{q_i} L_y^{r_i}$ to $L_\xi^{\alpha_i} L_\eta^{\beta_i}$ for $i = 0, 1$, $q_i, r_i, \alpha_i, \beta_i \in [1, +\infty]$, with constants M_i. Then the operator T is bounded from $L_x^{q_\theta} L_y^{r_\theta}$ to $L_\xi^{\alpha_\theta} L_\eta^{\beta_\theta}$, where indices are computed as in Theorem 1, i.e. $\gamma_\theta^{-1} = \theta\gamma_0^{-1} + (1-\theta)\gamma_1^{-1}$.

Therefore we obtain that for all pair (q, r) with $q > \tilde{q}$,

$$\| \int_{s<t} S(t-s)F(s)ds\|_{L_t^q L_x^r} \lesssim \|F(s)\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

and the inequality holds for $\int_{t<s} S(t-s)F(s)ds$ as well. In order to obtain the remaining cases $q < \tilde{q}$, notice this last operator $(t < s)$ is the adjoint of the previous one $(s < t)$:

$$\begin{aligned}< \int_{s<t} S(t-s)F(s)ds, G(t) >_{\mathbb{R}^{n+1}} &= \int_{s<t} < S(t-s)F(s), G(t) >_{\mathbb{R}^n} dsdt, \\ &= \int_{s<t} < F(s), S(s-t)G(t) >_{\mathbb{R}^n} dsdt, \\ &= < F(s), \int_{s<t} S(s-t)G(t)dt >_{\mathbb{R}^{n+1}} .\end{aligned}$$

Dualizing estimate (1.1), we obtain the case $q < \tilde{q}$, which ends the proof. $\square$

REMARK 7
We didn't actually prove the time continuity and leave it as an exercise to the reader !

We now deal with the endpoint case. The proof follows almost verbatim [38]. First, we rewrite the L^2 mass conservation and dispersion in a bilinear fashion: let F, G be two functions of x, t,

$$|< S(-s)F(s), S(-t)G(t) >| \leq \|F(s)\|_{L^2_x} \|G(t)\|_{L^2_x}, \tag{1.15}$$

and

$$|< S(-s)F(s), S(-t)G(t) >| \lesssim \frac{1}{|t-s|^{\frac{n}{2}}} \|F(s)\|_{L^1_x} \|G(t)\|_{L^1_x} \tag{1.16}$$

as well as

$$|< S(-s)F(s), S(-t)G(t) >| \lesssim \frac{1}{|t-s|^{1+\beta(r,r)}} \|F(s)\|_{L^{r'}_x} \|G(t)\|_{L^{r'}_x} \tag{1.17}$$

where $\beta(r, \rho) = \frac{n}{2}(1 - \frac{1}{r} - \frac{1}{\rho}) - 1$. By symmetry (and positivity) we only need to prove the estimate with the restriction $s < t$,

$$|T(F, G) = \int_{s<t} < S(t-s)F(s), G(t) > dsdt| \lesssim \|F\|_{L^2_t(L_x^{\frac{2n}{n+2}})} \|G\|_{L^2_t(L_x^{\frac{2n}{n+2}})}.$$

We split the (t, s) half-plane defined by $s < t$ into diagonal strips, defined by $2^j \leq t - s < 2^{j+1}$. Therefore

$$T(F, G) = \sum_{j \in \mathbb{Z}} \mathbb{T}_j(F, G) = \sum_{j \in \mathbb{Z}} \int_{2^j \leq t-s < 2^{j+1}} < S(-s)F(s), S(-t)G(t) > \, ds \, dt,$$

and to conclude it would be sufficient to bound $|T_j(F, G)|$ by a convergent series in j. Denote $r' = \frac{2n}{n+2}$, we will prove the following proposition.

PROPOSITION 4
For all $j \in \mathbb{Z}$, and all pairs (a, b) in a neighborhood of (r, r), we have

$$|T_j(F, G)| \lesssim 2^{-j\beta(a,b)} \|F\|_{L^2_t(L^{a'}_x)} \|G\|_{L^2_t(L^{b'}_x)}. \tag{1.18}$$

Notice (1.18) is scale invariant. Therefore without loss of generality we can restrict ourselves to $j = 0$. Given that we are integrating over $1 < t - s < 2$, we split $F = \sum_n F_n$ and $G = \sum_m G_m$ where $F_n = \chi_{n \le t \le n+1} F$ and $G_m = \chi_{m \le s \le m+1} G$. This, in turn, allows to rewrite

$$T_0(F, G) = \sum_{m,n} T_0(F_n, G_m) = \sum_{n \sim m} T_0(F_n, G_m),$$

which means we can restrict ourselves to the diagonal case $n = m$. Let us deal with $T_0(F_n, G_n)$: we need to prove two kind of estimates:

- Case $a = b = \infty$. Integrating the bilinear dispersion, we get

  $$|T_0(F_n, G_n)| \lesssim \|F_n\|_{L^1_t(L^1_x)} \|G_n\|_{L^1_t(L^1_x)},$$

  and by Hölder,

  $$|T_0(F_n, G_n)| \lesssim \|F_n\|_{L^2_t(L^1_x)} \|G_n\|_{L^2_t(L^1_x)}.$$

- Case $2 \le a < r$, $b = 2$. We rewrite

  $$|T_0(F_n, G_n)| \le \sup_t \| \int_{1<t-s<2} S(-s) F_n(s) ds \|_{L^2_x} \int_t \|S(-t) G_n(t)\|_{L^2_x} dt,$$

  and for the first term we use the non endpoint estimate we already proved: $a' > r'$ therefore (A, a) is a pair, $A' < 2$ and

  $$|T_0(F_n, G_n)| \le \|F_n(s)\|_{L^{A'}_t(L^{a'}_x)} \|G_n(t)\|_{L^1_t L^2_x}.$$

  Finally, by Hölder

  $$|T_0(F_n, G_n)| \le \|F_n(s)\|_{L^2_t(L^{a'}_x)} \|G_n(t)\|_{L^2_t L^2_x}.$$

We need to sum over n, but given that $F_n(t) = \chi_{n<t<n+1} F(t)$, this follows easily from Hölder, as the sequence $(\|F_n(t)\|_{L^2_t})_n$ belongs to l^2 with sum $\|F(t)\|_{L^2_t}$. This ends the proof of Proposition 4.
In order to complete the proof, we need yet another version of interpolation, which we state without detailing the meaning of real interpolation spaces.

THEOREM 3
Let $A_0, A_1, B_0, B_1, C_0, C_1$ be Banach spaces, T a bounded bilinear operator from $A_0 \times B_0 \to C_0$, $A_1 \times B_0 \to C_1$ and $A_0 \times B_1 \to C_1$. Then, if $0 < \theta_0, \theta_1 < \theta < 1$, $1 \le p, q, r \le +\infty$ are such that $1 \le 1/p + 1/q$ and $\theta = \theta_0 + \theta_1$, the operator T is bounded from $(A_0, A_1)_{\theta_0, pr} \times (B_0, B_1)_{\theta_1, qr} \to (C_0, C_1)_{\theta, r}$.

The only thing we need to know with respect to real interpolation is the following:

- Let $1/p = \theta/p_0 + (1-\theta)/p_1$, then
$$(L_t^2, L_x^{p_0})_{\theta,2} = L_t^2 L_x^{p,2},$$
where $L_x^{p,2} \hookrightarrow L_x^p$ if $p \geq 2$.
- We have
$$(l_\infty^{s_0}, l_\infty^{s_1})_{\theta,1} = l_1^s$$
with $s = \theta s_0 + (1-\theta)s_1$, and $l_p^s = l^p(2^{js}dj)$ is a weighted version of l^p.

Now, we can rewrite Proposition 4:

$$T \text{ is bounded from } L_t^2 L_x^{a'} \times L_t^2 L_x^{b'} \text{ to } l_\infty^{\beta(a,b)},$$

where T denotes the (vector-valued) operator with coordinates T_j, $j \in \mathbb{Z}$. Applying the bilinear interpolation theorem with $p = q = 2$, $r = 1$ and a_0, a_1, b_0, b_1 such that $\beta(a_0, a_1) = \beta(a_1, b_0) \neq \beta(a_0, b_0)$, we obtain that in a neighborhood of (r, r),

$$T \text{ is bounded from } L_t^2 L_x^{a',2} \times L_t^2 L_x^{b',2} \text{ to } l_1^{\beta(a,b)},$$

and with the values $a = b = r$, we are done. $\square$

Remark 8
The "abstract" proof by real interpolation hides the rather simple property which follows: let F, G be such that $F(t) = 2^{-\frac{k}{r'}} f(t)\chi_{E(t)}$ and $G(s) = 2^{-\frac{l}{r'}}(s)\chi_{H(s)}$, with$|E(t)| = 2^k$, $|G(s)| = 2^l$. Then

$$|T_j(F,G)| \lesssim 2^{(k-j\frac{n}{2})(\frac{1}{r}-\frac{1}{a})+(l-j\frac{n}{2})(\frac{1}{r}-\frac{1}{b})} \|f\|_{L_t^2} \|g\|_{L_t^2}.$$

Given it holds for any pair (a, b) close to (r, r), we may optimize and obtain

$$|T_j(F,G)| \lesssim 2^{-\varepsilon(|k-j\frac{n}{2}||l-j\frac{n}{2}|} \|f\|_{L_t^2} \|g\|_{L_t^2},$$

which is summable in j.

Let us finish this section by giving a quick list of applications where Strichartz estimates are a crucial tool:

- Low regularity well-posedness for semilinear Schrödinger equations with a nonlinear term $|u|^{p-1}u$, see e.g. [18], [37].

- Well-posedness in the energy class of the same type of equation, provided $p < 1 + 4/n - 2$, globally in time in the defocusing case, see [30], [36].

- Scattering, same references as above, with the addition of [50] for low dimensions. See also [51] for a nice unified presentation.

Very recently, the existence of globally smooth solutions to the 3D quintic ($p = 5$) Schrödinger was obtained in [21], and Strichartz estimates play an important role.

1.2 Strichartz and its connection with the restriction problem in harmonic analysis

The present section attempts to briefly explain the aforementioned connection, and certainly does not to justice to the harmonic analysis side of the story (which deserves an entire book by itself !).
As explained before, this deep connection was established clearly in [66], where estimates for the wave and Schrödinger equations are deduced from restriction estimates to curved hypersurfaces (cones or paraboloïd).
Let us describe what the restriction problem is. Let us consider a Schwartz class function f. Its Fourier transform is well-defined and Schwartz class as well. Obviously,

$$\|\hat{f}\|_{L^\infty} \lesssim \|f\|_1,$$

and we can meaningfully restrict $\hat{f}$ to any hypersurface H, with

$$\|\hat{f}_{|H}\|_{L^\infty} \lesssim \|f\|_1.$$

A natural question (for an harmonic analyst !!) is then the following: can we replace L^1 by some higher L^p norm, perhaps at the expense of lowering the L^∞ norm to an L^q norm. Certainly taking $p = 2$ is not possible: we only control the L^2 norm of $\hat{f}$ by $\|f\|_{L^2}$, and there is no reason one can restrict an arbitrary L^2 function to an arbitrary hypersurface (take a plane to convince yourself). However, if one is allowed half a normal derivative relative to the hypersurface, a trace theorem gives control of an $L^p(H)$ norm by, say, $B_p^{\frac{1}{2},1}(\mathbb{R}^{n+1})$.Assuming we apply this to $\hat{f}$, and forgetting about q, the last norm is controlled by an $L^{p'}(|x|^{p'/2}dx)$ norm of f (if $p \geq 2$, but since all of our discussion is heuristic anyway, we forget about it and simply play with rescaling) Finally, this last norm controls a certain L^q norm, with $q \leq p$. It turns out (a fact first noticed by E. Stein) that whenever the hypersurface H

has non vanishing curvature, one may sometimes control $\|\hat{f}\|_{L^p(H)}$ directly by $\|f\|_{L^q}$. Rescaling gives the purposely optimal values of (p, q) (which depend on the curvature properties of H); non-vanishing curvature is required, as the simple plane example illustrates. The restriction conjecture states that these optimal values are indeed reached. The only settled case is currently $n+1=2$; For $n+1=2$ and with H being the sphere or a paraboloïd (essentially the same thing by parabolic rescaling), the optimal exponents are $4, 4/3$, and as often, 4 is a magic exponent as square of 2.
From our (PDE) point of view, and as noticed in [66], a particular case of the Strichartz estimates we proved earlier is exactly a restriction theorem for the paraboloïd, with $p=2$, or more accurately, the dual result ("extension problem"). Let us denote by g a density measure on a curved hypersurface H, or equivalently, if $d\sigma$ denotes the single layer distribution on H, the distribution $gd\sigma$; we would like to say something about the (inverse) Fourier transform of $gd\sigma$, like

$$\|\mathcal{F}^{-1}(gd\sigma)\|_{L^a(\mathbb{R}^{n+1})} \lesssim \|g\|_{L^b(H)}.$$

The restriction operator and this new operator (extension) are obviously adjoint, and therefore proving this estimate amounts to proving a restriction estimate with a pair (b', a').
Now we can easily establish the link with Schrödinger: From $\hat{u}_0(\xi)$ with $\xi \in \mathbb{R}^n$, we can define a single layer distribution on the paraboloïd $\tau = |\xi|^2$, namely the distribution $\delta(\tau - |\xi|^2)\hat{u}_0(\xi)$ and the other way around. By Plancherel $\|\hat{u}_0\|_{L^2}$ equals $\|\hat{u}_0\|_{L^2}$, and the inverse Fourier transform (in time-space) of $\delta(\tau - |\xi|^2)\hat{u}_0(\xi)$ is nothing but our solution $u(x, t)$ to the Schrödinger equation $i\partial_t u + \Delta u = 0$ with datum u_0. Therefore we have

$$\|u\|_{L^{\frac{2(n+2)}{n}}_{t,x}} \lesssim \|\hat{u}_0\|_{L^2_\xi},$$

or, as an extension estimate,

$$\|\mathcal{F}^{-1}(gd\sigma)\|_{L^{\frac{2(n+2)}{n}}(\mathbb{R}^{n+1})} \lesssim \|g\|_{L^2(P)},$$

where P is the n dimensional paraboloïd.
The near optimal result for $p=2$ is due to Tomas ([71]) in 1975 in a very short and clever paper. The extremal result stated above is due to Stein shortly after, and is unpublished. The 'modern" proof we give through the PDE is simpler and of wider scope than Stein's original argument, which was used by Strichartz in [66] and which requires a generalized version (due to Stein !!) of the complex interpolation theorem, allowing the operator to depend on a complex parameter as well.

1.3 Smoothing estimates

In the first section, we saw how, for, say, an L^2 datum, the solution to the Schrödinger equation gains integrability if one is willing to average in time. On the other hand, there is another interesting property which follows another conservation law for the Schrödinger equation: consider the (vector-valued) vector field $C = x/2 - it\nabla$. An easy sequence of computation shows C to commute with the flow. Hence, if we assume the datum to be in a weighted L^2 space, it somehow gains regularity when $t \neq 0$. Again, the weighted norms are not particularly convenient, and one would like to have an estimate with $\phi_0 \in L^2_x$. Estimates of this type are often referred to as "local smoothing" estimates, and can be traced back to a seminal paper by T. Kato [35] on the KdV equation. Later works ([58, 73, 23]) establish the same type of properties for a general class of dispersive models, including both KdV and Schrödinger. Roughly speaking, one gains half a derivative, but only locally in L^2_x, and averaged in the L^2_t sense:

$$\sup_{x_0} \int_{B(x_0,1)} \||\nabla|^{\frac{1}{2}} S(t)\phi_0\|_{L^2_t(L^2(B(x_0,1)))} \lesssim \|\phi_0\|_2.$$

Remark 9

Notice how this is an $L^2_{t,x}$ estimate. It turns out one can obtain it by some form or another of "integration by parts": see e.g. [46] where a nice connection with moment and averaging lemma for kinetic equations is established. At some abstract level, such local smoothing estimates are consequences of resolvent estimates; such estimates are the subject of a huge literature in spectral theory, and in particular are known to hold for much more general situations than the flat Laplacian. We only give a few references: [5], [74], [27, 26], [10].

It turns out that in 1D one can actually slightly improve the smoothing effect.

Theorem 4

Let $\phi = S(t)\phi_0$ be the solution to the Schrödinger equation, and $\phi_0 \in L^2$. Then we have

$$\|\sqrt{-\Delta}^{\frac{1}{4}} S(t)\phi_0\|_{L^\infty_x(L^2_t)} \lesssim \|\phi_0\|_2. \tag{1.19}$$

Proof: Let us write

$$\begin{aligned}
\|\sqrt{-\Delta}^{\frac{1}{4}} S(t)\phi_0\|^2_{L^\infty_x(L^2_t)} &= \int_t |\int_\xi e^{ix\xi - it|\xi|^2} |\xi|^{\frac{1}{2}} \hat{\phi}_0(\xi)\, d\xi|^2 dt \\
&= \int_t |\int_0^{+\infty} e^{-it|\xi|^2} |\xi|^{\frac{1}{2}} (e^{ix\xi}\hat{\phi}_0(\xi) + e^{-ix\xi}\hat{\phi}_0(-\xi))\, d\xi|^2 \\
&= \int_t |\int_\eta e^{-it\eta} \eta^{-\frac{1}{4}} (e^{ix\sqrt{\eta}}\hat{\phi}_0(\sqrt{\eta}) \\
&\qquad\qquad + e^{-ix\sqrt{\eta}}\hat{\phi}_0(-\sqrt{\eta}))\chi_{\eta>0} \frac{d\eta}{2}|^2 dt, \\
&= c\int_\eta \eta^{-\frac{1}{2}} |e^{ix\sqrt{\eta}}\hat{\phi}_0(\sqrt{\eta}) + e^{-ix\sqrt{\eta}}\hat{\phi}_0(-\sqrt{\eta})|^2 \chi_{\eta>0}\, d\eta, \\
&\lesssim \int |\hat{\phi}_0(\xi)|^2\, d\xi,
\end{aligned}$$

where we used a change of variable, Plancherel in time, and finally reverted the change of variable. $\square$

Now, we want inhomogeneous estimates as well.

THEOREM 5

Let ϕ be the solution to the Schrödinger equation

$$i\partial_t \phi + \partial_x^2 \phi = F(x,t),$$

with zero Cauchy datum. Then we have

$$\|\sqrt{-\Delta}^{\frac{1}{4}}\phi\|_{C_t(L^2_x)} \lesssim \|F(x,t)\|_{L^1_x(L^2_t)}, \tag{1.20}$$

and

$$\|\partial_x \phi\|_{L^\infty_x(L^2_t)} \lesssim \|F(x,t)\|_{L^1_x(L^2_t)}. \tag{1.21}$$

Proof: Obviously the first inequality is the adjoint version of (1.19). By splitting F is necessary and regularization, we can reduce ourselves to a situation where F is time supported in the future, $t > 0$. Taking the time Fourier transform, one has

$$(-\tau + \partial_x^2)\tilde{\phi} = \tilde{F},$$

and

$$\partial_x \tilde{\phi} = -\int_{\mathbb{R}} e^{ix\xi} i\xi \frac{\hat{\tilde{F}}(\xi,\tau)}{\tau + \xi^2} d\xi.$$

Certainly this expression makes sense whenever $\tau > 0$, which corresponds to a situation where the operator at hand is in fact elliptic. An easy sequence of computation shows that

$$\mathcal{F}_\xi(\frac{\xi}{\tau+\xi^2}) = Ce^{-\tau|x|},$$

and using Plancherel ($\xi \to x$)

$$|\partial_x \tilde{\phi}| = C|\int_{\mathbb{R}} e^{-\tau|y-x|}\tilde{F}(y,\tau)\,dy|,$$

and we obtain that at fixed τ,

$$\|\partial_x \tilde{\phi}\|_{L^\infty_x} \lesssim \|\tilde{F}\|_{L^1_x}.$$

Consider now the hyperbolic case $-\tau > 0$ and set $\sigma^2 = -\tau$: then

$$\frac{2\xi}{\tau+\xi^2} = \frac{1}{\xi+\sigma} + \frac{1}{\xi-\sigma},$$

and given that

$$\mathcal{F}_\xi(\frac{1}{\xi+\sigma}) = Ce^{ix\sigma}\mathrm{sgn}x,$$

we can conclude in the same way. Therefore, for all $\tau \in \mathbb{R}$,

$$\|\partial_x \tilde{\phi}\|_{L^\infty_x} \lesssim \|F\|_{L^1_x}.$$

But we can take L^2_τ, use Plancherel in time, and switch space and time norms using Minkowski (which goes in the appropriate direction on both sides), to get

$$\|\partial_x \phi\|_{L^\infty_x(L^2_t)} \lesssim \|F\|_{L^1_x(L^2_t)},$$

which is the desired estimate. □

Remark 10
We have been rather sloppy in term of justifying any of the formal computations. We refer to e.g. [39, 41] for a detailed proof along the same lines. Moreover, we point out that the previous proof is without doubt dependant on the space Fourier transform and explicit representation of the solution, which makes it rather difficult to generalize.

1.4 Maximal function estimates

The estimates of the previous section have one particularly useful feature: the inhomogeneous estimate allows to "recover" a derivative in a source term. As such, it can be used to deal with semilinear problems with a derivative term in the nonlinearity. However, no matter what, it will requires to have another set of estimates at hand, providing something like

$$\|S(t)\phi_0\|_{L^p_x(L^\infty_T)} \lesssim \|\phi_0\|_{H^s}$$

for appropriate values of p and s.
It turns out that such estimates do exist, and we will state and prove the 1D version which is known to be sharp. Higher dimensional versions are available (see again [73, 58, 23]) in the L^2_x setting but are non sharp. Further progress ($p < 2$) has been made recently using a rather heavy machinery based on restriction estimates, see the appendix to [8], and later bilinear restriction estimates, [68, 67]. We won't touch this subject which is very active but has more to do with harmonic analysis than PDEs (and deserve probably an entire book...).

THEOREM 6
Let $\phi = S(t)\phi_0$ be the solution to the 1D Schrödinger equation, with $\phi_0 \in L^2$. Then

$$\|\sqrt{-\Delta}^{-\frac{1}{4}}\phi\|_{L^4_x(L^\infty_t)} \lesssim \|\phi_0\|_2. \tag{1.22}$$

Proof: This result goes back to [44] though the statement there is different and one has to check the proof to realize it trivially implies the above theorem. The proof which follows is a simplification of the argument from [44] (most likely known and similar to the one alluded to in the original paper and attributed to Nagel and Stein, but we couldn't find a reference).
Recall that the solution writes

$$\phi(x,t) = \int e^{ix\xi - it|\xi|^2}\hat{\phi}_0(\xi)\,d\xi.$$

In order to prove (1.22), it suffices to consider the operator

$$T\phi_0 = \int e^{ix\xi - it(x)|\xi|^2}\frac{\hat{\phi}_0(\xi)}{|\xi|^{\frac{1}{4}}}\,d\xi,$$

where $t(x)$ is a bounded function, $|t(x)| \leq T$. Using Fourier inversion formula, we have

$$T\phi_0 = \int e^{i(x-y)\xi - it(x)|\xi|^2}\frac{1}{|\xi|^{\frac{1}{4}|}}\phi_0(y)\,d\xi dy,$$

and its adjoint $T^\star$ reads

$$T^\star \psi_0 = \int e^{i(x-z)\xi + it(z)|\xi|^2} \frac{1}{|\xi|^{\frac{1}{4}|}} \psi_0(z)\, d\xi dz.$$

Therefore we can proceed with a $TT^\star$ argument:

$$TT^\star f = \int e^{i((x-y)\xi + (y-z)\eta - t(x)\xi^2 + t(z)\eta^2)} \frac{1}{|\xi|^{\frac{1}{4}}|\eta|^{\frac{1}{4}}} f(z)\, dz dy d\eta d\xi.$$

We need to prove that $TT^\star$ is bounded from $L_x^{\frac{4}{3}}$ to L_x^4: in order to achieve this, we consider the kernel

$$K(x,z) = \int e^{i((x-y)\xi + (y-z)\eta - t(x)\xi^2 + t(z)\eta^2)} \frac{1}{|\xi|^{\frac{1}{4}}|\eta|^{\frac{1}{4}}}\, dy d\eta d\xi. \tag{1.23}$$

LEMMA 2
Let K be defined by (1.23). *Then*

$$|K(x,z)| \leq \frac{C}{|x-z|^{\frac{1}{2}}},$$

where C is independent of the function $t(x)$.

Let us postpone the proof of the lemma: using the Hardy-Littlewood-Sobolev inequality, we readily obtain the desired estimate, which concludes the proof of Theorem 6.
Let us go back to the lemma: from $\int e^{iy(\eta-\xi)}\, dy = \delta(\xi - \eta)$, one can rewrite

$$K(x,z) = \int e^{i(x-z)\xi + (t(z)-t(x))\xi^2} \frac{d\xi}{|\xi|^{\frac{1}{2}}}.$$

The result follows from a Van der Corput type estimate: let $a, b \in \mathbb{R}$, then

$$|\int e^{ia\xi + b\xi^2} \frac{d\xi}{|\xi|^{\frac{1}{2}}}| \leq \frac{C}{|a|^{\frac{1}{2}}},$$

where C doesn't depend on a, b. Indeed, changing variables we can reduce to $a = 1$: set $\eta = a\xi$,

$$|\int e^{i\eta + \frac{b}{a^2}\eta^2} \frac{d\eta}{|a|^{\frac{1}{2}}|\eta|^{\frac{1}{2}}}| \leq \frac{C}{|a|^{\frac{1}{2}}},$$

hence we only need to prove (taking $a = -2$ for convenience)

$$|\int e^{i-2\eta+\lambda\eta^2}\frac{d\eta}{|\eta|^{\frac{1}{2}}}| \leq C.$$

Without loss of generality we can reduce to $\lambda > 0$ ($\lambda = 0$ is trivial). The phase is stationary when $\lambda\eta = 1$. We rewrite the integral as

$$\int e^{i-2\eta+\lambda\eta^2}\frac{d\eta}{|\eta|^{\frac{1}{2}}} = e^{-i\frac{4}{\lambda}}\int e^{iy^2}\frac{dy}{\lambda^{\frac{1}{4}}|y+\frac{2}{\sqrt{\lambda}}|^{\frac{1}{2}}}.$$

Switch $\lambda = 1/\sigma^4$ and $y \to -x$, we have

$$I(\sigma) = \int e^{ix^2}\frac{\sigma\, dx}{|x-\sigma^2|^{\frac{1}{2}}} = \int_{-\infty}^{-\mu} + \int_{|x|<\mu} + \int_{\mu}^{\sigma^2-\varepsilon} + \int_{|x-2\sigma^2|<\varepsilon} + \int_{2\sigma^2+\varepsilon}^{+\infty},$$

and notice the trouble is with large σ. we integrate by parts the first, third and fourth term and get a bound

$$I(\sigma) \lesssim \frac{\sigma}{\mu\sqrt{\sigma^2-\mu}} + \frac{\mu\sigma}{\sqrt{\sigma^2-\mu}} + \frac{\sigma}{(\sigma^2-\varepsilon)\sqrt{\varepsilon}},$$

which is obviously bounded whenever $\sigma > 1$. For $\sigma < 1$, we can simply split

$$I(\sigma) = \int_{|x|<2} + \int_{|x|>2},$$

and the first integral is trivially bounded. Let us deal with the second one by IBP,

$$|\int_2^{+\infty} e^{ix^2}\frac{\sigma\, dx}{|x-\sigma^2|^{\frac{1}{2}}}| \lesssim \frac{\sigma}{2\sqrt{2-\sigma^2}} + \int_2^{\infty}\frac{\sigma}{x^2\sqrt{x-\sigma^2}}\, dx \leq C.$$

This ends the proof. □

There are other estimates which are of interest. For example, we have a variant of the previous theorem.

Theorem 7

Let $\phi = S(t)\phi_0$ be the solution to the 1D Schrödinger equation, with $\phi_0 \in L^2$ and compact support of the Fourier transform: $\operatorname{supp}\hat{\phi}_0 \subset [-1,1]$. *Then*

$$\|\phi\|_{L^2_x(L^\infty_T)} \lesssim T^{\frac{3}{4}+}\|\phi_0\|_2. \tag{1.24}$$

Proof: We omit the proof which proceeds along the same lines, replacing in the kernel K the $|\xi|^{-1/4}$ factor by $\varphi(\xi)$ where φ is a smooth cut-off function. We are then left to prove that the kernel sends L^2_x to L^2_x, which follows again from stationary phase estimates... $\square$

As we mentioned earlier, an important application of smoothing and maximal function estimates is the Cauchy problem for nonlinear Schrödinger equations. There is a long history on this topic, and we refer to [41], [33], [19] and [43] and references therein.

However, the maximal function estimates serve another purpose, which we briefly describe now: recall $\phi = S(t)\phi_0$ is the solution to the Schrödinger equation. Given ϕ_0 in a Sobolev space H^s, what are the requirements on s which imply almost everywhere convergence of $\phi(t)$ toward ϕ_0 ? We only have a complete answer to this question in 1D, where $s \geq 1/4$ is necessary and sufficient. The positive result is due to Carleson [16] and the negative result may be found in [24]. Even in 1D, possibly extending such a result for non Sobolev initial data is highly non trivial.

Let us consider a smooth datum ψ_0: then one has trivially everywhere convergence of the solution $\psi = S(t)\psi_0$ to the datum (actually, uniform convergence !). We then introduce

$$N\phi_0(x) = \limsup_{t\to 0}|S(t)\phi_0 - \phi_0|.$$

Next, notice that if ψ_0 is smooth,

$$N\phi_0(x) = N(\phi_0 - \psi_0)(x),$$

from the preceding remark. Therefore,

$$N\phi_0(x) \leq \sup_t |\phi - \psi|(x) + |\phi_0 - \psi_0|(x),$$

and as such,

$$\{N\phi_0(x) > \alpha\} \subset \{|\phi - \psi|(x) > \alpha/2\} \cup \{|\phi_0 - \psi_0|(x) > \alpha/2.$$

Now, by Chebyshev

$$\begin{aligned} |\{N\phi_0(x) > \alpha\}| &\lesssim \frac{\|\phi - \psi|_4^4}{\alpha^4} + \frac{\|\phi_0 - \psi_0|_2^2}{\alpha^2} \\ &\lesssim \frac{\|\phi_0 - \psi_0|^4_{\dot H^{\frac14}}}{\alpha^4} + \frac{\|\phi_0 - \psi_0|_2^2}{\alpha^2}. \end{aligned}$$

Now, we can pick ψ_0 such that the RHS is smaller than an arbitrary ε, and therefore at a given α, the set $\{N\phi_0(x) > \alpha\}$ has zero Lebesgue measure. In turn, this being true for any $\alpha > 0$, $N\phi_0(x) = 0$ almost everywhere.

Chapter 2

Smoothing for the 1D variable coefficients Schrödinger equation

Introduction

We would like to generalize the estimates proven in the first chapter to an equation like

$$i\partial_t \phi + g^{ij}\partial_i\partial_j\phi = 0, \quad \phi(x, t = 0) = \phi_0(x), \tag{2.1}$$

where g^{ij} is a given metric on a manifold. Before giving an historical perspective, let us make a digression and say a word about the wave equation: the main difference stands in the finite speed of propagation enjoyed by a wave equation, a fact which holds irrespective of the variable coefficients. Hence, as long as one is interested in local in time estimates, one may localize estimates in a neighborhood of a given point. In other words, the global geometry of the manifold doesn't play a role, and the requirements on the metric have merely to do with regularity (of course, getting global in time estimates will require additional assumptions "at infinity" and we will meet the same problems as for Schrödinger). For smooth metrics, Strichartz estimates for the wave equation hold locally (this can be obtained from a classical parametrix construction). This was lowered to C^2 coefficients in [60], and later to $\partial^2 g \in L^1_t L^\infty_x$ in [70]. Anything below this regularity leads to losses w.r.t. the flat case (however, one can certainly solve interesting problems despite the loss, see [3], [45] and [59].
In contrast, the picture for the Schrödinger equation is less complete, to say the least. The case of C^2 compact perturbations of the Laplacian was addressed in [62] and may be seen as an analog of Smith's result for the wave. However, we already see that a strong hypothesis is made "at infinity": the

metric is flat. Moreover, one has to make a non-trapping assumption: classical trajectories have to escape to infinity. This later point can be traced back to the use of local smoothing estimates in the proof (in a rather essential way). In dimensions $n \geq 2$, Doï [26, 28] and Burq [12] proved that the non trapping assumption is necessary for the optimal smoothing effect to hold. the expectation is that whenever there are trapped rays, we can hope for Strichartz estimates with a loss, and this is known to be the case for compact manifolds ([11]). Very recent work deal with short range smooth perturbations of the flat metric, with symbol-like decay at infinity ([55]). The idea to use local smoothing to derive Strichartz, however, was already present in [34] where a potential perturbation was treated. Dealing with a flat metric and a potential has a longer history, starting essentially with the aforementioned reference. One should also mention [75] and references therein, [56], [31] and references therein. All works on this topic make definitive use of resolvent estimates for the elliptic operator, which are closely related to local smoothing, as we hinted at earlier.

Now, consider the 1D equation,

$$i\partial_t u + \partial_x(a(x)\partial_x u) = 0, \quad u(x, t = 0) = u_0(x). \tag{2.2}$$

The 1D situation had not been investigated further until recently (to the knowledge of the author !). However, it does have some advantages, the first one being that a variable coefficients metric is just a fancy name for only one coefficient: either $g(x)\partial_x^2$ or $\partial_x a(x)\partial_x$, with a ellipticity condition. Moreover, whenever this coefficient is smooth, the metric is always non-trapping, as can be easily seen by looking at classical trajectories. V. Banica ([4]) considered the case where the metric a is piecewise constant (with a finite number of discontinuities). In [4], it is proved that the solutions of the Schrödinger equation associated to such a metric enjoy the same dispersion estimates (implying Strichartz) as in the case of the constant metric, and it is conjectured it would hold true for general $a \in \mathrm{BV}$ as well. Unfortunately, her method of proof (which consists in writing a complete description for the evolution problem) leads to constants depending upon the *number* of discontinuities rather than on the norm in BV of the metric and consequently does not extend to the general case. On the other hand, Castro and Zuazua [17] show that the space BV is more or less optimal: they construct metrics $a \in C^{0,\beta}$ for all $\beta \in [0, 1[$ (but not in BV) and solutions of the corresponding Schrödinger equation for which any *local* dispersive estimate of the type

$$\|u(t,x)\|_{L^1_{\mathrm{loc},t}(L^q_{\mathrm{loc},x})} \leq C\|u_0\|_{H^s}$$

fail if $1/p < 1/2 - s$ (otherwise, the estimate is a trivial consequence of Sobolev embeddings).

As such, BV or one its close cousins like $\dot{W}^1_1$ seems like a reasonable candidate: such functions have some smoothness, some decay at infinity (which insures we get close to the flat metric), and they have exactly the right scaling. We intend to prove this natural conjecture, namely that for BV metrics, the Schrödinger equation enjoys the same smoothing, Strichartz and maximal function estimates as for the constant coefficient case, globally in time. In the context of variable coefficients, this appears to be the first case where such a low regularity (including discontinuous functions) is allowed, together with a translation invariant formulation of the decay at infinity (no pointwise decay). Before briefly explaining our strategy, we note that Salort [57] recently obtained dispersion (hence, Strichartz) (locally in time) for 1D Schrödinger equations with C^2 coefficients through a completely different approach involving commuting vector fields.
A shorter presentation of what follows can be found in [13].

- In this chapter, we prove a smoothing estimate which is the key to all subsequent results, by an elementary integration by parts argument, reminiscent of the time-space symmetry for the 1D wave equation. Transferring results from the wave to Schrödinger is sometimes called a transmutation and has been used in different contexts ([47]). All the estimates we prove hold for a smooth coefficient $a(x)$, and the usual limiting argument yields $a \in W^1_1$, but we prove that one may extend it to $a \in$ BV by purely functional analytic methods (with the help of another blackbox, Reed-Simon [54]).

- In the next chapter, we deal with Strichartz and maximal function estimates, by combining our smoothing estimate with known estimates for the flat case. While the heuristic argument is (relatively) straightforward, its implementation requires a bit of paradifferential calculus which we detail along the way.

- Next, we consider the optimality of our result: we prove that the BV regularity threshold is optimal in a different direction from [17]: there exist a metric $a(x)$ which is in $L^\infty \cap W^{s,1}$ for any $0 \leq s < 1$, bounded from below by $c > 0$, and such that no smoothing effect nor (non trivial) Strichartz estimates are true (even with derivatives loss). This construction is very close in spirit to the one by Castro-Zuazua [17].

- The last chapter will be devoted to a nonlinear application. We obtain sharp wellposedness for a generalized Benjamin-Ono equation.

- Finally, there are several appendices; the first appendix is a recollection of some results of Auscher-Tchamitchian [2] and Auscher-MacIntosh-

Tchamitchian [1] which imply that the spectral localization with respect to the operators $\partial_x a(x)\partial_x$ and ∂_x^2 are reasonably equivalent. This is important if one wants to obtain estimates with derivatives (a crucial fact for nonlinear applications).

- In a second appendix we give a self-contained proof of a suitably modified version of Christ-Kiselev Lemma (see [20]).

2.1 Functional spaces

Up to now, we have avoided discussing complicated issues related to spaces: we only use Sobolev spaces (mainly L^2 !) and Lebesgue spaces. When dealing with the flat Schrödinger equation, this is made easier by a specific property of the flow: it commutes with the Laplacian, hence with its spectral localization, and more generally with any pseudo-differential operator defined by a Fourier multiplier $m(\xi)$. However, when we no longer have a constant coefficients Laplacian but a second order operator A, formally the flow will commute with "functions" of A. It turns out that under mild assumptions on A, its spectral localization has good commutation properties with the Fourier one, and this will be of use later (and detailed in the Appendix). Note that if one wants to prove an analog of (1.19), certainly we need a replacement of $\sqrt{-\Delta}^{\frac{1}{4}}$, and $\sqrt{-A}^{\frac{1}{4}}$ immediately comes to mind. However, we can temporarily avoid this discussion by using Besov spaces, and we take the opportunity to list basic facts.

2.1.1 Heuristic and Sobolev spaces

Sobolev spaces appear in a rather natural way: through Plancherel it is easy to evaluate the action of a constant coefficients differential operator, as it translates into multiplication by a polynomial. Now, we can try to interpret this in a slightly different way: on what functions does derivation becomes multiplication by a constant (when measured in L^2 norm) ? Take $f \in \mathcal{S}$; we know $\mathcal{F}(\nabla f) \approx \xi \hat{f}(\xi)$. Assume moreover that $\hat{f}$ has most of its mass around $|\xi| \approx \lambda$: say $\operatorname{supp} \hat{f} \subset \{\xi \text{ t.q. } \lambda - \mu \leq |\xi| \leq \lambda + \eta\}$, with $\mu, \eta > 0$. Plancherel yields

$$(\lambda - \mu)\|\hat{f}\|_2 \leq \|\partial f\|_2 \leq (\lambda + \eta)\|\hat{f}\|_2.$$

Therefore, to get $\|\nabla f\|_2 \approx \lambda \|f\|_2$, we just need $|[\lambda - \mu, \lambda + \eta] \approx \lambda$, in other words, $\mu \approx \lambda$.
Hence, it only makes sense to split any function f as a sum of pieces which possess the property above: e.g., let $\hat{f}_j(\xi) = \chi_{2^j \leq |\xi| \leq 2^{j+1}} \hat{f}(\xi)$, then

Proposition 5
An equivalent norm of the usual H^s norm is

$$\|u\|_{H^s}^2 \approx \|\chi_{|\xi|<1} f\|_2^2 + \sum_{j\geq 0} 2^{2js}\|f_j\|_2^2. \tag{2.3}$$

whose proof follows trivially the previous line of reasoning (applied to $|\xi|^s$). The next step becomes "what about L^p rather than L^2 ?". We lose Plancherel, and moreover, the Fourier multiplier $\chi_{2^j\leq|\xi|\leq 2^{j+1}}$ is not bounded on L^p, except for $n=1$ (surprising (at the time of its proof) and deep harmonic analysis result, [29]). Certainly, Fourier multipliers $m(\xi)$ are more difficult to handle on L^p.

2.1.2 Littlewood-Paley analysis

Recall we want to split the frequency space in octaves: zones where $|\xi| \approx \lambda$ and with a size variation at most $\lambda/2$. A dyadic partition should do, $\{2^j \leq |\xi| < 2^{j+1}\}$, but as we said, we run into problems with L^p: the solution is to smooth out the partition, while retaining almost orthogonality.

Definition 1
Set $\varepsilon < 1$, take $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\phi} = 1$ for $|\xi| \leq 1$ and $\widehat{\phi} = 0$ for $|\xi| > 1+\varepsilon$. Denote by $\phi_j(x) = 2^{nj}\phi(2^j x)$.

- *Define S_j as the convolution with ϕ_j. Note that* $\operatorname{supp}\mathcal{F}(S_j f) \subset \{|\xi| \leq (1+\varepsilon)2^j\}$*, and $|\xi| \leq 2^j \Longrightarrow \widehat{S_j f}(\xi) = \hat{f}(\xi)$.*
- *Let $\varphi(\xi) = \widehat{\phi}(\xi/2) - \widehat{\phi}(\xi)$, $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. We define a frequency localization operator, around 2^j, by $\Delta_j = S_{j+1} - S_j$, in other words $\Delta_j f = \mathcal{F}^{-1}(\varphi_j(\xi)\hat{f}(\xi))$.* $\operatorname{supp}\mathcal{F}(\Delta_j f) \subset \{2^j \leq \xi \leq 2(1+\varepsilon)2^j\}$*, and $2^j(1+\varepsilon) \leq |\xi| \leq 2^{j+1} \Longrightarrow \widehat{\Delta_j f}(\xi) = \hat{f}(\xi)$.*

Remark that $S_j = \sum_{k\in\mathbb{Z},\, k<j} \Delta_j$, and we have a resolution of the identity, $1 = \sum_{j\in\mathbb{Z}} \varphi_j(\xi)$.

Remark that the operators Δ_j are close to the $\chi_{2^j\leq|\xi|<2^{j+1}}$, and due to the support condition, they retain some orthogonality: if $|j-j'| \geq 2$, then $\Delta_j\Delta_{j'} = 0$. Playing with Plancherel, one gets easily

$$\|f\|_{H^s}^2 \approx \|S_0 f\|_2^2 + \sum_{j\in\mathbb{N}} 2^{2js}\|\Delta_j f\|_2^2. \tag{2.4}$$

We need a substitute for Plancherel in order to deal with L^p.

Lemma 3 (Bernstein's inequalities)
Let $f \in \mathcal{S}'(\mathbb{R}^n)$, such that $\operatorname{supp} \hat{f} \subset B(0,\lambda)$. Let $p \in \mathbb{R}_+ \cup \{+\infty\}$ with $1 \le p$, and suppose $f \in L^p$. There exists $C = C(n)$ such that

- *For all $q \ge p$, $f \in L^q$, with*

$$\|f\|_q \le C\lambda^{n(\frac{1}{p}-\frac{1}{q})}\|f\|_p. \tag{2.5}$$

- *For all $\alpha = (\alpha_1, \dots, \alpha_n)$, we have $\partial^\alpha f \in L^p$ with*

$$\|\partial^\alpha f\|_p \le C^{|\alpha|}\lambda^{|\alpha|}\|f\|_p. \tag{2.6}$$

- *If $\operatorname{supp} \hat{f} \subset \mathbb{R}^n \setminus B(0,\lambda/2)$, then*

$$C^{-|\alpha|}\lambda^{|\alpha|}\|f\|_p \le \sup_{\alpha=|\alpha|} \|\partial^\alpha f\|_p \le C^{|\alpha|}\lambda^{|\alpha|}\|f\|_p. \tag{2.7}$$

Proof: By rescaling, one may reduce to $\lambda = 1$. Let $m \in \mathcal{S}$, such that $\widehat{m} = 1$ on $B(0,1)$ and $\hat{m} = 0$ outside $B(0,2)$. Then

$$\hat{f} = \widehat{m}\hat{f}, \text{ or equivalently } f(x) = m * f(x) = \int m(x-y)f(y)dy.$$

By Young's inequality, it is enough to prove that $m \in L^r$, for all $1 < r < +\infty$, which is trivial as $m \in \mathcal{S}$. So we proved (2.5). Similarly, we prove (2.6), given

$$\partial^\alpha f = \partial^\alpha m * f.$$

For (2.6), we just proved half the inequality already. Now, we know moreover that $\operatorname{supp} \hat{f}$ is away from $\xi = 0$. Let us still denote by $m \in \mathcal{S}$ a function which is 1 on $\operatorname{supp} \hat{f}$ and 0 outside of $1/4 \le |\xi| \le 2$. We smoothly cut $\widehat{m}$ in angular neighborhoods around each ξ_i, $1 \le i \le n$.

$$\widehat{m}(\xi) = \sum_i m_i(\xi),$$

and on $\operatorname{supp} m_i(\xi)$, we have $\xi_i \ge 1/8$ (draw a picture...). Therefore

$$\hat{f}(\xi) = \sum_i \frac{m_i(\xi)}{\xi_i}\xi_i\hat{f}(\xi),$$

and each m_i is Schwartz class, leading to

$$\|f\|_p \le \sum_i C\|\partial_i f\|_p.$$

and up to constants,

$$\|f\|_p \le C \sup_i \|\partial_i f\|_p,$$

This ends the proof of Bernstein's inequalities. ∎

Remark 11

Remark that if Λ is defined by $\widehat{\Lambda f}(\xi) = |\xi|\hat{f}(\xi)$, we have $\widehat{\Lambda^s f}(\xi) = |\xi|^s \hat{f}(\xi)$. If $f \in L^p$ and $\operatorname{supp} \hat{f} \subset B(0,1) \cup \mathbb{R}^n \setminus B(0,1/2)$, we trivially get

$$\|f\|_p \approx \|\Lambda^s f\|_p,$$

as the symbol never vanishes on $\operatorname{supp} \hat{f}$. It turns out that controlling $\|\Lambda f\|_p$ is sufficient to control $\|\partial_i f\|_p$ (continuity of the Riesz transforms on L^p), but this is again a non trivial fact of harmonic analysis. The previous proof, however, shows exactly this when f is spectrally localized...

Now we are ready to define Besov spaces, taking our inspiration on (2.4). We refer to [7, 72, 52] for detailed exposition and finer details (most notably, equivalent definitions).

Definition 2

Let $1 \leq p, q \leq +\infty$, $s \in \mathbb{R}$, $f \in \mathcal{S}'(\mathbb{R}^n)$. We say f belongs to the homogeneous Besov space $\dot{B}^{s,q}_p$ if and only if

- *when $s < \frac{n}{p}$ or $s = \frac{n}{p}$ and $q = 1$, $\sum_{-m}^{m} \Delta_j(f)$ converges to f in $\sigma(\mathcal{S}, \mathcal{S}')$. In all other situations ($s > \frac{n}{p}$ and $s = n/p$, $q > 1$) we ask for the same convergence, but modulo polynomials.*
- *The series $\epsilon_j = 2^{js}\|\Delta_j(f)\|_p$ is l^q-summable.*

The norm on $\dot{B}^{s,q}_p$ will be

$$\|f\|_{\dot{B}^{s,q}_p} = \left(\sum_{j \in \mathbb{Z}} 2^{qjs} \|\Delta_j(f)\|_p^q \right)^{\frac{1}{q}}. \tag{2.8}$$

On can define an inhomogeneous version as well.

Definition 3

$1 \leq p, q \leq +\infty$, $s \in \mathbb{R}$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, f belongs to the inhomogeneous Besov space $B^{s,q}_p$ if and only if

- *we have $S_0 f \in L^p$.*
- *For $j \in \mathbb{N}$, the series $\epsilon_j = 2^{js}\|\Delta_j(f)\|_p$ is l^q-summable.*

The norm on $B^{s,q}_p$ will be

$$\|f\|_{B^{s,q}_p} = \|S_0 f\|_p + \left(\sum_{j \in \mathbb{N}} 2^{qjs} \|\Delta_j(f)\|_p^q \right)^{\frac{1}{q}}. \tag{2.9}$$

Remark 12
Notice that inhomogeneous Sobolev spaces are just a particular case: $H^s = B_2^{s,2}$.

The main advantage of homogeneous versus inhomogeneous is the (almost) invariance by scaling: for all $\lambda > 0$,

$$\|f(\lambda\cdot)\|_{\dot{B}_p^{s,q}} \approx \lambda^{s-\frac{n}{p}}\|f\|_{\dot{B}_p^{s,q}}, \tag{2.10}$$

while for the inhomogeneous norm, this holds only for large λ.
A few elementary properties (exercise !).

- Besov are Banach spaces.
- The dual space of $\dot{B}_p^{s,q}$ is $\dot{B}_{p'}^{-s,q'}$ (when $p < +\infty$).
- The operator Λ^σ, $\sigma \in \mathbb{R}$, is bounded from $\dot{B}_p^{s,q}$ to $\dot{B}_p^{s-\sigma,q}$.

2.1.3 Sobolev's embeddings

Let $f \in \dot{B}_p^{s,1}$, with $s > 0$. If r is defined by $s - n/p = -n/r$, by Bernstein,

$$\|f\|_r \leq \sum_{j\in\mathbb{Z}} \|\Delta_j f\|_r \leq \|f\|_{\dot{B}_p^{s,1}}.$$

We want to do better: trivially, we have the following injection.

Proposition 6
Let $f \in \dot{B}_p^{s,q}$, *and* $\tilde{s}, \tilde{p}, \tilde{q}$, *such that* $\tilde{s} < s$, $\tilde{q} \geq q$ *and* $s - n/p = \tilde{s} - n/\tilde{p}$. *Then*

$$\|f\|_{\dot{B}_{\tilde{p}}^{\tilde{s},\tilde{q}}} \lesssim \|f\|_{\dot{B}_p^{s,q}}. \tag{2.11}$$

In fact, one can prove the following.

Proposition 7 (Sobolev's embedding)
Let $s > 0$ *and* $s - n/p = -n/r$, $r, p \in (1, +\infty)$, $f \in \dot{B}_p^{s,p}$. *Then*

$$\|f\|_r \lesssim \|f\|_{\dot{B}_p^{s,p}}. \tag{2.12}$$

Proof: Let f be Schwartz class, we have

$$|f(x)| \leq \sum_{j\in\mathbb{Z}} |\Delta_j f|(x).$$

Fix $J \in \mathbb{Z}$. Then

$$\begin{aligned}\sum_{j\geq J}|\Delta_j f|(x) &\leq \sum_{j\geq J}2^{-sJ}\sup_j(2^{sj}|\Delta_j f|(x)) \\ &\leq 2^{-sJ}H(x),\end{aligned}$$

where $H(x) \in L^p$, as $f \in \dot{B}^{s,p}_p$:

$$\|\sup_j(2^{sj}|\Delta_j f|(x))\|_{L^p} \leq \|(\sum_j(2^{js}|\Delta_j f|(x))^p\|_{L^1}.$$

On the other hand,

$$\begin{aligned}\sum_{j<J}|\Delta_j f|(x) &\lesssim \sum_{j<J}2^{-(s-\frac{n}{p})J}\sup_j(2^{(s-\frac{n}{p})j}|\Delta_j f|(x)) \\ &\lesssim 2^{(\frac{n}{p}-s)J}L(x),\end{aligned}$$

where $L(x) \in L^\infty$, as $f \in \dot{B}_\infty^{-(\frac{n}{p}-s),\infty}$ by a previous proposition. Finally,

$$|f(x)| \lesssim 2^{(\frac{n}{p}-s)J}L(x) + 2^{-sJ}H(x).$$

We optimize J, and

$$|f(x)| \lesssim H^{\frac{p}{r}}(x)L^{1-\frac{p}{r}},$$

so that

$$\|f\|^r_{L^r} \lesssim \|H\|^p_{L^p}\|L\|^{r-p}_{L^\infty}.$$

Note that we have a restriction with this proof, namely $q \leq p$. One can dispose of it using real interpolation (and/or square functions).

Remark 13
We did slightly better:

$$\|f\|_{L^r} \lesssim \|f\|^{\frac{p}{r}}_{\dot{B}^{s,p}_p}\|f\|^{\frac{r-p}{r}}_{\dot{B}_\infty^{-(\frac{n}{p}-s),\infty}},$$

which is non only scale invariant, but modulation invariant (modulation is multiplication by $e^{i\omega\cdot x}$).

To end this quick survey of Besov spaces, we introduce a suitable modification, to handle the time variable.

Definition 4
Let $u(x,t) \in \mathcal{S}'(\mathbb{R}^{n+1})$, Δ_j be a frequency localization with respect to the x variable. We will say that $u \in \dot{B}^{s,q}_p(\mathcal{L}^\rho_t)$ iff

$$2^{js}\|\Delta_j u\|_{L^p_x(L^\rho_t)} = \varepsilon_j \in l^q, \tag{2.13}$$

and other requirements are the same as in Definition 2.

Notice that whenever $q = \rho$, the Besov space $\dot{B}^{s,q}_p(\mathcal{L}^q_t)$ is nothing but the usual "Banach valued" Besov space $\dot{B}^{s,q}_p(F)$ with $F = L^q_t$.

2.2 Local smoothing

Through the rest of these notes, we will denote by $\mathrm{S}(t) = e^{it\partial_x^2}$ and $\mathrm{S}_a(t) = e^{it\partial_x a(x)\partial_x}$ the (1D) group-evolution defined by the constant and variable coefficients equations respectively.
Recall that for the (flat) Schrödinger equation on the real line, we have the following estimate:

$$\|\partial_x \mathrm{S}(t)\phi_0\|_{L^\infty_x L^2_t} \simeq \|\phi_0\|_{\dot{H}^{\frac{1}{2}}},$$

by shifting regularity by 1/2 up. One may ask what is the equivalent for the 1D wave equation: assume Cauchy data ϕ_0 and $\partial_t\phi_0 = 0$, then $\phi = (\phi_0(x-t)+\phi_0(x+t))/2$ and as such,

$$\|\nabla\phi\|_{L^\infty_x(L^2_t)} \lesssim \|\nabla\phi_0\|_2. \tag{2.14}$$

The interesting point here is that this looks very close to the energy estimate for the wave equation, except for the switch in x,t on the left. But in 1D, the wave operator is symmetrical w.r.t. x and t. Consider the inhomogeneous equation (zero Cauchy data)

$$\partial_t^2\phi - \partial_x^2\phi = f,$$

the energy estimate reads

$$\|\partial\phi\|_{L^\infty_t(L^2_x)} \lesssim \|f\|_{L^1_t(L^2_x)}.$$

Due to the symmetry, we can switch roles, to get

$$\|\partial\phi\|_{L^\infty_x(L^2_t)} \lesssim \|f\|_{L^1_x(L^2_t)}.$$

Now, a $TT^\star$ argument gives back (2.14). The whole point is that we can prove the inhomogeneous estimate by a simple integration by parts and extend it

to variable coefficients $\partial_t^2 - \partial - xa(x)\partial_x$: one will pick an amplification factor $\exp \int_x a$.
Such a simple observation will lead to our key result, through an appropriate procedure to transfer the reasoning for the wave to the Schrödinger case (an instance of what is sometimes called transmutation, see [47]). Before stating a theorem, we set the hypothesis on $a(x)$.

DEFINITION 5
We call a an m-admissible coefficient when the following requirements are met:

- *the function a is real-valued, belongs to* BV*, namely*

$$\partial_x a \in \mathcal{M} = \{\mu \text{ t.q. } \int_{\mathbb{R}} d|\mu| < +\infty\},$$

- *the function a is bounded from below almost everywhere by m.*

We will denote by M its maximum and $\|a\|_{\mathrm{BV}}$ its bounded variation ($a(x) \leq M \leq \|a\|_{\mathrm{BV}}$).

After this preliminary definition, we can state the main theorem in this chapter.

THEOREM 8
Let $m > 0$ and a be an m-admissible coefficient. There exist $C(\|a\|_{\mathrm{BV}}, m) > 0$ such that

- *If u, f are solutions of*

$$(i\partial_t + \partial_x a(x)\partial_x)u = f, \tag{2.15}$$

  with zero Cauchy data then

$$\|\partial_x u\|_{L^\infty_x L^2_t} + \|(-\partial_t^2)^{1/4} u\|_{L^\infty_x L^2_t} \leq C\|f\|_{L^1_x L^2_t}. \tag{2.16}$$

- *If*

$$(i\partial_t + \partial_x a(x)\partial_x)u = 0, \text{ with } u_{|t=0} = u_0 \in L^2$$

  then

$$\|u\|_{\dot{B}^{\frac{1}{2},2}_\infty(\mathcal{L}^2_t)} \leq C\|u_0\|_{L^2}. \tag{2.17}$$

Remark 1
One may wonder why we chose to consider $\partial_x a(x)\partial_x$ as opposed to, say, $g(x)\partial_x^2$. It turns out that one may obtain one from another through an easy change of variable, and we elected to keep the divergence form as the most convenient for integration by parts. The astute reader will check that $b(x)\partial_x a(x)\partial_x$ can be dealt with as well, and the additional requirement will be for b to be m-admissible. Remark also that our method can handle first order terms of the kind $b(x)\partial_x$ with $b \in L^1$.

Proof: As explained, in order to obtain (2.16), we reduce ourselves to a situation akin to a wave equation and perform an integration by parts. Obtaining (2.17) from (2.16) is then a simple interpolation and $TT^\star$ argument.

2.2.1 Reduction to smooth a

We first reduce the study to smooth a.

Proposition 8
Denote by $A = \partial_x a(x)\partial_x$. Assume that the evolution semi-group $\mathrm{S}_a(t)$ satisfies for any smooth (C^∞) m-admissible a:

$$\forall u_0 \in L^2, \qquad \|\mathrm{S}_a(t)u_0\|_B \leq C\|u_0\|_{L^2}$$

with B a Banach space (weakly) continuously embedded in $\mathcal{D}'(\mathbb{R}^2)$, whose unit ball is weakly compact, and C a constant depending only on m and $\|\partial_x a\|_{L^1}$. Then the same result holds (with the same constant) for any m-admissible a.

Proof: Let us consider a resolution of the identity, i.e. $\rho \in C_0^\infty(\mathbb{R})$ a non negative function such that $\int \rho = 1$, and $\rho_\varepsilon = \varepsilon^{-1}\rho(x/\varepsilon)$. Denote by $a_\varepsilon = \rho_\varepsilon \star a$ and $A_\varepsilon = -\partial_x a_\varepsilon(x)\partial_x$. Obviously, the sequence a_ε is bounded in $\dot{W}^{1,1}$ (true for $\varepsilon = 1$, then rescale). Furthermore, a_ε converges to a for the L^∞ weak $\star$ topology. According to the weak compactness of the unit ball of B, taking a subsequence, we can assume that $\mathrm{S}_{a_\varepsilon}(t)u_0$ converges weakly to a limit v in B (and consequently in $\mathcal{D}'(\mathbb{R}^2)$). To conclude, it is enough to show that $v = \mathrm{S}_a(t)u_0$ in $\mathcal{D}'(\mathbb{R}^2)$. We first remark that as a (multiplication) operator on L^2, a_ε converges strongly to a (but of course not in operator norm): this follows from

$$\int (a_\varepsilon - a)^2 f^2 = \int a_\varepsilon(a_e - a)f^2 + \int a(a_e - a)f^2,$$

$f^2 \in L^1$, the weak convergence (recall that g converges weakly in L^∞ implies $|g|$ converges as well) and the boundedness of a and a_ε. Consequently $\partial_x a_\varepsilon(x)\partial_x$ converges strongly to $\partial_x a(x)\partial_x$ as an operator from H^1 to H^{-1}. On the other hand the bound $0 < m \leq a(x) \leq M$ and the fact that ρ is non negative imply that a_ε satisfy the same bound and consequently that the family $(A_\varepsilon + i)^{-1}$ is bounded from H^{-1} to H^1 by $1/m$:

$$(A_\varepsilon + i)u = f \Longrightarrow \int_x a(x)|\partial_x u|^2 \leq \|\partial_x u\|_2 \|f\|_{H^{-1}}.$$

From the resolvent formula

$$(A_\varepsilon + i)^{-1} - (A + i)^{-1} = (A_\varepsilon + i)^{-1}(A - A_\varepsilon)(A + i)^{-1},$$

given $(A_\varepsilon + i)^{-1}$ is uniformly bounded from H^{-1} to H^1, we obtain that $(A_\varepsilon + i)^{-1}$ converges strongly to $(A + i)^{-1}$ as an operator from H^{-1} to H^1, and consequently as an operator on L^2. This convergence implies (see [54, Vol I, Theorem VIII.9]) that A_ε converges to A in the strong resolvent sense and (see [54, Vol I, Theorem VIII.21]) that for any $t \in \mathbb{R}$, $\mathrm{S}_{a_\varepsilon}(t)$ converges strongly to $\mathrm{S}_a(t)$. Finally, from the boundedness of $\mathrm{S}_{a_\varepsilon}(t)u_0$ in $L^\infty_t(L^2_x)$, we deduce by dominated convergence that $\mathrm{S}_{a_\varepsilon}(t)u_0$ converges to $\mathrm{S}_a(t)u_0$ in $L^1_{t,\mathrm{loc}}(L^2)$ and hence in $\mathcal{D}'$. Similarly, we can handle non-homogeneous estimates. $\square$

2.2.2 A resolvent estimate

We are now considering the following equation (for $a \in C_0^\infty$):

$$-\sigma v + \partial_x(a(x)\partial_x v) = g. \tag{2.18}$$

where v, g will be chosen later to be the time Fourier transform of u, f (there lies the trick to pass from the wave to the Schrödinger equation, in some sense).

Proposition 9
There exist $C(m, \|a\|_{\mathrm{BV}})$ such that for any $\sigma = \tau + i\varepsilon, \varepsilon \neq 0$ the resolvent $(-\sigma + \partial_x a(x)\partial_x)^{-1}$, which is a well defined operator from $L^1 \subset H^{-1}$ to $H^1 \subset L^\infty$ and from L^2 to H^2 satisfies

$$\|(-\sigma + \partial_x a(x)\partial_x)^{-1}\|_{L^1 \to L^\infty} \leq C. \tag{2.19}$$

It should be noticed that since this and all further estimates are scale invariant (including the constants which are dependent on scale invariant quantities of a), we could reduce the study to the case $\tau = \pm 1$ by changing $a(x)$ into

$a(\sqrt{\pm\tau}^{-1}x)$. We elected to keep τ through the argument as it helps doing book keeping. However, this ability to kill a parameter "for free" is sometimes of great importance.
Now, notice that there are two distinct situations here:

- whenever $-\tau > 0$, the operator is hyperbolic: this is really a wave equation (one should think $-\tau = \eta^2$ and η a Fourier variable associated to a time t). Consequently we expect the heuristic alluded to above to hold.

- whenever $\tau > 0$, the operator cannot be reduced to a wave operator, but then again, it is elliptic: one should be ok by elliptic methods.

Remark 2
In fact, the elliptic case ($\tau > 0$) is more or less understood and as a corollary, the associated heat equation as well. In fact these results apply to a larger class of a than the one we consider here: $a \in L^\infty$, $\mathrm{Re} a > 0$. More specifically, the heat kernel (and its derivatives) associated to the operator $A = -\partial_x(a(x)\partial_x)$ is known to be of Gaussian type, a fact which will be of help later to handle derivatives. A very nice and thorough presentation of this (and a lot more !) can be found in [1]. We refer to Appendix A for a short recollection of the facts we will need later.

We now perform the long awaited integration by parts. We can assume $g \in L^2$ (or even C_0^∞ !), which implies $v \in H^2$ and all these integrations by parts can be carried out (most notably, the boundary terms near $\pm\infty$ vanish). We first multiply (2.18) by $\overline{v}$, integrate by parts and take the imaginary and real parts. This yields
(2.20)

$$|\varepsilon| \int_{\mathbb{R}} |v|^2 \leq \|g\|_{L^1} \|v\|_{L^\infty}$$

$$|\varepsilon| \int_{\mathbb{R}} a(x)|\partial_x v|^2 \leq |\varepsilon||\tau| \int_{\mathbb{R}} |v|^2 + |\varepsilon|\|g\|_{L^1}\|v\|_{L^\infty} \leq (|\varepsilon| + |\tau|)\|g\|_{L^1}\|v\|_{L^\infty}$$

We now proceed in the hyperbolic region $-\tau > 0$. Multiplying (2.18) by $a(x)\partial_x \overline{v}$ and integrating, we get

$$\int_{-\infty}^{x} -\sigma a v \partial_x(\overline{v}) + \int_{-\infty}^{x} \partial_x(a\partial_x v) a \partial_x \overline{v} = \int_{-\infty}^{x} g a \partial_x \overline{v}. \tag{2.21}$$

Integration by parts and taking the real part yields

$$(2.22)\qquad -\tau a|v|^2(x)+|a\partial_x v|^2(x)+2\int_{-\infty}^{x}\tau(\partial_x a)|v|^2 \le 2|\varepsilon|\int_{\mathbb{R}} a|v||\partial_x v| + 2\|g\|_{L^1}\|a\partial_x v\|_{L^\infty}.$$

We now use (2.20) to estimate the first term in the right hand side in (2.22) and obtain

$$-\tau a|v|^2(x)+|a\partial_x v|^2(x)+2\int_{-\infty}^{x}\tau(\partial_x a)|v|^2 \le 2\max(1,\|a\|_{L^\infty}^{\frac{1}{2}})\|g\|_{L^1}\big(\|a\partial_x v\|_{L^\infty}+(|\varepsilon|+|\tau|)^{1/2}\|v\|_{L^\infty}\big).$$

On the other hand we are in 1D and,

$$(2.23)\qquad \|v\|_{L^\infty}^2 \le 2\|v\|_{L^2}\|\partial_x v\|_{L^2}$$

which implies, using (2.20),

$$\varepsilon\|v\|_{L^\infty}^2 \le 2\frac{\|g\|_{L^1}}{\sqrt{m}}\sqrt{|\varepsilon|+|\tau|}\|v\|_{L^\infty}$$

Consequently we get

$$(|\varepsilon|+|\tau|)a|v|^2(x)+|a\partial_x v|^2(x)+2\int_{-\infty}^{x}\tau(\partial_x a)|v|^2 \le C(m,\|a\|_{\mathrm{BV}})\|g\|_{L^1}\big(\|a\partial_x v\|_{L^\infty}+(|\varepsilon|+|\tau|)^{1/2}\|v\|_{L^\infty}\big).$$

Setting

$$\begin{aligned}\Omega_+(x) &= \sup_{y<x}\big((|\varepsilon|+|\tau|)a(y)|v|^2(y)+|a(y)\partial_x v|^2(y)\big)\\ k(x) &= a(x)^{-1}|\partial_x a|,\end{aligned}$$

we have

$$(2.24)\qquad \Omega_+(x)\le C(m,\|a\|_{\mathrm{BV}})\sqrt{\Omega}_+(+\infty)\|g\|_{L^1_x}+2\int_{-\infty}^{x}k(y)\Omega_+(y)dy.$$

Given that Ω_+ is positive, we obtain by Gronwall inequality

$$\begin{aligned}\int_{-\infty}^{x}k(y)\Omega_+(y)dy &\le C(m,\|a\|_{\mathrm{BV}})\left(\int_{-\infty}^{x}e^{\int_y^x 2k(z)dz}k(y)dy\right)\|g\|_{L^1_x}\sqrt{\Omega}_+(+\infty)\\ &\le 2C(m,\|a\|_{\mathrm{BV}})e^{\int_{-\infty}^{x}2k(y)dy}\|g\|_{L^1_x}\sqrt{\Omega}_+(+\infty)\end{aligned}$$

and consequently, coming back to (2.24)

$$\sqrt{\Omega_+(+\infty)} \le C(m, \|a\|_{\mathrm{BV}})\|g\|_{L^1}\big(2 + 8e^{2\|k(x)\|_{L^1}}\big). \tag{2.25}$$

Now we proceed with the elliptic region $\tau > 0$, for which the above line of reasoning fails. We perform the usual elliptic regularity estimate and multiply the equation by $\overline{v}$, to obtain

$$\int_{\mathbb{R}} \tau|v|^2 + a|\partial_x v|^2 = -\mathrm{Re}\int_{\mathbb{R}} g\overline{v}\,, \qquad \varepsilon\int_{\mathbb{R}} |v|^2 = -\mathrm{Im}\int_{\mathbb{R}} g\overline{v}$$

which gives

$$\int_{\mathbb{R}} \big((|\tau| + |\varepsilon|)|v|^2 + a|\partial_x v|^2\big) \le 2\|g\|_{L^1_x}\|v\|_{L^\infty}. \tag{2.26}$$

In order to conclude, we go back to the (beginning of) the estimate we made in the hyperbolic case, i.e. (2.21) and integrate by parts only the second term in the left hand side,

$$|a\partial_x v|^2(x) \le 2\int_{-\infty}^{x} |g|a|\partial_x v| + 2\int_{-\infty}^{x} |\sigma|a|v||\partial_x v|$$

and to bound the last term we use (2.26),

$$\|a\partial_x v\|^2_{L^\infty} \le \|g\|_{L^1_x}\big(2\|a\partial_x v\|_{L^\infty} + 4|\tau|^{1/2}\|v\|_{L^\infty}\big). \tag{2.27}$$

Adding $\tau a|v|^2$ to (2.27) and using (2.26), (2.23), we obtain

$$\begin{aligned}
\Omega_-(x) &= \sup_{y\le x}(|\varepsilon| + |\tau|)a|v|^2(y) + |a\partial_y v|^2(y)\\
&\le 2M|\tau\|v\|_{L^2}\|\partial_x v\|_{L^2} + 4(|\varepsilon| + |\tau|)^{1/2}\|g\|_{L^1_x}\|v\|_{L^\infty}\\
&\le \|g\|_{L^1}(2M+4)\|(|\varepsilon| + |\tau|)^{1/2}v\|_{L^\infty}
\end{aligned}$$

which gives again

$$\sup_x \Omega_-(x) \le \frac{(\|a\|_{L^\infty} + 4)^2}{m}\|g\|^2_{L^1}. \tag{2.28}$$

This ends the proof of Proposition 9. $\square$

Remark 3

Notice that for this elliptic estimate, we only used $a \in L^\infty$ and nothing else.

We now come back to the proof of Theorem 8. Consider u, f solutions of (2.15). We can assume that f (and consequently u) is supported in $t > 0$ (the contribution of negative t being treated similarly)). Then for any $\varepsilon > 0$ $u_\varepsilon = e^{-\varepsilon t} u$ is solution of

$$(i\partial_t + i\varepsilon + \partial_x a(x)\partial_x)u_\varepsilon = f,$$

Assuming that f has compact support (in time), we can consider the Fourier transforms with respect to t of f and u_ε, $g(\tau)$ and $v_\varepsilon(\tau)$ which satisfy

$$(-\tau + i\varepsilon + \partial_x a(x)\partial_x)v_\varepsilon = g.$$

We may now apply Proposition 9, take L^2_τ norms, switch norms and revert back to time by Plancherel, and get

$$\begin{aligned}\|\partial_x u_\varepsilon\|_{L^\infty_x(L^2_t)} + \|(-\partial_t^2)^{\frac{1}{4}} u_\varepsilon\|_{L^\infty_x(L^2_t)} &= \|\partial_x v_\varepsilon\|_{L^\infty_x(L^2_\tau)} + \|(-\partial_t^2)^{\frac{1}{4}} v_\varepsilon\|_{L^\infty_x(L^2_\tau)} \\ &\leq \|\partial_x v_\varepsilon\|_{L^2_\tau(L^\infty_x)} + \|(-\partial_t^2)^{\frac{1}{4}} v_\varepsilon\|_{L^2_\tau(L^\infty_x)} \\ &\leq C\|g_\varepsilon\|_{L^2_\tau(L^1_x)} \leq C\|g_\varepsilon\|_{L^1_x(L^2_\tau)} = C\|f_\varepsilon\|_{L^1_x(L^2_t)},\end{aligned}$$

where $C = C(m, \|\partial_x a\|_{L^1_x})$ is uniform with respect to $\varepsilon > 0$. Letting $\varepsilon > 0$ tend to 0, we obtain the same estimate for u, which is exactly (2.16) in Theorem 8 (up to replacement of BV by $\dot{W}^{1,1}$, which was dealt with in Proposition 8). Finally we easily drop the compact in time assumption for f by a density argument.
We are left with proving the homogeneous estimate (2.17). As usual, estimates on the homogeneous problem follow from the estimate with a fractional time derivative: by a $TT^\star$ argument, and using the commutation between time derivatives and the flow, we get

$$\|(-\partial_t^2)^{\frac{1}{8}} u\|_{L^\infty_x(L^2_t)} \lesssim \sqrt{C}\|u_0\|_{L^2_x}.$$

Then, using the equation, $i\partial_t u = Au$ where $A = -\partial_x a(x)\partial_x$, we can replace $(i\partial_t)^{1/4}$ by $A^{1/4}$ (notice that we would have to properly define what a fractional power of A is, and deal with its action on various functional spaces...). However, we will need real derivatives later, rather than powers of A. We postpone the issue of equivalence between the two and take another road: notice that we obtained (2.19) for solutions of (2.18)

$$\|\partial_x v\|_{L^\infty_x} \lesssim \|g\|_{L^1_x},$$

which immediately implies

$$\|v\|_{\dot{B}^{1,\infty}_\infty} \lesssim \|g\|_{\dot{B}^{0,1}_1}. \tag{2.29}$$

Call $R_\sigma = (\partial_x a(x)\partial_x - \sigma)^{-1}$. Its adjoint is $R_{\overline{\sigma}}$: Proposition 9 does not care about the sign of ε and if we apply it to $\overline{\sigma} = \tau - i\varepsilon$), we get (2.29) for $\mathbb{R}_{\overline{\sigma}}$, and by duality,

$$\|v\|_{\dot{B}^{0,\infty}_\infty} \lesssim \|g\|_{\dot{B}^{-1,1}_1}. \tag{2.30}$$

By real interpolation ([7]) we have $\left(\dot{B}^{s_1,q_1}_p, \dot{B}^{s_2,q_2}_p\right)_{\theta,r} = \dot{B}^{s,r}_p$; in our situation, we obtain (with $\theta = 1/2, r = 2$)

$$\|v\|_{\dot{B}^{\frac{1}{2},2}_\infty} \lesssim \|g\|_{\dot{B}^{-\frac{1}{2},2}_1}.$$

Given that the third index is 2, we can again take L^2_τ norms, switch them (Minkowski) and by Plancherel (and letting ε tend to 0), we get the desired estimate:

$$\|u\|_{\dot{B}^{\frac{1}{2},2}_\infty(L^2_t)} \lesssim \|f\|_{\dot{B}^{-\frac{1}{2},2}_1(L^2_t)}.$$

denote by $\mathrm{S}_a(t)$ the evolution group for the homogeneous equation, we have

$$u = \int_{s<t} \mathrm{S}_a(t-s)f(s)ds,$$

solution of the inhomogeneous problem, and we can as well treat the $s > t$ case (the time direction is irrelevant !). Hence we have obtained

$$\|\int \mathrm{S}_a(t-s)f(s)ds\|_{\dot{B}^{\frac{1}{2},2}_2(L^2_t)} \lesssim \|f\|_{\dot{B}^{-\frac{1}{2},2}_1(L^2_t)}.$$

The usual $TT^\star$ argument applies and gives

$$\|\mathrm{S}_a(t)u_0\|_{\dot{B}^{\frac{1}{2},2}_\infty(L^2_t)} \lesssim \|u_0\|_{L^2_x}.$$

This ends the proof of Theorem 8. □

Notice that up to this point we avoided to use any of the machinery presented in Appendix A, thus keeping the proof self-contained. However, a rather natural question is now how one can handle (fractional) derivatives: i.e., replace $u_0 \in L^2$ by $u_0 \in \dot{H}^s$. In order to deal with commutation, we will rely in a very natural way on Appendix A.

Proposition 10

Assuming a is m-admissible, we have:

- *if u, f are solutions of*

$$(i\partial_t + \partial_x a(x)\partial_x)u = f,$$

then, for $0 < s < 1$,

$$\|u\|_{\dot{B}^{s,2}_\infty(\mathcal{L}^2_t)} \lesssim \|f\|_{\dot{B}^{s-1,2}_1(\mathcal{L}^2_t)}. \tag{2.31}$$

- *if* $-1 < s < \frac{1}{2}$ *and*

$$(i\partial_t + \partial_x a(x)\partial_x)u = 0, \text{ with } u_{|t=0} = u_0$$

then

$$\|u\|_{\dot{B}^{s+\frac{1}{2},2}_\infty(\mathcal{L}^2_t)} \lesssim \|u_0\|_{\dot{H}^s}. \tag{2.32}$$

Proof: Recall that by real interpolation between (2.29) and (2.30) we have, choosing now $(\theta = s, r = 2)$,

$$\|v\|_{\dot{B}^{s,2}_\infty} \lesssim \|g\|_{\dot{B}^{s-1,2}_1},$$

for all $0 < s < 1$, which immediately gives (2.31). For the homogeneous problem, we simply rely on the equivalence properties stated in Appendix A.2: we apply (2.17) to $\Delta_j^A u_0$, a datum localized with respect to A (see the Appendix for a definition with Gaussians, here we assume compact support spectrally w.r.t. the A operator) and use commutation between Δ_j^A and $S_a(t)$ to obtain

$$\|\Delta_j^A u\|_{\dot{B}^{\frac{1}{2},2}_\infty(\mathcal{L}^2_t)} \lesssim \|\Delta_j^A u_0\|_{L^2_x}.$$

Equivalence between $\dot{B}^{\frac{1}{2},2}_\infty(\mathcal{L}^2_t)$ and $\dot{B}^{\frac{1}{2},2}_{\infty,A}(\mathcal{L}^2_t)$ yields

$$2^{\frac{1}{2}j}\|\Delta_j^A u\|_{L^\infty_x L^2_t} \lesssim \|\Delta_j^A u_0\|_{L^2_x},$$

for which multiplying by 2^{js} and summing over j provides the desired result

$$\|u\|^2_{\dot{B}^{s+\frac{1}{2},2}_{\infty,A}(\mathcal{L}^2_t)} = \sum_{j\in\mathbb{Z}} 2^{(2s+1)j}\|\Delta_j^A u\|^2_{L^\infty_x L^2_t} \lesssim \sum_{j\in\mathbb{Z}} 2^{2sj}\|\Delta_j^A u_0\|^2_{L^2_x} = \|u_0\|_{\dot{B}^{\frac{1}{2},2}_{2,A}},$$

and we can switch back from A based Besov spaces to the usual ones, provided $s > -1$ (from the right hand side) and $s + 1/2 < 1$ (from the left hand side).
□

Chapter 3

Strichartz and maximal function estimates

Introduction

We intend to prove Strichartz and maximal function estimates by making use of the smoothing effect from the previous chapter, together with known estimates with the flat case. This idea goes back at least to [34] (though one could obtain the dispersion result without the smoothing effect just by using weighted estimates) in the context of the Laplacian plus a potential. If one just wants Strichartz estimates rather than the full dispersion, a simple use of Duhamel allows to conclude: assume

$$i\partial_t\phi + \Delta\phi - V\phi = 0,$$

one writes

$$\phi = S_0(t)\phi_0 + \int_0^t S_0(t-s)(V\phi)ds,$$

and, if we resort to Christ-Kiselev (which is unnecessary but simplifies the exposition), we need to study

$$T\phi_0 = \int S(-s)(V\phi)ds,$$

and prove it sends L^2 to L^2. Suppose through resolvent estimates, one can get a weighted L^2 estimate like

$$\int_{t,x} \frac{|\phi|^2}{|x|^2|}\,dxdt \lesssim \|\phi_0\|_2^2, \tag{3.1}$$

and moreover that

$$|V(x)| \lesssim \frac{1}{1+|x|^2},$$

then we will have

$$\int_{t,x} |x|^2 |V\phi|^2 \, dxdt \lesssim \|\phi_0\|_2^2,$$

and by the dual estimate obtained from (3.1), we get our result.

Remark 4
Note that we quietly slipped under the rug issues about the spectrum and/or resonances of the $-\Delta + V$ operator... These problems show up when trying to prove (3.1).

The above strategy is essentially the one used in [56], see [15] for an interesting borderline case and [14] for generalizations.
Of course, the above "source term" strategy seems doomed from the start when considering variable coefficients (though for compactly supported perturbations, it efficiently disposes of the region away from the perturbation). Similarly, even a lower order perturbation, say $-\Delta + B \cdot \nabla$ is already not so easy to treat. One can nevertheless say something in such a situation: assuming we have a smoothing effect, like

$$\sup_j \int_{2^j < |x| < 2^{j+1}} 2^{-j} ||\nabla|^{\frac{1}{2}} \phi|^2 \, dxdt \lesssim \|\phi_0\|_2^2, \tag{3.2}$$

one could get something for A such that

$$\sum_j \sup_{2^j < |x| < 2^{j+1}} 2^j |B(x)| < +\infty,$$

and this is assuming that we could make sense of

$$|\nabla|^{-\frac{1}{2}} (B \cdot \nabla \phi) \approx B |\nabla|^{\frac{1}{2}} \phi,$$

for which the only hope can come from spectral and paradifferential calculus. However, in 1D, the picture looks slightly better, because we can dispose with the weights, and use (1.19) instead (assuming we can prove it for $-\partial_x^2 + b\partial_x$, or that b is small). Obviously, and up to the sloppy fractional differential rule, we get Strichartz, provided that $b \in L^1_x$.
Then, we can use yet another specificity of the 1D case: a simple of change of variable,

$$A = -\partial_x a(x) \partial_x \longrightarrow -\partial_y^2 + b(y)\partial_y,$$

and this new operator is exactly of the aforementioned type, for which we have a (heuristic) strategy.

3.1 The estimates

We first state our main results.

THEOREM 9
Let a be an m-admissible coefficient. Let u be a solution of (2.2) with $u_0 \in L^2$. Then for $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$, $p \geq 4$, we have

$$\|\mathrm{S}_a(t)u_0\|_{L^p_t(\dot{B}^{0,2}_q)} \lesssim \|u_0\|_{L^2}. \tag{3.3}$$

When $p > 4$ ($q < +\infty$),

$$\|\mathrm{S}_a(t)u_0\|_{L^p_t(L^q_x)} \lesssim \|\mathrm{S}_a(t)u_0\|_{L^p_t(\dot{B}^{0,2}_q)} \lesssim \|u_0\|_{L^2}. \tag{3.4}$$

REMARK 5
Notice that the end-point $(4, \infty)$ is missing. This can be seen as an artifact of the proof. It will be clear that in this section, we only use $a \in L^\infty \cap \dot{B}^{1,\infty}_1$ and bounded from below (together with the estimates of Theorem 8). Adding a technical hypothesis like $a \in \dot{B}^{1,2}_1$ (which does not follow from $a \in \mathrm{BV}$) would allow to recover the end-point, at the expense of extra technicalities which we elected to keep out.

One may state a corollary including fractional derivatives as well.

PROPOSITION 11
Let u be a solution of (2.2), and $u_0 \in \dot{H}^s$, $|s| < 1$. Then for $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$, $p \geq 4$, we have

$$\|\mathrm{S}_a(t)u_0\|_{L^p_t(\dot{B}^{s,2}_q)} \lesssim \|u_0\|_{\dot{H}^s}. \tag{3.5}$$

Similarly, we also obtain maximal function estimates.

THEOREM 10
Let u be a solution of (2.2), and $u_0 \in \dot{H}^s$, $-3/4 < s < 1$. Then

$$\|\mathrm{S}_a(t)u_0\|_{\dot{B}^{s-\frac{1}{4},2}_4(\mathcal{L}^\infty_t)} \lesssim \|u_0\|_{\dot{H}^s}. \tag{3.6}$$

Proof: As explained in the introduction, we aim at taking advantage of an appropriate new formulation for our original problem and proving Theorems 9 and 10 at once. The operator $\partial_x a \partial_x$ may be rewritten as $(\sqrt{a}\partial_x)^2 + (\partial_x\sqrt{a})\partial_x$, and one would like to "flatter out" the higher order term through a change of variable. However, performing directly a change of variable leads to problems when dealing with the newly appeared first order term (this is where the heuristic with commuting b and the half-derivative needs to be worked upon). In order to remedy this problem, we will paralinearize the equation. Let us rewrite a:

$$a = \frac{m}{2} + b^2, \text{ with } \partial_x b \in L^1_x,$$

given that a is m-admissible. Now, for any given function such that

$$\lim_{j\to+\infty} S_j f = f, \quad \lim_{j\to-\infty} S_j f = 0$$

(for example $f \in \dot{B}^{s,q}_p$ with $s - n/p < 0$), we can rewrite f as a telescopic series,

$$f = \sum_j S_{j+1} f - S_j f.$$

An obvious extension to products leads to the following rewriting in our situation (notice the shift in indices, which is irrelevant for now but important later on):

$$\begin{aligned} b^2 \partial_x u &= \sum_k (S_{k-3} b)^2 \partial_x S_k u - (S_{k-4} b)^2 \partial_x S_{k-1} u \\ &= \sum_k (S_{k-3} b)^2 \partial_x \Delta_k u + \sum_k \Delta_{k-3} b (S_{k-3} + S_{k-4}) b \partial_x S_{k-1} u. \end{aligned}$$

Now, we go back to the equation and apply Δ_j: by taking advantage of the support conditions, the term

$$\partial_x (\Delta_j (S_{k-3} b)^2 \partial_x \Delta_j u)$$

has a non empty support in Fourier space only if $k \sim j$. In a similar way, the support of the third term is non empty only if $j \lesssim k$. Hence,

$$i\partial_t \Delta_j u + \frac{m}{2} \partial_x^2 \Delta_j u + \Delta_j \partial_x \sum_{k\sim j} \left((S_{k-3} b)^2 \partial_x \Delta_k u\right) + \Delta_j \partial_x \sum_{j \lesssim k \sim l} (\Delta_{k-3} b S_{l-3} b \partial_x S_{k-1} u) = 0.$$

We would like do deal

$$\frac{m}{2} \partial_x^2 \Delta_j u + \partial_x \left((S_{j-3} b)^2 \partial_x \Delta_j u\right) = \partial_x \left((\frac{m}{2} + (S_{j-3} b)^2) \partial_x \Delta_j u \right),$$

and inserting the appropriate terms, we get

$$i\partial_t \Delta_j u + \left(\sqrt{\frac{m}{2} + (S_{j-3} b)^2} \right) \partial_x \left(\left(\sqrt{\frac{m}{2} + (S_{j-3} b)^2} \right) \partial_x \Delta_j u \right) = R_j,$$

where, with $\tilde{\Delta}_j$ an enlargement of the localization,

$$R_j = -\Delta_j \partial_x \sum_{j \lesssim k} \Delta_k b S_k b \partial_x S_k u - \tilde{\Delta}_j \partial_x \sum_{k \sim j} [\Delta_j, (S_{j-3} b)^2] \partial_x \Delta_k u$$
$$- S_{j-3} b (\partial_x S_{j-3} b)(\partial_x \Delta_j u).$$

Note that the third term comes from commuting $\sqrt{a}$ with ∂_x, the second one from a commutation in order to get $\Delta_j(\sum_{k\sim j} \Delta_k u) = \Delta_j u$. The first term is what is left over: from the equation and from replacing $(S_{k-3}b)^2$ by $(S_{j-3}b)^2$, which produces factors

$$(S_{j-3}b)^2 - (S_{k-3}b)^2 \approx \Delta_k b S_{k-3} b,$$

which can be safely incorporated into the term coming from the equation. Note that we are obviously abusing the notation by keeping only one term where all indices are just k. One should write $j \lesssim k_1 \sim k_2 \sim k_3$. Anyway, the very important point is that, while we only have a sum of pieces which are frequency localized in balls, we always have a $\Delta_k b$ term, which is the key to subsequent resummation.
To go further, we will need a few lemma which are stated and proved in the last section: lemmata 5, 6 and 7.

We now go back to our main proof: assuming the smoothing effect from Theorem 8, we can effectively estimate the reminder $\sum_j R_j$.

Proposition 12
Assume that the hypothesis of Theorem 8 hold: then

$$\sum_j R_j \in \dot{B}_1^{-\frac{1}{2},2}(\mathcal{L}_t^2).$$

Proof: Let us do the first term,

$$R_j^1 = -\Delta_j \partial_x \sum_{j \lesssim k} \Delta_k b S_k b \partial_x S_k u.$$

From Theorem 8, we know that

$$2^{-\frac{k}{2}} \partial_x \Delta_k u \in l_k^2 L_x^\infty L_t^2,$$

which implies by Lemma 6

$$2^{-\frac{k}{2}} \partial_x S_k u \in l_k^2 L_x^\infty L_t^2.$$

On the other hand, we obviously have

$$S_k b \in l_k^\infty L_x^\infty, \text{ and } 2^k \Delta_k b \in l_k^\infty L_x^1 \,(\text{notice that } \dot{W}_1^1 \hookrightarrow \dot{B}_1^{1,\infty}).$$

Before applying the remaining ∂_x, we have a sum over k, $= \sum_{j \lesssim k} P_k$ which is such that $2^{\frac{k}{2}} P_k \in l_j^2 L_x^1 L_t^2$ and frequency localized in a ball of size 2^k, hence by lemma 7, $\sum_k P_k \in \dot{B}_1^{\frac{1}{2},2}(\mathcal{L}_t^2)$; of course the same holds for $\sum_j \Delta_j (\sum_{k \lesssim j} P_k) \approx \sum_k P_k$, and by derivation, we get $\sum_j R_j^1 \in \dot{B}_1^{\frac{1}{2},2}(\mathcal{L}_t^2)$.
Next, we turn to the commutator term:

$$R_j^2 = -\tilde{\Delta}_j \partial_x \sum_{k \sim j} [\Delta_j, (S_{j-3} b)^2] \partial_x \Delta_k u.$$

We will deal with it in a very similar way, thanks to the following lemma.

Lemma 4
Let $g(x,t)$ be such that $\|\partial_x g\|_{L_x^{p_1}(L_t^{q_\infty})} < +\infty$, and $f(x,t) \in L_x^{p_\infty}(L_t^{q_2})$, with $\frac{1}{p_1} + \frac{1}{p_\infty} = 1$ and $\frac{1}{q_\infty} + \frac{1}{q_2} = \frac{1}{2}$, then $h(x,t) = [\Delta_j, g] f$ is in $L_x^1(L_t^2)$.

Proof: We first take $p_1 = 1$, $p_\infty = \infty$: set $h(x) = [\Delta_j, g] f$, recall Δ_j is a convolution by $2^j \phi(2^j \cdot)$, and denote $\psi(z) = z|\phi|(z)$:

$$\begin{aligned} h(x) &= \int_y 2^j \phi(2^j(x-y))(g(y) - g(x)) f(y) dy \\ &= \int_{y,\theta \in [0,1]} 2^j \phi(2^j(x-y))(x-y) g'(x + \theta(y-x)) f(y) d\theta dy \\ |h(x)| &\le 2^{-j} \int_{y,\theta \in [0,1]} 2^j \psi(2^j(x-y)) |g'(x + \theta(y-x))| |f(y)| d\theta dy \end{aligned}$$

and then take successively time norms and space norms,

$$\begin{aligned} \|h(x,t)\|_{L_t^2} &\le 2^{-j} \int_{y,\theta \in [0,1]} 2^j \psi(2^j(x-y)) \|g'(x + \theta(y-x,t))\|_{L_t^{q_\infty}} \|f(y,t)\|_{L_t^{q_2}} d\theta dy \\ \int_x \|h(x)\|_{L_t^2} dx &\le 2^{-j} \|f\|_{L_x^\infty(L_t^{q_2})} \int_{\theta \in [0,1], x, y} 2^j \psi(2^j(x-y)) \|g'(x + \theta(y-x))\|_{L_t^{q_\infty}} dx dy d\theta \\ &\le 2^{-j} \|f\|_{L_x^\infty(L_t^{q_2})} \int_{\theta \in [0,1], z, x} 2^j \psi(2^j z) \|g'(x + \theta z)\|_{L_t^{q_\infty}} dx dz d\theta \\ &\le 2^{-j} \|f\|_{L_x^\infty(L_t^{q_2})} \int_z 2^j \psi(2^j z) dz \|g'(x)\|_{L_x^1(L_t^{q_2})}. \end{aligned}$$

The case $p_1 = \infty$, $p_\infty = 1$ is identical, exchanging f and g' (in fact, this would be the usual commutator estimate !). The general case then follows by bilinear complex interpolation. $\square$

Thus, the lemma allows us to effectively proceed with the second term in R_j as if the derivative on $\Delta_k u$ was in fact on an $S_{k-3}b$ factor, and then it becomes a term "like"

$$\partial_x \sum_{k\sim j} S_k b \partial_x S_k b \Delta_k u,$$

for which a computation similar to the one done with the first term holds as well: we have

$$2^{\frac{k}{2}}\Delta_k u \in l^2_k L^\infty_x L^2_t,\ \ S_k b \in l^\infty_k L^\infty_x,\ \text{and}\ \partial_x S_k b \in l^\infty_k L^1_x\ (\text{given that}\ b \in \dot{W}^1_1).$$

We are left with the third term: this is nothing but a paraproduct $\sum_j R^3_j$, as the support conditions imply that

$$R^3_j = -S_{j-3}b(\partial_x S_{j-3}b)(\partial_x \Delta_j u)$$

has support in a corona $|\xi| \approx 2^j$. We then easily estimate this term using lemma 5.

$$\partial_x S_{j-3}b \in l^\infty_j L^1_x,\ S_{j-3}b \in l^\infty_j L^\infty_x \text{ and } 2^{-\frac{j}{2}}\partial_x \Delta_j u \in L^\infty_x L^2_t.$$

This completes the proof of Proposition 12. $\square$

After the paralinearization step, we perform a change of variable. We have, denoting by $\omega = \sqrt{\frac{m}{2} + (S_{j-3}b)^2}$, and $u_j = \Delta_j u$,

$$i\partial_t u_j + \omega(x)\partial_x(\omega(x)\partial_x u_j) = R_j.$$

Now we set $x = \phi(y)$ through $\partial_y = \omega(x)\partial_x$, in other words

$$\omega(x) = \frac{dx}{dy},\ \ y = \int_0^x \omega(\rho)\, d\rho = \phi^{-1}(x),$$

which is clearly a C^1 diffeomorphism, uniformly with respect to j: ω is bounded in the range $[\frac{m}{2}, 2M]$. Denote by $v_j(y) = u_j \circ \phi(y)$ and $T_j(y) = R_j \circ \phi(y)$,

$$i\partial_t v_j + \partial^2_y v_j = T_j(y).$$

Given that our change of variable leaves L^p spaces invariant, from Proposition 12, we have that

$$T_j \in L^1_y L^2_t,\ \text{with}\ \|T_j\|_{L^1_y L^2_t} \lesssim 2^{\frac{j}{2}}\mu_j, \qquad (\mu_j)_j \in l^2. \tag{3.7}$$

By using Duhamel,

$$v_j = S(t)v_j(0) + \int_0^t S(t-s)T_j(y,s)ds \tag{3.8}$$

for which we can apply Christ-Kiselev Lemma; first, let us obtain Strichartz estimates: according to (3.7), (3.8) and Theorem 13, we obtain

$$\|v_j\|_{L_t^4 \dot{B}_\infty^{\frac{1}{2},2}} \lesssim \|v_j(0)\|_{\dot{H}^{\frac{1}{2}}} + 2^{\frac{j}{2}}\mu_j. \tag{3.9}$$

Now we would like to go back to u_j from v_j. While frequency localizations wrt x and y do not commute, they "almost" commute.

PROPOSITION 13
Let $x = \phi(y)$ be our diffeomorphism, $|s| < 1$ and $1 \le p, q \le +\infty$. Then the Besov spaces $\dot{B}_p^{s,q}(x)$ and $\dot{B}_p^{s,q}(y)$ are identical, with equivalent norms.

Proof: For any $p \in [1, +\infty]$, the $\dot{W}_p^1$ norms are equivalent: the two Jacobians $|\partial_y \phi(y)|$ or $|\partial_x \phi^{-1}(x)|$ are bounded. Therefore, with obvious notations,

$$\|\Delta_j^y \Delta_k^x \varphi\|_p \sim 2^{-j}\|\Delta_j^y \Delta_k^x \varphi\|_{\dot{W}_p^1(y)} \lesssim 2^{-j}\|\Delta_k^x \varphi\|_{\dot{W}_p^1(x)} \lesssim 2^{k-j}\|\Delta_k^x \varphi\|_p \sim 2^{k-j}\|\varphi\|_p.$$

Since x and y play the same part, by duality we obtain

$$\|\Delta_j^y \Delta_k^x \varphi\|_p \lesssim 2^{-|k-j|}\|\varphi\|_p.$$

This essentially allows to exchange x and y in Besov spaces, as long as we are using spaces involving strictly less than one derivative: say $\varphi(x) \in \dot{B}_p^{s,q}(x)$, then $\varphi(y) \in \dot{B}_p^{s,q}(y)$, as

$$\|\Delta_j^y \varphi\|_p \lesssim \sum_k 2^{-|k-j|}\|\Delta_k^x \varphi\|_p \lesssim \sum_k 2^{-|k-j|}2^{-sk}\varepsilon_k$$
$$2^{js}\|\Delta_j^y \varphi\|_p \lesssim \sum_k 2^{-(1-s)|k-j|}\varepsilon_k \lesssim \mu_j,$$

where $(\mu_j)_j \in l^q$ as an $l^1 - l^q$ convolution. $\square$

REMARK 6
Proposition 13 is nothing but the invariance of Besov spaces under diffeomorphism. Given that we only have a C^1 diffeomorphism, we are restricted to Besov spaces with $|s| < 1$ regularity.

Going back to (3.9), we immediately obtain by inverting the change of variable,

$$\|u_j\|_{L_t^4(\dot{B}_\infty^{\frac{1}{2},2})} \lesssim \|u_j(0)\|_{\dot{H}^{\frac{1}{2}}} + 2^{\frac{j}{2}}\mu_j,$$

and given that $u_j = \Delta_j u$,

$$\|u_j\|_{L_t^4(L_x^\infty)} \lesssim \|\Delta_j u_0\|_2 + \mu_j,$$

which, by summing over j, gives the desired Strichartz estimate. All other Strichartz estimates are obtained directly in the same way or by interpolation with the conservation of mass. This ends the proof of Theorem 9.
The strategy is exactly similar for the maximal function estimate. Recall Theorem 6; for the (flat) Schrödinger equation, we have

$$\|(-\partial_y^2)^{-\frac{1}{8}} S(t)u_0\|_{L_y^4(L_t^\infty)} \simeq \left\| \int_{\mathbb{R}} e^{iy\xi - it|\xi|^2} \frac{\hat{u}_0}{|\xi|^{\frac{1}{4}}} \, d\xi \right\|_{L_y^4(L_t^\infty)} \lesssim \|u_0\|_{L^2(\mathbb{R})}. \tag{3.10}$$

By combining (3.10), dual smoothing (1.20) and Christ-Kiselev, we get the following inhomogeneous estimate for the flat case:

$$\|(-\partial_y^2)^{\frac{1}{8}} \int_0^t S(t-s)f(s)\, ds\|_{L_y^4(L_t^\infty)} \lesssim \|f\|_{L_y^2(L_t^1)}.$$

Therefore applying this estimate on (3.8) (at the frequency-localized scale) we get

$$\|v_j\|_{\dot{B}_4^{\frac{1}{4},2}(L_t^\infty)} \lesssim \|v_j(0)\|_{\dot{H}^{\frac{1}{2}}} + 2^{\frac{j}{2}}\mu_j,$$

and then, inverting the change of variable as before,

$$\|u_j\|_{L_x^4(L_t^\infty)} \lesssim 2^{\frac{j}{4}}(\|\Delta_j u_0\|_2 + \mu_j),$$

which we can then sum up.

Remark 7
Here we are using an equivalence between Besov spaces wrt x and Besov spaces wrt y with value in L_t^∞. The reader will easily check that the argument we used to obtain Proposition 13 applies with any Besov spaces with value in L_t^q for any $1 \leq q \leq +\infty$. As an alternative, one could use the definition with moduli of continuity (which is the usual way to prove invariance by diffeomorphism) to obtain the $0 < s < 1$ range (and duality if one needs $-1 < s < 0$).

This completes the proof of Theorem 10 for the special case $s = \frac{1}{4}$. We are left with shifting regularity in the appropriate range: but this is again nothing but a consequence of the equivalence from Appendix A. We therefore obtain the full range in Theorem 10 as well as Proposition 11, where the restriction on s follows from book keeping. $\square$

3.2 Paraproducts

As we just saw, very often we have to deal with sums of spectrally localized pieces. Firstly, we have the following important lemma.

LEMMA 5
Let $f = \sum_j f_j$, where $\operatorname{supp} \hat{f}_j \subset B(0, \gamma 2^j) \backslash B(0, \gamma^{-1} 2^j)$, with $(\eta_j = 2^{js} \|f_j\|_p)_j \in l^q$. Then $f \in \dot{B}^{s,q}_p$

$$\|f\|_{B^{s,q}_p} \lesssim \|(\eta_j)_j\|_{l^q}. \tag{3.11}$$

Proof: One just need to control $\Delta_j f$: let K be such that $2^{K-1} \leq \gamma < 2^K$, from the support condition, we get

$$\Delta_j \sum_k f_k = \sum_{-K-1 \leq m \leq K+1} \Delta_j f_{j+m},$$

meaning we have a finite sum on the right handside (which depends on γ), and

$$\begin{aligned}
2^{js} \|\Delta_j f\|_p &\lesssim \sum_{-K-1 \leq m \leq K+1} 2^{-ms} 2^{s(j+m)} \|f_{j+m}\|_p, \\
2^{js} \|\Delta_j f\|_p &\lesssim (2K+1) 2^{sK} 2^{sj} \|f_j\|_p, \\
\|(2^{js} \|\Delta_j f\|_p)_j\|_{l^q} &\lesssim \|(2^{sj} \|f_j\|_p)_j\|_{l^q},
\end{aligned}$$

which ends the proof of the lemma. $\square$
We need more, and the following two lemmata are useful generalizations of the previous one.

LEMMA 6
Let $s < 0$, $1 \leq p, q \leq +\infty$. Then $f \in \dot{B}^{s,q}_p$ if and only if

$$(\eta_j = 2^{js} \|S_j f\|_p)_{j \in \mathbb{Z}} \in l^q, \tag{3.12}$$

and $\|(\eta_j)_j\|_{l^q}$ is an equivalent norm.

Proof: One direction is trivial: if $(\eta_j)_j \in l^q$, then, as

$$2^{js}\|\Delta_j f\|_p = 2^{js}\|(S_{j+1} - S_j)f\|_p \leq 2^{js}(\|(S_{j+1}f\|_p + \|S_j)f\|_p),$$

we have $2^{js}\|\Delta_j f\|_p \lesssim \eta_j$ and one may sum over j. In the other direction, write

$$2^{js}\|S_j f\|_p \leq \sum_{k<j} 2^{s(j-k)} 2^{sk}\|\Delta_k f\|_p.$$

Denote by $\alpha_k = 2^{sk}\|\Delta_k f\|_p$. We know that $(\alpha_j)_j \in l^q$. The right handside above is nothing but a convolution between $(\alpha_k)_k$ and $(0)_{m<0} \cup (2^{sm})_{m\geq 0}$, which, as $s < 0$, is l^1. By Young for sequences, $l^1 * l^q \to l^q$ and

$$\|(2^{js}\|S_j f\|_p)_j\|_{l^q} \lesssim \|(\alpha_j)_j\|_{l^q},$$

which is the desired estimate. □

Remark 14
Essentially, this lemma allows to replace Δ_j by S_j in the definition of Besov spaces with strictly negative regularity.

The next lemma is a dual version.

Lemma 7
Let $s > 0$, $f = \sum_j f_j$, where $\operatorname{supp} \hat{f}_j \subset B(0, \gamma 2^j)$, with $(\eta_j = 2^{js}\|f_j\|_p)_j \in l^q$. Then $f \in \dot{B}^{s,q}_p$ and

$$\|f\|_{B^{s,q}_p} \lesssim \|(\eta_j)_j\|_{l^q}. \tag{3.13}$$

Proof: Again, we need to estimate $\Delta_j f$: from the support condition, and choosing $1 \leq \gamma < 2$ (which is always possible up to renumbering),

$$\begin{aligned}
\Delta_j f &= \sum_{j\leq k} \Delta j f_k,\\
\|\Delta_j f\|_p &\lesssim \sum_{k\geq j} \|f_k\|_p,\\
2^{js}\|\Delta_j f\|_p &\lesssim \sum_{k\geq j} 2^{s(j-k)}\eta_k.
\end{aligned}$$

As above we have a convolution between l^q and l^1, which ends the proof. □
For the remaining of this section, we briefly formalize what underlines the paralinearization step.

Definition 6

Take $f, g \in \mathcal{S}$. We call paraproduct of f and g the operator

$$\pi_g f = \sum_j S_{j-1} g \Delta_j f. \tag{3.14}$$

One may split a product fg in the following manner:

$$fg = \pi_g f + \pi_f g + \sum_{|k-k'|\leq 1} \Delta_k f \Delta_{k'} g. \tag{3.15}$$

The gain from introducing $\pi_g f$ has to do with the spectral properties of $S_{j-1} g \Delta_j f$: as

$$\operatorname{supp} \widehat{\Delta_j f} \subset \{2^j \leq |\xi| \leq (1+\varepsilon)2^{j+1}\} \text{ et } \operatorname{supp} \widehat{S_{j-1} f} \subset \{|\xi| \leq (1+\varepsilon)2^{j-1}\},$$

and $\operatorname{supp} \mathcal{F}(S_{j-1} g \Delta_j f)$ is contained in the algebraic sum of the respective supports,

$$\operatorname{supp} \mathcal{F}(S_{j-1} g \Delta_j f) \subset \{2^{j-1}(1-\varepsilon) \leq |\xi| \leq 2^{j+1}\frac{5+\varepsilon}{4}\} \subset \frac{1}{4}2^j \leq |\xi| \leq 4.2^j,$$

(assuming $\varepsilon < 1/2$ for convenience).

In order to split fg, write

$$\begin{aligned}
fg &= \sum_j \Delta_j f \sum_k \Delta_k g \\
&= \sum_j (\sum_{k<j-1} \Delta_k g) \Delta_j f + \sum_k \sum_{k>j-2} \Delta_j f \Delta_k g \\
&= \sum_j S_{j-1} g \Delta_j f + \sum_k (\sum_{j<k-1} \Delta_j f) \Delta_k g + \sum_{k-2<j<k+2} \Delta_k g \Delta_j f \\
&= \pi_g f + \pi_f g + \sum_{|j-k|\leq 1} \Delta_k g \Delta_j f.
\end{aligned}$$

Unlike for the two paraproducts, the last sum is only a sum of pieces localized in balls $\{|\xi| \leq 4.2^j\}$. This is generally the deadly term if one is considering non positive regularity.

Chapter 4

A singular metric: counterexamples

Introduction

In this chapter, we construct a metric on $\mathbb{R}$, which is in $W^{s,1}$ for any $0 \leq s < 1$ (but not in BV), bounded from below and above and for which no smoothing estimate and no (non trivial) Strichartz estimates hold. In fact this construction is a simplification of an argument of Castro and Zuazua [17] (whose proof relies in turn upon some related works in semi-classical analysis and unique continuation theories), who, in the context of wave equations, provide counter examples with $C^{0,\alpha}, 0 \leq \alpha < 1$ metrics (i.e. Hölder continuous of exponent α). As noticed by Castro and Zuazua, these counter examples extend to our setting. Figure 4.1 shows the range where full Strichartz/smoothing are true or no Strichartz/smoothing holds.

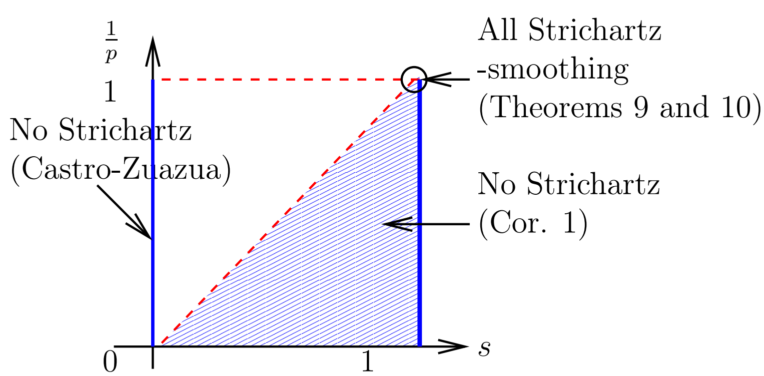

Figure 4.1: Range of regularity $W^{s,p}$

A most interesting range of regularity is $a \in \dot{W}^{s,\frac{1}{s}}$ and in particular $\dot{H}^{1/2} = \dot{W}^{1/2,2}$ because these regularities are scale invariant. Note of course that one has to assume $a \in L^\infty$, bounded from below. A natural question would be to ask whether some Strichartz/smoothing estimates might hold (possibly with derivatives loss) at these levels of regularity. Remark that neither our counter examples nor Castro-Zuazua's lie in this range (except for $s = 0$).

4.1 Construction of the metric

We will construct a metric which has the desired properties in term of regularity, together with the existence of a family $(\phi_k)_k$ which are "almost" eigenvectors (quasimodes). Certainly, a metric which has eigenvectors cannot satisfy any dispersive inequalities which would improve on Sobolev embeddings: if (ϕ_λ, λ) is a couple eigenfunction/eigenvalue, $\phi = e^{i\lambda t}\phi_\lambda$ will be a solution, and all estimates reduce to properties of ϕ_λ.

Proposition 14
There exist a metric $\beta(x) \in W^{s,1}$ for any $0 \leq s < 1$, bounded from below and above $0 < m \leq \beta(x) \leq M$, a sequence of functions $\phi_k \in C_0^\infty(]2^{-k-1/2}, 2^{k+1/2}[)$ and a sequence $(x_k = 2^{-k}, \lambda_k = 2^k k)$ such that

$$(\partial_x \beta(x) \partial_x + \lambda_k)\phi_k = \mathcal{O}(\lambda_k^{-\infty})_{H^1} \tag{4.1}$$
$$\|\phi_k\|_{L^2} = 1,$$

Remark 8
Note that from $\beta \in W^{s,1} \cap L^\infty$, we have by interpolation $\beta \in W^{s,p}$, with $s < \frac{1}{p}$.

Proof: The starting point of the proof is the interval instability of the Hill equation.

Lemma 8
There exist $w, \alpha \in C^\infty$ such that

$$w'' + \alpha w = 0 \ \ \text{on} \ \ \mathbb{R}, \tag{4.2}$$

α is 1-periodic on $\mathbb{R}^+$ and $\mathbb{R}^-$, equal to $4\pi^2$ in a neighborhood of 0, and

$$|\alpha - 4\pi^2| \leq 1, \tag{4.3}$$

together with

$$w(x) = pe^{-|x|} \text{ where } p \text{ is 1-periodic on } \mathbb{R}^+ \text{ and } \mathbb{R}^-, \ \|w\|_{L^2} = 1. \tag{4.4}$$

Proof: We refer to [22] and [17] for detailed explanations and further references to the Hill equation. Note that if we look at (4.2) on $\mathbb{R}$, ODE techniques provide us with a solution $p(x)e^{-\varepsilon x}$. We just need to choose α in such a way that we can make an even extension (hence, the requirement for α to be constant in a neighborhood of 0). One may choose

$$\alpha(x) = 4\pi^2(1-4\varepsilon\gamma(2\pi x)\sin(4\pi x)+2\varepsilon\gamma'(2\pi x)\cos^2(2\pi x)-4\varepsilon^2\gamma^2(2\pi x)\cos^4(2\pi x),$$

and

$$w(x) = \cos(2\pi x)e^{-2\varepsilon\int_0^{2\pi x}\gamma(y)\cos^2(y)\,dy},$$

when γ is a positive, smooth, 2π periodic function, such that $\gamma = 0$ in a neighborhood of 0 and

$$\int_0^{2\pi}\gamma(y)\cos^2(y)\,dy = \frac{1}{2}, \int_0^{2\pi}\gamma(y)\cos^2(y)\sin y\,dy > 0.$$

One may then verify that our chosen α and w verify all the requirements...
□

We now go back to the proof of Proposition 14. We change variables in our Hill equation and set

$$y(x) = \int_0^x \alpha(s)ds, \qquad v(y) = w(x(y)), \qquad \beta(y) = \alpha(x(y)).$$

Remark that from (4.3), $y(x)$ is indeed a diffeormorphism. We have

$$\frac{\partial}{\partial y} = \alpha^{-1}(x)\frac{\partial}{\partial x}$$

and the equation becomes

$$\partial_y(\beta(y)\partial_y v) + v = 0.$$

Then we translate and rescale: denote by

$$v^{\lambda,m}(y) = v(\lambda(y-m)), \qquad \beta^{\lambda,m}(y) = \beta(\lambda(y-m))$$

solutions of

$$(\partial_y(\beta^{\lambda,m}(y)\partial_y) + \lambda^2)v^{\lambda,m} = 0, \tag{4.5}$$

from the exponential decay of w (4.4), they satisfy

$$|v^{\lambda,m}(y)| \leq Ce^{-\lambda|y-m|}. \tag{4.6}$$

Consider

$$\Psi_1 \in C_0^\infty(]\frac{-1}{4},\frac{1}{4}[) \text{ t.q. } \Psi_1 = 1 \text{ on } [\frac{-1}{5},\frac{1}{5}]$$
$$\Psi_2 \in C_0^\infty(]\frac{-1}{5},\frac{1}{5}[) \text{ t.q. } \Psi_2 = 1 \text{ on } [\frac{-1}{6},\frac{1}{6}],$$

and the two sequences $m_n = 2^{-n}$, $\lambda_n = n2^n$. Notice we choose an oscillation parameter λ_n which is slightly bigger than the inverse of the localization m_n. Using (4.6), we see that $v_n = v^{\lambda_n,m_n}(y)\Psi_2(2^n(y-m_n))$ is a solution of

$$(\partial_y \beta^{\lambda_n,m_n}(y)\partial_y + \lambda_n^2)v_n = \mathcal{O}(\lambda_n e^{-cn})_{H^1}. \tag{4.7}$$

Remark also that on the support of v_n, $\Psi_1(2^n(y-m_n)) = 1$ and consequently we can replace in (4.7) $\beta^{\lambda_n,m_n}(y)$ by $\beta_n(y) = \beta^{\lambda_n,m_n}(y)\Psi_1(2^n(y-m_n))$, which amounts to localizing the coefficient β. Remark also that for $p \neq n$, the support of v_n is disjoint from the support of $\Psi_1(2^p(y-m_p))$: as such, we can add any β_p to β_n. Consequently, we can replace in (4.7) $\beta^{\lambda_n,m_n}(y)$ by

$$\beta(y) = \sum_{n\in\mathbb{N}} \beta_n(y) + 4\pi(1 - \sum_{n\in\mathbb{N}} \Psi_1(2^n(y-m_n))$$

where the last term was added only to ensure that β is bounded from below ($\beta(y) \geq 2\pi$).

To achieve the proof of Proposition 14, it is now enough to show that β is in $W^{s,1}$ for any $0 \leq s < 1$. A direct calculation yields

$$\|\beta_n\|_{W^{1,1}} \sim n, \qquad \|\beta_n\|_{L^1} \sim 2^{-n},$$

which together with interpolation ensures

$$\|\beta_n\|_{\dot{W}^{s,1}} \leq Cn^s 2^{1-s}.$$

This last bound implies that the series defining β converges in $W^{s,1}$. $\square$

4.2 Counterexamples

Having obtained a coefficient β with the desired properties, we use the associated quasi-modes ϕ_k in order to prove there exist no Strichartz estimates.

Corollary 1 *For the coefficient β constructed in Proposition 14, we have for any $r < (q-2)/2q$ (recall that by the usual Sobolev embedding, $H^{(q-2)/2q} \to L^q$),*

$$\lim_{k\to+\infty} \frac{\|e^{it(\partial_x\beta(x)\partial_x)}\phi_k\|_{L^1(-\varepsilon,\varepsilon);L^q(\mathbb{R})}}{\|\phi_k\|_{H^r}} = +\infty \tag{4.8}$$

Proof: According to (4.1), $\|\phi_k\|_{H^1} \leq C\lambda_k$ and, by interpolation,

$$\|\phi_k\|_{H^r} \leq C\lambda_k^r \qquad (0 \leq r \leq 1). \tag{4.9}$$

According to (4.1),

$$e^{it(\partial_x\beta(x)\partial_x)}\phi_k = e^{it\lambda_k^2}\phi_k + v$$

where $\|v\|_{L^\infty_{t,\mathrm{loc}}(H^1(\mathbb{R}))} = \mathcal{O}(\lambda_k^{-\infty})$. Using the Sobolev embedding $H^1 \to L^q$, we can drop the contribution of v in (4.8). Using Hölder inequality (and the fact that ϕ_k is supported in a ball of radius 2^{-k}), we obtain

$$1 = \|\phi_k\|_{L^2} \leq C2^{-k(q-2)/q}\|\phi_k\|_{L^q}$$

and consequently, according to (4.9),

$$\frac{\|e^{it(\partial_x\beta(x)\partial_x)}\phi_k\|_{L^1(-\varepsilon,\varepsilon);L^q}}{\|\phi_k\|_{H^r(\mathbb{R})}} \geq \frac{c2^{k(q-2)/q}}{(k2^k)^r} \to +\infty, \text{ when } k \to +\infty,$$

which ends the proof. □

Chapter 5

A generalized Benjamin-Ono equation

Introduction

The Benjamin-Ono family of equations reads

$$(\partial_t + H\partial_x^2)u \pm u^p \partial_x u = 0, \tag{5.1}$$

where $p \geq 2$ is usually an integer. Moreover, the data u_0 at time $t = 0$ is assumed to be real-valued, and as such, the solution u will be real-valued as well. Here H denotes the Hilbert transform,

$$Hf(x) \approx \int \frac{f(y)}{x-y} dy = \mathrm{pv}\frac{1}{x} * f,$$

and can be seen as well as a Fourier multiplier, $-i\mathrm{sign}(\xi)$.
Given that the solution is real-valued, we can recover it from its positive spectrum; most interestingly, if one looks at the linear Benjamin-Ono equation,

$$(\partial_t + H\partial_x^2)\phi = 0,$$

by projecting on positive frequencies, we get a Schrödinger equation,

$$(i\partial_t + \partial_x^2)\phi^+ = 0.$$

This immediately implies that all the known estimates for the Schrödinger equation extend to the linear Benjamin-Ono equation: smoothing, Strichartz, maximal function estimates.
There are several cases of interest: mainly $p = 1$, $p = 2$ and $p = 4$. We will restrict ourselves to $p = 4$. Any higher p could be treated in a similar

way. For $p = 2$, one could obtain local well-posedness in $H^{\frac{1}{2}+}$, recovering the recent result of [48] by a different method. The case $p = 1$ is the most interesting one, but it falls out of scope of this presentation.
The study of the IVP for (5.1) with low regularity data was initiated in [40, 42]. The best results to date for $p = 4$ were obtained recently in [48], where they prove (among other results for different p) (5.1) to be locally wellposed in $H^{\frac{1}{2}^+}$. The authors were able to remove the (rather natural with the techniques at hand) restriction on the size of the data by adapting the renormalization procedure from [69] (where global wellposedness for the $p = 1$ case is obtained in H^1). The same authors proved earlier in [49] that (5.1) was globally wellposed for small data in $\dot{B}_4^{1/4,1}$ (and extended this result to $\dot{H}^{\frac{1}{4}}$ in [48]). We refer to [48] for a very nice presentation of the Benjamin-Ono family of equations and of the context in which they arise.

5.1 Well-posedness

We will prove local well-posedness for our generalized Benjamin-Ono by a method which is different from the gauge transform: essentially, if one considers a nonlinear Schrödinger equation with a first order term,

$$i\partial_t\phi + \Delta\phi - B \cdot \nabla\phi = N(\phi),$$

on can think of two different ways to obtain estimates:

- Gauge away the first order term. This is in essence the idea in [33] in order to solve 1D equations with derivative in the nonlinearity (there, the gauge transform reduces the equation to an equation which can be dealt with by energy methods), and both [69] and [48] make use of a similar idea.

- Prove estimates for the linear part, including the first order term. This is the approach we will use, and we refer to [43] and references therein for an interesting discussion on the topic and the connection with the first approach.

We intend to remove the restriction on the size of the data, all the way down to $s = 1/4$ (which is the scaling exponent). If one thinks of the result from [49] and earlier ([42]), the restriction on the size of the data is a direct consequence of the use of smoothing together with the maximal function estimate for the nonlinear term: working at regularity $s = 1/4$ and considering what may be seen as the worst term (which is a paraproduct, low frequency/high frequency interacting)

$$W = \sum_j (S_{j-10}u)^4 \partial_x \Delta_j u,$$

the loss of derivative can only be recovered by using the inhomogeneous smoothing estimate on the linear flow (which we denote by $S(t)$)

$$\|\partial_x \int_0^t S(t-s) f\, ds\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^1_x(L^2_t)},$$

which becomes (after frequency localization)

$$\| \int_0^t S(t-s) f\, ds\|_{\dot{B}^{\frac{3}{4},2}_\infty(\mathcal{L}^2_t)} \lesssim \|f\|_{\dot{B}^{-\frac{1}{4},2}_1(\mathcal{L}^2_t)}.$$

Thus, we are forced to use

$$\partial_x \Delta_j u \in 2^{-\frac{j}{4}} l^2_j L^\infty_x(L^2_t),\ S_{j-10}u \in L^4_x(L^\infty_t),$$

and to contract the $\dot{B}^{\frac{3}{4},2}_\infty(\mathcal{L}^2_t)$ norm, we will need $\|u\|_{L^4_x(L^\infty_t)}$ small with no recourse. Therefore, a very natural idea is to replace u by $u - u_0$, at the expense of considering a new linear operator, which is basically (up to possibly a paralinearization)

$$\partial_t \phi + H\partial_x^2 \phi \pm u_0^4 \partial_x \phi.$$

It turns out that one can deal with this linear operator with the help of the estimates proved earlier for the variable coefficient operator.

THEOREM 11
Let $u_0 \in \dot{H}^{\frac{1}{4}}$, then the generalized Benjamin-Ono equation (5.1), *for $p = 4$ is locally wellposed, i.e. there exists a time $T(u_0)$ such that a unique solution u exists with*

$$u \in C_T(\dot{H}^{\frac{1}{4}}) \cap \dot{B}^{\frac{3}{4},2}_\infty(L^2_T) \cap \dot{L}^4_x(L^\infty_T).$$

Moreover, the flow map is locally Lipschitz.

Combining this local wellposedness result, which is subcritical with respect to the "energy norm" $\dot{H}^{\frac{1}{2}}$, with the conservation of mass and energy,

$$\|u(t)\|_2 = \|u_0\|_2 \text{ and } E(u) = \|u\|^2_{\dot{H}^{\frac{1}{2}}} \mp \frac{1}{15}\int_{\mathbb{R}} u^6 = E(u_0)$$

and Gagliardo-Nirenberg, we also obtain global wellposedness in the energy space when the energy controls the $\dot{H}^{\frac{1}{2}}$ norm, which occurs in the defocusing case (minus sign in (5.1)) or if the L^2 norm is small enough (focusing: plus sign in (5.1)).

THEOREM 12
Let $u_0 \in H^{\frac{1}{2}}$*, then the defocusing generalized Benjamin-Ono equation (5.1), for* $p = 4$*, is globally wellposed, i.e. there exists a unique solution* u *such that*

$$u \in C_T(H^{\frac{1}{2}}) \cap \dot{B}_{\infty}^{\frac{3}{4},2}(L^2_{t,loc}) \cap \dot{L}^4_x(L^\infty_{t,loc}).$$

Proof: We first prove Theorem 11. For local well-posedness, the sign in (5.1) is irrelevant and we take $+$ for convenience. Let us sketch our strategy: the restriction on small data is induced by the maximal function estimate (3.10): even on the linear part, $\|S(t)u_0\|_{L^4_x(L^\infty_t)}$ will be small only if $\|u_0\|_{\dot{H}^{\frac{1}{4}}}$ is small as well. Recall $S(t)$, the linear operator, reduces to the Schrödinger group on positive frequencies. Now, if we consider instead the difference $S(t)u_0 - u_0$, then the associated maximal function is small provided we restrict ourselves to a small time interval $[0, T]$:

LEMMA 9
Let $u_0 \in \dot{H}^{\frac{1}{4}}$*, then for any* $\varepsilon > 0$*, there exists* $T(u_0)$ *such that*

$$\| \sup_{|t|<T} |S(t)u_0 - u_0| \|_{L^4_x} < \varepsilon. \tag{5.2}$$

Proof: For the linear flow,

$$\begin{aligned}
\|S(t)u_0 - u_0\|_{L^4_x(L^\infty_T)} &\leq \sum_{|j|<N} \|\Delta_j(S(t)u_0 - u_0)\|_{L^4_x(L^\infty_T)} \\
&\qquad + 2\Big(\sum_{|j|>N} 2^{\frac{j}{2}} \|\Delta_j u_0\|_2^2\Big)^{\frac{1}{2}} \\
&\leq \sum_{|j|<N} 2^{2j} \| \int_0^t S(s)\Delta_j u_0 ds\|_{L^4_x(L^\infty_T)} \\
&\qquad + 2\Big(\sum_{|j|>N} 2^{\frac{j}{2}} \|\Delta_j u_0\|_2^2\Big)^{\frac{1}{2}} \\
&\leq T \sum_{|j|<N} 2^{2j} \|\Delta_j u_0\|_{L^4_x(L^\infty_T)} + 2\Big(\sum_{|j|>N} 2^{\frac{j}{2}} \|\Delta_j u_0\|_2^2\Big)^{\frac{1}{2}} \\
&\leq T 2^{2N} \|u_0\|_{\dot{H}^{\frac{1}{4}}} + 2\Big(\sum_{|j|>N} 2^{\frac{j}{2}} \|\Delta_j u_0\|_2^2\Big)^{\frac{1}{2}}
\end{aligned}$$

and by choosing first N large enough and then T accordingly, we get arbitrary smallness. Of course the parameter T depends on u_0 and cannot be taken uniform. □

Given that local in time solutions do exist ([40]), we could set up an a priori estimate and pass to the limit. However, in order to get the flow to be Lipschitz, one has essentially to estimate differences of solutions, and in turn this provides the required estimates to set up a fixed point procedure.
Firstly, we proceed with an appropriate paralinearization of the equation itself. All computations which follow are justified if we consider smooth solutions. We have, denoting $u_j = \Delta_j u$, $u_{\prec j} = S_{j-10}u$ and $u_{\preceq j} = S_j u$

$$\partial_t u_j + H\Delta u_j + \Delta_j(u^4\partial_x u) = 0.$$

Rewriting $u^4\partial_x u = \partial_x(u^5)/5$ and using a telescopic series $u = \sum_k S_k u - S_{k-1}u$, we get by standard paraproduct-like rearrangements

$$\begin{aligned}5\Delta_j(u^4\partial_x u) = \Delta_j\partial_x(u^5) &= \Delta_j\Big((u_{\prec j})^4\sum_{k\sim j}\partial_x u_k\Big) + \partial_x\Delta_j\Big(\sum_{k\sim k'\sim j}(u_{k'})^2(u_{\preceq k'})^3\Big)\\ &\quad + \Delta_j\Big(\sum_{k\sim j}(u_{\prec j})^3 u_k\partial_x u_{\prec j}\Big)\\ &= \Delta_j\Big((u_{\prec j})^4\sum_{k\sim j}\partial_x u_k\Big) - R_j(u).\end{aligned}$$

We will now consider the original equation as a system of frequency localized equations,

$$\partial_t u_j + H\Delta u_j + \Delta_j\Big(\sum_{k\sim j}(u_{\prec j})^4\partial_x u_k\Big) = R_j(u).$$

If we set $\pi(f_1, f_2, f_3, f_4, g) = \sum_j \Delta_j\Big(\sum_{k\sim j} f_{1,\prec j}f_{2,\prec j}f_{3,\prec j}f_{4,\prec j}g_k\Big)$ we can rewrite our model (abusing notations for π)

$$\partial_t u + H\Delta u + \pi(u^{(4)}, \partial_x u) = R(u), \tag{5.3}$$

and we intend to solve (5.3) by Picard iterations.
Now, let us consider u_L the solution to the linear BO equation, and the following linear equation:

$$\partial_t v + H\Delta v + \pi(u_L^{(4)}, \partial_x v) = 0, \text{ and } v_{t=0} = u_0.$$

At the frequency localized level, this is almost what we can handle, except for a commutator term. Therefore we have

$$\begin{aligned}\partial_t v_j + H\Delta v_j + (u_{L,\prec j})^4\partial_x v_j &= -\Big(\sum_{k\sim j}[\Delta_j, (u_{L,\prec j})^4]\partial_x v_k\Big),\\ \partial_t v_j + H\Delta v_j + (u_{0,\prec j})^4\partial_x v_j &= \Big((u_{0,\prec j})^4 - (u_{L,\prec j})^4\Big)\partial_x v_j\\ &\quad - \Big(\sum_{k\sim j}[\Delta_j, (u_{L,\prec j})^4]\partial_x v_k\Big),\end{aligned}$$

for which we aim at using the estimates from Section 2.2.
The iteration map will therefore be

$$\partial_t u_{n+1} + H\Delta u_{n+1} + \pi(u_L^{(4)}, \partial_x u_{n+1}) = \pi(u_L^{(4)}, \partial_x u_n) - \pi(u_n^{(4)}, \partial_x u_n) + R(u_n).$$

Hence we need estimates for the linear equation

$$\partial_t v + H\Delta v + \pi(u_L^{(4)}, \partial_x v) = f(x,t), \text{ and } v_{t=0} = u_0. \tag{5.4}$$

Restrict time to $[0,T]$ with T to be chosen later, let 0^+ denote a small number close to 0, and define

$$E_s = \cap_{0^+\le\theta\le 1} \dot{B}^{s+\frac{5\theta-1}{4},2}_{\frac{4}{1-\theta}}(\mathcal{L}_t^{\frac{2}{\theta}}) \text{ as well as } F_s = \sum_{0^+\le\theta\le 1,\text{finite}} \dot{B}^{s+\frac{1-3\theta}{4},2}_{\frac{4}{3+\theta}}(\mathcal{L}_t^{\frac{2}{2-\theta}})$$

(we left out the maximal function part, $\theta = 0$ because we need a slightly different estimate).

Proposition 15
Let v be a solution of equation (5.4), $u_0 \in \dot{H}^s \cap \dot{H}^{\frac14}$ with $-3/4 < s < 1/2$ and $f \in F_s$. Then there exists $T(u_0)$ such that on the time interval $[-T,T]$, we have

$$\|v\|_{E_s} \lesssim_T \|u_0\|_{\dot{H}^s} + \|f\|_{F_s}.$$

Moreover,

$$\|v-u_0\|_{\dot{B}_4^{s-\frac14,2}(\mathcal{L}_t^\infty)} \lesssim_T \|S(t)u_0 - u_0\|_{\dot{B}_4^{s-\frac14,2}(\mathcal{L}_t^\infty)} + \|f\|_{\dot{B}_1^{s-\frac12,2}(\mathcal{L}_t^2)} + \|u_0\|_4^4 \|v\|_{\dot{B}_\infty^{s+\frac12,2}(\mathcal{L}_t^2)},$$

and

$$\|v-u_0\|_{L_x^4(L_T^\infty)} \lesssim_T \|S(t)u_0-u_0\|_{L_x^4 L_T^\infty} + \|f\|_{\dot{B}_1^{-\frac14,2}(\mathcal{L}_t^2)} + \|u_0\|_4^4\|v\|_{\dot{B}_\infty^{\frac34,2}(\mathcal{L}_t^2)}.$$

Proof: Let us consider the equation at the frequency localized level,

$$\begin{aligned}\partial_t v_j + H\Delta v_j + (u_{0,\prec j})^4 \partial_x v_j =& \Big((u_{0,\prec j})^4 - (u_{L,\prec j})^4\Big)\partial_x v_j \\ &- \Big(\sum_{k\sim j}[\Delta_j, (u_{L,\prec j})^4]\partial_x v_k\Big) + f_j,\end{aligned}$$

and we will denote by R_j the right hand side. Notice R_j is spectrally localized. In order to connect this equation with the model worked upon in Section 2.2, denote by

$$b(x) = (u_{0,\prec j})^4 \in L_x^1, \quad \text{and consider} \quad i\partial_t w + \partial_x^2 w + b(x)\partial_x w = g.$$

We remark that the distinction between the BO linear equation and the Schrödinger is irrelevant at this point: we can simply deal with the positive frequencies projection of v_j, and the projection commutes with the product by $u_{0,\prec j}$ due to support conditions. Hence, one should really see w as $v_j^+$with an obvious notation. By reversing the procedure we used in Section 3, we can reduce the operator $\partial_x^2 + b(x)\partial_x$ to $\partial_y a(y)\partial_y$ and apply all the estimates we already know: set

$$\frac{dy}{dx} = A(x) = \phi'(x), \quad \text{with} \quad A(x) = \exp\Big(\int_{-\infty}^{x} (u_{0,\prec j})^4(\rho)\, d\rho\Big),$$

then $y = \phi(x)$ is a diffeomorphism and $\sqrt{a}(y) = A \circ \phi^{-1}(y)$ which insures $a \in \dot{W}^{1,1}$, and a is 1-admissible. A simple calculation shows that under this change of variables,

$$\partial_y a(y)\partial_y \to \partial_x^2 + b(x)\partial_x$$

Note that everything is uniform wrt j. Interpolation between all the various bounds which one can deduce from Proposition 10 and Theorem 10 yields estimates for $w\circ\phi^{-1}$ which are identical to the flat case (or, to get a better sense of perspective, to linear estimates for the linear Benjamin-Ono equation, see e.g. [49]):

$$\|w \circ \phi^{-1}\|_{E_s} \lesssim \|w_0 \circ \phi^{-1}\|_{\dot{H}^s} + \|g \circ \phi^{-1}\|_{F_s},$$

with $-3/4 < s < 1/2$. Using Proposition 13, we can revert back to the x variable and obtain the *exact* same estimates for w:

$$\|w\|_{E_s} \lesssim \|w_0\|_{\dot{H}^s} + \|g\|_{F_s}.$$

Recalling that $w = v_j = \Delta_j v$ and $g = R_j$ is frequency localized as well, hence for any $0^+ \leq \theta \leq 1$,

$$2^{j(s+\frac{5\theta-1}{4})}\|v_j\|_{L_x^{\frac{4}{1-\theta}}(L_t^{\frac{2}{\theta}})} \lesssim 2^{js}\|u_{0,j}\|_2 + \sum_{k,\text{finite}} 2^{j(s+\frac{1-3\theta_k}{4})}\|R_j\|_{L_x^{\frac{4}{3+\theta_k}}(L_t^{\frac{2}{2-\theta_k}})}.$$

All is left is to estimate R_j in order to contract the v_j term:

$$R_j = \Big((u_{0,\prec j})^4 - (u_{L,\prec j})^4\Big)\partial_x v_j - \Big(\sum_{k\sim j}[\Delta_j, (u_{L,\prec j})^4]\partial_x v_k\Big) + f_j.$$

From the smoothing estimate for the flat Schrödinger equation and Lemma 9, there exist $T(u_0)$ such that

$$\Big(\sum_j (2^{\frac{-j}{4}}\|\partial_x u_{L,\preceq j}\|_{L_x^\infty(L_T^2)})^2\Big)^{\frac{1}{2}} + \|u_{L,\prec j} - u_{0,\prec j}\|_{L_x^4(L_T^\infty)}) < \eta(u_0), \tag{5.5}$$

where $\eta(u_0)$ can be made as small as needed by choice of a smaller $T(u_0)$. This allows to write, picking a θ_k close to 1 and abusing notations,

$$2^{js}\|v_j\|_{L^\infty_x(L^2_T)}) + 2^{(s^-)j}\|v_j\|_{L^{\infty^-}_x(L^{2^+}_T)} \leq \frac{1}{2}2^{(s-1)j}\|\partial_x v_j\|_{L^\infty_x(L^2_T)} + \frac{1}{2K}\sum_{|l-j|<K} 2^{(s^-)l}\|w_l\|_{L^{\infty^-}_x(L^{2^+}_T)} + 2^{(s-1)j}\|f_j\|_{L^1_x(L^2_T)},$$

where we used Lemma 4 to estimate the commutator with $2^{-(1^-)j}\partial_x u_{L,\prec j} \in L^{4^+}_x(L^{\infty^-}_T)$ small enough by (5.5) and interpolation with $u_L \in L^4_x(L^\infty_T)$. We have therefore obtained, after summing over j,

$$\|v\|_{E_s} \leq C(u_0)(\|u_0\|_{\dot H^s_2} + \|f\|_{F_s}).$$

We only have a local in time estimate for the linearized equation, but it depends only on the data and nothing else, through lemma 9. At our desired level of regularity, namely $s = 3/4$,

$$\|v\|_{\dot B^{\frac{3}{4},2}_\infty(\mathcal{L}^2_t)} \leq C(u_0)(\|f\|_{\dot B^{-\frac{1}{4},2}_1(\mathcal{L}^2_t)} + \|u_0\|_{\dot H^{\frac{1}{2}}}).$$

We also need the maximal function, or more accurately, $v - u_0$: but this is now very easy, simply reverting back to writing ($S(t)$ being here the group associated to the linear BO)

$$v = u_L + \int_0^t S(t-s)(f - \pi(u_L, \partial_x v))ds,$$

and we therefore get (using the third case in Theorem 13 for the special case $s = 0$)

$$\|v - u_0\|_{\dot B^{s-\frac{1}{4},2}_4(\mathcal{L}^\infty_t)} \lesssim \|u_L - u_0\|_{\dot B^{s-\frac{1}{4},2}_4(\mathcal{L}^\infty_t)} + \|f\|_{\dot B^{s-\frac{1}{2},2}_1(\mathcal{L}^2_t)} + \|u_0\|_4^4\|v\|_{\dot B^{s-\frac{1}{4},2}_\infty(\mathcal{L}^2_t)},$$

and

$$\|v - u_0\|_{L^4_x(L^\infty_T)} \lesssim \|u_L - u_0\|_{L^4_x(L^\infty_T)} + \|f\|_{\dot B^{-\frac{1}{4},2}_1(\mathcal{L}^2_t)} + \|u_0\|_4^4\|v\|_{\dot B^{\frac{3}{4},2}_\infty(\mathcal{L}^2_t)}.$$

This achieves the proof of Proposition 15. □

Everything is now ready for a contraction in a complete metric space, which will be the intersection of two balls,

$$B_M(u_0, T) = \{u \text{ s.t. } \|u - u_0\|_{\dot B^{0,1}_4(L^\infty_T)} < \varepsilon(u_0)\},$$

and

$$B_S(u_0,T)=\{u \text{ s.t. } \|u\|_{\dot B^{\frac34,1}_\infty(L^2_T)}<\varepsilon(u_0)\}.$$

We first check that the mapping K is from $B_M\cap B_S$ to itself, where $K(v)=u$ with

$$\partial_t u+H\Delta u+\pi(u_L^{(4)},\partial_x u)=\pi(u_L^{(4)},\partial_x v)-\pi(v^{(4)},\partial_x v)+R(v).$$

For this we use Proposition 15 with $s=3/4$ and standard (para)product estimates. The B_S part is trivial (one doesn't even need to take advantage of the difference on the right). The B_M part follows from the ability to factor an u_L-u while rewriting the difference of the π on the right.
The next step is then to contract, i.e. estimate $K(v_1)-K(v_2)$ in terms of v_1-v_2. But this is again trivial given we have a multilinear operator, it will be exactly as the $v\to u$ mapping. This ends the proof of Theorem 11. $\square$
We now briefly sketch the proof of Theorem 12. We now have a minus sign in (5.1) but this doesn't change the local in time contraction. Given a datum in the (inhomogeneous) space H^s, with $s>1/4$, a standard modification of the fixed point provides that the solution u is $C_t(H^s)$. In order to iterate whenever $s=1/2$, we need to check that the local time $T(u_0)$ can be repeatedly chosen in a uniform way. All is required is an appropriate modification of Lemma 9: recall we can write

$$\sum_j \|\Delta_j(S(t)u_0-u_0)\|_{L^4_x(L^\infty_T)}\leq T2^{2N}\|u_0\|_{\dot H^{\frac14}}+2\Big(\sum_{|j|>N}2^{\frac j4}\|\Delta_j u_0\|_2\Big),$$

from which we get, taking advantage of $u_0\in L^2\cap \dot H^{\frac12}$,

$$\sum_j \|\Delta_j(S(t)u_0-u_0)\|_{L^4_x(L^\infty_T)}\leq T2^{2N}\Big(\|u_0\|_2\|u_0\|_{\dot H^{\frac12}}\Big)^{\frac12}+2^{-\frac N4}(\|u_0\|_2+\|u_0\|_{\dot H^{\frac12}}).$$

Obviously, picking $T=2^{-\frac94 N}$ gives the bound $2^{-\frac N4}(\|u_0\|_2+\|u_0\|_{\dot H^{\frac12}})$, which by an appropriate choice of N can be made as small as we need with respect to $(\|u_0\|_2+\|u_0\|_{\dot H^{\frac12}})$. However, both the L^2 and $\dot H^{\frac12}$ norms are controlled, thus the local time $T(u_0)$ is uniform and we can iterate the local existence result to a global result. $\square$

Appendix A

Localization with respect to ∂_x versus localization with respect to $(-\partial_x(a(x)\partial_x))^{\frac{1}{2}}$

A.1 The heat flow associated with $-\partial_x(a(x)\partial_x$

We would like to define an analog of the Littlewood-Paley operator Δ_j, but using $A = -\partial_x(a(x)\partial_x$ rather than $-\partial_x^2$. In the second chapter, this turns out to be useful because such a localization wrt A will commute with the Schrödinger flow. Through spectral calculus, we can easily define $\phi(A)$ for a smooth ϕ, but we need various properties on L^p spaces for all $1 \leq p \leq +\infty$, which requires a bit more of real analysis. Fortunately, all the results we need are more or less direct consequences of (part of) earlier work related to the Kato conjecture, and we simply give a short recollection of the main facts we need, skipping details and referring to [2, 1]. We call $S_A(t)$ the heat flow, namely $S_A(t)f$ solves

$$\partial_t g + Ag = 0, \text{ with } g(0) = f, \tag{A.1}$$

and define $\Delta_j^A f = 4^{-j} A S_A(4^{-j}) f$. Again, in L^2 all of this makes sense through spectral considerations, and were a to be just 1, we would just get a localization operator based on the Mexican hat $\xi^2 \exp -\xi^2$. In [2], such a semi-group $S_A(t)$ is proved to be analytic, and moreover the square-root of A can be factorized as $R\partial_x$, where R is a Calderon-Zygmund operator, under rather mild hypothesis: $a \in L^\infty$, complex valued, with $\mathrm{Re} a > 1$. On the other hand, in [1], the authors prove Gaussian bounds for the kernel of the semi-group as well as its derivatives, and this provides everything which is needed here. Such bounds are obtained through the following strategy:

- Derive bounds for the operator $(1+A)^{-1}$: given that it maps H^{-1} to H^1, it follows that it maps L^1 to L^∞ by Sobolev embeddings.

- Obtain bounds for $(\lambda+A)^{-1}$, $\mathrm{Re}\lambda > 0$, by rescaling, given the hypothesis on a are invariant.

- Obtain bounds for $A(1+A)^{-1}$ by algebraic manipulations, proving it maps L^1 to L^∞.

- Obtain again an $L^1 - L^\infty$ bound for $\partial_x(1+A)^{-1}$ by "interpolation" between the two previous bounds. This specific bound we did prove directly in Section 2.2, namely (2.28).

- Use a nifty trick (see Davies ([25])): remark that provided ω is sufficiently small (wrt the lower bound of $\mathrm{Re}a$), all previous estimates hold as well for
$$A_\omega = \exp(\omega\cdot)A\exp(-\omega\cdot).$$
Then any of the new kernels $K_\theta(x,y)$ are just $K(x,y)\exp(-\omega|x-y|)$, which gives exponential decay pointwise from the $L^1 - L^\infty$ bound.

- Use the representation of $S_A(t)$ in term of $R_\lambda(A) = (\lambda+A)^{-1}$ to obtain that $S_A(t)$ maps L^1 to L^∞ and that its kernel verifies Gaussian bounds, as well as its derivatives.

We can summarize with the following proposition.

Proposition 16 ([1])
Let $K_A(x,y,t)$ be the kernel of the heat flow $S_A(t)$. There exists c depending only on the lower bound of $\mathrm{Re}a$ *and its L^∞ norm, such that*

$$|K_A(x,y,t)| \lesssim \frac{1}{\sqrt{t}} e^{c\frac{-|x-y|^2}{t}}, \tag{A.2}$$

$$|\partial_y K_A(x,y,t)| + |\partial_x K_A(x,y,t)| \lesssim \frac{1}{t} e^{c\frac{-|x-y|^2}{t}}, \tag{A.3}$$

and

$$|AK_A(x,y,t)| \lesssim \frac{1}{t^{\frac{3}{2}}} e^{c\frac{-|x-y|^2}{t}}. \tag{A.4}$$

Once we have all the Gaussian bounds, it becomes very easy to prove that $S_A(t)$ is continuous on L^p (from (A.2)), as well as Δ_j^A (from (A.4)). We are, in effect, reduced to the usual heat equation, with appropriate Bernstein type inequalities.

A.2 Equivalence of Besov norms

We first define Besov spaces using the A localization rather the usual one:

DEFINITION 7
Let f be in $\mathcal{S}'(\mathbb{R}^n)$, $s < 1$. We say f belongs to $\dot{B}^{s,q}_{p,A}$ if and only if

- *The partial sum $\sum_{-m}^{m} \Delta_j^A(f)$ converges to f as a tempered distribution (modulo constants if $s \geq 1/p, q > 1$).*
- *The sequence $\varepsilon_j = 2^{js}\|\Delta_j^A(f)\|_{L^p}$ belongs to l^q.*

Alternatively, one could replace the discrete sum with a continuous one, which is somewhat more appropriate when using the heat flow. Both can be proved to be equivalent, exactly as in the usual situation.
Now, our aim is to prove these spaces to be equivalent to the ones defined by Definition 2. In order to achieve this, we would like to estimate $\Pi_{jk} = \Delta_j^A \Delta_k$ and its adjoint. The adjoint can be dealt with by duality, so we focus on Π_{jk}: there are obviously 2 cases,

- when $j > k$, we write
$$\Pi_{jk} = 4^{-j} S_A(4^{-j}) \partial_x a(x) \partial_x \Delta_k,$$
which immediately yields, for any $1 \leq p \leq +\infty$,
$$\begin{aligned}\|\Pi_{jk} f\|_p &= 2^{-j}\|S_A(4^{-j}) 2^{-j} \partial_x a(x) \partial_x \Delta_k f\|_p \\ &\lesssim 2^{-j}\|a(x)\partial_x \Delta_k f\|_p \\ &\lesssim 2^{-j}\|\partial_x \Delta_k f\|_p \\ &\lesssim 2^{k-j}\|\Delta_k f\|_p,\end{aligned}$$
where we used the bound (A.3) on $S_A(1)\partial_x$.
- In the same spirit, when $k > j$,
$$\Pi_{jk} = 4^{-j} S_A\left(\frac{4^{-j}}{2}\right) S_A\left(\frac{4^{-j}}{2}\right) \partial_x (\partial_x)^{-1} \Delta_k,$$
and then
$$\begin{aligned}\|\Pi_{jk} f\|_p &\lesssim 2^{j}\|S_A(\frac{4^{-j}}{2}) 2^{-j} \partial_x (\partial_x)^{-1} \Delta_k f\|_p \\ &\lesssim 2^{j}\|(\partial_x)^{-1} \Delta_k f\|_p \\ &\lesssim 2^{j-k}\|\Delta_k f\|_p,\end{aligned}$$
where we used (again) the bound (A.3) on $S_A(1)\partial_x$.

Therefore,

Proposition 17
Let $|s| < 1$, $1 \leq p, q \leq +\infty$, then $\dot{B}_p^{s,q}$ and $\dot{B}_{p,A}^{s,q}$ are identical, with equivalence of norms.

Remark 9
We actually used repeatedly Besov spaces taking values in the separable Hilbert space L_t^2: as a matter of fact, one can reduce to the scalar case by projecting over an Hilbert basis, hence all the results can be translated to the Hilbert-valued situation as well.

Appendix B

Christ-Kiselev lemma

Introduction

This result, which is somewhat technical in nature, was proved in [20]. Its relevance in the context of PDEs was noticed shortly thereafter, see e.g. [61]. In this context, one may summarize it as follows (the original result is stronger and deals with a maximal function estimate). Consider and operator T defined on functions of one (time) variable by a kernel K:

$$Tf(t) = \int K(t,s)f(s)ds.$$

Assume moreover that T sends L^p_t to L^q_t.
Then, if one consider the restricted operator

$$T_< f(t) = \int_{s<t} K(t,s)f(s)ds,$$

it will be bounded as well from L^p to L^q, provided that $p < q$. Note that we had to deal several times with operators and situations like this one (f and Tf would be Banach valued but this turns out to be mostly irrelevant). For Strichartz estimates, we proved an estimate for a $TT^\star$ operator, and then observed that for the desired retarded estimate, the same proof holds, and then interpolation between various cases gave all we needed. There are situations which turn out to be a lot more complicated than this, and where having the Christ-Kiselev property turns out to be valuable. One particular example is when space and time norms are reversed, like for smoothing and maximal function estimates. This was observed in [53] and further exploited in [49], to bypass lenghty and non necessarily sharp arguments. In this appendix, we prove the versions of this result we need in the previous sections.

B.1 The main theorem

The proof is very much inspired from [20], but given we do not seek the maximal function estimate, we do not need to use a Whitney decomposition, which renders the argument more readable.

THEOREM 13

- *Let* $1 \leq \max(p,q) < r \leq +\infty$, *$B$ a Banach space, and T a bounded operator from $L^p(\mathbb{R}_y; L^q(\mathbb{R}_s))$ to $L^r(\mathbb{R}_t; B)$ with norm C:*

$$\|Tf\|_{L^r(\mathbb{R}_t;B)} \leq C\|f\|_{L^p(\mathbb{R}_y;L^q(\mathbb{R}_s))}.$$

  Let $K(y,s,t)$ be its kernel, and $K \in L^1_{loc}(\mathbb{R}^3_{y,s,t})$ taking values in the class of bounded operators on B. Define T_R to be the operator with kernel $\chi_{s<t}K(y,s,t)$,

$$T_R f = \int_{s<t} K(y,s,t)f(s,y)\,dyds.$$

  Then T_R is bounded from $L^p(\mathbb{R}_y; L^q(\mathbb{R}_s))$ to $L^r(\mathbb{R}_t; B)$, and

$$\|T_R f\|_{L^r(\mathbb{R}_t;B)} \leq \frac{C}{(1-2^{\frac{1}{r}-\frac{1}{\max(p,q)}})}\|f\|_{L^p(\mathbb{R}_y;L^q(\mathbb{R}_s))}.$$

- *If* $\max(p,q) < \min(\alpha,\beta)$ *and T is a bounded operator from the space $L^p(\mathbb{R}_y; L^q(\mathbb{R}_s))$ to $L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))$ with norm C. Let $K(y,s,x,t)$ be its kernel, and $K \in L^1_{loc}(\mathbb{R}^3_{y,s,x,t})$. Define T_R to be the operator with kernel $1_{s<t}K(y,s,x,t)$. Then T_R is bounded from $L^p(\mathbb{R}_y; L^q(\mathbb{R}_s))$ to $L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))$ and*

$$\|T_R f\|_{L^\alpha(\mathbb{R}_x;L^\beta(\mathbb{R}_t))} \leq \frac{C}{(1-2^{\frac{1}{\min(\alpha,\beta)}-\frac{1}{\max(p,q)}})}\|f\|_{L^p(\mathbb{R}_y;L^q(\mathbb{R}_s))}.$$

- *If T is a bounded operator from $\dot{B}^{0,2}_1(\mathcal{L}^2_t)$ to $L^4(\mathbb{R}_x; L^\infty(\mathbb{R}_t))$ with norm C. Let $K(y,s,x,t)$ be its kernel, and $K \in L^1_{loc}(\mathbb{R}^3_{y,s,x,t})$. Define T_R to be the operator with kernel $1_{s<t}K(y,s,x,t)$. Then T_R is bounded from $\dot{B}^{0,2}_1(L^2_t)$ to $L^4(\mathbb{R}_x; L^\infty(\mathbb{R}_t))$ with norm smaller than $C/(1-2^{-1/4})$.*

Proof: Let us start with the first case in Theorem 13. For any (smooth) function $f \in L^p(\mathbb{R}_y; L^q(\mathbb{R}_s))$ such that $\|f\|_{L^p(\mathbb{R}_y;L^q(\mathbb{R}_s))} = 1$, the function $F(t) = \|1_{s<t}f(s,y)\|^p_{L^p(\mathbb{R}_y;L^q(\mathbb{R}_s))}$ is an increasing function from $\mathbb{R}$ to $[0,1]$, and without loss of generality we can take it to be injective (hence, invertible). We have

Lemma 10
For any $f \in L^p(\mathbb{R}_y; L^q(\mathbb{R}_s))$, *such that* $\|f\|_{L^p(\mathbb{R}_y;L^q(\mathbb{R}_s))} = 1$,

$$\|1_{F^{-1}(]a,b[)} f\|_{L^p(\mathbb{R}_y;L^q(\mathbb{R}_s))} \leq C|b-a|^{\frac{1}{\max(p,q)}} \tag{B.1}$$

Indeed denote by $]t_a, t_b[= F^{-1}(]a,b[)$ and

$$G(t,x) = \left(\int_{s<t} |f(s,x)|^q ds\right)^{\frac{1}{q}}$$

1. If $p \geq q$, using that for $a, b \geq 0$ we have $(a+b)^{p/q} \geq a^{p/q} + b^{p/q}$ we obtain

$$\begin{aligned}
\|1_{F^{-1}(]a,b[)} f\|^p_{L^p(\mathbb{R}_y;L^q(\mathbb{R}_s))} &= \int_x \left(\int_{t_a \leq s \leq t_b} |f(s,x)|^q ds\right)^{\frac{p}{q}} dx \\
&= \int_x \left(\int_{s \leq t_b} |f(s,x)|^q ds - \int_{s \leq t_a} |f(s,x)|^q ds\right)^{\frac{p}{q}} dx \\
&\leq \int_x \left(\int_{s \leq t_b} |f(s,x)|^q ds\right)^{\frac{p}{q}} \\
&\qquad - \left(\int_{s \leq t_a} |f(s,x)|^q ds\right)^{\frac{p}{q}} dx \\
&\leq F(t_b) - F(t_a) = b - a
\end{aligned}$$

2. If $p \leq q$, using that for $x, y \geq 0$, $(x^q - y^q) \leq \frac{q}{p}(x^p - y^p)(\max(x,y)^{q-p}$, we obtain

$$\begin{aligned}
&\|1_{F^{-1}(]a,b[)} f\|^p_{L^p(\mathbb{R}_y;L^q(\mathbb{R}_s))} = \int_x \left(\int_{t_a \leq s \leq t_b} |f(s,x)|^q ds\right)^{\frac{p}{q}} dx \\
&= \int_x (G(t_b,x)^q - G(t_a,x)^q)^{\frac{p}{q}} dx \\
&\leq \frac{p}{q} \int_x (G(t_b,x)^p - G(t_a,x)^p)^{\frac{p}{q}} \left(G(t_b,x)^{q-p}\right)^{\frac{p}{q}} dx \\
&\leq \frac{p}{q} \left(\int_x G(t_b,x)^p - G(t_a,x)^p dx\right)^{\frac{p}{q}} \left(\int_x G(t_b,x)^{(q-p)\frac{q}{(q-p)}} dx\right)^{\frac{q-p}{q}} \\
&\leq \frac{p}{q} (b-a)^{\frac{p}{q}} \|f\|^{\frac{q-p}{q}}_{L^p(\mathbb{R}_y;L^q(\mathbb{R}_s))} \leq \frac{p}{q} (b-a)^{\frac{p}{q}}.
\end{aligned}$$

Consider now the (level set) dyadic decomposition of the real axis given by

$$\mathbb{R} =]-\infty, t_{n,1}] \cup]t_{n,1}, t_{n,2}] \cup \cdots \cup]t_{n,2^n-1}, +\infty[= \cup_{j=1}^{2^n} I_j$$

such that

$$\|f\|^{r'}_{L^{r'}(]t_{n,j},t_{n,j+1}];B)} = 2^{-n}$$

with the convention $t_{n,0} = -\infty$ and $t_{n,2^{n+1}} = +\infty$. Remark that $F(t_{n,j}) = j2^{-n}$ is the usual dyadic decomposition of the interval $[0,1[$. We have

$$1_{s<t} = \sum_{n=1}^{+\infty} \sum_{j=1}^{2^{n-1}} 1_{(s,t)\in Q_{n,j}}$$

where $(s,t) \in Q_{n,j} \Leftrightarrow (F(s), F(t)) \in \widetilde{Q}_{n,j}$ and $\widetilde{Q}_{n,j}$ is as in Figure B.1.

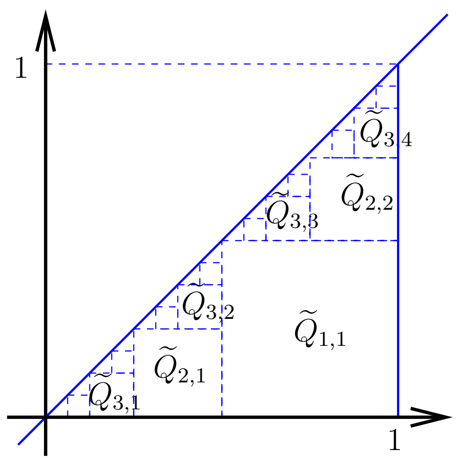

Figure B.1: Decomposition of a triangle as a union of squares

Remark that $1_{(s,t)\in Q_{n,j}} = 1_{t\in I_{n,j}} 1_{s\in I'_{n,j}}$ for suitable dyadic intervals $I_{n,j}$ and $I'_{n,j}$.
We are now ready to prove the main estimate, by rewriting T_R as follows:

$$\|T_R f\|_{L^r(\mathbb{R}_t;B)} = \|\sum_n \sum_{j=1}^{2^{n-1}} T_{n,j} f\|_{L^r(\mathbb{R}_t;B)}$$

where the kernel of the operator $T_{n,j}$ is equal to $K(y,s,t) \times 1_{(s,t)\in Q_{n,j}}$. Consequently $T_{n,j}$ is (uniformly) bounded from $L^{r'}(\mathbb{R}_t;B)$ to $L^{p'}(\mathbb{R}_y;L^{q'}(\mathbb{R}_s))$ with norm smaller than C.

Since $p' \geq q'$ and for fixed n, the functions $T_{n,j}f$ have disjoint support (in the variable t) we have

$$
\begin{aligned}
\|T_R f\|_{L^r(\mathbb{R}_t;B)} &\leq C \sum_n \Big(\sum_{j=1}^{2^{n-1}} \|1_{s\in]t_{n,j},t_{n,j+1}[} f\|^r_{L^p(\mathbb{R}_y;L^q(\mathbb{R}_s))} \Big)^{\frac{1}{r}} \\
&\leq C \sum_n \Big(\sum_{j=1}^{2^{n-1}} 2^{-\frac{nr}{\max(p,q)}} \Big)^{\frac{1}{r}} = (1 - 2^{\frac{1}{r} - \frac{1}{\max(p,q)}})^{-1}.
\end{aligned}
$$

We now study the second case in Theorem B. The proof relies on

Lemma 11
Assume that $(f_k)_{k\in\mathbb{N}}$ have disjoint supports in t. Then

$$
\| \sum_k f_k \|_{L^\alpha(\mathbb{R}_x;L^\beta(\mathbb{R}_t))} \leq \Big(\sum_k \|f_k\|^{\min(\alpha,\beta)}_{L^\alpha(\mathbb{R}_x;L^\beta(\mathbb{R}_t))} \Big)^{\frac{1}{\min(\alpha,\beta)}}
$$

We distinguish two cases:

- $\beta \geq \alpha$

$$
\| \sum_k f_k \|_{L^\alpha(\mathbb{R}_x;L^\beta(\mathbb{R}_t))} = \Big(\int_x \Big(\sum_k \int_t |f_k|(t,x)^\beta dt \Big)^{\alpha/\beta} dx \Big)^{1/\alpha}
$$

  but since $\alpha \leq \beta$, we have $(\sum_k a_k)^{\alpha/\beta} \leq \sum_k a_k^{\alpha/\beta}$ and we obtain

$$
\| \sum_k f_k \|_{L^\alpha(\mathbb{R}_x;L^\beta(\mathbb{R}_t))} \leq \Big(\int_x \Big(\sum_k \int_t |f_k|(t,x)^\beta dt \Big)^{\alpha/\beta} dx \Big)^{1/\alpha}
$$

- $\beta \leq \alpha$

$$
\begin{aligned}
\| \sum_k f_k \|_{L^\alpha(\mathbb{R}_x;L^\beta(\mathbb{R}_t))} &= \Big\| \Big(\sum_k \int_t |f_k|^\beta \Big) \|^{1/\beta}_{L_x^{\alpha/\beta}} \\
&\leq \Big(\sum_k \| \Big(\int_t |f_k|^\beta \Big) \|_{L_x^{\alpha/\beta}} \Big)^{\frac{1}{\beta}} \\
&\leq \Big(\sum_k \|f_k\|^\beta_{L^\alpha(\mathbb{R}_x;L^\beta(\mathbb{R}_t))} \Big)^{\frac{1}{\beta}}
\end{aligned}
$$

To prove the second case in Theorem B, we use the same dyadic decomposition of $\mathbb{R}$ as before and use Lemma 11 to estimate $\|T_R f\|_{L^\alpha_x, L^\beta_t}$. This gives

$$\|T_R f\|_{L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))} \le \sum_n \Big(\sum_{j=1}^{2^{n-1}} \|T_{n,j} f\|^{\min(\alpha,\beta)}_{L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))}\Big)^{\frac{1}{\min(\alpha,\beta)}}$$

and we conclude as in the previous case.
Finally, to prove the last case in Theorem B, we need to combine Lemma 11 with $\alpha = 4, \beta = +\infty$ to deal with the $L^4(\mathbb{R}_x; L^\infty(\mathbb{R}_t))$ norm with a choice of a suitable dyadic decomposition and prove the analog of Lemma 10 for the Besov space $\dot{B}^{0,2}_1(\mathbb{R}_x)$. The dyadic decomposition is based on

$$F(t) = \sum_j \|\Big(\int_{s<t} |\Delta_j f(s)|^2 ds\Big)^{1/2}\|^2_{L^1_x} = \sum_j \gamma_j(t)^2.$$

LEMMA 12
For any function f such that $\|f\|_{B^{0,2}_1(\mathcal{L}^2_t)} = \Big(\sum_j \|\Delta_j f\|^2_{L^1_x; L^2_t}\Big)^{1/2} = 1$ we have $\|1_{F^{-1}(]a,b[)} f\|_{B^{0,2}_1(\mathcal{L}^2_t)} \le C(b-a)^{\frac{1}{2}}$.

Proof: Denote by $J_j(t,x) = (\int_{s<t} |\Delta_j f(s)|^2\, ds)^{\frac{1}{2}}$. Then (using $2 \ge 1$)

$$\text{(B.2)}\quad \begin{aligned} \|\Delta_j \chi_{F^{-1}(I)}(s) f(s)\|^2_{L^2_t} &= \int_{t_a}^{t_b} |\Delta_j f(s)|^2\, ds = J_j(t_b,x)^2 - J_j(t_a,x)^2, \\ &\le (J_j(t_b,x) - J_j(t_a,x))(J_j(t_b,x) + J_j(t_a,x)). \end{aligned}$$

Then we add the L^1_x norm, to get (using Cauchy-Schwarz at the second line)

(B.3)
$$\begin{aligned} \int_x \|\Delta_j \chi_{F^{-1}(I)}(s) f(s)\|_{L^2_t}\, dx &\lesssim \int_x (J_j(t_b,x) - J_j(t_a,x))^{\frac{1}{2}} (J_j(t_b,x) + J_j(t_a,x))^{\frac{1}{2}}\, dx \\ &\lesssim \Big(\int_x J_j(t_b,x) - J_j(t_a,x)\, dx\Big)^{\frac{1}{2}} \Big(\int_x (J_j(t_b,x) + J_j(t_a,x))\, dx\Big)^{\frac{1}{2}} \end{aligned}$$

and consequently

(B.4)
$$\begin{aligned} \sum_j \Big(\int_x \|\Delta_j \chi_{F^{-1}(I)}(s) f(s)\|_{L^2_t}\, dx\Big)^2 &\lesssim \sum_j (\gamma_j(t_b) - \gamma_j(t_a))(\gamma_j(t_b) + \gamma_j(t_a)) \\ &\lesssim \sum_j (\gamma_j^2(t_b) - \gamma_j^2(t_a)) = F(t_b) - F(t_a) = |I|. \end{aligned}$$

The rest of the proof of Theorem B is as in the previous cases. $\square$

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On the blow up phenomenon for the L^2 critical non linear Schrödinger Equation

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The aim of these notes is to provide a self contained presentation of recent developments concerning the singularity formation for the L^2 critical non linear Schrödinger equation

$$(NLS) \quad \begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}}u, & (t,x) \in [0,T) \times \mathbf{R}^N \\ u(0,x) = u_0(x), & u_0 : \mathbf{R}^N \to \mathbf{C} \end{cases} \tag{1}$$

with $u_0 \in H^1 = \{u, \nabla u \in L^2(\mathbf{R}^N)\}$ in dimension $N \geq 1$. This equation for $N = 2$ appears in physics as a universal model to describe self trapping of waves propagating in non linear media. The physical expectation for large smooth data is the concentration of part of the L^2 mass in finite time corresponding to the focusing of the laser beam. If some explicit examples of this phenomenon are known, and despite a number of both numerical and mathematical works, a general description of blow up dynamics is mostly open.

(NLS) is an infinite dimensional Hamiltonian system with energy space H^1 without any space localization property. It is in this context together with the critical generalized KdV equation the only example where blow up is known to occur. For (NLS), an elementary proof of existence of blow up solutions is known since the 60's but is based on energy constraints and is not constructive. In particular, *no qualitative information of any type on the blow up dynamics is obtained this way.*

The natural questions we address regarding blow up dynamics in the energy space are the following:
-Does there exist a Hamiltonian characterization of blow up solutions, or at least necessary conditions for blow up simply expressed from the Hamiltonian invariants?
-Assuming blow up, does there exist a universal blow up speed, or are there several possible regimes? Among these regimes, which ones are stable?
-Does there exist a universal space time structure for the formation of singularities independent at the first order of the initial data?
We will present precise answers to these issues in the setting of a perturbative analysis close to the exceptional solution to (1): the ground state solitary wave.

These notes are organized as follows. In a first section, we recall main standard results about non linear Schrödinger equations. In the second section, we focus onto the critical blow up problem and recall the few known results in the field. The next section is devoted to an exposition of the recent results obtained in collaboration with F.Merle in [20], [21], [22], [23], [24] and [32]. In the last section, we present a detailed proof of the first of these results which is the exhibition of a sharp upper bound on blow up rate for a suitable class of initial data. We expect the presentation to be essentially self contained provided the prior knowledge of standard tools in the study of non linear PDE's.

1 Hamiltonian structure and global well posedness

In this section, we recall main classical facts regarding the global well posedness in the energy space of non linear Schrödinger equations. We will also introduce one of the fundamental objects for the study of (1): the ground state solitary wave.

1.1 Local wellposedness, symmetries and Hamiltonian structure

Let us consider the general non linear Schrödinger equation:

$$\left\{ \begin{array}{l} iu_t = -\Delta u - |u|^{p-1}u \\ u(0,x) = u_0(x) \in H^1 \end{array} \right. \tag{2}$$

with

$$1 < p < +\infty \ \text{ for } \ N = 1, 2, \ \ 1 < p < 2^* - 1 \ \text{ for } \ N \geq 3, \tag{3}$$

where $2^* = \frac{2N}{N-2}$ is the Sobolev exponent. The first fundamental question arising when dealing with a non linear PDE like (2) is the existence of a solution locally in time in the given Cauchy space which we have chosen here to be the energy space H^1. This type of results relies on the theory of oscillatory integrals and the well known Strichartz estimates for the propagator $e^{it\Delta}$ of the linear group. Local well posedness of (2) in H^1 is in this frame a well known result of Ginibre, Velo, [8]. See also [10]. Thus, for $u_0 \in H^1$, there exists $0 < T \leq +\infty$ such that $u(t) \in \mathcal{C}([0,T), H^1)$. Moreover, the life time of the solution can be proved to be lower bounded by a function depending on the H^1 size of the solution only, $T(u_0) \geq f(\|u_0\|_{H^1})$. A corollary of these techniques is the global wellposedness for small data in H^1. The idea is that small data remain small through the iterative scheme used to construct the solution which may thus be continued up to any arbitrary time.

On the contrary, for large H^1 data, three possibilities may occur:
(i) $T = +\infty$ and $\limsup_{t\to+\infty} |u(t)|_{H^1} < +\infty$, we say the solution is global and bounded.
(ii) $T = +\infty$ and $\limsup_{t\to+\infty} |u(t)|_{H^1} = +\infty$, we say the solution blows up in infinite time.
(iii) $0 < T < +\infty$, but then from local well posedness theory:

$$|u(t)|_{H^1} \to +\infty \ \text{ as } \ t \to T,$$

we say the solution blows up finite time.

To prove global posedness of the solution, it thus suffices to control the size of the solution in H^1. This is achieved in some cases thanks to the *Hamiltonian structure* of (2). Indeed, (2) admits the following invariants in H^1:

- L^2-norm:
$$\int |u(t,x)|^2 = \int |u_0(x)|^2; \tag{4}$$

- Energy:
$$E(u(t,x)) = \frac{1}{2}\int |\nabla u(t,x)|^2 - \frac{1}{p+1}\int |u(t,x)|^{p+1} = E(u_0); \tag{5}$$

- Momentum:
$$Im\left(\int \nabla u\overline{u}(t,x)\right) = Im\left(\int \nabla u_0\overline{u_0}(x)\right). \tag{6}$$

Note that the growth condition on the non linearity (3) ensures from Sobolev embedding that the energy is well defined, and this is why H^1 is referred to as the energy space.

From Ehrenfest law, these invariants are related to the group of symmetry of (2) in H^1:

- Space-time translation invariance: if $u(t,x)$ solves (2), then so does $u(t+t_0, x+x_0)$, $t_0 \in \mathbf{R}$, $x_0 \in \mathbf{R}^N$.
- Phase invariance: if $u(t,x)$ solves (2), then so does $u(t,x)e^{i\gamma}$, $\gamma \in \mathbf{R}$.
- Scaling invariance: if $u(t,x)$ solves (2), then so does $\lambda^{\frac{2}{p-1}}u(\lambda^2 t, \lambda x)$, $\lambda > 0$.
- Galilean invariance: if $u(t,x)$ solves (2), then so does $u(t, x-\beta t)e^{i\frac{\beta}{2}(x-\frac{\beta}{2}t)}$, $\beta \in \mathbf{R}^N$.

Let us point out that this group of H^1 symmetries is the same like for the linear Schrödinger equation.

As an outcome, we have the following result:

Theorem 1 (Global wellposedness in the subcritical case) *Let $N \geq 1$ and $1 < p < 1 + \frac{4}{N}$, then all solutions to (2) are global and bounded in H^1.*

Proof of Theorem 1

The proof is elementary and relies on the Hamiltonian structure and the Gagliardo-Nirenberg interpolation inequality. Indeed, let $u_0 \in H^1$, $u(t)$ the corresponding solution to (2) with $[0,T)$ its maximum time interval existence in H^1, we then have the a priori estimate: there exists $C(u_0)>0$ such that,

$$\forall t\in[0,T),\quad |\nabla u(t)|_{L^2}\leq C(u_0). \tag{7}$$

From the conservation of the L^2 norm, we conclude: $\forall t\in[0,T)$, $|u(t)|_{H^1}\leq C(u_0)$, and this uniform bound on the solution implies global well posedness from the local Cauchy theory in H^1.
It remains to prove (7) which is a consequence of the conservation of the energy and the Gagliardo-Nirenberg interpolation estimate: let $N=1,2$ and $1\leq p<+\infty$ or $N\geq 3$ and $1\leq p\leq 2^*-1$, then there holds for some universal constant $C(N,p)>0$,

$$\forall v\in H^1,\quad \int |v|^{p+1}\leq C(N,p)\left(\int|\nabla v|^2\right)^{\frac{N(p-1)}{4}}\left(\int|v|^2\right)^{\frac{p+1}{2}-\frac{N(p-1)}{4}}. \tag{8}$$

Applying this with $v=u(t)$, we get from the conservation of the energy and the L^2 norm:

$$\forall t\in[0,T),\quad E_0\geq\frac{1}{2}[\int|\nabla v|^2-C(u_0)\left(\int|\nabla v|^2\right)^{\frac{N(p-1)}{4}}],$$

from which (7) follows from subcriticality assumption $1\leq p<1+\frac{4}{N}$. This concludes the proof of Theorem 1.

The physical meaning of Theorem 1 is that for waves propagating in a too weakly focusing medium, the potential term in the energy is dominated by the kinetic term according to (8) and no focusing can occur. A critical exponent arises from this analysis for which these two terms balance exactly, and we shall concentrate from now on on this case alone which is referred to as *the critical case*:

$$p=1+\frac{4}{N}.$$

1.2 Minimizers of the energy

The criticality of equation (1) may be understood from the exact balance in this case between the kinetic and the potential energy. This may be quantified in a sharp way from the knowledge of the exact constant in the Gagliardo-Nirenberg inequality (8).

Theorem 2 (Minimizers of the energy) *Let the H^1 functional:*

$$J(v)=\frac{(\int|\nabla v|^2)(\int|v|^2)^{\frac{2}{N}}}{\int|v|^{2+\frac{4}{N}}}. \tag{9}$$

The minimization problem

$$\min_{v\in H^1,\quad v\neq 0} J(v)$$

is attained on the three parameters family:

$$\lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0) e^{i\gamma_0}, \quad (\lambda_0, x_0, \gamma_0) \in \mathbf{R}_*^+ \times \mathbf{R}^N \times \mathbf{R},$$

where Q is the unique positive radial solution to the system:

$$\left\{ \begin{array}{l} \Delta Q - Q + Q^{1+\frac{4}{N}} = 0 \\ Q(r) \to 0 \quad as \quad r \to +\infty. \end{array} \right. \tag{10}$$

In particular, there holds the following Gagliardo-Nirenberg inequality with best constant:

$$\forall v \in H^1, \quad E(v) \geq \frac{1}{2} \int |\nabla v|^2 \left(1 - \left(\frac{|v|_{L^2}}{|Q|_{L^2}} \right)^{\frac{4}{N}} \right). \tag{11}$$

The existence of a positive solution to (10) is a result obtained from the theory of calculus of variations by Berestycki-Lions, [1], and lies within the range of the concentration compactness techniques introduced by P.L Lions at the beginning of the 80's, see [13], [14]. An ODE type of approach is also available from [2]. The fact that the positive solution to (10) is necessarily radial is a deep and general result by Gidas, Ni, Nirenberg, [7]. Uniqueness of the ground state in the ODE sense is a result by Kwong, [11]. Last, the fact that the minimization problem is attained is due to Weinstein, [37].

From standard elliptic theory, the ground state Q is C^3_{loc} and exponentially decreasing at infinity in space:

$$Q(r) \leq e^{-C(N)r},$$

and one should think of Q as a smooth well localized bump. In dimension $N = 1$, equation (10) may even be integrated explicitly for:

$$Q(x) = \left(\frac{3}{\mathrm{ch}^2(x)} \right)^{\frac{1}{4}}.$$

In higher dimension on the contrary, equation (10) admits excited solutions $(Q_i)_{i \geq 1}$ with growing L^2 norm: $|Q_i|_{L^2} \to +\infty$ as $i \to +\infty$.

A reformulation of (11) is the following variational characterization of Q which we will mostly use:

Proposition 1 (Variational characterization of the ground state) *Let $v \in H^1$ such that $\int |v|^2 = \int Q^2$ and $E(v) = 0$, then*

$$v(x) = \lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0) e^{i\gamma_0},$$

for some parameters $\lambda_0 \in \mathbf{R}_+^*$, $x_0 \in \mathbf{R}^N$, $\gamma_0 \in \mathbf{R}$.

To sum up, the situation is as follows: let $v \in H^1$, then if $|v|_{L^2} < |Q|_{L^2}$ ie for "small" v, the kinetic energy dominates the potential energy and (11) yields $E(v) > C(v) \int |\nabla v|^2$ and the energy is in particular non negative; at the critical mass level $|v|_{L^2} = |Q|_{L^2}$, the only zero energy function, ie for which the kinetic and the potential energies exactly balance, is Q up to the symmetries of scaling, phase and translation which generate the three dimensional manifold of minimizers of (9).

A fundamental generalization of Theorem 1 has been obtained by Weinstein [37]:

Theorem 3 (Global well posedness for subcritical mass) *Let $u_0 \in H^1$ with $|u_0|_{L^2} < |Q|_{L^2}$, the corresponding solution $u(t)$ to (1) is global and bounded in H^1.*

Proof of Theorem 3

As for the proof of Theorem 1, it suffices from local well posedness theory to prove a priori estimate (7). But from the conservation of the L^2 norm, $|u(t)|_{L^2} < |Q|_{L^2}$ for all $t \in [0,T)$, and (7) follows from the conservation of the energy and the sharp Gagliardo-Nirenberg inequality (11) applied to $v = u(t)$. This concludes the proof of Theorem 3.

2 General blow up results

Our aim in this section is to recall some known blow up criterions and qualitative properties of the blow up solutions. On the contrary to the results in the preceding section which could be extended to more general non linearities, we shall now focus onto the very specific algebraic structure of (1).

2.1 Solitary waves and the critical mass blow up

Weinstein's criterion for global solutions given by Theorem 3 is sharp. On the one hand, from (10),

$$W(t,x) = Q(x)e^{it}$$

is a solution to (1) with critical mass $|W|_{L^2} = |Q|_{L^2}$. Note that W keeps its shape in time and *does not disperse*. It is the minimal object in L^2 sense for which dispersion -measured by the kinetic term- and concentration -measured by the potential term- exactly compensate. This exceptional solution is called the ground state solitary wave. H^1 symmetries of (1) generate in fact a three parameter family of solitary waves:

$$W_{\lambda_0,x_0,\gamma_0}(t,x) = \lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0) e^{i(\gamma_0 + \lambda_0^2 t)}, \quad (\lambda_0, x_0, \gamma_0) \in \mathbf{R}_*^+ \times \mathbf{R}^N \times \mathbf{R}. \tag{12}$$

Now a fundamental remark is the following: in the critical case $p = 1 + \frac{4}{N}$,

all H^1 symmetries of (1) are L^2 isometries.

This is why (1) is called L^2 critical. All the solitary waves (12) thus have critical L^2 mass:

$$|W_{\lambda_0,x_0,\gamma_0}|_{L^2} = |Q|_{L^2}.$$

Moreover, from explicit computation and $E(Q) = 0$, $Im(\int \nabla Q\overline{Q}) = 0$, we have:

$$E(W_{\lambda_0,x_0,\gamma_0}) = 0, \quad Im(\int \nabla W_{\lambda_0,x_0,\gamma_0}\overline{W_{\lambda_0,x_0,\gamma_0}}) = 0.$$

In other words, the L^2 criticality of the equation implies the existence of a three parameters family of solitary waves with arbitrary size in H^1 but frozen Hamiltonian invariants. The consideration of these invariants only is thus no longer enough to estimate the size of the solution nor to separate within these different solitary waves.

In general, the L^2 scaling invariance of the solitary waves is a known criterion of instability, see [35]. In our case, it may be precised by exhibiting an *explicit blow up solution*. Existence of this object is based on the pseudo-conformal symmetry of (1) which is not in the energy space H^1 but in the so called virial space:

$$\Sigma = H^1 \cap \{xu \in L^2\},$$

and which writes: if $u(t,x)$ is a solution to (1), then so is

$$v(t,x) = \frac{1}{|t|^{\frac{N}{2}}}\overline{u}(\frac{1}{t}, \frac{x}{t})e^{i\frac{|x|^2}{4t}}. \tag{13}$$

An equivalent but more enlightening way of seeing this symmetry is the following: for any parameter $a \in \mathbf{R}$, the solution to (1) with initial data $v_a(0,x) = u(0,x)e^{ia\frac{|x|^2}{4}}$ is

$$v_a(t,x) = \frac{1}{(1+at)^{\frac{N}{2}}}u(\frac{t}{1+at}, \frac{x}{1+at})e^{ia\frac{|x|^2}{4(1+at)}}. \tag{14}$$

Note that this symmetry is also a symmetry of the linear equation. Nevertheless, the fundamental difference between the linear and the non linear equation is that all solutions to the linear equation are dispersive and go to zero when time evolves for example in L^2_{loc}, whereas the non linear problem admits non dispersive solutions: the solitary waves. *The pseudo-conformal transformation applied to the non dispersive solution now yields a finite time blow up solution*:

$$S(t,x) = \frac{1}{|t|^{\frac{N}{2}}}Q(\frac{x}{t})e^{-i\frac{|x|^2}{4t}+\frac{i}{t}}. \tag{15}$$

This solution should be viewed at the solution to (1) with Cauchy data at $t = -1$:

$$S(-1,x) = Q(x)e^{i\frac{|x|^2}{4}-i}.$$

It blows up at time $T = 0$ with the following explicit properties:

- First observe that pseudo-conformal symmetry (13) is again an L^2 isometry. Thus $|S|_{L^2} = |Q|_{L^2}$ and global wellposedness criterion given by Theorem 3 is sharp.

- From explicit computation, S has non negative energy:

$$E(S) > 0. \tag{16}$$

- The blow up speed, measured by the L^2 norm of the gradient -as the L^2 norm itself is conserved-, is given by:

$$|\nabla u(t)|_{L^2} \sim \frac{1}{|t|}. \tag{17}$$

- The solution leaves the Cauchy space H^1 by forming a Dirac mass in L^2:

$$|S(t)|^2 \rightharpoonup \left(\int Q^2 \right) \delta_{x=0} \quad \text{as} \quad t \to 0. \tag{18}$$

Like the solitary wave is a non dispersive global solution, $S(t)$ is a non dispersive blow up solution in the sense that *it accumulates all its L^2 mass into blow up*: no L^2 mass is lost in the focusing process. This property should be understood as a fundamental feature of a critical mass blow up solution, and indeed the critical mass blow up dynamic is very constrained according to the following fundamental classification result by F. Merle, [17]:

Theorem 4 (Uniqueness of the critical mass blow up solution) *Let $u_0 \in H^1$ with $|u_0|_{L^2} = |Q|_{L^2}$, and assume that the corresponding solution $u(t)$ to (1) blows up in finite time $0 < T < +\infty$. Then*

$$u(t) = S(t - T)$$

up to the H^1 symmetries.

2.2 Blow up for large data: the virial identity

Let us now consider super critical mass initial data $u_0 \in H^1$ with $|u_0|_{L^2} > |Q|_{L^2}$, and ask the question of the existence of finite time blow up solutions. The answer is surprisingly simple in the case when the virial law applies. This identity first derived by Zakharov and Shabat, [38], is a consequence of the pseudo-conformal symmetry. Let a data in the virial space $u_0 \in \Sigma$, then the corresponding solution $u(t)$ to (1) on $[0, T)$ satisfies:

$$\forall t \in [0, T), \quad u(t) \in \Sigma \quad \text{and} \quad \frac{d^2}{dt^2} \int |x|^2 |u(t)|^2 = 16 E_0. \tag{19}$$

Let us now observe that if from (11) subcritical mass functions have non negative energy, the sign of the energy is no longer prescribed for super critical mass functions. For example, an explicit computation ensures $\frac{d}{d\eta} E\left((1+\eta)Q\right)_{|\eta=0} < 0$, and from $E(Q) = 0$, any neighborhood of Q in H^1 contains data with non positive energy. Let then $u_0 \in \Sigma$ with

$E_0 < 0$, then from virial law (19) and the conservation of the energy, the positive quantity $\int |x|^2|u(t)|^2$ is an inverted parabola which must thus become negative in finite time, and therefore the solution cannot exist for all time and blows up in finite time $0 < T < +\infty$. Note that this argument can be generalized to the energy space via an H^1 regularization of (19), and we have the following:

Theorem 5 (Virial blow up for $E_0 < 0$) *Let $u_0 \in H^1$ with*

$$E_0 < 0,$$

then:
(i) Ogawa, Tsutsumi, [29]: if $N = 1$, then $0 < T < +\infty$.
(ii) Nawa, [28]: if $N \geq 2$ and u_0 is radial, then $0 < T < +\infty$.

This blow up argument is extraordinary for at least two reasons:
(i) It provides a blow up criterion based on a pure Hamiltonian information $E_0 < 0$ which applies to arbitrarily large initial data in H^1. In particular, it exhibits an open region of the energy space where blow up is known to be a stable phenomenon.
(ii) This argument also applies in Σ to the super critical case $1 + \frac{4}{N} < p < 2^* - 1$ where it is essentially the only known blow up result.

Now this argument has two major weaknesses:
(i) It heavily relies on the pseudo-conformal symmetry, and thus is unstable by perturbation of the equation.
(ii) More fundamentally, *this argument is purely obstructive and says nothing on the singularity formation.*

2.3 The L^2 concentration phenomenon

In the general setting, little is known regarding the description of the singularity formation. This is mainly a consequence of the fact that the virial blow up argument does not provide any insight into the blow up dynamics. Nevertheless, a general result of L^2 concentration obtained by Merle, Tsutsumi, [25], in the radial case, and generalized by Nawa, [28], provides a first description of the singularity formation: at blow up time, the solution leaves the Cauchy space H^1 by forming a Dirac mass in L^2.

Theorem 6 (L^2 concentration phenomenon) *Let $u_0 \in H^1$ such that the corresponding solution $u(t)$ to (1) blows up in finite time $0 < T < +\infty$. Then there exists some continuous function of time $x(t) \in \mathbf{R}^N$ such that:*

$$\forall R > 0, \quad \liminf_{t \to T} \int_{|x - x(t)| \leq R} |u(t,x)|^2 dx \geq \int Q^2. \tag{20}$$

To enlighten the meaning of this theorem, let us first recall from Cazenave, Weissler [6], that the Cauchy problem for (1) is locally wellposed in L^2. This space is the scaling invariant Sobolev space for (1), and in some sense the lowest one in term of Sobolev regularity for which local wellposedness can be expected. In this case, the lifetime of the solution cannot be lower bounded by a function of the size of the data only -which is conserved by the flow-, but fundamentally depends on the full initial profile. In this sense, to understand the way the solution may leave the Cauchy space in the critical space L^2 is a fundamental -and difficult- problem. Theorem 6 implies that a blow up data in H^1 leaves also L^2 when it leaves H^1, and thus the lifetimes in L^2 and H^1 are the sames, and this is done by creating a Dirac mass in L^2 with a *minimal universal amount of mass*.

Proof of Theorem 6

We prove the result in the radial case for $N \geq 2$. The general case follows from concentration compactness techniques, see [27].
Let $u_0 \in H^1$ radial and assume that the corresponding solution $u(t)$ to (1) blows up at time $0 < T < +\infty$, or equivalently:

$$\lim_{t \to T} |\nabla u(t)|_{L^2} = +\infty. \tag{21}$$

We need to prove (20) and argue by contradiction: assume that for some $R > 0$ and $\varepsilon > 0$, there holds on some sequence $t_n \to T$,

$$\lim_{n \to +\infty} \int_{|y| \leq R} |u(t_n, y)|^2 \leq \int Q^2 - \varepsilon. \tag{22}$$

Let us rescale the solution by its size and set:

$$\lambda(t_n) = \frac{1}{|\nabla u(t_n)|_{L^2}}, \quad v_n(y) = \lambda^{\frac{N}{2}}(t_n) u(t_n, \lambda(t_n) y),$$

then from explicit computation:

$$|\nabla v_n|_{L^2} = 1 \quad \text{and} \quad E(v_n) = \lambda^2(t_n) E(u). \tag{23}$$

First observe that v_n is H^1 bounded and we may assume on a sequence $n \to +\infty$:

$$v_n \rightharpoonup V \quad \text{in} \quad H^1.$$

We first claim that V is non zero. Indeed, from (21), (23) and the conservation of the energy for $u(t)$, $E(v_n) \to 0$ as $n \to +\infty$, and thus:

$$\frac{1}{2 + \frac{4}{N}} \int |v_n|^{2+\frac{4}{N}} = \frac{1}{2} \int |\nabla v_n|^2 - E(v_n) = \frac{1}{2} - E(v_n) \to \frac{1}{2} \quad \text{as} \quad n \to +\infty.$$

Now from compact embedding of H^1_{radial} to $L^{2+\frac{4}{N}}$, $v_n \to V$ in $L^{2+\frac{4}{N}}$ up to a subsequence, and thus $\frac{1}{2+\frac{4}{N}}\int |V|^{2+\frac{4}{N}} \geq \frac{1}{2}$ and V is non zero. Moreover, from weak H^1 convergence and strong $L^{2+\frac{4}{N}}$ convergence,

$$E(V) \leq \liminf_{n\to+\infty} E(v_n) = 0.$$

Last, we have from (21), (22) and weak H^1 convergence: $\forall A > 0$

$$\begin{aligned}\int_{|y|\leq A} |V(y)|^2 dy &\leq \liminf_{n\to+\infty} \int_{|y|\leq A} |v_n(y)|^2 dy \leq \lim_{n\to+\infty} \int_{|y|\leq \frac{R}{\lambda(t_n)}} |v(t_n,y)|^2 dy \\ &= \lim_{n\to+\infty} \int_{|x|\leq R} |u(t_n,x)|^2 dx \leq \int Q^2 - \varepsilon.\end{aligned}$$

Thus $\int |V|^2 \leq \int Q^2 - \varepsilon$, what together with V non zero and $E(V) \leq 0$ contradicts sharp Gagliardo-Nirenberg inequality (11). This concludes the proof of Theorem 6.

Remark 1 *Above argument is fundamentally based on the Hamiltonian structure of the equation in H^1. If one restricts itself to pure L^2 data which is a much more difficult situation, a similar concentration result has been proved in dimension $N = 2$ by Bourgain, [3], and then precised by Merle, Vega, [26]. Nevertheless, to obtain in L^2, or even in H^s, $0 \leq s < 1$, the sharp constant $\int Q^2$ of minimal focused mass is open.*

Remark 2 *The non radial case in H^1 is handled by Nawa in [27] using standard concentration compactness techniques to overcome the non compact injection of H^1 into $L^{2+\frac{4}{N}}$. Further refined use of these techniques has also allowed Nawa to precise the singularity formation by proving in a very weak sense a profile type of decomposition, see [28].*

Two fundamental questions following Theorem 6 are still open in the general case:
(i) Does the function $x(t)$ has a limit as $t \to T$ defining then at least one exact blow up point in space where L^2 concentration takes place?
(i) Which is the exact amount of mass which is focused by the blow up dynamic?

An explicit construction of blow up solutions due to Merle, [16], is the following: let k points $(x_i)_{1\leq i\leq k} \in \mathbf{R}^N$, then there exists a blow up solution $u(t)$ which blows up in finite time $0 < T < +\infty$ exactly at these k points and behaves locally near x_i like $S(t)$ given by (15). In particular, it satisfies:

$$|u(t)|^2 \rightharpoonup \Sigma_{1\leq i\leq k} |Q|^2_{L^2} \delta_{x=x_i} \quad \text{as} \quad t \to T,$$

in the sense of measures. Let us observe first that from the construction, one could place at x_i instead of $S(t)$ any pseudo-conformal transformation of an excited ground state solution Q_i solution to (10). The solution focuses then at x_i exactly the mass $|Q_i|_{L^2}$ which is quantized but arbitrarily large. Second, similarly as for $S(t)$, such a solution is non dispersive as it accumulates all its initial L^2 mass into blow up.

A general conjecture concerning L^2 concentration is formulated in [24] and states that a blow up solution focuses a quantized and universal amount of mass at a finite number of points in $\mathbf{R}^N$, the rest of the L^2 mass being purely dispersed. The exact statement is the following:

Conjecture (*): *Let $u(t) \in H^1$ a solution to (1) which blows up in finite time $0 < T < +\infty$. Then there exist $(x_i)_{1\leq i\leq L} \in \mathbf{R}^N$ with $L \leq \frac{\int |u_0|^2}{\int Q^2}$, and $u^* \in L^2$ such that: $\forall R > 0$,*

$$u(t) \to u^* \quad in \quad L^2(\mathbf{R}^N - \bigcup_{1\leq i\leq L} B(x_i, R))$$

$$and \quad |u(t)|^2 \rightharpoonup \Sigma_{1\leq i\leq L} m_i \delta_{x=x_i} + |u^*|^2 \quad with \quad m_i \in [\int Q^2, +\infty).$$

The set M of admissible focused mass m_i for $N \geq 2$ is known to contain the unbounded set of the L^2 masses of excited bound states Q^i solutions to (10) from [16], and these are the only known examples.

2.4 Orbital stability of the ground state

From now on and for the rest of these notes, we restrict ourselves to considering small super critical initial data, that is:

$$u_0 \in \mathcal{B}_{\alpha^*} = \{u_0 \in H^1 \ \text{ with } \int Q^2 \leq \int |u_0|^2 \leq \int Q^2 + \alpha^*\},$$

for some parameter $\alpha^* > 0$ small enough. This situation is conjectured to locally describe the generic blow up dynamic around one blow up point.

In this case, we have a fundamental property which is the so called *orbital stability of the solitary wave*. We will give precise statements later, and we just underline here the main facts. Let $u_0 \in \mathcal{B}_{\alpha^*}$ for some $\alpha^* > 0$ universal and small enough, and assume that the corresponding solution to (1) blows up in finite time $0 < T < +\infty$, then there exist continuous parameters $(x(t), \gamma(t)) \in \mathbf{R}^N \times \mathbf{R}$ such that for t close enough to T, $u(t)$ admits a decomposition:

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q+\varepsilon)(t, \frac{x-x(t)}{\lambda(t)})e^{i\gamma(t)}, \tag{24}$$

where

$$|\varepsilon(t)|_{H^1} \leq \delta(\alpha^*), \quad \delta(\alpha^*) \to 0 \ \text{ as } \ \alpha^* \to 0,$$

and

$$\lambda(t) \sim \frac{1}{|\nabla u(t)|_{L^2}}. \tag{25}$$

In other words, finite time blow up solutions to (1) with small super critical mass are closed to the ground state in H^1 up to the set of H^1 symmetries. This property is again purely based on the Hamiltonian structure and the variational characterization of Q, and not on refined properties of the flow.

The main point of this non linear decomposition is that it now allows a perturbative analysis by studying the equation governing the H^1 small excess of mass $\varepsilon(t)$.

To describe the blow up dynamic is now equivalent to understand in the perturbative regime how to extract from the infinite dimensional dynamic of (1) a finite dimensional and possibly universal dynamic for the evolution of the geometrical parameters $(\lambda(t), x(t), \gamma(t))$ *which is coupled to the dispersive dynamic which drives the small excess of mass* $\varepsilon(t)$.

Indeed, to estimate for example the blow up speed is now equivalent to estimating the size of $\lambda(t)$, or to prove the existence of the blow up point is equivalent to proving the existence of a strong limit $x(t) \to x(T) \in \mathbf{R}^N$ as $t \to T$. Similarly, the structure in space of the singularity relies on the dispersive behavior of ε as t approaches blow up time.

2.5 Explicit construction of blow up solutions

As it allows a perturbative approach of the blow up problem, the existence of the geometrical decomposition (24) is a first step for the construction of blow up solutions to (1). We already mentioned a blow up construction by Merle, [16], which build non L^2 dispersive blow up solutions. There are two other fundamental results of construction of blow up solutions.

A first natural question is the existence in the super critical case of a blow up dynamic similar to the one of the explicit critical mass blow up solution $S(t)$. In [5], Bourgain and Wang construct indeed in dimension $N = 1, 2$ solutions $u(t)$ to (1) which blow up in finite time and behave locally like explicit blow up solution $S(t)$ given by (15). More precisely, given a limiting profile $u^* \in H^1$ sufficiently decaying at infinity -for technical reason- and *flat near zero* -this is not a technical point...- in the sense that for some $A > 0$ large enough,

$$\frac{d^i}{dx^i} u^*(0) = 0, \quad 1 \leq i \leq A, \tag{26}$$

they build a solution to (1) which blows up in finite time $0 < T < +\infty$ at $x = 0$ and satisfies:

$$u(t) - S(t-T) \to u^* \ \text{ in } \ H^1 \ \text{ as } \ t \to T. \tag{27}$$

Note that flatness assumption (26) is not open in H^1, and this statement ensures thus the stability of the $S(t)$ dynamic on a finite codimensional manifold. The meaning of this flatness assumption is to decouple in space the regular part of the solution which will evolve to u^*, and the singular part which will consist of $S(t)$ only. Recall that the Schrödinger propagator a priori allows infinite speed of propagation for waves, so it is a very non trivial fact to be able to control somehow the decoupling in space of the regular and the singular parts of the solution. As a corollary, these solutions have the same blow up speed like $S(t)$:

$$|\nabla u(t)|_{L^2} \sim \frac{1}{T-t}. \tag{28}$$

Now this rate of blow up turns out not to be the "generic" one. First, we have the following *universal lower bound on the blow up rate* known as the scaling lower bound:

Proposition 2 (Scaling lower bound on blow up rate) *Let $u_0 \in H^1$ such that the corresponding solution $u(t)$ to (1) blows up in finite time $0 < T < +\infty$, then there holds for some constant $C(u_0) > 0$:*

$$\forall t \in [0,T), \quad |\nabla u(t)|_{L^2} \geq \frac{C(u_0)}{\sqrt{T-t}} \tag{29}$$

Proof of Proposition 2

The proof is elementary and based on the scaling invariance of the equation and the local well posedness theory in H^1. Indeed, consider for fixed $t \in [0,T)$

$$v^t(\tau, z) = |\nabla u(t)|_{L^2}^{-\frac{N}{2}} u\left(t + |\nabla u(t)|_{L^2}^{-2}\tau, |\nabla u(t)|_{L^2}^{-1} z\right).$$

v^t is a solution to (1) by scaling invariance. We have $|\nabla v^t|_{L^2} + |v^t|_{L^2} \leq C$, and so by the resolution of the Cauchy problem locally in time by fixed point argument, there exists $\tau_0 > 0$ independent of t such that v^t is defined on $[0, \tau_0]$. Therefore, $t + |\nabla u(t)|_{L^2}^{-2}\tau_0 \leq T$ which is the desired result. This concludes the proof of Proposition 2.

Because it is related to the scaling symmetry of the problem, on which in other instances formal arguments indeed rely to derive the correct blow up speed, lower bound (29) has long been conjectured to be optimal. Yet, in the mid 80's, numerical simulations, see Landman, Papanicolaou, Sulem, Sulem, [12], have suggested that the correct and stable blow up speed is a slight correction to the scaling law:

$$|\nabla u(t)|_{L^2} \sim \sqrt{\frac{\log|\log(T-t)|}{T-t}}, \tag{30}$$

which is referred to as *the log-log law*. Solutions blowing up with this speed indeed appeared to be stable with respect to perturbation of the initial data. Quite an amount of formal work has been devoted to understanding the exact nature of the double log correction to the scaling estimate. We refer to the excellent monograph by Sulem, Sulem, [34], for further discussions on this subject. In this frame, for $N = 1$, Perelman in [31] rigorously proves the existence of one solution which blows up according to (30) and its stability in some space strictly included in H^1.

These two constructions of blow up solutions thus imply the following: there are at least two blow up dynamics for (1) with two different speeds, one which is a continuation of the explicit $S(t)$ blow up dynamic with the $1/(T-t)$ speed (28), and which is suspected to be unstable because it is not seen numerically; one with the log-log speed (30) which is conjectured to be stable from numerics.

2.6 Structural instability of the log-log law

We have so far exhibited two important features of the blow up dynamics for (NLS):
(i) there exists a critical mass blow up solution;
(ii) there are at least two blow up speeds.

These two facts are somehow fundamental difficulties for the analysis. The existence of the critical mass blow up solution implies that the set of initial data which yields a finite time blow up solution is not open, and thus blow up is not a stable phenomenon a priori. On the contrary, only one blow up regime is from numerics expected to be stable.

These facts are somehow believed to be closely related to the very specific algebraic structure of (1), and in particular to the existence of the pseudo-conformal symmetry.

An important result in this direction is the so called *structural instability* of the log-log law in the following sense. Consider in dimension $N = 2$ the Zakharov system:

$$\begin{cases} iu_t = -\Delta u + nu \\ \frac{1}{c_0^2} n_{tt} = \Delta n + \Delta |u|^2 \end{cases} \tag{31}$$

for some fixed constant $0 < c_0 < +\infty$. This system is the previous step in the asymptotic expansion of Maxwell equations which leads to (1), see [34]. In the limit $c_0 \to +\infty$, we formally recover (1). This system is still a Hamiltonian system and shares many of the variational structure of (1). In particular, a virial law in the spirit of (19) holds and yields finite time blow up for radial non positive energy initial data, see Merle, [19]. Moreover, a one parameter family of blow up solution may be constructed and should be understood as a continuation of the exact $S(t)$ solution for (1), see Glangetas, Merle, [9], and these explicit solutions have blow up speed:

$$|\nabla u(t)|_{L^2} \sim \frac{C(u_0)}{T-t}.$$

They moreover appear to be stable from numerics, see Papanicolaou, Sulem, Sulem, Wang, [30]. Now from Merle, [18], *all finite time blow up solutions to (31) satisfy*

$$|\nabla u(t)|_{L^2} \geq \frac{C(u_0)}{T-t}.$$

In particular, there will be no log-log blow up solutions for (31). This fact suggests that in some sense, the Zakharov system provides a much more stable and robust blow up dynamics than its asymptotic limit (NLS). This fact enlightens the belief that the log-log law heavily relies on the specific algebraic structure of (1), and some non linear degeneracy properties will indeed be at the heart of the understanding of the blow up dynamics.

3 Blow up dynamics of small super critical mass initial data

In this section and for the rest of these notes, we restrict ourselves to initial data with small super critical mass, that is:

$$u_0 \in \mathcal{B}_{\alpha^*} = \{u_0 \in H^1 \ \text{ with } \ \int Q^2 \leq \int |u_0|^2 \leq \int Q^2 + \alpha^*\},$$

for some parameter $\alpha^* > 0$ small enough. We present the results on the blow up dynamics obtained in the series of papers [20], [21], [32], [22], [23], [24] and which allow a precise understanding of the blow up dynamics in this setting. The description of the blow up dynamic involves two different type of questions:

- In [20], [21], we consider non positive energy initial data and address the question of an upper bound on the blow up rate. In [32], the dynamically richer case of non negative energy is addressed together with the issue of the stability of the blow up regimes.
- In [22], using as a starting point the point of view and the estimates in [20], [21], [32], we investigate the question of the shape of the solution in space and the existence of a universal asymptotic profile which attracts blow up solutions. These questions rely on Liouville type of theorems to classify the non dispersive dynamics of solitary waves. Further understanding of these issues will then allow one as in [23], [24], to prove sharp lower bounds on the blow up rate related to the expected log-log law and then quantization results on the focused mass -or equivalently Conjecture (*) for data $u_0 \in \mathcal{B}_{\alpha^*}$-.

First, we introduce for notational purpose the following invariant which sign is preserved by the H^1 symmetries:

$$E_G(u) = E(u) - \frac{1}{2}\left(\frac{Im(\int \nabla u \overline{u})}{|u|_{L^2}}\right)^2. \tag{32}$$

Next, we will assume in all our results a *Spectral Property* which amounts counting the number of negative directions of an explicit Schrödinger operator $-\Delta + V$ where the well localized potential V is stationary and build from the ground state. This property was proved in dimension $N = 1$ in [20] using the explicit formula for the ground state Q, and checked numerically in dimension $N = 2, 3, 4$ to which we will thus restrict ourselves, see Proposition 4. Note that this property is the only part of the proof where the restriction on the dimension is needed.

3.1 Finite time blow up for non positive energy initial data

In this subsection, we address the question of the blow up dynamics for non positive energy solutions. The result is the following:

Theorem 7 ([20],[21]) *Let $N = 1, 2, 3, 4$. There exist universal constants $\alpha^*, C^* > 0$ such that the following holds true. Given $u_0 \in \mathcal{B}_{\alpha^*}$ with*

$$E_G(u_0) < 0,$$

the corresponding solution $u(t)$ to (1) blows up in finite time $0 < T < +\infty$ and there holds for t close to T:

$$|\nabla u(t)|_{L^2} \leq C^* \left(\frac{\log|\log(T-t)|}{T-t} \right)^{\frac{1}{2}}. \tag{33}$$

Comments on Theorem 7

1. Galilean invariance: From Galilean invariance, blow up criterion $E_G(u_0) < 0$ is equivalent to

$$E_0 < 0 \ \text{ and } \ Im\left(\int \nabla u_0 \overline{u_0}\right) = 0. \tag{34}$$

Indeed, let u_0 with $E_G(u_0) < 0$ and set

$$(u_0)_\beta = u_0 e^{i\frac{\beta}{2}\cdot x} \ \text{ with } \ \beta = -2\frac{Im(\int \nabla u_0 \overline{u_0})}{\int |u_0|^2},$$

then from explicit computation, $(u_0)_\beta$ satisfies (34). Now from Galilean invariance, $u_\beta(t,x) = u(t, x - \beta t) e^{i\frac{\beta}{2}\cdot(x - \frac{\beta}{2}t)}$, so that blow behavior of $u(t,x)$ and $u_\beta(t,x)$ are the same.

2. Blow up criterion: The blow up criterion is in H^1 and thus improves the virial result which holds in virial space Σ only -up to results of Theorem 5-. In this region of the energy space, blow up is thus a stable phenomenon. Moreover, the result also holds for $t < 0$ by considering $\overline{u}(-t)$ which is also a solution to (1), and thus strictly negative energy solutions blow up in finite time on both sides in time.

3. Instability of $S(t)$: The major fact of Theorem 7 is that it removes for non positive energy solutions the possibility of $S(t)$ type of blow up as log-log upper bound (33) is below the $1/(T-t)$ speed. We will indeed later prove that there is in this case only one blow up regime which speed is given by the exact log-log law (30). Now a fundamental corollary of Theorem 7 obtained using the pseudo-conformal transformation is the instability of $S(t)$ in a strong sense. $S(t)$ is the critical mass blow up solution, so it is unstable in a trivial sense: any H^1 neighborhood of $S(-1)$ contains initial data $u(-1)$ which solution $u(t)$ is global in time, it suffices to take subcritical mass initial data. We claim a much stronger statement which is that the blow up dynamic of $S(t)$ itself is unstable in the following sense: any H^1 neighborhood of $S(-1)$ contains initial data $u(-1)$ which solution $u(t)$ blows up in finite time but with the log-log speed.
Indeed, let the initial data at time $t=1$: $u_\eta(1,x) = (1+\eta)Q(x)$ for $\eta > 0$ and small, and $u_\eta(t)$ the corresponding solution to (1). From explicit computation, $E(u_\eta) < 0$ and thus $u_{0\eta}$ satisfies the hypothesis of Theorem 7. It thus blows up in finite time $1 < T_\eta < +\infty$. We now apply the pseudo-conformal symmetry and consider the solutions

$$v_\eta(t) = \frac{1}{|t|^{\frac{N}{2}}} u_\varepsilon \left(\frac{-1}{t}, \frac{x}{t} \right) e^{-i\frac{|x|^2}{4t} - i}.$$

First observe that

$$v_\eta(-1) \to S(-1) \quad \text{as} \quad \eta \to 0$$

in some strong sense. Next, from its definition, $v_\eta(t)$ blows up in finite time $T'_\eta = \frac{-1}{T_\eta} < 0$. Now $T'(\eta) < 0$ and the uniform space time bound on $|xu_\varepsilon(t)|_{L^2}$ given by the virial law (19) ensure that $v_\eta(t)$ satisfies upper bound (33) for t close enough to T'_η as desired.

The above example also illustrates a standard feature: upper bound (33) is satisfied asymptotically near blow up time, that is for $t \in [t(u_0), T)$, and the time $t(u_0)$ depends on the full profile of the initial data.

3.2 H^1 stability of the log-log law

Let us now investigate the dynamics for positive energy initial data. In this case, three different dynamics are known to possibly occur:

- $S(t)$ behavior: results in [5] yield existence of finite time blow up solutions $u(t)$ satisfying $u_0 \in \mathcal{B}_{\alpha^*}$, $E_0^G > 0$ and $|\nabla u(t)|_{L^2} \sim \frac{1}{T-t}$ near blow up time.
- log-log behavior: Using the pseudo-conformal symmetry and Theorem 7, we can easily exhibit strictly positive energy solutions satisfying the log-log upper bound (33). Indeed, for $\eta > 0$ small enough, let $u_0(\eta) = (1+\eta)Q$ and $u_\eta(t)$ the corresponding solution to (1). We have $E(u_0(\eta)) \sim -C\eta < 0$, and thus $u_\eta(t)$ blows up in finite time $T(\eta)$ with upper bound (33). Applying now the pseudo conformal transformation to

this solution, we let $v_0(\eta)=u_0(\eta^4)e^{-i\eta\frac{|y|^2}{4}}$ and compute its energy. From:

$$E\left(ve^{-i\eta\frac{|y|^2}{4}}\right)=E(v)-\frac{\eta}{2}Im\left(\int x\cdot\nabla v\overline{v}\right)+\frac{\eta^2}{8}\int|y|^2|v|^2, \tag{35}$$

we have $E(v_0(\eta))>0$ for η small enough. Now pseudo-conformal formula (14) yields:

$$v(\eta)(t)=\frac{1}{(1-\eta t)^{\frac{N}{2}}}u_\eta\left(\frac{t}{1-\eta t},\frac{x}{1-\eta t}\right)e^{-i\eta\frac{|x|^2}{4(1-\eta t)}},$$

so that $v(t)$ is defined on $[0,T_v(\eta))$, $T_v(\eta)=\frac{T_\eta}{1+\eta T_\eta}<\frac{1}{\eta}$, and blows up at $T_v(\eta)$ with upper bound (33) as wanted.

- Global solutions: Given $u(t)\in\Sigma$ a solution to (1) which blows up at $0<T<+\infty$, pseudo conformal symmetry (14) applied with parameter $a=\frac{1}{T}$ yields a solution $v(t)$ to (1) globally defined on $[0,+\infty)$.

There certainly is a poor understanding in general of which conditions on the initial data are enough to select one of the above dynamics. Nevertheless, we have the following:

Theorem 8 ([32]) *Let $N=1,2,3,4$. There exist universal constants $C^*,C_1^*>0$ such that the following is true:*

(i) Rigidity of blow up rate: Let $u_0\in\mathcal{B}_{\alpha^*}$ with

$$E_G(u_0)>0,$$

and assume the corresponding solution $u(t)$ to (1) blows up in finite time $T<+\infty$, then there holds for t close to T either

$$|\nabla u(t)|_{L^2}\leq C^*\left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}$$

or

$$|\nabla u(t)|_{L^2}\geq\frac{C_2^*}{(T-t)\sqrt{E_G(u_0)}}. \tag{36}$$

(ii) Stability of the log-log law: Moreover, the set of initial data $u_0\in\mathcal{B}_{\alpha^*}$ such that $u(t)$ blows up in finite time with upper bound (33) is open in H^1.

Comments on Theorem 8

1. Optimal criterion of stability: A slightly more self contained statement if that the set

$$\mathcal{O}=\{u_0\in\mathcal{B}_{\alpha^*},\ \int_0^{T_u}|\nabla u(t)|_{L^2}dt<+\infty\}\ \text{ is open in }\ H^1,$$

and that $\mathcal{O}$ is exactly the set of initial data which blow up in finite time with log-log upper bound (33). We will later refer to $\mathcal{O}$ as the open set of log-log blow up.

2. *Size of the log-log set*: $\mathcal{O}$ is known to contain non positive energy initial data from Theorem 7. Then the pseudo-conformal invariance allows one to obtain non negative energy solutions which satisfy log-log upper bound (33). One can prove that this procedure does not describe all $\mathcal{O}$, and that there exist initial data $u_0 \in \Sigma \cap \mathcal{O}$ with non negative energy which cannot be obtained using the pseudo-conformal symmetry from a non positive energy initial data, see [32].

4. *Stability versus instability*: Let us recall that the existence of critical mass blow up solution $S(t)$ implies that the set of initial data which lead to a finite time blow up solution is not open in H^1. In this setting, the fact that the blow up speed is a sufficient criterion of stability in the energy space is a new feature in the non linear dispersive setting. Now if stability of the log-log regime is proved, instability in the strong sense of solutions satisfying lower bound (36) is proved only for $S(t)$ itself, see Comment 3 of the previous subsection. Dynamical instability in this sense of these solutions is open. A simpler result would be to prove the strong instability of solutions build in [5], this is also open.

5. *Universal upper bound on the blow up speed*: Upper bound (33) corresponds to the stable blow up dynamic, while lower bound (36) is obtained by assumption of escaping this stable blow up regime. In this sense, these estimates correspond to two different asymptotic blow up regimes which each require a specific analysis. This is why no general upper bound on blow up rate of any type holds so far. Let us recall that solutions build in [5] satisfy exact law

$$|\nabla u(t)|_{L^2} \sim \frac{C(u_0)}{T-t},$$

and it seems reasonable to conjecture that this law is sharp. Let us remark that this would imply from the pseudo-conformal symmetry that blow up in Σ always occurs in finite time, this is also an open problem.

We have not addressed so far the question of blow up dynamics of zero energy initial data. This question turns out to be very fundamental but requires different type of ideas. Note that the solitary wave $Q(x)e^{it}$ is a global in time zero energy solution to (1). We will come back later to the issue of classifying this dynamic among the set of zero energy solutions.

3.3 Universality of the blow up profile

We now turn to the question of the dispersive properties of blow up solutions to (1). Recall from existence of geometrical decomposition (24) that blow up solutions write near blow up time:

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q+\varepsilon)(t, \frac{x-x(t)}{\lambda(t)})e^{i\gamma(t)}$$

from some H^1 small excess of mass $\varepsilon(t)$. We ask the question of the dispersive behavior of $\varepsilon(t)$ as $t \to T$. The result is the following:

Theorem 9 ([22]) *Let $N = 1,2,3,4$. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$. Then there exist parameters $\lambda_0(t) = \frac{|\nabla Q|_{L^2}}{|\nabla u(t)|_{L^2}}$, $x_0(t) \in \mathbf{R}^N$ and $\gamma_0(t) \in \mathbf{R}$ such that*

$$e^{i\gamma_0(t)}\lambda_0^{\frac{N}{2}}(t)u(t, \lambda_0(t)x + x_0(t)) \to Q \quad \text{in} \quad L^2_{loc} \quad \text{as} \quad t \to T.$$

In the variables of the decomposition (24), this means:

$$\varepsilon(t) \to 0 \quad \text{as} \quad t \to T \quad \text{in} \quad L^2_{loc}.$$

Let us observe that this is the typical dispersive behavior for Schrödinger group: the L^2 mass is conserved, so L^2 convergence to zero is forbidden, but it happens locally in space meaning that the excess of mass is dispersed away. From geometrical decomposition (24), this theorem thus asserts that in rescaled variables, blow up solutions admit a universal asymptotic profile in space which is given by the ground state Q itself.

This type of questions goes beyond blow up issues and is related to a wide range of problems regarding the asymptotic stability of solitary waves in non linear dispersive PDE's. For blow up problems in the non linear dispersive setting, the first result in this direction has been obtained by Martel and Merle, [15], for the generalized critical KdV equation:

$$(KdV) \quad \begin{cases} u_t + (u_{xx} + u^5)_x = 0, & (t,x) \in [0,T) \times \mathbf{R}, \\ u(0,x) = u_0(x), \in H^1 & u_0 : \mathbf{R} \to \mathbf{R}. \end{cases} \tag{37}$$

This equation shares a lot of the variational structure of (1), and in particular finite time blow up solutions admit a geometrical decomposition similar to (24). In [15], Martel and Merle also prove the asymptotic stability of Q as the blow up profile. One of the fundamental observation of their proof is to show that this result is essentially equivalent to proving a lower bound on blow up rate which avoids the self similar regime. We similarly have:

Theorem 10 ([22]) *Let $N = 1,2,3,4$. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$. Then:*

$$|\nabla u(t)|_{L^2}\sqrt{T-t} \to +\infty \quad \text{as} \quad t \to T. \tag{38}$$

Let us recall that there always holds the scaling lower bound (29):

$$|\nabla u(t)|_{L^2} \geq \frac{C(u_0)}{\sqrt{T-t}}.$$

This lower bound is thus never sharp for data $u_0 \in \mathcal{B}_{\alpha^*}$. Bourgain in [4] conjectured that there indeed are no self similar solutions, that is blowing up with the exact scaling law $|\nabla u(t)|_{L^2} \sim \frac{C(u_0)}{\sqrt{T-t}}$, *in the energy space* H^1.

Now a fundamental fact which will enlighten our further analysis is that there do exist self similar solutions, but they never belong to L^2. More precisely, a standard way of exhibiting self similar solutions is to look for a blow up solution with the form:

$$U_b(t,x) = \frac{1}{(2b(T-t))^{\frac{N}{4}}} Q_b\left(\frac{x}{\sqrt{2b(T-t)}}\right) e^{-i\frac{\log(T-t)}{2b}}$$

for some fixed parameter $b>0$ and a fixed profile Q_b solving the elliptic ODE:

$$\Delta Q_b - Q_b + ib\left(\frac{N}{2}Q_b + y\cdot\nabla Q_b\right) + Q_b|Q_b|^{\frac{4}{N}} = 0. \tag{39}$$

Now from [33], solutions Q_b *never belong to* L^2 from a logarithmic divergence at infinity:

$$|Q_b(y)| \sim \frac{C(b)}{|y|^{\frac{N}{2}}} \quad \text{as} \quad |y| \to +\infty$$

and thus always miss the energy space. Nevertheless, for any given parameter $b>0$ small enough, one can exhibit a solution to (39) which will be in $\dot{H}^1$ and will satisfy:

$$Q_b \to Q \quad \text{as } b \to 0 \quad \text{in} \quad \dot{H}^1 \cap L^2_{loc}.$$

In other words, one can build self similar solutions to (1) which on compacts sets will look like a smooth solution, but then display an oscillatory behavior at infinity in space which induces *a non* L^2 *tale escaping the soliton core.*

Now to prove Theorems 9 or 10, one needs to understand how to use the information that the solution we consider is in L^2, and one thus needs to exhibit L^2 *dispersive estimates on the solution.* Now recall that L^2 is the scaling invariant space for this equation, and thus any dispersive information in L^2 is in fact a global information in space. Now the only given global information in L^2 is the conservation of the L^2 norm, and somehow the task here is *to be able to use the conservation of the energy in a dynamical way.*

Following the analysis in [15], the strategy used to prove Theorem 9 is by contradiction. Assuming that the blow up profile is not Q, we prove using compactness type of arguments based on the estimates on the blow up dynamic proved in [20], [21], [32], that it implies the existence of a *self similar solution* $v(t)$ in H^1 which is non dispersive in the sense that:

$$|v(t)|^2 \rightharpoonup \left(\int |v(0)|^2\right) \delta_{x=0} \quad \text{as} \quad t \to T \tag{40}$$

in the weak sense of measures. In other words, if the excess mass is not dispersed, one can extract a fully non L^2 dispersive blow up solution in the sense that it accumulates whole its L^2 mass into blow up. A crucial point in this step is the proof of the continuity of the blow up time with respect to the initial data *in the open set* $\mathcal{O}$.

The second step of the analysis is now to classify the non L^2 dispersive solutions. The proof of this step involves the expected new type of dispersive estimates in L^2. The result is the following.

Theorem 11 ([22]) *Let $N = 1, 2, 3, 4$. Let an initial data $v_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution to (1) blows up in finite time $0 < T < +\infty$ and does not disperse in L^2 in the sense that it satisfies (40), then*

$$v(t) = S(t)$$

up to the set of H^1 symmetries of (1).

In other words, the only non dispersive blow up solution in $\mathcal{B}_{\alpha^*}$ is the critical mass blow up solution, which of course cannot loose mass at blow. This result should be seen as the dispersive super critical version of Theorem 4.

A key in the proof is that non L^2 dispersive information (40) together with the fact that $v(t)$ satisfies the self similar law implies estimates on the solution in the virial space Σ. This allows us to use the pseudo-conformal symmetry, and in then turns out that Theorem 11 is equivalent to classification results of zero energy solutions to (1):

Theorem 12 ([22]) *Let $N = 1, 2, 3, 4$. Let $u_0 \in \mathcal{B}_{\alpha^*} \cap \Sigma$ with*

$$E_G^0 = 0,$$

$u(t)$ the corresponding solution to (1). Assume that $u(t)$ is not a soliton up to the symmetries in H^1, then $u(t)$ blows up in finite time on both sides in time with upper bound (33).

Observe that the solitary wave is a global in time zero energy solution in Σ. In other words, to classify non L^2 dispersive solutions is equivalent from pseudo-conformal symmetry to dynamically classify the solitary wave in the set of zero energy solutions in Σ. This kind of Liouville theorems and dynamical classification is a completely new -and unexpected- feature for (NLS).

3.4 Exact log-log law and the mass quantization conjecture

L^2 dispersive estimates needed for the proof of Theorem 9 are exhibited for the proof of the classification result of Theorem 11. In this sense, these estimates are not proved for a "true" blow up solution but are exhibited as specific properties of a non dispersive blow up solution.

Now a further understanding of these properties in fact allows one to obtain direct dispersive estimates in L^2 on a blow up solution. More specifically, let a finite time blow up solution $u(t) \in \mathcal{O}$. We exhibit a global in space information on the solution by proving that in rescaled variables, the space divides in three specific regions:
(i) on compact sets, the solution looks like Q in a strong sense;
(ii) a radiative regime then takes place where the solution looks like the *non L^2 tale* of explicit self similar solutions to (39);
(iii) this regime cannot last forever in space because the tale of self similar solutions is not L^2, while the solution is. We then exhibit a third regime further away in space where a purely linear dispersive dynamic takes place.

Another way of viewing the picture is the following: the non linear dynamic represented by Q on compact set is connected to a linear dispersive dynamic at infinity in space by a universal radiative regime given by the tale of explicit self similar solutions. This radiation is the mechanism which takes the L^2 mass out of the soliton core on compact sets to disperse it to infinity in space: this is an "outgoing radiation" process corresponding to the so-called dynamical metastability of self similar profiles Q_b solutions to (39). Now the rate at which the mass is extracted is submitted to one global constraint in time: the conservation of the L^2 norm. Moreover, this mechanism quantifies how the L^2 constraint implies the non persistence of the self similar regime, that is how the radiation is connected to the dispersive dynamic at infinity. And we have seen that this is related to obtaining lower bounds on the blow up rate.

The outcome of this analysis is the following sharp lower bound on blow up rate.

Theorem 13 ([23]) *Let $N = 1, 2, 3, 4$. There exists a universal constants $C_3^* > 0$ such that the following holds true. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$, then one has the following lower bound on blow up rate for t close to T:*

$$|\nabla u(t)|_{L^2} \geq C_3^* \left(\frac{\log|\log(T-t)|}{T-t} \right)^{\frac{1}{2}}. \tag{41}$$

In the log-log regime, the blow up speed may in fact be exactly evaluated according to:

$$\frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}} \left(\frac{T-t}{\log|\log(T-t)|} \right)^{\frac{1}{2}} \to \frac{1}{\sqrt{2\pi}} \quad \text{as} \quad t \to T. \tag{42}$$

In addition, we may extend the dynamical characterization of solitons in the zero energy manifold to the full energy space H^1:

Theorem 14 ([23]) *Let $N = 1, 2, 3, 4$. Let $u_0 \in \mathcal{B}_{\alpha^*}$ with $E_0^G = 0$ and assume u_0 is not a soliton up to fixed scaling, phase, translation and Galilean invariances.*
Then u blows up both for $t < 0$ and $t > 0$, and (42) holds.

It is a surprising fact somehow that the analysis needed to obtain lower and upper bounds in the log-log regime requires different type of informations:
(i) The proof of the upper bound on blow up rate (33) requires only local in space information on the soliton core, and global in $\dot{H}^1$. But nothing is needed in L^2 and indeed explicit self similar profiles solutions to (39) in $\dot{H}^1$ would fit into this analysis.
(ii) The proof of the lower bound on blow up rate (41) requires global in space dispersive informations in L^2, that is estimates on the solution in the different regimes in space. One may then estimate the flux of L^2 norm in between these different regimes which is submitted to the L^2 conservation constraint. This yields the exact log-log law.

Moreover, and this certainly is the main motivation to go through the whole log-log analysis, the precise understanding of the L^2 structure in space of the solution in rescaled variables now allows us to investigate the behavior of the solution in the original non rescaled variables.

Indeed, let us make the following simple observation. From geometrical decomposition (24), a blow up solution $u(t)$ writes near blow up time:

$$u(t,x) = Q_{sing}(t,x) + \tilde{u}(t,x)$$

with

$$Q_{sing}(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}} Q(t, \frac{x - x(t)}{\lambda(t)}) e^{i\gamma(t)}, \quad \tilde{u}(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}} \varepsilon(t, \frac{x - x(t)}{\lambda(t)}) e^{i\gamma(t)}.$$

Q_{sing} is the singular part of the solution. We address the following natural question: does the excess of mass $\tilde{u}(t,x)$ remain smooth up to blow up time? A first answer to this question has been obtained in rescaled variables. Indeed, Theorem 9 asserts:

$$\varepsilon(t) \to 0 \ \text{ as } \ t \to T \ \text{ in } \ L^2_{loc}.$$

But as $\lambda(t) \to 0$ as $t \to T$, this is very far from obtaining regularity control on $\tilde{u}(t)$. In particular, it does not prevent a priori the excess of mass $\tilde{u}$ from focusing some small mass at blow up time. The regularity of $\tilde{u}$ is thus deeply related both to the shape in space of $\varepsilon(t)$ and the rate at which it is dispersed. Both these questions are now precisely the ones addressed in the proof of Theorem 13. Further use of the obtained estimates then allow one to prove the following result.

Theorem 15 ([24]) *Let $N = 1, 2, 3, 4$. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution to (1) blows up in finite time $0 < T < +\infty$. Then there exist parameters $(\lambda(t), x(t), \gamma(t)) \in \mathbf{R}_+^* \times \mathbf{R}^N \times \mathbf{R}$ and an asymptotic profile $u^* \in L^2$ such that*

$$u(t) - \frac{1}{\lambda(t)^{\frac{N}{2}}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \quad in \quad L^2 \quad as \quad t \to T. \tag{43}$$

Moreover, blow up point is finite in the sense that

$$x(t) \to x(T) \in \mathbf{R}^N \quad as \quad t \to T.$$

In other words, up to a singular part which has a universal space time structure, blow up solutions remain smooth in L^2 up to blow up time.

A fundamental corollary is the so called *quantization phenomenon* for (1): blow up solutions in $\mathcal{B}_{\alpha^*}$ focus the universal amount of mass $\int Q^2$ into blow up, the rest is purely dispersed, or in other words:

$$|u(t)|^2 \rightharpoonup \left(\int Q^2\right) \delta_{x=x(T)} + |u^*|^2 \quad \text{as} \quad t \to T \quad \text{with} \quad \int |u_0|^2 = \int Q^2 + \int |u^*|^2.$$

This is in contrast with the Zakharov model (31) where explicit blow up solutions build by Glangetas, Merle, [9], accumulate a continuum of mass into blow up.

A second outcome of Theorem 15 is the fact that the formation of the singularity is a well localized in space phenomenon. Indeed, blow up occurs at a well defined blow up point $x(T)$ where a fixed amount of mass is focused, but outside $x(T)$, the solution has a strong L^2 limit. It means in particular that the phase of the solution is not oscillatory outside blow up point, whereas the phase $\gamma(t)$ of the singularity is known to satisfy $\gamma(t) \to +\infty$ as $t \to T$. This strong regularity of the solution outside the blow up point was not expected. From the proof also, one can prove that the blow up point $x(T)$ and the asymptotic profile u^* are in the log-log regime continuous functions of the initial data.

Observe now that Theorem 15 includes both blow up regimes which would in particular be characterized by a different law for $\lambda(t)$ in the singular part of the solution. We now claim that the difference between the two blow up regimes may be seen on the asymptotic profile u^* which in fact *connects in a universal way depending on the blow up regime the regular and singular parts of the solution.*

Theorem 16 ([24]) *Let $N = 1, 2, 3, 4$. There exists a universal constant $C^* > 0$ such that the following holds true. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ to (1) blows up in finite time $0 < T < +\infty$. Let $x(T)$ its blow up point and $u^* \in L^2$*

its profile given by Theorem 15, then for $R > 0$ small enough, we have:
(i) Log-log case: if $u_0 \in \mathcal{O}$, then

$$\frac{1}{C^*(\log|\log(R)|)^2} \leq \int_{|x-x(T)|\leq R} |u^*(x)|^2 dx \leq \frac{C^*}{(\log|\log(R)|)^2}, \tag{44}$$

and in particular:

$$u^* \notin H^1 \quad and \quad u^* \notin L^p \quad for \quad p > 2. \tag{45}$$

(ii) $S(t)$ case: if $u(t)$ satisfies (36), then

$$\int_{|x-x(T)|\leq R} |u^*|^2 \leq C^* E_0 R^2, \tag{46}$$

and

$$u^* \in H^1.$$

The fact that one can separate within the two blow up dynamics and see the different blow up speeds on asymptotic profile u^* is a completely new feature for (NLS) and was not even expected at the formal level. Moreover, this results strengthens our belief that $S(t)$ type of solutions are in some sense on the boundary of the set of finite time blow up solutions:

- The stable log-log blow up scenario is based on the ejection of a radiative mass which strongly couples the singular and the regular parts of the solution and induces singular behavior (44) of the profile at blow up point. The universal singular behavior (44) is the "trace" of the radiative regime in the rescaled variables which couples the blow up dynamic on compact sets to the dispersive dynamic at infinity.

- On the contrary, the $S(t)$ regime corresponds to formation of a minimal mass blow up bubble very decoupled from the regular part which indeed remains smooth in the Cauchy space. This blow up scenario somehow corresponds to the "minimal" blow up configuration. Observe that in this last regime, (46) in dimension $N = 1$ implies $u^*(0) = 0$. Now in [5], Bourgain and Wang construct for a given radial profile u^* smooth with $\frac{d^i}{dr^i}u^*(r)_{|r=0} = 0$, $1 \leq i \leq A$, a solution to (1) with blow up point $x = 0$ and asymptotic profile u^*. In their proof, A is very large, what is used to decouple the regular and the singular parts of the solution. In this sense, estimate (46) proves in general a decoupling of this kind for the $S(t)$ dynamic. It is an open problem to estimate the exact degeneracy of u^*.

4 Log-log upper bound on the blow up rate

This section is devoted to a presentation of the main results needed for the proof of the log-log upper bound on blow up rate in the non positive energy case, that is Theorem 7.

We will in particular focus onto the proof of the key dispersive controls in $\dot{H}^1$ which are at the heart of the control from above on the blow up speed. More detailed proofs are to be found in [20], [21].

The heart of our analysis will be to exhibit as a consequence of dispersive properties of (1) close to Q strong rigidity constraints for the dynamics of non positive energy solutions. These will in turn imply monotonicity properties, that is the existence of a Lyapounov function. The corresponding estimates will then allow us to prove blow up in a dynamical way and the sharp upper bound on the blow up speed will follow.

In the whole section, we consider a data

$$u_0 \in \mathcal{B}_{\alpha^*}$$

for some small universal $\alpha^* > 0$ and let $u(t)$ the corresponding solution to (1) with maximal time interval existence $[0, T)$ in H^1, $0 < T \leq +\infty$. We further assume $E_G(u_0) < 0$. According to Comment 1 of Theorem 7, we equivalently have up to a fixed Galilean Transformation:

$$E_0 < 0 \quad \text{and} \quad Im\left(\int \nabla u_0 \overline{u_0}\right) = 0. \tag{47}$$

For a given function f, we will note

$$f_1 = \frac{N}{2}f + y \cdot \nabla f, \quad f_2 = \frac{N}{2}f_1 + y \cdot \nabla f_1.$$

Note that from integration by parts:

$$(f_1, g) = -(f, g_1).$$

4.1 Existence of the geometrical decomposition

We recall in this section the orbital stability of the solitary wave which implies the existence of geometrical decomposition (24). The argument is based only on the conservation of the energy and the L^2 norm and the small super critical mass assumption $u_0 \in \mathcal{B}_{\alpha^*}$. The idea is the following. Recall that the ground state minimizes the energy according to Proposition 1: let $v \in H^1$ such that $\int |v|^2 = \int Q^2$ and $E(v) = 0$, then

$$v(x) = \lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0) e^{i\gamma_0},$$

for some parameters $\lambda_0 \in \mathbf{R}_+^*$, $x_0 \in \mathbf{R}^N$, $\gamma_0 \in \mathbf{R}$. From standard concentration compactness type of arguments, this implies in particular that functions with negative energy and small super critical mass have a very specific shape, and indeed, are close to the three dimensional minimizing manifold. The result is the following:

Lemma 1 (Orbital stability of the ground state) *There exists a universal constant $\alpha^* > 0$ such that the following is true. For all $0 < \alpha' \leq \alpha^*$, there exists $\delta(\alpha')$ with $\delta(\alpha') \to 0$ as $\alpha' \to 0$ such that $\forall v \in H^1$, if*

$$\int Q^2 \leq \int |v|^2 < \int Q^2 + \alpha' \quad \textit{and} \quad E(v) \leq \alpha' \int |\nabla v|^2,$$

then there exist parameters $\gamma_0 \in \mathbf{R}$ and $x_0 \in \mathbf{R}^N$ such that

$$|Q - e^{i\gamma_0} \lambda_0^{\frac{N}{2}} v(\lambda_0 x + x_0)|_{H^1} < \delta(\alpha') \tag{48}$$

with $\lambda_0 = \frac{|\nabla Q|_{L^2}}{|\nabla v|_{L^2}}$.

Proof of Lemma 1

We prove the claim in the radial case for $N \geq 2$. The general case follows from standard concentration compactness techniques.
Arguing by contradiction, we equivalently need to prove the following: let a sequence $v_n \in H^1$ such that:

$$\forall n, \quad \int |\nabla v_n|^2 = \int |\nabla Q|^2, \tag{49}$$

$$\int |v_n|^2 \to \int Q^2 \quad \text{as} \quad n \to +\infty, \tag{50}$$

and

$$\limsup_{n \to +\infty} E(v_n) \leq 0, \tag{51}$$

then there exist $\gamma_n \in \mathbf{R}$ such that

$$e^{i\gamma_n} v_n \to Q \quad \text{in} \quad H^1 \quad \text{as} \quad n \to +\infty. \tag{52}$$

Let us consider $w_n = |v_n|$. First observe from $\int |\nabla w_n|^2 \leq \int |\nabla v_n|^2$ that the sequence w_n is H^1 bounded, thus

$$w_n \rightharpoonup W \quad \text{in} \quad H^1$$

up to a subsequence. We first claim that W is non zero. Indeed, from (49):

$$\frac{1}{2 + \frac{4}{N}} \int |w_n|^{2+\frac{4}{N}} = \frac{1}{2 + \frac{4}{N}} \int |v_n|^{2+\frac{4}{N}} = \frac{1}{2} \int |\nabla v_n|^2 - E(v_n) = \frac{1}{2} \int |\nabla Q|^2 - E(v_n),$$

and thus from (51):

$$\liminf_{n \to +\infty} \int |w_n|^{2+\frac{4}{N}} > 0.$$

Now from compact embedding of H^1_{radial} into $L^{2+\frac{4}{N}}$,

$$\int |w_n|^{2+\frac{4}{N}} \to \int |W|^{2+\frac{4}{N}}, \tag{53}$$

and thus W is non zero. Now from (51),

$$E(W) \le \liminf_{n \to +\infty} E(w_n) \le \liminf_{n \to +\infty} E(v_n) \le 0, \tag{54}$$

and from (49),

$$\int |W|^2 \le \liminf_{n \to +\infty} \int |w_n|^2 = \int Q^2.$$

Thus W is a non zero negative energy function with subcritical mass, what from sharp Gagliardo-Nirenberg inequality (11) and Proposition 1 characterizes the ground state up to fixed scaling and phase invariances. Now W is real so the phase is zero. Moreover, this yields $\int W^2 = \int Q^2$ and thus $\int W_n^2 \to \int W^2$ from (50). Similarly, $E(W) = 0$ and thus from (54), $E(w_n) \to E(W) = 0$, and from (53), $\int |\nabla w_n|^2 \to \int |\nabla W|^2$. This implies from (49) that $W = Q$ and so $w_n \to W = Q$ strongly in H^1. It is now an easy task to conclude to (52). This ends the proof of Lemma 1.

The small critical mass assumption $u_0 \in \mathcal{B}_{\alpha^*}$ and the negative energy assumption now allow us to apply Lemma 1 to $v = u(t)$ for all fixed $t \in [0, T)$, and thus to exhibit parameters $\gamma_0(t) \in \mathbf{R}$, $x_0(t) \in \mathbf{R}^N$ and $\lambda_0(t) = \frac{|\nabla Q|_{L^2}}{|\nabla u(t)|_{L^2}}$ such that $u(t)$ satisfies (48) for all time. Let us observe that this geometrical decomposition is by no mean unique, and the parameters $(\lambda_0(t), \gamma_0(t), x_0(t))$ build from Lemma 1 are a priori no better than continuous functions of time. Nevertheless, one can freeze and regularize this decomposition by choosing a set of orthogonality conditions on the excess of mass: this is the so-called modulation theory which will be examined later on. Let us so far assume that we have a smooth decomposition of the solution: $\forall t \in [0, T)$,

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q + \varepsilon)(t, \frac{x - x(t)}{\lambda(t)})e^{i\gamma(t)}$$

with

$$\lambda(t) \sim \frac{C}{|\nabla u(t)|_{L^2}} \quad \text{and} \quad |\varepsilon(t)|_{H^1} \le \delta(\alpha^*) \to 0 \quad \text{as} \quad \alpha^* \to 0.$$

To study the blow up dynamic is now equivalent to understanding the coupling between the finite dimensional dynamic which governs the evolution of the geometrical parameters $(\lambda(t), \gamma(t), x(t))$ and the infinite dimensional dispersive dynamic which drives the excess of mass $\varepsilon(t)$.

To enlighten the main issues, let us rewrite (1) in the so-called rescaled variables. Let us introduce the rescaled time:

$$s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)}.$$

It is elementary to check that whatever is the blow up behavior of $u(t)$, one always has:

$$s([0, T)) = \mathbf{R}^+.$$

Let us set:

$$v(s,y) = e^{i\gamma(t)}\lambda(t)^{\frac{N}{2}}u(\lambda(t)x + x(t)),$$

then from direct computation, $u(t,x)$ solves (1) on $[0,T)$ iff $v(s,y)$ solves: $\forall s \geq 0$,

$$iv_s + \Delta v - v + v|v|^{\frac{4}{N}} = i\frac{\lambda_s}{\lambda}\left(\frac{N}{2}v + y \cdot \nabla v\right) + i\frac{x_s}{\lambda}\cdot\nabla v + \tilde{\gamma}_s v, \tag{55}$$

where $\tilde{\gamma} = -\gamma - s$. Now from (48), $v(s,y) = Q + \varepsilon(s,y)$, so we may linearize (55) close to Q. The obtained system has the form:

$$i\varepsilon_s + L\varepsilon = i\frac{\lambda_s}{\lambda}\left(\frac{N}{2}Q + y\cdot\nabla Q\right) + \gamma_s Q + i\frac{x_s}{\lambda}\cdot\nabla Q + R(\varepsilon), \tag{56}$$

$R(\varepsilon)$ formally quadratic in ε, and $L = (L_+, L_-)$ is the matrix linearized operator closed to Q which has components:

$$L_+ = -\Delta + 1 - \left(1 + \frac{4}{N}\right)Q^{\frac{4}{N}}, \quad L_- = -\Delta + 1 - Q^{\frac{4}{N}}.$$

A standard approach is to think of equation (56) in the following way: it is essentially a linear equation forced by terms depending on the law for the geometrical parameters. The classical study of this kind of system relies on the understanding of the dispersive properties of the propagator e^{isL} of the linearized operator close to Q. In particular, one needs to exhibit its spectral structure. This has been done by Weinstein, [36], using the variational characterization of Q. The result is the following: L is a non self adjoint operator with a generalized eigenspace at zero. The eigenmodes are explicit and generated by the symmetries of the problem:

$$L_+\left(\frac{N}{2}Q + y\cdot\nabla Q\right) = -2Q \ \ \text{(scaling invariance)},$$

$$L_+(\nabla Q) = 0 \ \ \text{(translation invariance)},$$

$$L_-(Q) = 0 \ \ \text{(phase invariance)}, \quad L_-(yQ) = -2\nabla Q \ \ \text{(Galilean invariance)}.$$

An additional relation is induced by the pseudo-conformal symmetry:

$$L_-(|y|^2 Q) = -4\left(\frac{N}{2}Q + y\cdot\nabla Q\right),$$

and this in turns implies the existence of an additional mode ρ solution to

$$L_+\rho = -|y|^2 Q.$$

These explicit directions induce "growing" solutions to the homogeneous linear equation $i\partial_S\varepsilon + L\varepsilon = 0$. More precisely, there exists a (2N+3) dimensional space S spanned by the above directions such that $H^1 = M \oplus S$ with $|e^{isL}\varepsilon|_{H^1} \leq C$ for $\varepsilon \in M$ and $|e^{isL}\varepsilon|_{H^1} \sim s^3$

for $\varepsilon \in S$. As each symmetry is at the heart of a growing direction, a first idea is to use the symmetries from modulation theory to a priori ensure that ε is orthogonal to S. Roughly speaking, the strategy to construct blow up solutions is then: chose the parameters λ, γ, x so as to get good a priori dispersive estimates on ε in order to build it from a fixed point scheme. Now the fundamental problem is that one has (2N+2) symmetries, but (2N+3) bad modes in the set S. Both constructions in [5] and [31] develop non trivial strategies to overcome this fundamental difficulty of the problem.

Our strategy will be more non linear. On the basis of decomposition (48), we will prove dispersive estimates on ε induced by the virial structure (19). The proof will rely on non linear degeneracies of the structure of (1) around Q. Using then the Hamiltonian information $E_0 < 0$, we will inject these estimates into the finite dimensional dynamic which governs $\lambda(t)$ -which measures the size of the solution- and prove rigidity properties of Lyapounov type. This will then allow us to prove finite time blow up together with the control of the blow up speed.

4.2 Choice of the blow up profile

Before exhibiting the modulation theory type of arguments, we present in this subsection a formal discussion regarding explicit solutions of equation (55) which is inspired from a discussion in [34].

First, let us observe that the key geometrical parameters is λ which measures the size of the solution. Let us then set

$$-\frac{\lambda_s}{\lambda} = b$$

and look for solutions to a simpler version of (55):

$$iv_s + \Delta v - v + ib\left(\frac{N}{2}v + y \cdot \nabla v\right) + v|v|^{\frac{4}{N}} \sim 0.$$

Moreover, from orbital stability property, we want solutions which remain close to Q in H^1. Let us look for solutions of the form $v(s,y) = Q_{b(s)}(y)$ where the mappings $b \to Q_b$ and the law for $b(s)$ are the unknown. We think of b remaining uniformly small and $Q_{b=0} = Q$. Injecting this ersatz into the equation, we get:

$$i\frac{db}{ds}\left(\frac{\partial \overline{Q}_b}{\partial b}\right) + \Delta \overline{Q}_{b(s)} - \overline{Q}_{b(s)} + ib(s)\left(\frac{N}{2}\overline{Q}_{b(s)} + y \cdot \nabla \overline{Q}_{b(s)}\right) + \overline{Q}_{b(s)}|\overline{Q}_{b(s)}|^{\frac{4}{N}} = 0.$$

To handle the linear group, we let $\overline{P}_{b(s)} = e^{i\frac{b(s)}{4}|y|^2}\overline{Q}_{b(s)}$ and solve:

$$i\frac{db}{ds}\left(\frac{\partial \overline{P}_b}{\partial b}\right) + \Delta \overline{P}_{b(s)} - \overline{P}_{b(s)} + \left(\frac{db}{ds} + b^2(s)\right)\frac{|y|^2}{4}\overline{P}_{b(s)} + \overline{P}_{b(s)}|\overline{P}_{b(s)}|^{\frac{4}{N}} = 0. \tag{57}$$

A remarkable fact related to the specific algebraic structure of (1) around Q is that (57) admits three solutions:

- The first one is $(b(s), \overline{P}_{b(s)}) = (0, Q)$, that is the ground state itself. This is just a consequence of the scaling invariance.

- The second one is $(b(s), \overline{P}_{b(s)}) = (\frac{1}{s}, Q)$. This non trivial solution is a rewriting of the explicit critical mass blow up solution $S(t)$ and is induced by the pseudo-conformal symmetry.

- The third one is given by $(b(s), \overline{P}_{b(s)}) = (b, \overline{P}_b)$ for some fixed non zero constant b and $\overline{P}_b$ satisfies:
$$\Delta \overline{P}_b - \overline{P}_b + \frac{b^2}{4}|y|^2\overline{P}_b + \overline{P}_b|\overline{P}_b|^{\frac{4}{N}} = 0. \tag{58}$$
Solutions to this non linear elliptic equation are those who produce the explicit self similar profiles solutions to (39). A simple way to see this is to recall that we have set $b = -\frac{\lambda_s}{\lambda}$, so if b is frozen, we have from $\frac{ds}{dt} = \frac{1}{\lambda^2}$:
$$b = -\frac{\lambda_s}{\lambda} = -\lambda\lambda_t \ \text{ ie } \ \lambda(t) = \sqrt{2b(T-t)},$$
this is the scaling law for the blow up speed.

Now the fundamental point is, see [33], that solutions to (58) never belong to L^2 from a logarithmic divergence at infinity:
$$|P_b(y)| \sim \frac{C(P_b)}{|y|^{\frac{N}{2}}} \ \text{ as } \ |y| \to +\infty.$$
This behavior is a consequence of the oscillations induced by the linear group after the turning point $|y| \geq \frac{2}{|b|}$. Nevertheless, in the ball $|y| < \frac{2}{|b|}$, the operator $-\Delta + 1 - \frac{b^2|y|^2}{4}$ is coercive, and no oscillations will take place in this zone.

Because we track a log-log correction to the self similar law as an upper bound on the blow up speed, profiles $\overline{Q}_b = e^{-i\frac{b}{4}|y|^2}\overline{P}_b$ are natural candidates as refinements of the Q profile in the geometrical decomposition (24). Nevertheless, as they are not in L^2, we need to build a smooth localized version avoiding the non L^2 tale, what according to the above discussion is doable in the coercive zone $|y| < \frac{2}{|b|}$.

Proposition 3 (Localized self similar profiles) *There exist universal constants $C > 0$, $\eta^* > 0$ such that the following holds true. For all $0 < \eta < \eta^*$, there exist constants $\nu^*(\eta) > 0$, $b^*(\eta) > 0$ going to zero as $\eta \to 0$ such that for all $|b| < b^*(\eta)$, let*
$$R_b = \frac{2}{|b|}\sqrt{1-\eta}, \quad R_b^- = \sqrt{1-\eta}R_b,$$

$B_{R_b} = \{y \in \mathbf{R}^N, \quad |y| \leq R_b\}$. *Then there exists a unique radial solution* Q_b *to*

$$\begin{cases} \Delta Q_b - Q_b + ib\left(\frac{N}{2}Q_b + y \cdot \nabla Q_b\right) + Q_b|Q_b|^{\frac{4}{N}} = 0, \\ P_b = Q_b e^{i\frac{b|y|^2}{4}} > 0 \quad in \quad B_{R_b}, \\ Q_b(0) \in (Q(0) - \nu^*(\eta), Q(0) + \nu^*(\eta)), \quad Q_b(R_b) = 0. \end{cases}$$

Moreover, let a smooth radially symmetric cut-off function $\phi_b(x) = 0$ *for* $|x| \geq R_b$ *and* $\phi_b(x) = 1$ *for* $|x| \leq R_b^-$, $0 \leq \phi_b(x) \leq 1$ *and set*

$$\tilde{Q}_b(r) = Q_b(r)\phi_b(r),$$

then

$$\tilde{Q}_b \to Q \quad as \quad b \to 0$$

in some very strong sense, and $\tilde{Q}_b$ *satisfies*

$$\Delta\tilde{Q}_b - \tilde{Q}_b + ib(\tilde{Q}_b)_1 + \tilde{Q}_b|\tilde{Q}_b|^{\frac{4}{N}} = -\Psi_b$$

with

$$Supp(\Psi_b) \subset \{R_b^- \leq |y| \leq R_b\} \quad and \quad |\Psi_b|_{\mathcal{C}^1} \leq e^{-\frac{C}{|b|}}.$$

The meaning of this proposition is that one can build localized profiles $\tilde{Q}_b$ on the ball B_{R_b} which are a smooth function of b and approximate Q in a very strong sense as $b \to 0$, and these profiles satisfy the self similar equation up to an exponentially small term Ψ_b supported around the turning point $\frac{2}{b}$. The proof of this Proposition uses standard variational tools in the setting of non linear elliptic problems. A similar statement is also to be found in [31].

Now one can think of making a formal expansion of $\tilde{Q}_b$ in terms of b, and the first term is non zero:

$$\frac{\partial \tilde{Q}_b}{\partial b}_{|b=0} = -\frac{i}{4}|y|^2 Q.$$

A fundamental degeneracy property is now that the energy of $\tilde{Q}_b$ is degenerated in b at all orders:

$$|E(\tilde{Q}_b)| \leq e^{-\frac{C}{|b|}}, \tag{59}$$

for some universal constant $C > 0$.

The existence of a one parameter family of profiles satisfying the self similar equation up to an exponentially small term and having an exponentially small energy is an algebraic property of the structure of (1) around Q which is at the heart of the existence of the log-log regime.

4.3 Modulation theory

We now are in position to exhibit the sharp decomposition needed for the proof of the log-log upper bound. From Lemma 1 and the proximity of $\tilde{Q}_b$ to Q in H^1, the solution $u(t)$ to (1) is for all time close to the four dimensional manifold

$$\mathcal{M} = \{e^{i\gamma}\lambda^{\frac{N}{2}}\tilde{Q}_b(\lambda y + x), \quad (\lambda, \gamma, x, b) \in \mathbf{R}_+^* \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}\}.$$

We now sharpen the decomposition according to the following Lemma.

Lemma 2 (Non linear modulation of the solution close to $\mathcal{M}$) *There exist $\mathcal{C}^1$ functions of time $(\lambda, \gamma, x, b) : [0, T) \to (0, +\infty) \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}$ such that:*

$$\forall t \in [0, T), \quad \varepsilon(t, y) = e^{i\gamma(t)}\lambda^{\frac{N}{2}}(t)u(t, \lambda(t)y + x(t)) - \tilde{Q}_{b(t)}(y) \tag{60}$$

satisfies:
(i)

$$\left(\varepsilon_1(t), (\Sigma_{b(t)})_1\right) + \left(\varepsilon_2(t), (\Theta_{b(t)})_1\right) = 0, \tag{61}$$

$$\left(\varepsilon_1(t), y\Sigma_{b(t)}\right) + \left(\varepsilon_2(t), y\Theta_{b(t)}\right) = 0, \tag{62}$$

$$-\left(\varepsilon_1(t), (\Theta_{b(t)})_2\right) + \left(\varepsilon_2(t), (\Sigma_{b(t)})_2\right) = 0, \tag{63}$$

$$-\left(\varepsilon_1(t), (\Theta_{b(t)})_1\right) + \left(\varepsilon_2(t), (\Sigma_{b(t)})_1\right) = 0, \tag{64}$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $\tilde{Q}_b = \Sigma_b + i\Theta_b$ in terms of real and imaginary parts;

$$\textit{(ii)} \quad |1 - \lambda(t)\frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}}| + |\varepsilon(t)|_{H^1} + |b(t)| \leq \delta(\alpha^*) \quad \textit{with} \quad \delta(\alpha^*) \to 0 \quad \textit{as} \quad \alpha^* \to 0.$$

Let us insist onto the fact that the reason for this precise choice of orthogonality conditions is a fundamental issue which will be addressed in the next section.

Proof of Lemma 2

This Lemma follows the standard frame of modulation theory and is obtained from Lemma 1 using the implicit function theorem.
From Lemma 1, there exist parameters $\gamma_0(t) \in \mathbf{R}$ and $x_0(t) \in \mathbf{R}^N$ such that with $\lambda_0(t) = \frac{|\nabla Q|_{L^2}}{|\nabla u(t)|_{L^2}}$,

$$\forall t \in [0, T), \quad \left|Q - e^{i\gamma_0(t)}\lambda_0(t)^{\frac{N}{2}}u(\lambda_0(t)x + x_0(t))\right|_{H^1} < \delta(\alpha^*)$$

with $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$. Now we sharpen this decomposition using the fact that $\tilde{Q}_b \to Q$ in H^1 as $b \to 0$, i.e. we chose $(\lambda(t), \gamma(t), x(t), b(t))$ close to $(\lambda_0(t), \gamma_0(t), x_0(t), 0)$ such that

$$\varepsilon(t, y) = e^{i\gamma(t)}\lambda^{1/2}(t)u(t, \lambda(t)y + x(t)) - \tilde{Q}_{b(t)}(y)$$

is small in H^1 and satisfies suitable orthogonality conditions (61), (62), (63) and (64). The existence of such a decomposition is a consequence of the implicit function Theorem. For $\delta > 0$, let $V_\delta = \{v \in H^1(\mathbf{C}); \quad |v - Q|_{H^1} \leq \delta\}$, and for $v \in H^1(\mathbf{C})$, $\lambda_1 > 0$, $\gamma_1 \in \mathbf{R}$, $x_1 \in \mathbf{R}^N$, $b \in \mathbf{R}$ small, define

$$\varepsilon_{\lambda_1,\gamma_1,x_1,b}(y) = e^{i\gamma_1}\lambda_1^{\frac{N}{2}} v(\lambda_1 y + x_1) - \tilde{Q}_b. \tag{65}$$

We claim that there exists $\overline{\delta} > 0$ and a unique C^1 map : $V_{\overline{\delta}} \to (1 - \overline{\lambda}, 1 + \overline{\lambda}) \times (-\overline{\gamma}, \overline{\gamma}) \times B(0, \overline{x}) \times (-\overline{b}, \overline{b})$ such that if $v \in V_{\overline{\delta}}$, there is a unique $(\lambda_1, \gamma_1, x_1, b)$ such that $\varepsilon_{\lambda_1,\gamma_1,x_1,b} = (\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1 + i(\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2$ defined as in (65) satisfies

$$\rho^1(v) = ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1, (\Sigma_b)_1) + ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2, (\Theta_b)_1) = 0,$$

$$\rho^2(v) = ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1, y\Sigma_b) + ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2, y\Theta_b) = 0,$$

$$\rho^3(v) = -((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1, (\Theta_b)_2) + ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2, (\Sigma_b)_2) = 0,$$

$$\rho^4(v) = ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1, (\Theta_b)_1) - ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2, (\Sigma_b)_1) = 0.$$

Moreover, there exists a constant $C_1 > 0$ such that if $v \in V_{\overline{\delta}}$, then $|\varepsilon_{\lambda_1,\gamma_1,x_1}|_{H^1} + |\lambda_1 - 1| + |\gamma_1| + |x_1| + |b| \leq C_1\overline{\delta}$. Indeed, we view the above functionals $\rho^1, \rho^2, \rho^3, \rho^4$ as functions of $(\lambda_1, \gamma_1, x_1, b, v)$. We first compute at $(\lambda_1, \gamma_1, x_1, b, v) = (1, 0, 0, 0, v)$:

$$\frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial x_1} = \nabla v, \quad \frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial \lambda_1} = \frac{N}{2}v + x\cdot\nabla v, \quad \frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial \gamma_1} = iv, \quad \frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial b} = -\left(\frac{\partial \tilde{Q}_b}{\partial b}\right)_{|b=0}.$$

Now recall that $(\tilde{Q}_b)_{|b=0} = Q$ and $\left(\frac{\partial \tilde{Q}_b}{\partial b}\right)_{|b=0} = -i\frac{|y|^2}{4}Q$. Therefore, we obtain at the point $(\lambda_1, \gamma_1, x_1, b, v) = (1, 0, 0, 0, Q)$,

$$\begin{aligned}
&\frac{\partial \rho^1}{\partial \lambda_1} = |Q_1|_2^2, \quad \frac{\partial \rho^1}{\partial \gamma_1} = 0, \quad \frac{\partial \rho^1}{\partial x_1} = 0, \frac{\partial \rho^1}{\partial b} = 0,\\
&\frac{\partial \rho^2}{\partial \lambda_1} = 0, \quad \frac{\partial \rho^2}{\partial \gamma_1} = 0, \quad \frac{\partial \rho^2}{\partial x_1} = -\frac{1}{2}|Q|_2^2, \frac{\partial \rho^2}{\partial b} = 0,\\
&\frac{\partial \rho^3}{\partial \lambda_1} = 0, \quad \frac{\partial \rho^3}{\partial \gamma_1} = -|Q_1|_2^2, \quad \frac{\partial \rho^3}{\partial x_1} = 0, \frac{\partial \rho^3}{\partial b} = 0,\\
&\frac{\partial \rho^4}{\partial \lambda_1} = 0, \quad \frac{\partial \rho^4}{\partial \gamma_1} = 0, \quad \frac{\partial \rho^4}{\partial x_1} = 0, \frac{\partial \rho^4}{\partial b} = \frac{1}{4}|yQ|_2^2.
\end{aligned}$$

The Jacobian of the above functional is non zero, thus the implicit function Theorem applies and conclusion follows. This concludes the proof of Lemma 2.

Let us now write down the equation satisfied by ε in rescaled variables. To simplify notations, we note

$$\tilde{Q}_b = \Sigma + \Theta$$

in terms of real and imaginary parts. We have: $\forall s\in\mathbf{R}_+$, $\forall y\in\mathbf{R}^N$,

$$\begin{aligned}
b_s\frac{\partial\Sigma}{\partial b}+\partial_s\varepsilon_1-M_-(\varepsilon)+b\left(\frac{N}{2}\varepsilon_1+y\cdot\nabla\varepsilon_1\right) &= \left(\frac{\lambda_s}{\lambda}+b\right)\Sigma_1+\tilde{\gamma}_s\Theta+\frac{x_s}{\lambda}\cdot\nabla\Sigma \\
&+ \left(\frac{\lambda_s}{\lambda}+b\right)\left(\frac{N}{2}\varepsilon_1+y\cdot\nabla\varepsilon_1\right)+\tilde{\gamma}_s\varepsilon_2+\frac{x_s}{\lambda}\cdot\nabla\varepsilon_1 \\
&+ Im(\Psi)-R_2(\varepsilon)
\end{aligned}\tag{66}$$

$$\begin{aligned}
b_s\frac{\partial\Theta}{\partial b}+\partial_s\varepsilon_2+M_+(\varepsilon)+b\left(\frac{N}{2}\varepsilon_2+y\cdot\nabla\varepsilon_2\right) &= \left(\frac{\lambda_s}{\lambda}+b\right)\Theta_1-\tilde{\gamma}_s\Sigma+\frac{x_s}{\lambda}\cdot\nabla\Theta \\
&+ \left(\frac{\lambda_s}{\lambda}+b\right)\left(\frac{N}{2}\varepsilon_2+y\cdot\nabla\varepsilon_2\right)-\tilde{\gamma}_s\varepsilon_1+\frac{x_s}{\lambda}\cdot\nabla\varepsilon_2 \\
&- Re(\Psi)+R_1(\varepsilon),
\end{aligned}\tag{67}$$

with $\tilde{\gamma}(s)=-s-\gamma(s)$. The linear operator close to $\tilde{Q}_b$ is now a deformation of the linear operator L close to Q and writes $M=(M_+,M_-)$ with

$$M_+(\varepsilon)=-\Delta\varepsilon_1+\varepsilon_1-\left(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1-\left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_2,$$

$$M_-(\varepsilon)=-\Delta\varepsilon_2+\varepsilon_2-\left(\frac{4\Theta^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_2-\left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_1.$$

Interaction terms are formally quadratic in ε and write:

$$R_1(\varepsilon)=(\varepsilon_1+\Sigma)|\varepsilon+\tilde{Q}_b|^{\frac{4}{N}}-\Sigma|\tilde{Q}_b|^{\frac{4}{N}}-\left(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1-\left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_2,$$

$$R_2(\varepsilon)=(\varepsilon_2+\Theta)|\varepsilon+\tilde{Q}_b|^{\frac{4}{N}}-\Theta|\tilde{Q}_b|^{\frac{4}{N}}-\left(\frac{4\Theta^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_2-\left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_1.$$

Two natural estimates may now be performed:

- First, we may rewrite the conservation laws in the rescaled variables and linearize obtained identities close to Q. This will give crucial degeneracy estimates on some specific order one in ε scalar products.
- Next, we may inject orthogonality conditions of Lemma 2 into equations (66), (67). This will compute the geometrical parameters in their differential form $\frac{\lambda_s}{\lambda},\tilde{\gamma}_s,\frac{x_s}{\lambda},b_s$ in terms of ε: these are the so called modulation equations. This step requires estimating the non linear interaction terms. A crucial point here is to use the fact that the ground state Q is exponentially decreasing in space.

The outcome is the following:

Lemma 3 (First estimates on the decomposition) *We have for all $s \geq 0$:*
(i) Estimates induced by the conservation of the energy and the momentum:

$$|(\varepsilon_1, Q)| \leq \delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}} + C\lambda^2|E_0|, \tag{68}$$

$$|(\varepsilon_2, \nabla Q)| \leq C\delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}}. \tag{69}$$

(ii) Estimate on the geometrical parameters in differential form:

$$\left|\frac{\lambda_s}{\lambda} + b\right| + |b_s| + |\tilde{\gamma}_s| \leq C\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-1|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}}, \tag{70}$$

$$\left|\frac{x_s}{\lambda}\right| \leq \delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-1|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}}, \tag{71}$$

where $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$.

Remark 3 *The exponentially small term in degeneracy estimate (68) is in fact related to the value of $E(\tilde{Q}_b)$, so we use here in a fundamental way non linear degeneracy estimate (59).*

Here are two fundamental comments on Lemma 3:

- First, the norm which appears in the estimates of Lemma 3 is essentially a local norm in space. The conservation of the energy indeed relates the $\int |\nabla\varepsilon|^2$ norm with the local norm. These two norms will turn out to play an equivalent role in the analysis. A key is that no global L^2 norm is needed so far.
- Comparing estimates (70) and (71), we see that the term induced by translation invariance is smaller than the ones induced by scaling and phase invariances. This non trivial fact is an outcome of our use of the Galilean transform to ensure the zero momentum condition (47).

4.4 The virial type dispersive estimate

Our aim in this subsection is to exhibit the dispersive virial type inequality at the heart of the proof of the log-log upper bound. This information will be obtained as a consequence of the virial structure of (1) in Σ.

Let us first recall that virial identity (19) corresponds to two identities:

$$\frac{d^2}{dt^2}\int |x|^2|u|^2 = 4\frac{d}{dt}Im(\int x \cdot \nabla u\overline{u}) = 16E_0. \tag{72}$$

We want to understand what information can be extracted from this dispersive information in the variables of the geometrical decomposition.

To clarify the claim, let us consider an ε solution to the linear homogeneous equation

$$i\partial_s\varepsilon + L\varepsilon = 0 \tag{73}$$

where $L = (L_+, L_-)$ is the linearized operator close to Q. A dispersive information on ε may be extracted using a similar virial law as (19):

$$\frac{1}{2}\frac{d}{ds}Im(\int y\cdot\nabla\varepsilon\overline{\varepsilon}) = H(\varepsilon,\varepsilon), \tag{74}$$

where $H(\varepsilon,\varepsilon) = (\mathcal{L}_1\varepsilon_1,\varepsilon_1) + (\mathcal{L}\varepsilon_2,\varepsilon_2)$ is a Schrödinger type quadratic form decoupled in the real and imaginary parts with explicit Schrödinger operators:

$$\mathcal{L}_1 = -\Delta + \frac{2}{N}\left(\frac{4}{N}+1\right)Q^{\frac{4}{N}-1}y\cdot\nabla Q \quad , \quad \mathcal{L}_2 = -\Delta + \frac{2}{N}Q^{\frac{4}{N}-1}y\cdot\nabla Q.$$

Note that both these operators are of the form $-\Delta + V$ for some smooth well localized time independent potential $V(y)$, and thus from standard spectral theory, they both have a finite number of negative eigenvalues, and then continuous spectrum on $[0,+\infty)$. A simple outcome is then that given an $\varepsilon\in H^1$ which is orthogonal to all the bound states of $\mathcal{L}_1,\mathcal{L}_2$, then $H(\varepsilon,\varepsilon)$ is coercive, that is

$$H(\varepsilon,\varepsilon)\geq\delta_0\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right)$$

for some universal constant $\delta_0>0$. Now assume that for some reason -it will be in our case a consequence of modulation theory and the conservation laws-, ε is indeed for all times orthogonal to the bound states, then injecting the coercive control of $H(\varepsilon,\varepsilon)$ into (74) yields:

$$\frac{1}{2}\frac{d}{ds}Im(\int y\cdot\nabla\varepsilon\overline{\varepsilon})\geq\delta_0\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right). \tag{75}$$

Integrating this in time yields a standard dispersive information: a space time norm is controlled by a norm in space.

We want to apply this strategy to the full ε equation. There are two main obstructions.

First, it is not reasonable to assume that ε is orthogonal to the exact bound states of H. In particular, due to the right hand side in the ε equation, other second order terms will appear which will need be controlled. We thus have to exhibit a set of orthogonality conditions which ensures both the coercivity of the quadratic form H and the control of these other second order interactions. Note that the number of orthogonality conditions we can ensure on ε is the number of symmetries plus the one from b. A first key is the following Spectral Property which is precisely the property which has been proved in dimension $N = 1$ in [20] using the explicit value of Q and checked numerically for $N = 2,3,4$.

Proposition 4 (Spectral Property) *Let $N = 1,2,3,4$. There exists a universal constant $\delta_0 > 0$ such that $\forall \varepsilon \in H^1$,*

$$\begin{aligned} H(\varepsilon,\varepsilon) \quad &\geq \quad \delta_0\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right)-\frac{1}{\delta_0}\left\{(\varepsilon_1,Q)^2+(\varepsilon_1,Q_1)^2+(\varepsilon_1,yQ)^2\right. \\ &+ \quad \left.(\varepsilon_2,Q_1)^2+(\varepsilon_2,Q_2)^2+(\varepsilon_2,\nabla Q)^2\right\}. \end{aligned} \tag{76}$$

To prove this property amounts first counting exactly the number of negative eigenvalues of each Schrödinger operator, and then prove that the specific chosen set of orthogonality conditions, which is not exactly the set of the bound states, is enough to ensure the coercivity of the quadratic form. Both these issues appear to be non trivial when Q is not explicit.

Then, the second major obstruction is the fact that the right hand side $Im(\int y\cdot\nabla\varepsilon\overline{\varepsilon})$ is an unbounded function of ε in H^1. This is a priori a major obstruction to the strategy, *but an additional non linear algebra inherited from virial law (19) rules out this difficulty.*

The formal computation is as follows. Given a function $f\in\Sigma$, we let $\Phi(f) = Im(\int y\cdot\nabla f\overline{f})$. According to (74), we want to compute $\frac{d}{ds}\Phi(\varepsilon)$. Now from (72) and the conservation of the energy:

$$\forall t\in[0,T),\quad \Phi(u(t)) = 4E_0t+c_0$$

for some constant c_0. The key observation is that the quantity $\Phi(u)$ is scaling, phase and also translation invariant from zero momentum assumption (47). From geometrical decomposition (60), we get:

$$\forall t\in[0,T),\quad \Phi(\varepsilon+\tilde{Q}_b) = 4E_0t+c_0.$$

We now expand this according to:

$$\Phi(\varepsilon+\tilde{Q}_b) = \Phi(\tilde{Q}_b)-2(\varepsilon_2,\Sigma_1)+2(\varepsilon_1,\Theta_1)+\Phi(\varepsilon).$$

From explicit computation,

$$\Phi(\tilde{Q}_b) = -\frac{b}{2}|y\tilde{Q}_b|_2^2\sim -Cb$$

for some universal constant $C > 0$. Next, from explicit choice of orthogonality condition (64),

$$(\varepsilon_2,\Sigma_1)-(\varepsilon_1,\Theta_1) = 0.$$

We thus get using $\frac{dt}{ds} = \lambda^2$:
$$(\Phi(\varepsilon))_s \sim 4\lambda^2 E_0 + Cb_s.$$
In other words, to compute the a priori unbounded quantity $(\Phi(\varepsilon))_s$ *for the full non linear equation* is from the virial law equivalent to computing the time derivative of b_s, what of course makes now perfectly sense in H^1.

The virial dispersive structure on $u(t)$ in Σ thus induces a dispersive structure in $L^2_{loc} \cap \dot{H}^1$ on $\varepsilon(s)$ for the full non linear equation.

The key dispersive virial estimate is now the following.

Proposition 5 (Local viriel estimate in ε) *There exist universal constants $\delta_0 > 0$, $C > 0$ such that for all $s \geq 0$, there holds:*
$$b_s \geq \delta_0 \left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - \lambda^2 E_0 - e^{-\frac{C}{|b|}}. \tag{77}$$

Proof of Proposition 5

Using the heuristics, we can compute in a suitable way b_s using orthogonality condition (64). The computation -see Lemma 5 in [21]- yields:
$$\begin{aligned} & \frac{1}{4}|yQ|_2^2 b_s = H(\varepsilon,\varepsilon) + 2\lambda^2|E_0| - \frac{x_s}{\lambda}\{(\varepsilon_2,(\Sigma_1)_y) - (\varepsilon_1,(\Theta_1)_y)\} \quad (78) \\ - \quad & \left(\frac{\lambda_s}{\lambda} + b\right)\{(\varepsilon_2,\Sigma_2) - (\varepsilon_1,\Theta_2)\} - \tilde{\gamma}_s\{(\varepsilon_1,\Sigma_1) + (\varepsilon_2,\Theta_1)\} \\ - \quad & (\varepsilon_1, Re(\Psi)_1) - (\varepsilon_2, Im(\Psi)_1) + (l.o.t), \end{aligned}$$
where the lower order terms may be estimated from the smallogess of ε in H^1:
$$|l.o.t| \leq \delta(\alpha^*) \left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right).$$
We now explain how the choice of orthogonality conditions and the conservation laws allow us to deduce (77).

step 1 Modulation theory for phase and scaling.

The choice of orthogonality conditions (63), (61) has been made to cancel the two second order in ε scalar products in (78):
$$\left(\frac{\lambda_s}{\lambda} + b\right)\{(\varepsilon_2,\Sigma_2) - (\varepsilon_1,\Theta_2)\} + \tilde{\gamma}_s\{(\varepsilon_1,\Sigma_1) + (\varepsilon_2,\Theta_1)\} = 0.$$

step 2 Elliptic estimate on the quadratic form H.

We now need to control the negative directions in the quadratic form as given by Proposition 4. Directions (ε_1, Q_1), (ε_1, yQ), (ε_2, Q_2) and (ε_2, Q_1) are treated thanks to the choice of orthogonality conditions and the closeness of $\tilde{Q}_b$ to Q for $|b|$ small. For example,

$$\begin{aligned}(\varepsilon_2, Q_1)^2 &= |\{(\varepsilon_2, Q_1 - \Sigma_1) + (\varepsilon_1, \Theta_1)\} + (\varepsilon_2, \Sigma_1) - (\varepsilon_1, \Theta_1)|^2 \\ &= |(\varepsilon_2, Q_1 - \Sigma_1) + (\varepsilon_1, \Theta_1)|^2\end{aligned}$$

so that

$$(\varepsilon_2, Q_1)^2 \leq \delta(\alpha^*)(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}).$$

Similarly, we have:

$$(\varepsilon_1, yQ)^2 + (\varepsilon_2, Q_2)^2 + (\varepsilon_1, Q_1)^2 \leq \delta(\alpha^*)(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}). \tag{79}$$

The negative direction $(\varepsilon_1, Q)^2$ is treated from the conservation of the energy which implied (68). The direction $(\varepsilon_2, \nabla Q)$ is treated from the zero momentum condition which ensured (69). Putting this together yields:

$$(\varepsilon_1, Q)^2 + (\varepsilon_2, \nabla Q)^2 \leq \delta(\alpha^*)\left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} + \lambda^2 |E_0|\right) + e^{-\frac{C}{|b|}}.$$

step 3 Modulation theory for translation and use of Galilean invariance.

Galilean invariance has been used to ensure zero momentum condition (47) which in turn led together with the choice of orthogonality condition (62) to degeneracy estimate (71):

$$|\frac{x_s}{\lambda}| \leq C\delta(\alpha^*)(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-1|y|})^{\frac{1}{2}} + e^{-\frac{C}{|b|}}.$$

Therefore, we estimate the term induced by translation invariance in (78) as

$$\left|\frac{x_s}{\lambda}\{(\varepsilon_2, (\Sigma_1)_y) - (\varepsilon_1, (\Theta_1)_y)\}\right| \leq C\delta(\alpha^*)\left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) + e^{-\frac{C}{|b|}}.$$

step 4 Conclusion.

Injecting these estimates into the elliptic estimate (76) yields so far:

$$b_s \geq \delta_0\left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) - 2\lambda^2 E_0 - e^{-\frac{C}{|b|}} - \frac{1}{\delta_0}(\lambda^2 E_0)^2.$$

We now use in a crucial way *the sign of the energy* $E_0 < 0$ and the smallogess $\lambda^2 |E_0| \leq \delta(\alpha^*)$ which is a consequence of the conservation of the energy to conclude. This ends the proof of Proposition 5.

4.5 Monotonicity and control of the blow up speed

Virial dispersive estimate (77) means a control of the excess of mass ε by an exponentially small correction in b in time averaging sense. More specifically, this means that in rescaled variables, the solution writes $\tilde{Q}_b + \varepsilon$ where $\tilde{Q}_b$ is the regular deformation of Q and the rest is in a suitable norm exponentially small in b. This is thus an expansion of the solution with respect to an internal parameter in the problem, b.

This virial control is the first dispersive estimate for the infinite dimensional dynamic driving ε. Observe that it means little by itself if nothing is known about $b(t)$. We shall now inject this information into the finite dimensional dynamic driving the geometrical parameters. The outcome will be *a rigidity property for the parameter $b(t)$ which will in turn imply the existence of a Lyapounov functional in the problem.* This step will again heavily rely on the conservation of the energy. This monotonicity type of results will then allow us to conclude.

We start with exhibiting the rigidity property which proof is a maximum principle type of argument.

Proposition 6 (Rigidity property for b**)** *$b(s)$ vanishes at most once on $\mathbf{R}_+$.*

Note that the existence of a quantity with prescribed sign in the description of the dynamic is unexpected. Indeed, b is no more then the projection of some a priori highly oscillatory function onto a prescribed direction. It is a very specific feature of the blow up dynamic that this projection has a fixed sign.

Proof of Proposition 6

Assume that there exists some time $s_1 \geq 0$ such that $b(s_1) = 0$ and $b_s(s_1) \leq 0$, then from (77), $\varepsilon(s_1) = 0$. Thus from the conservation of the L^2 norm and $\tilde{Q}_b(s_2) = Q$, we conclude $\int |u_0|^2 = \int Q^2$ what contradicts the strictly negative energy assumption. This concludes the proof of Proposition 6.

The nest step is to get the exact sign of b. This is done by injecting virial dispersive information (77) into the modulation equation for the scaling parameter what will yield

$$-\frac{\lambda_s}{\lambda} \sim b. \tag{80}$$

The fundamental monotonicity result is then the following.

Proposition 7 (Existence of an almost Lyapounov functional) *There exists a time $s_0 \geq 0$ such that*

$$\forall s > s_0, \quad b(s) > 0.$$

Moreover, the size of the solution is in this regime an almost Lyapounov functional in the sense that:

$$\forall s_2 \geq s_1 \geq s_0, \quad \lambda(s_2) \leq 2\lambda(s_1). \tag{81}$$

Proof of Proposition 7

step 1 Equation for the scaling parameter.

The modulation equation for the scaling parameter λ inherited from choice of orthogonality condition (61) implied control (70):

$$\left|\frac{\lambda_s}{\lambda} + b\right| \leq C\left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-1|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}},$$

which implies (80) in a weak sense. Nevertheless, this estimate is not good enough to possibly use the virial estimate (77). We claim using extra degeneracies of the equation that (70) can be improved for:

$$\left|\frac{\lambda_s}{\lambda} + b\right| \leq C\left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-1|y|}\right) + e^{-\frac{C}{|b|}} \tag{82}$$

step 2 Use of the virial dispersive relation and the rigidity property.

We now inject virial dispersive relation (77) into (82) to get:

$$\left|\frac{\lambda_s}{\lambda} + b\right| \leq Cb_s + e^{-\frac{C}{|b|}}.$$

We integrate this inequality in time to get: $\forall 0 \leq s_1 \leq s_2$,

$$\left|\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) + \int_{s_1}^{s_2} b(s)ds\right| \leq \frac{1}{4} + \int_{s_1}^{s_2} e^{-\frac{C}{|b(s)|}}ds. \tag{83}$$

The key is now to use rigidity property of Proposition 6 to ensure that $b(s)$ *has a fixed sign for* $s \geq \tilde{s}_0$*, and thus*: $\forall s \geq \tilde{s}_0$,

$$\left|\int_{s_1}^{s_2} e^{-\frac{C}{|b(s)|}}ds\right| \leq \frac{1}{2}\left|\int_{s_1}^{s_2} b(s)ds\right|. \tag{84}$$

step 3 b is positive for s large enough.

Assume that $\left|\int_0^{+\infty} b(s)ds\right| < +\infty$, then b has a fixed sign for $s \geq \tilde{s}_0$ and $|b_s| \leq C$ is straightforward from the equation, so that we conclude: $b(s) \to 0$ as $s \to +\infty$. Now from (83) and (84), this implies that $|\log(\lambda(s))| \leq C$ as $s \to +\infty$, and in particular

$\lambda(s) \geq \lambda_0 > 0$ for s large enough. Injecting this into virial control (77) for s large enough yields:

$$b_s \geq \frac{1}{2}|E_0|\lambda_0^2.$$

Integrating this on large time intervals contradicts the uniform boundedness of b. Here we have used again assumption $E_0 < 0$.
We thus have proved: $\left|\int_0^{+\infty} b(s)ds\right| = +\infty$. Now assume that $b(s) < 0$ for all $s \geq \tilde{s}_1$, then from (83) and (84) again, we conclude that $\log(\lambda(s)) \to 0$ as $s \to +\infty$. Now from $\lambda(t) \sim \frac{1}{|\nabla u(t)|_{L^2}}$, this yields $|\nabla u(t)|_{L^2} \to 0$ as $t \to T$. But from Gagliardo-Nirenberg inequality and the conservation of the energy and the L^2 mass, this implies $E_0 = 0$, contradicting again the assumption $E_0 < 0$.

step 4 Almost monotonicity of the norm.

We now are in position to prove (81). Indeed, injecting the sign of b into (83) and (84) yields in particular: $\forall s_0 \leq s_1 \leq s_2$,

$$\frac{1}{4} + \frac{1}{2}\int_{s_1}^{s_2} b(s)ds \leq -\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) \leq \frac{1}{4} + 2\int_{s_1}^{s_2} b(s)ds, \tag{85}$$

and thus:

$$\forall s_0 \leq s_1 \leq s_2, \quad -\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) \geq \frac{1}{4},$$

what yields (81). This concludes the proof of Proposition 7.

Note that from the above proof, we have obtained $\int_0^{+\infty} b(s)ds = +\infty$, and thus from (85):

$$\lambda(s) \to 0 \quad \text{as} \quad s \to \infty, \tag{86}$$

that is finite or infinite time blow up. *On the contrary to the virial argument, the blow up proof is no longer obstructive but completely dynamical, and relies mostly on the rigidity property of Proposition 6.*

Let us conclude these notes by finishing the proof of Theorem 7. We need to prove finite time blow up together with the log-log upper bound (33) on blow up rate. The proof goes as follows.

step 1 Lower bound on $b(s)$.

We claim: there exist some universal constant $C > 0$ and some time $s_1 > 0$ such that $\forall s \geq s_1$,

$$Cb(s) \geq \frac{1}{\log|\log(\lambda(s))|}. \tag{87}$$

Indeed, first recall (77). Now that we know the sign of $b(s)$ for $s \geq s_0$, we may view this inequality as a differential inequality for b for $s > s_0$:

$$b_s \geq -e^{-\frac{C}{b}} \geq -b^2 e^{-\frac{C}{2b}} \quad \text{ie} \quad -\frac{b_s}{b^2} e^{\frac{C}{2b}} \leq 1.$$

We integrate this inequality from the non vanishing property of b and get for $s \geq \tilde{s}_1$ large enough:

$$e^{\frac{C}{b(s)}} \leq s + e^{\frac{C}{b(1)}} \leq 2Cs \quad \text{ie} \quad b(s) \geq \frac{C}{\log(s)}. \tag{88}$$

We now recall (85) on the time interval $[\tilde{s}_1, s]$:

$$\frac{1}{2}\int_{\tilde{s}_1}^{s} b \leq -\log\left(\frac{\lambda(s)}{\lambda(\tilde{s}_1)}\right) + \frac{1}{4} \leq -2\log(\lambda(s))$$

for $s \geq \tilde{s}_2$ large enough from $\lambda(s) \to 0$ as $s \to +\infty$. Inject (88) into the above inequality, we get for $s \geq \tilde{s}_3$

$$C\frac{s}{\log(s)} \leq \int_{\tilde{s}_2}^{s} \frac{Cd\tau}{\log(\tau)} \leq \frac{1}{4}\int_{\tilde{s}_2}^{s} b \leq -\log(\lambda(s)) \quad \text{ie} \quad |\log(\lambda(s))| \geq C\frac{s}{\log(s)}$$

for some universal constant $C > 0$, and thus for s large

$$\log|\log(\lambda(s))| \geq \log(s) - \log(\log(s)) \geq \frac{1}{2}\log(s)$$

and conclusion follows from (88). This concludes the proof of (87).

step 2 Finite time blow up and control of the blow up speed.

We first use the finite or infinite time blow up result (86) to consider a sequence of times $t_n \to T \in [0, +\infty]$ defined for n large such that

$$\lambda(t_n) = 2^{-n}.$$

Let $s_n = s(t_n)$ the corresponding sequence and $\bar{t}$ such that $s(\bar{t}) = s_0$ given by Proposition 7. Note that we may assume $n \geq \overline{n}$ such that $t_n \geq \bar{t}$. Remark that $0 < t_n < t_{n+1}$ from (81), and so $0 < s_n < s_{n+1}$. Moreover, there holds from (81)

$$\forall s \in [s_n, s_{n+1}], \quad 2^{-n-1} \leq \lambda(s) \leq 2^{-(n-1)}. \tag{89}$$

We now claim that (33) follows from a control from above of the size of the intervals $[t_n, t_{n+1}]$ for $n \geq \overline{n}$.
Let $n \geq \overline{n}$. (87) implies

$$\int_{s_n}^{s_{n+1}} \frac{ds}{\log|\log(\lambda(s))|} \leq C\int_{s_n}^{s_{n+1}} b(s)ds.$$

(85) with $s_1 = s_n$ and $s_2 = s_{n+1}$ yields:

$$\frac{1}{2}\int_{s_n}^{s_{n+1}} b(s) \leq \frac{1}{4} - |yQ|_{L^2}^2 \log(\frac{\lambda(s_{n+1})}{\lambda(s_n)}) \leq C.$$

Therefore,

$$\forall n \geq \overline{n}, \quad \int_{s_n}^{s_{n+1}} \frac{ds}{\log|\log(\lambda(s))|} \leq C.$$

Now we change variables in the integral at the left of the above inequality according to $\frac{ds}{dt} = \frac{1}{\lambda^2(s)}$ and estimate with (89):

$$C \geq \int_{s_n}^{s_{n+1}} \frac{ds}{\log|\log(\lambda(s))|} = \int_{t_n}^{t_{n+1}} \frac{dt}{\lambda^2(t)\log|\log(\lambda(t))|} \geq \frac{1}{10\lambda^2(t_n)\log|\log(\lambda(t_n))|}\int_{t_n}^{t_{n+1}} dt$$

so that

$$t_{n+1} - t_n \leq C\lambda^2(t_n)\log|\log(\lambda(t_n))|.$$

From $\lambda(t_n) = 2^{-n}$ and summing the above inequality in n, we first get

$$T < +\infty$$

and

$$\begin{aligned}
C(T - t_n) &\leq \sum_{k\geq n} 2^{-2k}\log(k) = \sum_{n\leq k\leq 2n} 2^{-2k}\log(k) + \sum_{k\geq 2n} 2^{-2k}\log(k)\\
&\leq C2^{-2n}\log(n) + 2^{-4n}\log(2n)\sum_{k\geq 0} 2^{-2k}\frac{\log(2n+k)}{\log(2n)}\\
&\leq C2^{-2n}\log(n) + C2^{-4n}\log(n) \leq C2^{-2n}\log(n) \leq C\lambda^2(t_n)\log|\log(\lambda(t_n))|.
\end{aligned}$$

From the monotonicity of λ (81), we extend the above control to the whole sequence $t \geq \overline{t}$. Let $t \geq \overline{t}$, then $t \in [t_n, t_{n+1}]$ for some $n \geq \overline{n}$, and from $\frac{1}{2}\lambda(t_n) \leq \lambda(t) \leq 2\lambda(t_n)$, we conclude

$$\lambda^2(t)\log|\log(\lambda(t))| \geq C\lambda^2(t_n)\log|\log(\lambda(t_n))| \geq C(T - t_n) \geq C(T - t).$$

Now remark that the function $f(x) = x^2\log|\log(x)|$ is non decreasing in a neighborhood at the right of $x = 0$, and moreover

$$f\left(\frac{C}{2}\sqrt{\frac{T-t}{\log|\log(T-t)|}}\right) = \frac{C^2}{4}\frac{(T-t)}{\log|\log(T-t)|}\log\left|\log\left(C\sqrt{\frac{T-t}{\log|\log(T-t)|}}\right)\right| \leq C(T-t)$$

for t close enough to T, so that we get for some universal constant C^*:

$$f(\lambda(t)) \geq f\left(C^*\sqrt{\frac{T-t}{\log|\log(T-t)|}}\right) \quad \text{ie} \quad \lambda(t) \geq C^*\sqrt{\frac{T-t}{\log|\log(T-t)|}}$$

and (33) is proved.

This concludes the proof of Theorem 7.

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