Legendrian dualities and spacelike hypersurfaces in the lightcone

Shyuichi IZUMIYA

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Abstract

We show four Legendrian dualities between pseudo-spheres in Minkowski space as a basic theorem. We can apply such dualities for constructing extrinsic differential geometry of spacelike hypersurfaces in pseudo-spheres. In this paper we stick to spacelike hypersurfaces in the lightcone and establish an extrinsic differential geometry which we call the *lightcone differential geometry.*

1 Introduction

In this paper we present some results of the project constructing the extrinsic differential geometry on submanifolds of pseudo-spheres in Minkowski space (cf., [16, 17, 18, 19, 20, 21, 22, 23]). In particular we stick to spacelike hypersurfaces in the lightcone here. It has been known in [2] (cf., Theorem 3.1) that a simply connected Riemannian manifold is conformally flat if and only if it can be embedded as a spacelike hypersurface in the lightcone. According to this result, if we study an extrinsic differential geometry of spacelike hypersurfaces in the lightcone, then we might obtain new extrinsic invariants of conformally flat Riemannian manifolds. This is a main motivation for the study of spacelike hypersurfaces in the lightcone. Moreover, the situation in this case is quite different from other submanifold theories because the metric on the lightcone is degenerate (cf., [3, 5, 6, 7, 8, 10, 11, 16, 33, 41, 43, 44, 46]). Therefore we cannot apply the ordinary submanifold theory of semi-Riemannian geometry (cf., [37]). Instead of such a theory we need a new method.

On the other hand, in the classical theory of hypersurfaces in Euclidean space the Gauss map plays a principal role to define geometric invariants. The derivation of the Gauss map (i.e., the Weingarten map) induces the principal curvatures, the Gauss-Kronecker curvature and the mean curvature of the hypersurface. In [5] Bleeker and Wilson studied the singularities of the Gauss map of a surface in Euclidean 3-space. In their paper, the main theorem asserts that the generic singularities of Gauss maps are folds or cusps. Later that Banchoff et al [3],Landis [26] and Platonova [40] have studied geometric meanings of cusps of the Gauss map

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of a surface. Bruce [7] and Romero-Fuster [42] have also independently studied the singularities of the Gauss map and the dual of a hypersurface in Euclidean space. The singularity of the dual of a hypersurface is deeply related to the singularity of the Gauss map of the hypersurface. Their main tool for the study is the family of height functions on a hypersurface. It has been classically known that the singular set of the Gauss map is the parabolic set of the surface and it can be interpreted as the criminant set of the family of height functions. This is the reason why they adopted the height function for the study of Gauss maps. They applied the deformation theory of smooth functions to the height function and derived geometric results on Gauss maps. We can interpret that these results on Gauss maps describe the contact between hypersurfaces and hyperplanes. It is called the "flat geometry" of hypersurfaces in Euclidean space. It also has been known that Gauss maps of hypersurfaces are Lagrangian maps. Moreover, the generic singularities of Gauss maps of hypersurfaces and Lagrangian maps are the same [1]. Singularities of projective Gauss maps are also studied by McCrory et al [31, 32]. There are many other articles concerning the singularities of Gauss maps, we only refer here to the book [3]. If a hypersurface is located in hyperbolic space, we can construct the unit normal vector field along the hypersurface by an analogous method to the case for hypersurfaces in Euclidean space. In [16] we have studied geometric properties of hypersurfaces in hyperbolic space associated to the contact with hyperhorospheres. We call this geometry the "horospherical geometry" of hypersurfaces in hyperbolic space. The main tool for the study of hypersurfaces in hyperbolic space is the notion of hyperbolic Gauss maps which has been independently introduced by Bryant [6] and Epstein [10] in the Poincaré ball model. Kobayashi [25] has also independently defined it for a hypersurface in $H^n(\mathbb{R}) = SO_0(n,1)/SO(n)$. It is a quite useful tool for the study of mean curvature one surfaces in hyperbolic space [6, 47]. For fundamental concepts and results in this area, please refer [6, 10, 11, 39]. The target of the hyperbolic Gauss map is the boundary sphere of the Poincaré ball in the original definition. In [16] we have studied hypersurfaces in the Minkowski space model of hyperbolic space (i.e., the pseudo-sphere with negative radius). In this case the corresponding hyperbolic Gauss map is a mapping from the hypersurface to the spacelike sphere on the lightcone. Instead of the notion of hyperbolic Gauss map we have defined the hyperbolic Gauss indicatrix on the lightcone whose singular set is the same as that of the hyperbolic Gauss map. Minkowski space is originally from the relativity theory in Physics (i.e., Lorentzian geometry in Mathematics). We refer to the book [37] for general properties of Minkowski space and Lorentzian geometry. We remark that we can also construct the similar geometry on spacelike hypersurfaces in de Sitter space (i.e, the pseudo-sphere with a positive radius) analogous to the hyperbolic case.

On the other hand, for a spacelike hypersurface in the lightcone (i.e., the pseudo-sphere with zero radius), we cannot construct "normal vector fields" in the tangent space of the lightcone. In this paper we show four Legendrian dualities between pseudo-spheres in Minkowski space as a basic theorem (cf., Theorem 2.2). The case for hypersurfaces in hyperbolic space in [16] can also be interpreted as an application of the basic theorem (cf., §2). We can obtain a kind of normal vector fields to a spacelike hypersurface in the lightcone as an application of the basic Legendrian duality theorem. By using this "normal vector field", we define a mapping to the lightcone which is called the *lightcone Gauss image* (cf., §3). It follows from the properties of the Legendrian duality that we show the derivation of the lightcone Gauss image can be interpreted as a linear transformation on the tangent space. We call it the *lightcone Weingarten map*. Therefore we can define the *lightcone principal curvature* κ_ℓ, the *lightcone Gauss-Kronecker curvature* K_ℓ and the *lightcone mean curvature* H_ℓ for a spacelike hypersurface in the lightcone. We study totally umbilic spacelike hypersurfaces under this framework and give a classification in §3. Such a spacelike hypersurface is a quadric hypersurface in the lightcone (i.e, the intersection of the lightcone with a hyperplane in Minkowski space). We briefly call it the *hyperquadric*. There are three kinds of hyperquadrics. The flat one is the *parabolic hyperquadric*. In §4–7 we study local differential geometry from the contact viewpoint of spacelike hypersurfaces with parabolic hyperquadrics as applications of the theory of Legendrian singularities (cf., the appendix). We consider generic properties in §8. In §9, we show the Gauss-Bonnet type theorem for the *normalized lightcone Gauss-Kronecker* curvature $\overline{K}_\ell$. Locally the normalized lightcone Gauss-Kronecker curvature has the similar properties as the lightcone Gauss-Kronecker curvature (cf., Corollary 9.3). We study spacelike surfaces in the 3-dimensional lightcone in §10. We can show the analogous result of *Theorema Egregium* of Gauss (cf., Proposition 10.2). However, as a corollary of Proposition 10.2, we show that the lightcone mean curvature is equal to the sectional curvature of the spacelike surface (cf., Theorem 10.3). This is really a "surprising theorem" because the lightcone Gauss-Kronecker curvature is an extrinsic invariant but the lightcone mean curvature is an intrinsic invariant. In the remaining part of §10, we study geometric meanings of generic singularities of the lightcone Gauss image and give a relationship between the Euler number of the global lightcone Gauss image and geometric invariants (cf., Theorem 10.7). We give some examples in §11. We give the definitions of parallels and evolutes of spacelike hypersurfaces in the lightcone in §12. Concerning those notions, we can easily recognize that the situation is a quite different from other geometry. Such parallels and evolutes cannot be located in the lightcone at any case. Moreover those definitions unify the notion of parallels and evolutes in other pseudo-spherical geometry. We will describe detailed properties of such unified notions of parallels and evolutes in the forthcoming paper.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 Basic notations and the duality theorem

In this section we prepare basic notions on Minkowski space and contact geometry. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) | x_i \in \mathbb{R},\ i = 0, 1, \dots, n\}$ be an $(n+1)$-dimensional vector space. For any vectors $\boldsymbol{x} = (x_0, \dots, x_n)$, $\boldsymbol{y} = (y_0, \dots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *Minkowski n+1-space* and denoted by $\mathbb{R}^{n+1}_1$.

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1} \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Given a vector $\boldsymbol{n} \in \mathbb{R}^{n+1}_1$ and a real number c, the hyperplane with pseudo normal $\boldsymbol{n}$ is given by

$$HP(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}.$$

We say that $HP(\boldsymbol{n}, c)$ is a *spacelike* , *timelike* or *lightlike hyperplane* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively. In this paper we use the following basic facts.

Lemma 2.1 *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n+1}_1$ be lightlike vectors. If $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$, then $\boldsymbol{x}, \boldsymbol{y}$ are linearly dependent.*

Proof. Suppose that $\boldsymbol{x}, \boldsymbol{y}$ are linearly independent. Let N be the two dimensional subspace of $\mathbb{R}^{n+1}_1$ generated by $\boldsymbol{x}, \boldsymbol{y}$. Then all vectors in N are lightlike. We consider the subspace

$\mathbb{R}^n_0 = \{\boldsymbol{x} = (x_0, x_1, \ldots, x_b) \in \mathbb{R}^{n+1}_1 \mid x_0 = 0\}$. Then $N \cap \mathbb{R}^n_0$ is a positive dimensional subspace in $\mathbb{R}^{n+1}_1$. However the vector $\boldsymbol{x} \in N \cap \mathbb{R}^n_0$ is lightlike and spacelike. This is a contradiction. $\square$

We have the following three kinds of pseudo-spheres in $\mathbb{R}^{n+1}_1$: The *Hyperbolic n-space* is defined by

$$H^n(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

the *de Sitter n-space* by

$$S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \}$$

and the (*open*) *lightcone* by

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \}.$$

We also define

$$LC^*_+ = \{\boldsymbol{x} \in LC^* \mid x_0 > 0\}$$

and call it the *future lightcone.* If $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ is a non-zero lightlike vector, then $x_0 \neq 0$. Therefore we have

$$\tilde{\boldsymbol{x}} = (1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in S^{n-1}_+ = \{\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0, \ x_0 = 1 \}.$$

We call S^{n-1}_+ the *lightcone* (*or, spacelike unit*) $(n-1)$-*sphere*. In this paper we stick to spacelike hypersurfaces in the lightcone LC^*. Typical such hypersurfaces are given by the intersection of LC^* with a hyperplane in $\mathbb{R}^{n+1}_1$:

$$HL(\boldsymbol{n}, c) = HP(\boldsymbol{n}, c) \cap LC^*.$$

We say that $HL(\boldsymbol{n}, c)$ is a *quadric hypersurface in the lightcone* (or briefly, *hyperquadric*). We also say that $HL(\boldsymbol{n}, c)$ is *elliptic*, *hyperbolic* or *parabolic* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively. These hyperquadrics are the candidates of totally umbilic spacelike hypersurfaces in the lightcone (cf., §3).

We now review some properties of contact manifolds and Legendrian submanifolds. The detailed properties is described in the appendix. Let N be a $(2n+1)$-dimensional smooth manifold and K be a tangent hyperplane field on N. Locally such a field is defined as the field of zeros of a 1-form α. The tangent hyperplane field K is *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of N. We say that (N, K) is a *contact manifold* if k is a non-degenerate hyperplane filed. In this case K is called a *contact structure* and α is a *contact form.* Let $\phi : N \longrightarrow N'$ be a diffeomorphism between contact manifolds (N, K) and (N', K'). We say that ϕ is a *contact diffeomorphism* if $d\phi(K) = K'$. Two contact manifolds (N, K) and (N', K') are *contact diffeomorphic* if there exists a contact diffeomorphism $\phi : N \longrightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if $\dim L = n$ and $di_x(T_xL) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \to M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \to M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \to M$ is called a *Legendrian map.* The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of i which is denoted by $W(i)$. For any $p \in E$, it is known that there is a local coordinate system $(x_1, \ldots, x_m, p_1, \ldots, p_m, z)$ around p such that

$$\pi(x_1, \ldots, x_m, p_1, \ldots, p_m, z) = (x_1, \ldots, x_m, z)$$

and the contact structure is given by the 1-form

$$\alpha = dz - \sum_{i=1}^{m} p_i dx_i$$

(cf. [1], 20.3).

One of the examples of Legendrian fibrations is given by the unit spherical tangent bundle of a Riemannian manifold. Let M be a Riemannian manifold and TM is its tangent bundle. Let $(x_1, \dots, x_n)$ be local coordinates on a neighbourhood U of M and $(v_1, \dots, v_n)$ coordinates on the fiber over U. Let g_{ij} be the components of the metric $\langle\ ,\ \rangle$ with respect to the above coordinates. Then the canonical one-form can be locally defined by $\theta = \sum_{i,j} g_{ij} v_j dq_i$ where $q_i = x_i \circ \pi$ for the projection $\pi : TM \longrightarrow M$. Let $\widetilde{\pi} : T_1M \longrightarrow M$ be the unit spherical tangent bundle with respect to the metric $\langle\ ,\ \rangle$ The the restriction of θ onto T_1M gives a contact structure and $\widetilde{\pi} : T_1M \longrightarrow M$ is a Legendrian fibration (cf., [4]).

We now show the basic theorem in this paper which is the fundamental tool for the study of hypersurfaces in pseudo-spheres in Minkowski space. We consider the following four double fibrations:

(1)(a) $H^n(-1) \times S^n_1 \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\ \}$,
(b) $\pi_{11} : \Delta_1 \longrightarrow H^n(-1)$,$\pi_{12} : \Delta_1 \longrightarrow S^n_1$,
(c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1$, $\theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_1$.

(2)(a) $H^n(-1) \times LC^* \supset \Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -1\ \}$,
(b) $\pi_{21} : \Delta_2 \longrightarrow H^n(-1)$,$\pi_{22} : \Delta_2 \longrightarrow LC^*$,
(c) $\theta_{21} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_2$, $\theta_{22} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_2$.

(3)(a) $LC^* \times S^n_1 \supset \Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 1\ \}$,
(b) $\pi_{31} : \Delta_3 \longrightarrow LC^*$,$\pi_{32} : \Delta_3 \longrightarrow S^n_1$,
(c) $\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_3$, $\theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_3$.

(4)(a) $LC^* \times LC^* \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -2\ \}$,
(b) $\pi_{41} : \Delta_4 \longrightarrow LC^*$,$\pi_{42} : \Delta_4 \longrightarrow LC^*$,
(c) $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_4$, $\theta_{42} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_4$.

Here,$\pi_{i1}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{i2}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$, $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = -w_0 dv_0 + \sum_{i=1}^n w_i dv_i$ and $\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = -v_0 dw_0 + \sum_{i=1}^n v_i dw_i$.

We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i. The basic theorem in this paper is the following theorem:

Theorem 2.2 *Under the same notations as the previous paragraph, each (Δ_i, K_i) $(i = 1, 2, 3, 4)$ is a contact manifold and both of π_{ij} $(j = 1, 2)$ are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic each other.*

Proof. By definition we can easily show that each Δ_i $(i = 1, 2, 3, 4)$ is a smooth submanifold in $\mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1$ and each π_{ij} $(i = 1, 2, 3, 4; j = 1, 2)$ is a smooth fibration.

We now show that (Δ_1, K_1) is a contact manifold. Since $H^n(-1)$ is a spacelike hypersurface in $\mathbb{R}^{n+1}_1$, $\langle\ ,\ \rangle | H^n(-1)$ is a Riemannian metric. Let $\pi : S(TH^n(-1)) \longrightarrow H^n(-1)$ be the unit tangent sphere bundle of $H^n(-1)$. For any $\boldsymbol{v} \in H^n(-1)$, we have the local coordinates

$(v_1, \ldots, v_n)$ such that $\boldsymbol{v} = (\pm\sqrt{v_1^2 + \cdots v_n^2 + 1}, v_1, \ldots, v_n)$. We can represent the tangent vector $\boldsymbol{w} \in T_v H^n(-1)$ by

$$\boldsymbol{w} = (\pm\frac{1}{v_0}\sum_{i=1}^{n} w_i v_i, w_1, \ldots, w_n).$$

It follows that $\langle \boldsymbol{w}, \boldsymbol{v}\rangle = \pm\frac{1}{v_0}\sum_{i=1}^{n} w_i v_i)(\mp v_0) + \sum_{i=1}^{n} w_i v_i = 0$. Therefore $\boldsymbol{w} \in S(T_v H^n(-1))$ if and only if

$$\langle \boldsymbol{w}, \boldsymbol{w}\rangle = 1 \text{ and } \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0.$$

The last conditions are equivalent to the condition that $(\boldsymbol{v}, \boldsymbol{w}) \in \Delta_1$. This means that we can canonically identify $S(TH^n(-1))$ with Δ_1. Moreover, the canonical contact structure on $S(TH^n(1))$ is given by the one-form $\theta(V) = \langle d\pi(V), \tau(V)\rangle$ where $\tau : TS(TH^n(-1)) \longrightarrow S(TH^n(-1))$ is the tangent bundle of $S(TH^n(-1))$ (cf., §2 and [4, 9]). It can be represented by $\langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_1$ by the above identification. Thus $(\Delta_1, \theta_{11}^{-1}(0))$ is a contact manifold.

On the other hand, let $X = (\boldsymbol{\xi}, \boldsymbol{\eta})$ be a tangent vector of $\mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1$ at $(\boldsymbol{v}, \boldsymbol{w})$ which is represented by

$$X = \sum_{i=0}^{n} \xi \frac{\partial}{\partial v_i} + \sum_{j=0}^{n} \eta_j \frac{\partial}{\partial w_j}.$$

We can show that $X \in T_{(v,w)}\Delta_1$ if and only if

$$\langle \boldsymbol{\xi}, \boldsymbol{v}\rangle = \langle \boldsymbol{\eta}, \boldsymbol{w}\rangle = \langle \boldsymbol{\xi}, \boldsymbol{w}\rangle + \langle \boldsymbol{\eta}, \boldsymbol{v}\rangle = 0.$$

It follows that

$$\theta_{11}(X) = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle(X) = \langle \boldsymbol{\xi}, \boldsymbol{w}\rangle = -\langle \boldsymbol{\eta}, \boldsymbol{v}\rangle = -\langle \boldsymbol{v}, d\boldsymbol{w}\rangle(X) = -\theta_{12}(X).$$

Therefore both of θ_{11} and θ_{12} give the common contact structure on Δ_1.

We now consider $\Delta_2 \subset H^n(-1) \times LC^*$. By the same reason as the above case, both of θ_{21} and θ_{22} give the common tangent hyperplane filed on Δ_2. We define a smooth mapping

$$\Phi_{21} : \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1$$

by $\Phi_{21}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, \boldsymbol{v} - \boldsymbol{w})$. We can easily check that $\Phi_{21}(\Delta_2) = \Delta_1$ and $\Phi_{21}(\Delta_1) = \Delta_2$. Since Φ_{21} is an involution, $\Phi_{21}|\Delta_2$ is a diffeomorphism onto Δ_1. Moreover, we have

$$\Phi_{21}^*\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{v} - \boldsymbol{w}\rangle|\Delta_2 = (\langle d\boldsymbol{v}, \boldsymbol{v}\rangle - \langle d\boldsymbol{v}, \boldsymbol{w}\rangle)|\Delta_2 = -\langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_2 = -\theta_{21},$$

so that θ_{21} gives a contact structure on Δ_2 and $\Phi_{21}|\Delta_2$ is a contact diffeomorphism.

We also consider $\Delta_3 \subset LC^* \times S^n_1$. We define a smooth mapping

$$\Phi_{31} : \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1$$

by $\Phi_{31}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v} - \boldsymbol{w}, \boldsymbol{w})$. We can also check that $\Phi_{31}(\Delta_3) = \Delta_1$. The converse mapping

$$\Phi_{13} : \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1$$

is given by $\Phi_{13}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v} + \boldsymbol{w}, \boldsymbol{w})$. It can be shown that $\Phi_{13}(\Delta_1) = \Delta_3$. Therefore $\Phi_{31}|\Delta_3$ is a diffeomorphism onto Δ_1. Moreover, we have

$$\Phi_{31}^*\theta_{11} = \langle d(\boldsymbol{v} - \boldsymbol{w}), \boldsymbol{w}\rangle|\Delta_3 = (\langle d\boldsymbol{v}, \boldsymbol{w}\rangle - \langle d\boldsymbol{w}, \boldsymbol{w}\rangle)|\Delta_3 = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_3 = \theta_{31},$$

so that θ_{31} gives a contact structure on Δ_3 and $\Phi_{31}|\Delta_3$ is a contact diffeomorphism. By the same reason as the previous cases, θ_{31} and θ_{32} give the common contact structure on Δ_3.

Finally we consider $\Delta_4 \subset LC^* \times LC^*$. We define a smooth mapping

$$\Phi_{14} : \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1$$

by $\Phi_{14}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}+\boldsymbol{w}, \boldsymbol{v}-\boldsymbol{w})$. The converse mapping is given by

$$\Phi_{41}(\boldsymbol{v}, \boldsymbol{w}) = \left(\frac{\boldsymbol{v}+\boldsymbol{w}}{2}, \frac{\boldsymbol{v}-\boldsymbol{w}}{2}\right).$$

We can also check that $\Phi_{14}(\Delta_1) = \Delta_4$ and $\Phi_{41}(\Delta_4) = \Delta_1$, so that $\Phi_{14}|\Delta_1$ and $\Phi_{41}|\Delta_4$ are diffeomorphisms. Moreover, we have

$$\begin{aligned}
\Phi_{14}^*\theta_{41} &= \langle d(\boldsymbol{v}+\boldsymbol{w}), \boldsymbol{v}-\boldsymbol{w}\rangle|\Delta_1 \\
&= (\langle d\boldsymbol{v}, \boldsymbol{v}\rangle - \langle d\boldsymbol{v}, \boldsymbol{w}\rangle + \langle d\boldsymbol{w}, \boldsymbol{v}\rangle - \langle d\boldsymbol{w}, \boldsymbol{w}\rangle)|\Delta_1 \\
&= -\langle d\boldsymbol{v}, \boldsymbol{w}\rangle + \langle d\boldsymbol{w}, \boldsymbol{v}\rangle|\Delta_1 \\
&= 2\theta_{12},
\end{aligned}$$

so that θ_{41} gives a contact structure on Δ_4 and $\Phi_{14}|\Delta_1$ is a contact diffeomorphism. By the same reason as the previous cases, θ_{41} and θ_{42} give the common contact structure on Δ_4. Other assertions are trivial by definition.

This completes the proof. $\square$

We give a quick review on the previous results on hypersurfaces in hyperbolic space (cf., [16]) here and interpret the results via the duality theorem.

Given n vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathbb{R}^{n+1}_1$, we can define the wedge product $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n$ as follows:

$$\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ a_0^1 & a_1^1 & \cdots & a_n^1 \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{vmatrix},$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{a}_i = (a_0^i, a_1^i, \ldots, a_n^i)$. We can easily check that

$$\langle \boldsymbol{a}, \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n\rangle = \det(\boldsymbol{a}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n),$$

so that $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n$ is pseudo orthogonal to $\boldsymbol{a}_i$, $\forall i = 1, \ldots, n$. In [16] we have studied the extrinsic differential geometry of hypersurfaces in $H^n(-1)$. Let $\boldsymbol{x} : U \longrightarrow H^n_+(-1)$ be a regular hypersurface (i.e., an embedding), where $U \subset \mathbb{R}^{n-1}$ is an open subset. We denote that $M = \boldsymbol{x}(U)$ and identify M with U by the embedding $\boldsymbol{x}$. Since $\langle \boldsymbol{x}, \boldsymbol{x}\rangle \equiv -1$, we have $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x}\rangle \equiv 0 \quad (i = 1, \ldots, n-1)$, where $u = (u_1, \ldots u_{n-1}) \in U$. Define a vector

$$\boldsymbol{e}(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}(u)\|},$$

then we have

$$\langle \boldsymbol{e}, \boldsymbol{x}_{u_i}\rangle \equiv \langle \boldsymbol{e}, \boldsymbol{x}\rangle \equiv 0, \quad \langle \boldsymbol{e}, \boldsymbol{e}\rangle \equiv 1.$$

Therefore the vector $\boldsymbol{x} \pm \boldsymbol{e}$ is lightlike. We define maps

$$\mathbb{E} : U \longrightarrow S^n_1 \text{ and } \mathbb{L}^\pm : U \longrightarrow LC^*$$

by $\mathbb{E}(u) = \boldsymbol{e}(u)$ and $\mathbb{L}^\pm(u) = \boldsymbol{x}(u) \pm \boldsymbol{e}(u)$ which are called the *de Sitter Gauss image* and the *lightcone Gauss image* of M which has been called the hyperbolic Gauss indicatrix of M in [16]. We have study the extrinsic differential geometry of $\boldsymbol{x}(U) = M$ by using both of the de Sitter Gauss image $\mathbb{E}$ and the lightcone Gauss image $\mathbb{L}^\pm$ like as the unit normal of a hypersurface in Euclidean space in [16, 17] . For any $p = \boldsymbol{x}(u_0) \in M$ and $\boldsymbol{v} \in T_pM$, we can show that $D_v\mathbb{E} \in T_pM$, where D_v denotes the *covariant derivative* with respect to the tangent vector $\boldsymbol{v}$. Since the derivative $d\boldsymbol{x}(u_0)$ can be identified with the identity mapping 1_{T_pM} on the tangent space T_pM, we have

$$d\mathbb{L}^\pm(u_0) = 1_{T_pM} \pm d\mathbb{E}(u_0),$$

under the identification of U and M via the embedding $\boldsymbol{x}$. We call the linear transformation $A_p = -d\mathbb{E}(u_0)$ the *de Sitter shape operator* and $S^\pm_p = -d\mathbb{L}^\pm(u_0) : T_pM \longrightarrow T_pM$ the *lightcone shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ which has been called the hyperbolic shape operator in [16]. The *de Sitter Gauss-Kronecker curvature* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is defined to be $K_d(u_0) = \det A_p$ and the *lightcone Gauss-Kronecker curvature* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is $K^\pm_\ell(u_0) = \det S^\pm_p$.

In [16] we have investigate the geometric meanings of the lightcone Gauss-Kronecker curvature from the contact viewpoint. On of the consequences of the results is that the lightcone Gauss-Kronecker curvature estimates the contact of hypersurfaces with hyperhorospheres. It has been also shown that the Gauss-Bonnet type theorem holds on the lightcone Gauss-Kronecker curvature [17].

We can interpret the above construction by using the Legendrian duality theorem (Theorem 2.2). For any regular hypersurface $\boldsymbol{x} : U \longrightarrow H^n(-1)$, we have $\langle \boldsymbol{x}(u), \mathbb{L}^\pm(u)\rangle = -1$. Therefore, we can define a pair of embeddings

$$\mathcal{L}^\pm_2 : U \longrightarrow \Delta_2$$

by $\mathcal{L}^\pm_2(u) = (\boldsymbol{x}(u), \mathbb{L}^\pm(u))$. Since $\langle \boldsymbol{x}_{u_i}(u), \mathbb{L}^\pm(u)\rangle = 0$, each of $\mathcal{L}^\pm_2$ is a Legendrian embedding.

On the other hand, $\pi_{21} : \Delta_2 \longrightarrow H^n(-1)$ is a Legendrian fibration. The fiber is the intersection of LC^* with a spacelike hyperplane (i.e., an elliptic hyperquadric). Therefore the intersection of the fiber with the normal plane (i.e., a timelike plane) in $\mathbb{R}^{n+1}_1$ of M consists of two points at each point of M. This is the reason why we have two such Legendrian embeddings. However, one of the results in the theory of Legendrian singularities (cf., the appendix) asserts that the Legendrian submanifold is uniquely determined by the wave front set at least locally. Here, $M = \boldsymbol{x}(U) = \pi_{21} \circ \mathcal{L}^\pm_4(U)$ is the wave front set of $\mathcal{L}^\pm_2(U)$ through the Legendrian fibration π_{21}. Therefore each of the Legendrian embeddings $\mathcal{L}^\pm_2$ is uniquely determined with respect to $M = \boldsymbol{x}(U)$. It follows that we have a unique pair of lightcone Gauss images $\mathbb{L}^\pm = \pi_{22} \circ \mathcal{L}^\pm_2$.

3 Geometry of spacelike hypersurfaces in the lightcone

In this section we construct the basic tools for the study of the extrinsic differential geometry on spacelike hypersurfaces in the lightcone LC^*. Before we start to develop the theory, we refer the following result [2] why this case is important and interesting.

Theorem 3.1 *Let M be a simply connected Riemannian manifold with* $\dim M \geq 3$. *Then M is conformally flat if and only if M can be isometrically embedded as a spacelike hypersurface in the lightcone.*

By this theorem, if we construct an extrinsic differential geometry on spacelike hypersurfaces in the lightcone, We might obtain some new extrinsic invariants of conformally flat Riemannian manifolds.

Let $\boldsymbol{x} : U \longrightarrow LC^*$ be a regular spacelike hypersurface (i.e., an embedding from an open subset $U \subset \mathbb{R}^{n-1}$ and $\boldsymbol{x}_{u_i}$, $(i = 1, \ldots, n-1)$ are spacelike vectors). If we consider the wedge product $\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}(u)$, then we can show that this vector is parallel to $\boldsymbol{x}$ by Lemma 2.1 (cf., the proof of Proposition 5.1). Therefore we have nothing new information from this construction. Instead of this construction, Theorem 2.2 supplies the lightcone normal vector to $M = \boldsymbol{x}(U)$. We consider the double Legendrian fibration $\pi_{4i} : \Delta_4 \longrightarrow LC^*$ $(i = 1, 2)$. The fiber of π_{41} is the intersection of LC^* with a lightlike hyperplane (i.e., a parabolic hyperquadric). Therefore the intersection of the fiber with the normal plane (i.e., a time like plane) in $\mathbb{R}^{n+1}_1$ of M consists of only one point at each point of M. Since $\pi_{41} : \Delta_4 \longrightarrow LC^*$ is a Legendrian fibration, there is a Legendrian submanifold $\mathcal{L}_4 : U \longrightarrow \Delta_4$ such that $\pi_{41} \circ \mathcal{L}_4(u) = \boldsymbol{x}(u)$. It follows that we have a smooth map $\boldsymbol{x}^\ell : U \longrightarrow LC^*$ such that $\mathcal{L}_4(u) = (\boldsymbol{x}(u), \boldsymbol{x}^\ell(u))$. Since $\mathcal{L}_4$ is a Legendrian embedding, we have $\langle d\boldsymbol{x}(u), \boldsymbol{x}^\ell(u)\rangle = 0$, so that $\boldsymbol{x}^\ell(u)$ belongs to the normal plane in $\mathbb{R}^{n+1}_1$. If we have another Legendrian embedding $\mathcal{L}^1_4(u) = (\boldsymbol{x}(u), \boldsymbol{x}^\ell_1(u))$, then $\boldsymbol{x}^\ell(u)$ and $\boldsymbol{x}^\ell_1(u)$ are parallel. However, we have a relation $\langle \boldsymbol{x}(u), \boldsymbol{x}^\ell(u)\rangle = \langle \boldsymbol{x}(u), \boldsymbol{x}^\ell_1(u)\rangle = -2$, so that $\boldsymbol{x}^\ell(u) = \boldsymbol{x}^\ell_1(u)$. This means that $\mathcal{L}_4$ is the unique (even in the global sense) Legendrian lift of $\boldsymbol{x}$. We call $\boldsymbol{x}^\ell(u) = \pi_{42} \circ \mathcal{L}_4$ the *lightcone normal vector* to $M = \boldsymbol{x}(U)$ at $\boldsymbol{x}(u)$. By the proof of Theorem 2.2, we have the canonical contact diffeomorphism

$$\Phi_{41} : \Delta_4 \longrightarrow \Delta_1$$

defined by

$$\Phi_{41}(\boldsymbol{v}, \boldsymbol{w}) = \left(\frac{\boldsymbol{v} + \boldsymbol{w}}{2}, \frac{\boldsymbol{v} - \boldsymbol{w}}{2}\right).$$

Therefore, we have a Legendrian submanifold $\mathcal{L}_1 : U \longrightarrow \Delta_1$ defined by $\mathcal{L}_1(u) = \Phi_{41} \circ \mathcal{L}_4(u)$. If we denote that $\mathcal{L}_1(u) = (\boldsymbol{x}^h(u), \boldsymbol{x}^d(u))$, then we have

$$\boldsymbol{x}^h(u) = \frac{\boldsymbol{x}(u) + \boldsymbol{x}^\ell(u)}{2},\ \boldsymbol{x}^d(u) = \frac{\boldsymbol{x}(u) - \boldsymbol{x}^\ell(u)}{2}.$$

We call $\boldsymbol{x}^h(u)$ the *hyperbolic normal vector* to $M = \boldsymbol{x}(U)$ at $\boldsymbol{x}(u)$ and $\boldsymbol{x}^d(u)$ the *de Sitter normal vector* to $M = \boldsymbol{x}(U)$ at $\boldsymbol{x}(u)$.

Since $\boldsymbol{x}(u), \boldsymbol{x}^\ell(u)$ are linearly independent lightlike vectors and $\boldsymbol{x}$ is a spacelike embedding, we have a basis

$$\boldsymbol{x}(u), \boldsymbol{x}^\ell(u), \boldsymbol{x}_{u_1}(u), \cdots, \boldsymbol{x}_{u_{n-1}}(u)$$

of $T_p\mathbb{R}^{n+1}_1$, where $p = \boldsymbol{x}(u)$. We call a mapping $\boldsymbol{x}^\ell : U \longrightarrow LC^*$ the *lightcone Gauss image* of $\boldsymbol{x}(U) = M$. We also respectively call $\boldsymbol{x}^h : U \longrightarrow H^n(-1)$ the *hyperbolic Gauss image* and $\boldsymbol{x}^d : U \longrightarrow S^n_1$ the *de Sitter Gauss image* of $\boldsymbol{x}(U) = M$. We also define the *lightcone Gauss map* $\widetilde{\boldsymbol{x}}^\ell : U \longrightarrow S^n_+$ by $\widetilde{\boldsymbol{x}}^\ell(u) = \widetilde{\boldsymbol{x}^\ell(u)}$. We can study the extrinsic differential geometry of $\boldsymbol{x}(U) = M$ by using $\boldsymbol{x}^\ell, \boldsymbol{x}^h, \boldsymbol{x}^d$ like as the Gauss map of a hypersurface in Euclidean space. For the purpose, we have the following fundamental lemma.

Lemma 3.2 *For any $p = \boldsymbol{x}(u_0) \in M$ and $\boldsymbol{v} \in T_pM$, we have $D_v\boldsymbol{x}^\ell(u_0) \in T_pM$, so that $D_v\boldsymbol{x}^h(u_0), D_v\boldsymbol{x}^d(u_0) \in T_pM$. Here D_v denotes the covariant derivative with respect to the tangent vector $\boldsymbol{v}$.*

Proof. We have

$$D_v\boldsymbol{x}^\ell = \lambda\boldsymbol{x} + \eta\boldsymbol{x}^\ell + \mu_1\boldsymbol{x}_{u_1} + \cdots + \mu_{n-1}\boldsymbol{x}_{u_{n-1}}$$

for some real numbers $\lambda, \eta, \mu_1, \dots, \mu_{n-1}$. It follows from the fact that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ we have $\boldsymbol{v}(\langle \boldsymbol{x}, \boldsymbol{x} \rangle) = 0$. Therefore we have $2\langle D_v\boldsymbol{x}, \boldsymbol{x} \rangle = \boldsymbol{v}(\langle \boldsymbol{x}, \boldsymbol{x} \rangle) = 0$, so that $-2\eta = 0$. By the same arguments for $\boldsymbol{x}^\ell$, we have $-2\lambda = 0$. Since $\boldsymbol{x}_{u_1}(u_0), \cdots, \boldsymbol{x}_{u_{n-1}}(u_0)$ are the basis of T_pM, $D_v\boldsymbol{x}(u_0) \in T_pM$.

Since $\boldsymbol{x}^h(u) = (\boldsymbol{x}(u)+\boldsymbol{x}^\ell(u))/2$ and $\boldsymbol{x}^d(u) = (\boldsymbol{x}(u)-\boldsymbol{x}^\ell(u))/2$, we have $D_v\boldsymbol{x}^h(u_0), D_v\boldsymbol{x}^d(u_0) \in T_pM$. $\square$

Here we identify U and M through the embedding $\boldsymbol{x}$. Under the identification, the derivatives $d\boldsymbol{x}^\ell(u_0), d\boldsymbol{x}^h(u_0), d\boldsymbol{x}^d(u_0)$ can be considered as linear transformations on the tangent space T_pM where $p = \boldsymbol{x}(u_0)$. We respectively call the linear transformations $S_p^\ell = -d\boldsymbol{x}^\ell(u_0) : T_pM \longrightarrow T_pM$ the *lightcone shape operator*, $S_p^h = -d\boldsymbol{x}^h(u_0) : T_pM \longrightarrow T_pM$ the *hyperbolic shape operator* and $S_p^d = -d\boldsymbol{x}^d(u_0) : T_pM \longrightarrow T_pM$ the *de Sitter shape operator.* We respectively denote the eigenvalues of S_p^ℓ by $\kappa_\ell(p)$, S_p^h by $\kappa_h(p)$ and S_p^d by $\kappa_d(p)$, which we call the *lightcone principal curvature*, the *hyperbolic principal curvature* and the *de Sitter principal curvature* of M at p respectively. We might consider that $d\boldsymbol{x}(u_0)$ is the identity mapping on T_pM under the identification between U and M through $\boldsymbol{x}$. By the relations between $\boldsymbol{x}, \boldsymbol{x}^\ell, \boldsymbol{x}^h, \boldsymbol{x}^d$, the principal directions of S_p^ℓ, S_p^h, S_p^d are the common and we have the following relations between the corresponding principal curvatures:

$$\kappa_h(p) = \frac{\kappa_\ell(p) - 1}{2} \text{ and } \kappa_d(p) = \frac{-\kappa_\ell(p) - 1}{2}.$$

We now define the notion of curvatures of $\boldsymbol{x}(U) = M$ at $p = \boldsymbol{x}(u_0)$ as follows:

$$\begin{aligned}
K_\ell(u_0) &= \det S_p^\ell\ ; \quad \textit{The lightcone Gauss-Kronecker curvature},\\
K_h(u_0) &= \det S_p^h\ ; \quad \textit{The hyperbolic Gauss-Kronecker curvature},\\
K_d(u_0) &= \det S_p^d\ ; \quad \textit{The de Sitter Gauss-Kronecker curvature},\\
H_\ell(u_0) &= \frac{1}{n-1}\mathrm{Trace}\, S_p^\ell\ ; \quad \textit{The lightcone mean curvature},\\
H_h(u_0) &= \frac{1}{n-1}\mathrm{Trace}\, S_p^h\ ; \quad \textit{The hyperbolic mean curvature},\\
H_d(u_0) &= \frac{1}{n-1}\mathrm{Trace}\, S_p^d\ ; \quad \textit{The de Sitter mean curvature}.
\end{aligned}$$

We can define the notion of umbilicity like as the case of hypersurfaces in Euclidean space. We say that a point $p = \boldsymbol{x}(u_0)$ (or u_0) is an *umbilic point* if $S_p^\ell = \kappa_\ell(p)1_{T_pM}$. Since the eigenvectors of S_p^ℓ, S_p^h and S_p^d are the same, the above condition is equivalent to both the conditions $S_p^h = \kappa_h(p)1_{T_pM}$ and $S_p^d = \kappa_d(p)1_{T_pM}$. We say that $M = \boldsymbol{x}(U)$ is *totally umbilic* if all points on M are umbilic. We have the following classification of totally umbilic hypersurfaces in LC^*.

Proposition 3.3 *Suppose that $M = \boldsymbol{x}(U)$ is totally umbilic. Then $\kappa_\ell(p)$ is constant κ_ℓ. Under this condition, we have the following classification.*

(1) *If $\kappa_\ell < 0$, then M is a part of hyperbolic hyperquadric $HL(\boldsymbol{c}, 1/\sqrt{-\kappa_\ell})$, where*

$$\boldsymbol{c} = \frac{-1}{2\sqrt{-\kappa_\ell}}(\kappa_\ell \boldsymbol{x}(u) + \boldsymbol{x}^\ell(u)) \in S^n_1$$

is a constant spacelike vector.

(2) *If $\kappa_\ell = 0$, then M is a part of parabolic hyperquadric $HL(\boldsymbol{c}, -2)$, where $\boldsymbol{c} = \boldsymbol{x}^\ell(u) \in LC^*$ is a constant lightlike vector.*

(3) *If $\kappa_\ell > 0$, then M is a part of elliptic hyperquadric $HL(\boldsymbol{c}, -1/\sqrt{\kappa_\ell})$, where*

$$\boldsymbol{c} = \frac{1}{2\sqrt{\kappa_\ell}}(\kappa_\ell \boldsymbol{x}(u) + \boldsymbol{x}^\ell(u)) \in H^n(-1)$$

is a constant timelike vector.

Proof. By definition, we have $-(\boldsymbol{x}^\ell)_{u_i} = \kappa_\ell(u)\boldsymbol{x}_{u_i}$ for $i = 1, \dots, n-1$. Therefore, we have

$$-(\boldsymbol{x}^\ell)_{u_i u_j} = (\kappa_\ell)_{u_j}(u)\boldsymbol{x}_{u_i} + \kappa_\ell(u)\boldsymbol{x}_{u_i u_j}.$$

Since $-(\boldsymbol{x}^\ell)_{u_i u_j} = -(\boldsymbol{x}^\ell)_{u_j u_i}$ and $\kappa_\ell(u)\boldsymbol{x}_{u_i u_j} = \kappa_\ell(u)\boldsymbol{x}_{u_j u_i}$, we have $(\kappa_\ell)_{u_j}(u)\boldsymbol{x}_{u_i} - (\kappa_\ell)_{u_i}(u)\boldsymbol{x}_{u_j}$. By definition $\{\boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_{n-1}}\}$ is linearly independent, so that κ_ℓ is constant. Under this condition, we distinguish three cases.

(Case 1). We assume that $\kappa_\ell < 0$: By definition, we have $-d\boldsymbol{x}^\ell = \kappa_\ell d\boldsymbol{x}$. Since κ_ℓ is constant, it follows from the above equality that $d(\kappa_\ell \boldsymbol{x} + \boldsymbol{x}^\ell) = \boldsymbol{0}$. Therefore $\boldsymbol{c} = \frac{-1}{2\sqrt{-\kappa_\ell}}(\kappa_\ell \boldsymbol{x}(u) + \boldsymbol{x}^\ell(u))$ is constant and we have $\langle \boldsymbol{c}, \boldsymbol{c} \rangle = 1$. On the other hand, we have

$$\langle \boldsymbol{x}(u), \boldsymbol{c} \rangle = \frac{-1}{2\sqrt{-\kappa_\ell}}\langle \boldsymbol{x}(u), \kappa_\ell \boldsymbol{x}(u) + \boldsymbol{x}^\ell(u) \rangle = -2 \times \frac{-1}{2\sqrt{-\kappa_\ell}} = \frac{1}{\sqrt{-\kappa_\ell}}.$$

This means that $M = \boldsymbol{x}(U) \subset HL(\boldsymbol{c}, 1/\sqrt{-\kappa_\ell})$.

(Case 2). We assume that $\kappa_\ell = 0$: By definition, we have $d\boldsymbol{x}^\ell(u) = \boldsymbol{0}$, so that $\boldsymbol{c} = \boldsymbol{x}$ is constant. We also have $\langle \boldsymbol{x}(u), \boldsymbol{c} \rangle = \langle \boldsymbol{x}(u), \boldsymbol{x}^\ell(u) \rangle = -2$. This means that $M = \boldsymbol{x}(U) \subset HL(\boldsymbol{c}, -2)$.

(Case 3). We assume that $\kappa_\ell > 0$: By the same reasons as the above cases,

$$\boldsymbol{c} = \frac{1}{2\sqrt{\kappa_\ell}}(\kappa_\ell \boldsymbol{x}(u) + \boldsymbol{x}^\ell(u))$$

is constant and $\langle \boldsymbol{c}, \boldsymbol{c} \rangle = -1$, so that $\boldsymbol{c} \in H^n(-1)$. Moreover, we have

$$\langle \boldsymbol{x}(u), \boldsymbol{c} \rangle = -2 \times \frac{1}{2\sqrt{\kappa_\ell}} = \frac{-1}{\sqrt{\kappa_\ell}}.$$

Therefore we have $M = \boldsymbol{x}(U) \subset HL(\boldsymbol{c}, -1/\sqrt{\kappa_\ell})$. This completes the proof. □

By the above proposition, we can classify the umbilic point as follows. Let $p = \boldsymbol{x}(u_0) \in \boldsymbol{x}(U) = M$ be an umbilic point; we say that p is a *timelike umbilic point* if $\kappa_\ell < 0$, a *lightcone flat point* if $\kappa_\ell = 0$ or a *spacelike umbilic point* if $\kappa_\ell > 0$.

In the last part of this section, we prove the lightcone Weingarten formula. Since $\boldsymbol{x}_{u_i}$ $(i = 1, \dots n-1)$ are spacelike vectors, we induce the Riemannian metric (the *lightcone first fundamental form*) $ds^2 = \sum_{i=1}^{n-1} g_{ij} du_i du_j$ on $M = \boldsymbol{x}(U)$, where $g_{ij}(u) = \langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}_{u_j}(u) \rangle$ for any $u \in U$. We also define the *lightcone second fundamental invariant* by $h^\ell_{ij}(u) = \langle -(\boldsymbol{x}^\ell)_{u_i}(u), \boldsymbol{x}_{u_j}(u) \rangle$ for any $u \in U$.

Proposition 3.4 *Under the above notations, we have the following lightcone Weingarten formula:*

$$\left(\boldsymbol{x}^{\ell}\right)_{u_i} = -\sum_{j=1}^{n-1} \left(h^{\ell}\right)_i^j \boldsymbol{x}_{u_j},$$

where $\left(h^{\ell}\right)_i^j = \left(h_{ik}^{\ell}\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = \left(g_{kj}\right)^{-1}$.

Proof. By Lemma 5.2, there exist real numbers Γ_i^j such that

$$\left(\boldsymbol{x}^{\ell}\right)_{u_i} = \sum_{j=1}^{n-1} \Gamma_i^j \boldsymbol{x}_{u_j}.$$

By definition, we have

$$-h_{i\beta}^{\ell} = \sum_{\alpha=1}^{n-1} \Gamma_i^{\alpha} \langle \boldsymbol{x}_{u_\alpha}, \boldsymbol{x}_{u_\beta} \rangle = \sum_{\alpha=1}^{n-1} \Gamma_i^{\alpha} g_{\alpha\beta}.$$

Hence, we have

$$-\left(h^{\ell}\right)_i^j = -\sum_{\beta=1}^{n-1} h_{i\beta}^{\ell} g^{\beta j} = \sum_{\beta=1}^{n-1}\sum_{\alpha=1}^{n-1} \Gamma_i^{\alpha} g_{\alpha\beta} g^{\beta j} = \Gamma_i^j.$$

This completes the proof of the lightcone Weingarten formula. □

As a corollary of the above proposition, we have an explicit expression of the lightcone Gauss-Kronecker curvature by Riemannian metric and the lightcone second fundamental invariant.

Corollary 3.5 *Under the same notations as in the above proposition, the lightcone Gauss-Kronecker curvature is given by*

$$K_{\ell} = \frac{\det\left(h_{ij}^{\ell}\right)}{\det\left(g_{\alpha\beta}\right)}.$$

Proof. By the lightcone Weingarten formula, the representation matrix of the lightcone shape operator with respect to the basis $\{\boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_{n-1}}\}$ is $\left(\left(h^{\ell}\right)_i^j\right) = \left(h_{i\beta}^{\ell}\right)\left(g^{\beta j}\right)$. It follows from this fact that

$$K_{\ell} = \det S_p^{\ell} = \det\left(\left(h^{\ell}\right)_i^j\right) = \det\left(h_{i\beta}^{\ell}\right)\left(g^{\beta j}\right) = \frac{\det\left(h_{ij}^{\ell}\right)}{\det\left(g_{\alpha\beta}\right)}.$$

□

We also have the following expressions on the hyperbolic Gauss-Kronecker curvature and the de Sitter Gauss-Kronecker curvature as a corollary of Proposition 3.4.

Corollary 3.6 *Under the same notations in the previous corollary, we have the following formulae:*

(1) $$K_h = \frac{1}{2^{n-1}} \frac{\det\left(h_{ij}^{\ell} - g_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}$$

(2) $$K_d = \frac{1}{2^{n-1}} \frac{\det\left(-h_{ij}^{\ell} - g_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}$$

Proof. (1) Since $\boldsymbol{x}^h = (\boldsymbol{x} + \boldsymbol{x}^\ell)/2$, we have

$$\left(\boldsymbol{x}^h\right)_{u_i} = \sum_{j=1}^{n-1} \frac{\left(\delta_i^j - \left(h^\ell\right)_i^j\right)}{2} \boldsymbol{x}_{u_j}.$$

It follows from the similar calculation as the proof of the above corollary that we have the desired formula. The second formula also follows from the equation that $\boldsymbol{x}^d = (\boldsymbol{x} - \boldsymbol{x}^\ell)/2$. □

We say that a point $p = \boldsymbol{x}(u)$ is a *lightcone parabolic point* if $K_\ell(u) = 0$ and a *lightcone flat point* if it is an umbilic point and $K_\ell(u) = 0$.

We also get in this context the *lightcone Gauss equations* as we shall see next and it will be used in §10. Since $\boldsymbol{x}(U) = M$ is a Riemannian manifold, it makes sense to consider the *Christoffel symbols*:

$$\left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\} = \frac{1}{2} \sum_m g^{km} \left\{ \frac{\partial g_{jm}}{\partial u_i} + \frac{\partial g_{im}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_m} \right\}.$$

Proposition 3.7 *Let $\boldsymbol{x} : U \longrightarrow LC^*$ be a spacelike hypersurface. Then we have the following lightcone Gauss equations:*

$$\boldsymbol{x}_{u_i u_j} = \sum_k \left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\} \boldsymbol{x}_{u_k} + \frac{1}{2}\left(g_{ij}\boldsymbol{x}^\ell - h_{ij}^\ell \boldsymbol{x}\right).$$

Proof. Since $\{\boldsymbol{x}, \boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_{n-1}}, \boldsymbol{x}^\ell\}$ is a basis of $\mathbb{R}_1^{n+1}$, we can write $\boldsymbol{x}_{u_i u_j} = \sum_k \Gamma_{ij}^k \boldsymbol{x}_{u_k} + \Gamma_{ij}\boldsymbol{x}^\ell + \Gamma^{ij}\boldsymbol{x}$. We now have

$$\langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x}_{u_m} \rangle = \sum_k \Gamma_{ij}^k \langle \boldsymbol{x}_{u_k}, \boldsymbol{x}_{u_m} \rangle = \sum_k \Gamma_{ij}^k g_{km}.$$

Since $\frac{\partial g_{i\ell}}{\partial u_j} = \langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x}_{u_\ell} \rangle + \langle \boldsymbol{x}_{u_i}, \boldsymbol{x}_{u_\ell u_j} \rangle$ and $\boldsymbol{x}_{u_i u_j} = \boldsymbol{x}_{u_j u_i}$, we get $\Gamma_{ij}^k = \Gamma_{ji}^k,\ \Gamma_{ij} = \Gamma_{ji},\ \Gamma^{ij} = \Gamma^{ji}$. By exactly the same calculations as those of the case for hypersurfaces in Euclidean space, $\Gamma_{ij}^k = \left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\}$.

On the other hand, we have $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x}^\ell \rangle = \langle \boldsymbol{x}^\ell, \boldsymbol{x}^\ell \rangle = 0$ and $\langle \boldsymbol{x}, \boldsymbol{x}^\ell \rangle = -2$, so that $-2\Gamma^{ij} = \langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x}^\ell \rangle = h_{ij}^\ell$. Moreover $\langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x} \rangle = -2\Gamma_{ij}$. and since $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x} \rangle = 0$, we have $\langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x} \rangle = -\langle \boldsymbol{x}_{u_i}, \boldsymbol{x}_{u_j} \rangle = -g_{ij}$. which implies that $2\Gamma_{ij} = g_{ij}$. □

Since $\boldsymbol{x}^h = (\boldsymbol{x} + \boldsymbol{x}^\ell)/2$ and $\boldsymbol{x}^d = (\boldsymbol{x} - \boldsymbol{x}^\ell)/2$, we have th following corollary.

Corollary 3.8 *Under the same assumption as the above proposition, we have*

$$\boldsymbol{x}_{u_i u_j} = \sum_k \left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\} \boldsymbol{x}_{u_k} + \frac{1}{2}\left(g_{ij} - h_{ij}^\ell\right)\boldsymbol{x}^h - \frac{1}{2}\left(g_{ij} + h_{ij}^\ell\right)\boldsymbol{x}^d.$$

4 Lightcone height functions

In this section we introduce a family of functions on a spacelike hypersurface in the lightcone which are useful for the study of singularities of lightcone Gauss images. Let $\boldsymbol{x} : U \longrightarrow LC^*$ be a spacelike hypersurface. We define a family of functions

$$H : U \times LC^* \longrightarrow \mathbb{R}$$

by $H(u, \boldsymbol{v}) = \langle \boldsymbol{x}(u), \boldsymbol{v} \rangle + 2$. We call H a *lightcone height function* on $\boldsymbol{x} : U \longrightarrow LC^*$.

Proposition 4.1 *Let $H : U \times LC^* \longrightarrow \mathbb{R}$ be a lightcone height function on $\boldsymbol{x} : U \longrightarrow LC^*$. Then*

(1) $H(u, \boldsymbol{v}) = 0$ *if and only if* $(\boldsymbol{x}(u), \boldsymbol{v}) \in \Delta_4$.

(2) $H(u, \boldsymbol{v}) = \dfrac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = 0$ $(i = 1, \ldots, n-1)$ *if and only if* $\boldsymbol{v} = \boldsymbol{x}^\ell(u)$.

Proof. The assertion (1) follows from the definition of H and Δ_4.

(2) There exist real numbers λ, μ, ξ_i $(i = 1, \ldots, n-1)$ such that $\boldsymbol{v} = \lambda \boldsymbol{x}^\ell + \mu \boldsymbol{x} + \sum_{i=1}^{n-1} \xi_i \boldsymbol{x}_{u_i}$. Since $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$, we have $\langle \boldsymbol{x}, \boldsymbol{x}_{u_i} \rangle = 0$. Therefore $0 = H(u, \boldsymbol{v}) = \langle \boldsymbol{x}, \lambda \boldsymbol{x}^\ell \rangle + 2 = -2\lambda + 2$ if and only if $\lambda = 1$. Since $\dfrac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = \langle \boldsymbol{x}_{u_i}, \boldsymbol{v} \rangle$, we have $0 = \langle \boldsymbol{x}_{u_i}, \boldsymbol{v} \rangle + \sum_{j=1}^{n-1} \xi_j g_{ij}(u)$. The equation $\langle d\boldsymbol{x}, \boldsymbol{x}^\ell \rangle = 0$ means that $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x}^\ell \rangle = 0$. It follows that $\sum_{j=1}^{n-1} \xi_j g_{ij}(u) = 0$. Since g_{ij} is positive definite, we have $\xi_j = 0$ $(j = 1, \ldots, n-1)$. We also have $0 = \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 2\mu \langle \boldsymbol{x}, \boldsymbol{x}^\ell \rangle = -4\mu$. This completes the proof. □

We denote the *Hessian matrix* of the lightcone height function $h_{v_0}(u) = H(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(h_{v_0})(u_0)$.

Proposition 4.2 *Let $\boldsymbol{x} : U \longrightarrow LC^*$ be a hypersurface in the lightcone and $\boldsymbol{v}_0 = \boldsymbol{x}^\ell(u_0)$. Then*

(1) $p = \boldsymbol{x}(u_0)$ *is a lightcone parabolic point if and only if* $\det \mathrm{Hess}(h_{v_0})(u_0) = 0$.

(2) $p = \boldsymbol{x}(u_0)$ *is a lightcone flat point if and only if* $\mathrm{rank}\, \mathrm{Hess}(h_{v_0})(u_0) = 0$.

Proof. By definition, we have

$$\mathrm{Hess}(h_{v_0})(u_0) = \left(\langle \boldsymbol{x}_{u_i u_j}(u_0), \boldsymbol{x}^\ell(u_0) \rangle \right) = \left(-\langle \boldsymbol{x}_{u_i}(u_0), \boldsymbol{x}^\ell_{u_j}(u_0) \rangle \right).$$

By definition, we have $-\langle \boldsymbol{x}_{u_i}, \boldsymbol{x}^\ell_{u_j} \rangle = h^\ell_{ij}$, so that we have

$$K_\ell(u_0) = \frac{\det \mathrm{Hess}(h_{v_0})(u_0)}{\det\ (g_{\alpha\beta}(u_0))}.$$

The first assertion follows from this formula.

For the second assertion, by the lightcone Weingarten formula, $p = \boldsymbol{x}(u_0)$ is an umbilic point if and only if there exists an orthogonal matrix A such that ${}^t A \left((h^\ell)_i^\alpha \right) A = \kappa_\ell I$. Therefore, we have $\left((h^\ell)_i^\alpha \right) = A \kappa_\ell I {}^t A = \kappa_\ell I$, so that

$$\mathrm{Hess}(h_{v_0}) = \left(h^\ell_{ij} \right) = \left((h^\ell)_i^\alpha \right) (g_{\alpha j}) = \kappa_\ell \left(g_{ij} \right).$$

Thus, p is a lightcone flat point (i.e., $\kappa_\ell(u_0) = 0$) if and only if $\mathrm{rank}\, \mathrm{Hess}(h_{v_0})(u_0) = 0$. □

5 Lightcone Gauss images as wave fronts

In this section we naturally interpret the lightcone Gauss image of a spacelike hypersurface in the lightcone as a wave front set in the framework of contact geometry. We consider a point $\boldsymbol{v} = (v_0, v_1, \ldots, v_n) \in LC^*$, then we have the relation $v_0 = \pm\sqrt{v_1^2 + \cdots + v_n^2}$. We have two components $LC^* = LC^*_+ \cup LC^*_-$, where $LC^*_+ = \{\boldsymbol{v} = (v_0, v_1, \ldots, v_n) \in LC^* \mid v_0 > 0\}$ which is called a *future component* and $LC^*_- = \{\boldsymbol{v} = (v_0, v_1, \ldots, v_n) \in LC^* \mid v_0 < 0\}$ which is called a *past component.* So we adopt the coordinate systems $(v_1, \dots, v_n)$ on both LC^*_+ and LC^*_-. We consider the projective cotangent bundle $\pi : PT^*(LC^*) \longrightarrow LC^*$ with the canonical contact structure. The basic properties of this space is described in the appendix. We only claim here that we have a trivialization :

$$\Phi : PT^*(LC^*) \equiv LC^* \times P(\mathbb{R}^{n-1})^* \ ; \ \Phi([\sum_{i=1}^{n} \xi dv_i]) = ((v_0, v_1, \dots, v_n), [\xi_1 : \dots : \xi_n]).$$

by using the above coordinate systems.

On the other hand, we define the following mapping:

$$\Psi : \Delta_4 \longrightarrow LC^* \times P(\mathbb{R}^{n-1})^* \ ; \ \Psi(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{w}, [v_0 w_1 - v_1 w_0 : \dots : v_0 w_n - v_n w_0]).$$

For the canonical contact form $\theta = \sum_{i=1}^n \xi_i dv_i$ on $PT^*(LC^*)$, we have

$$\begin{aligned} \Psi^*\theta &= (v_0 w_1 - v_1 w_0) dw_1 + \dots + (v_0 w_n - v_n w_0) dw_n | \Delta_4 \\ &= -w_0(-v_0 dw_0 + v_1 dw_1 + \dots + v_n dw_n) | \Delta_4 = -w_0 \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_4 = -w_0 \theta_{41}. \end{aligned}$$

Thus Ψ is a contact morphism.

In the appendix, we give a quick survey on the theory of Legendrian singularities. For notions and some basic results on generating families, please refer to the appendix.

Proposition 5.1 *The lightcone height function $H : U \times LC^* \longrightarrow \mathbb{R}$ is a Morse family of hypersurfaces.*

Proof. Without the loss of the generality, we consider on the future component LC^*_+. For any $\boldsymbol{v} = (v_0, v_1, \dots, v_n) \in LC^*_+$, we have $v_0 = \sqrt{v_1^2 + \dots + v_n^2}$, so that

$$H(u, \boldsymbol{v}) = -x_0(u)\sqrt{v_1^2 + \dots + v_n^2} + x_1(u)v_1 + \dots + x_n(u)v_n + 1,$$

where $\boldsymbol{x}(u) = (x_0(u), \dots, x_n(u))$. We have to prove that the mapping

$$\Delta^* H = \left(H, \frac{\partial H}{\partial u_1}, \dots, \frac{\partial H}{\partial u_{n-1}} \right)$$

is non-singular at any point on $(\Delta^* H)^{-1}(0)$. If $(u, \boldsymbol{v}) \in (\Delta^* H)^{-1}(0)$, then $\boldsymbol{v} = \boldsymbol{x}^\ell(u)$ by Proposition 4.1. The Jacobian matrix of $\Delta^* H$ is given as follows:

$$\begin{pmatrix} \langle \boldsymbol{x}_{u_1}, \boldsymbol{v} \rangle & \cdots & \langle \boldsymbol{x}_{u_{n-1}}, \boldsymbol{v} \rangle & -x_0 \dfrac{v_1}{v_0} + x_1 & \cdots & -x_0 \dfrac{v_n}{v_0} + x_n \\ \langle \boldsymbol{x}_{u_1 u_1}, \boldsymbol{v} \rangle & \cdots & \langle \boldsymbol{x}_{u_1 u_{n-1}}, \boldsymbol{v} \rangle & -x_{0u_1} \dfrac{v_1}{v_0} + x_{1u_1} & \cdots & -x_{0u_1} \dfrac{v_n}{v_0} + x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \boldsymbol{x}_{u_{n-1} u_1}, \boldsymbol{v} \rangle & \cdots & \langle \boldsymbol{x}_{u_{n-1} u_{n-1}}, \boldsymbol{v} \rangle & -x_{0u_{n-1}} \dfrac{v_1}{v_0} + x_{1u_{n-1}} & \cdots & -x_{0u_{n-1}} \dfrac{v_n}{v_0} + x_{nu_{n-1}} \end{pmatrix}.$$

We now show that the determinant of the matrix

$$A = \begin{pmatrix} -x_0 \dfrac{v_1}{v_0} + x_1 & \cdots & -x_0 \dfrac{v_n}{v_0} + x_n \\ -x_{0u_1} \dfrac{v_1}{v_0} + x_{1u_1} & \cdots & -x_{0u_1} \dfrac{v_n}{v_0} + x_{nu_1} \\ \vdots & \vdots & \vdots \\ -x_{0u_{n-1}} \dfrac{v_1}{v_0} + x_{1u_{n-1}} & \cdots & -x_{0u_{n-1}} \dfrac{v_n}{v_0} + x_{nu_{n-1}} \end{pmatrix}.$$

does not vanish at $(u, \boldsymbol{v}) \in \Sigma_*(H)$. We denote that

$$\boldsymbol{a} = \begin{pmatrix} x_0 \\ x_{0u_1} \\ \vdots \\ x_{0u_{n-1}} \end{pmatrix}, \boldsymbol{b}_1 = \begin{pmatrix} x_1 \\ x_{1u_1} \\ \vdots \\ x_{1u_{n-1}} \end{pmatrix}, \dots, \boldsymbol{b}_n = \begin{pmatrix} x_n \\ x_{nu_1} \\ \vdots \\ x_{nu_{n-1}} \end{pmatrix}.$$

Then we have

$$\det A = \frac{v_0}{v_0}\det(\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_n) - \frac{v_1}{v_0}\det(\boldsymbol{a}\ \boldsymbol{b}_2 \ \dots \boldsymbol{b}_n) - \dots - \frac{v_n}{v_0}\det(\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_{n-1}\ \boldsymbol{a}).$$

On the other hand, we have

$$\boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \dots \wedge \boldsymbol{x}_{u_{n-1}} = (-\det(\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_n), -\det(\boldsymbol{a}\ \boldsymbol{b}_2 \ \dots \boldsymbol{b}_n), \dots, -\det(\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_{n-1}\ \boldsymbol{a})).$$

We now consider a hyperplane $HP(\boldsymbol{c}, 0)$, where $\boldsymbol{c} = \boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \dots \wedge \boldsymbol{x}_{u_{n-1}}$. By definition, the basis of the vector subspace $HP(\boldsymbol{c}, 0)$ is $\{\boldsymbol{x}, \boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_{n-1}}\}$. Since $\boldsymbol{x}, \boldsymbol{x}_{u_i}$ $(i = 1, \dots, n-1)$ are tangent to the lightcone LC^*, the hyperplane $HP(\boldsymbol{c}, 0)$ is a lightlike hyperplane. By Lemma 2.1, $\boldsymbol{c}$ and $\boldsymbol{x}$ are linearly dependent, so that there exists a non-zero real number λ such that $\lambda\boldsymbol{x} = \boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \dots \wedge \boldsymbol{x}_{u_{n-1}}$. Therefore we have

$$\begin{aligned} \det A &= \left\langle \left(\frac{v_0}{v_0}, \dots, \frac{v_n}{v_0}\right), \boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \dots \wedge \boldsymbol{x}_{u_{n-1}} \right\rangle \\ &= \frac{1}{v_0}\langle \boldsymbol{x}^\ell, \boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \dots \wedge \boldsymbol{x}_{u_{n-1}} \rangle = \frac{1}{v_0}\langle \boldsymbol{x}^\ell, \lambda\boldsymbol{x} \rangle = -\frac{2\lambda}{v_0} \neq 0. \end{aligned}$$

□

We now show that H is a generating family of $\mathcal{L}_4(U) \subset \Delta_4$.

Theorem 5.2 *For any spacelike hypersurface $\boldsymbol{x} : U \longrightarrow LC^*$, the lightcone height function $H : U \times LC^* \longrightarrow \mathbb{R}$ of $\boldsymbol{x}$ is a generating family of the Legendrian immersion $\mathcal{L}_4$.*

Proof. We remember the contact morphism

$$\Psi : \Delta_4 \longrightarrow LC^* \times P(\mathbb{R}^{n-1})^*.$$

Since the lightcone height function $H : U \times LC^* \longrightarrow \mathbb{R}$ is a Morse family of hypersurfaces, we have a Legendrian immersion

$$\mathcal{L}_H : \Sigma_*(H) \longrightarrow LC^* \times P(\mathbb{R}^{n-1})^*$$

defined by

$$\mathcal{L}_H(u, \boldsymbol{v}) = \left(\boldsymbol{v}, \left[\frac{\partial H}{\partial v_0} : \dots : \frac{\partial H}{\partial v_n}\right]\right),$$

where $\boldsymbol{v} = (v_0, \dots, v_n)$. By Proposition 4.1, we have

$$\Sigma_*(H) = \{(u, \boldsymbol{x}^\ell(u)) \in LC^* \times P(\mathbb{R}^{n-1})^* \mid u \in U\}.$$

Since $\boldsymbol{v} = \boldsymbol{x}^\ell(u)$ and $v_0 = \pm\sqrt{v_1^2 + \cdots + v_n^2}$, we have

$$\frac{\partial H}{\partial v_i}(u, \boldsymbol{x}^\ell(u)) = -x_0(u)\frac{x_i^\ell(u)}{x_0^\ell(u)} + x_i(u),$$

where $\boldsymbol{x}(u) = (x_0(u), \dots, x_n(u))$ and $\boldsymbol{x}^\ell(u) = (x_0^\ell(u), \dots, x_n^\ell(u))$. It follows that

$$\mathcal{L}_H(u, \boldsymbol{x}^\ell(u)) = (\boldsymbol{x}^\ell(u), [x_0^\ell(u)x_1(u) - x_1^\ell(u)x_0(u) : \cdots : x_0^\ell(u)x_n(u) - x_n^\ell(u)x_0(u)]).$$

Therefore we have $\Psi \circ \mathcal{L}_4(u) = \mathcal{L}_H(u)$. This means that H is a generating family of $\mathcal{L}_4(U) \subset \Delta_4$. $\square$

6 The lightcone Gauss image and the lightcone Gauss map of a spacelike hypersurface in the lightcone

In this section we consider the relationship between the lightcone Gauss image and the lightcone Gauss map of a spacelike hypersurface in the lightcone. For any spacelike hypersurface $\boldsymbol{x} : U \longrightarrow LC^*$, we define a function $\mathfrak{H} : U \times S_+^{n-1} \longrightarrow \mathbb{R}$ by

$$\mathfrak{H}(u, \boldsymbol{v}) = -\frac{\langle \boldsymbol{x}(u), \boldsymbol{v}\rangle}{2}.$$

We call $\mathfrak{H}$ the *lightlike height function* of $\boldsymbol{x}(U) = M$. We also define a function $\widetilde{\mathfrak{H}} : U \times S_+^{n-1} \times \mathbb{R}^* \longrightarrow \mathbb{R}$ by

$$\widetilde{\mathfrak{H}}(u, \boldsymbol{v}, y) = \mathfrak{H}(u, \boldsymbol{v}) + y = -\frac{\langle \boldsymbol{x}(u), \boldsymbol{v}\rangle}{2} + y,$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. We call $\widetilde{\mathfrak{H}}$ the *extended lightlike height function* on $\boldsymbol{x}(U) = M$.

Using calculations similar to the proof of Proposition 4.1, we have

$$\mathcal{D}_{\widetilde{\mathfrak{H}}} = \left\{ \left(\widetilde{\boldsymbol{x}}^\ell(u), -\frac{\langle \boldsymbol{x}(u), \widetilde{\boldsymbol{x}}^\ell(u)\rangle}{2} \right) \in S_+^{n-1} \times \mathbb{R}^* \mid u \in U \right\}.$$

Let $\pi_1 : S_+^{n-1} \times \mathbb{R}^* \longrightarrow S_+^{n-1}$ be the canonical projection, then $\pi_1|\mathcal{D}_{\widetilde{\mathfrak{H}}}$ can be identified with the lightcone Gauss map of $\boldsymbol{x}(U) = M$.

We define a diffeomorphism $\psi : S^{n-1} \times \mathbb{R}^* \longrightarrow LC^*$ by $\psi(\boldsymbol{v}, y) = (1/y)\boldsymbol{v}$. Since

$$\boldsymbol{x}^\ell(u) = -\frac{2}{\langle \boldsymbol{x}(u), \widetilde{\boldsymbol{x}}^\ell(u)\rangle}\widetilde{\boldsymbol{x}}^\ell(u),$$

we have

$$\psi(\mathcal{D}_{\widetilde{\mathfrak{H}}}) = \{\boldsymbol{x}^\ell(u) \mid u \in U\ \} = \mathcal{D}_H.$$

By these arguments, we say that the lightcone Gauss image is the *lift* of the lightcone Gauss map. In fact, we also have

$$\Sigma_*(\widetilde{\mathfrak{H}}) = \left\{ \left(u, \widetilde{\boldsymbol{x}}^\ell(u), -\frac{\langle \boldsymbol{x}(u), \widetilde{\boldsymbol{x}}^\ell(u)\rangle}{2} \right) \,\middle|\, u \in U \right\}.$$

We now consider a local coordinate neighbourhood of S^{n-1}_+. Without the loss of generality, we choose

$$U_1 = \{\boldsymbol{v} = (1, v_1, \dots, v_n) \in S^{n-1}_+ \mid v_1 > 0\},$$

so that

$$\widetilde{\mathfrak{H}}(u, \boldsymbol{v}, y) = \frac{1}{2}\left(x_0(u) + \sqrt{1 - (v_1^2 + \cdots + v_n^2)}x_1(u) + v_2 x_2(u) + \cdots v_n x_n(u)\right) + y.$$

We can calculate that

$$\begin{aligned} \frac{\partial \widetilde{\mathfrak{H}}}{\partial v_i} &= \frac{1}{2}\left(-\frac{v_i}{v_1}x_1(u) + x_i(u)\right), \\ \frac{\partial \widetilde{\mathfrak{H}}}{\partial y} &= 1, \end{aligned}$$

where $i = 2, \dots, n$ and $v_1 = \sqrt{1 - (v_2^2 + \cdots + v_n^2)}$. Therefor we have a Legendrian embedding

$$\mathcal{L}_{\widetilde{\mathfrak{H}}} : (\widetilde{\boldsymbol{x}}^{\ell})^{-1}(U_1) \subset U \longrightarrow T^*U_1 \times \mathbb{R}^*$$

defined by

$$\mathcal{L}_{\widetilde{\mathfrak{H}}}(u) = \left(\widetilde{\boldsymbol{x}}^{\ell}(u), \left(\frac{1}{2}\left(-\frac{v_2}{v_1}x_1(u) + x_2(u)\right), \dots, \frac{1}{2}\left(-\frac{v_n}{v_1}x_1(u) + x_n(u)\right)\right), -\frac{\langle \boldsymbol{x}(u), \widetilde{\boldsymbol{x}}^{\ell}(u)\rangle}{2}\right).$$

We can also consider a Lagrangian embedding (for basic properties of Lagrangian singularities, see [1]):

$$\widetilde{\mathcal{L}}_{\mathfrak{H}} : (\widetilde{\boldsymbol{x}}^{\ell})^{-1}(U_1) \subset U \longrightarrow T^*U_1$$

defined by

$$\widetilde{\mathcal{L}}_{\mathfrak{H}}(u) = \left(\widetilde{\boldsymbol{x}}^{\ell}(u), \left(\frac{1}{2}\left(-\frac{v_2}{v_1}x_1(u) + x_2(u)\right), \dots, \frac{1}{2}\left(-\frac{v_n}{v_1}x_1(u) + x_n(u)\right)\right)\right)$$

whose generating family is the lightlike height function $\mathfrak{H}$. We now consider the canonical projection $\Pi_1 : T^*S^{n-1}_+ \times \mathbb{R}^* \longrightarrow T^*S^{n-1}_+$, then

$$\Pi_1 \circ \mathcal{L}_{\widetilde{\mathfrak{H}}} = \widetilde{\mathcal{L}}_{\mathfrak{H}}.$$

We remark that if we adopt other local coordinates on S^{n-1}_+, exactly the same results hold. Therefore we have the following proposition.

Proposition 6.1 *Under the same assumptions as in the previous paragraph, we have the following:*

(1) *The lightcone Gauss map is a Lagrangian map. The corresponding Lagrangian embedding is called the Lagrangian lift of the lightcone Gauss map.*

(2) *The Legendrian lift of the lightcone Gauss image* (*i.e.*, $\mathcal{L}_4$) *is a covering of the Lagrangian lift of the lightcone Gauss map.*

7 Contact with parabolic hyperquadrics

Before we start to consider the contact between spacelike hypersurfaces and parabolic hyperquadrics, we briefly review the theory of contact due to Montaldi [34]. Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact of* X_1 and Y_1 at y_1 is the same type as the *contact of* X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. In his paper [34], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory.

Theorem 7.1 *Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

We now consider a function $\mathcal{H} : LC^* \times LC^* \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + 2$. For any $\boldsymbol{v}_0 \in LC^*$, we denote that $\mathfrak{h}_{v_0}(\boldsymbol{u}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v}_0)$ and we have a parabolic hyperquadric $\mathfrak{h}_{v_0}^{-1}(0) = HP(\boldsymbol{v}_0, -2) \cap LC^* = HL(\boldsymbol{v}_0. - 2)$. For any $u_0 \in U$, we consider the lightlike vector $\boldsymbol{v}_0 = \boldsymbol{x}^\ell(u_0)$, then we have

$$\mathfrak{h}_{v_0} \circ \boldsymbol{x}(u_0) = \mathcal{H} \circ (\boldsymbol{x} \times 1_{LC^*})(u_0, \boldsymbol{v}_0) = H(u_0, \boldsymbol{x}^\ell(u_0)) = 0.$$

By Proposition 4.1, we also have relations that

$$\frac{\partial \mathfrak{h}_{v_0} \circ \boldsymbol{x}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0, \boldsymbol{x}^\ell(u_0)) = 0.$$

for $i = 1, \dots, n-1$. This means that the parabolic hyperquadric $\mathfrak{h}_{v_0}^{-1}(0) = HL(\boldsymbol{v}_0, -2)$ is tangent to $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. In this case, we call $HL(\boldsymbol{v}_0, -1)$ the *tangent parabolic hyperquadric* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ (or, u_0), which we write $TPH(\boldsymbol{x}, u_0)$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be lightlike vectors. If $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly dependent, then corresponding lightlike hyperplanes $HP(\boldsymbol{v}_1, -2), HP(\boldsymbol{v}_2, -2)$ are parallel. Therefore, we say that parabolic hyperquadrics $HL(\boldsymbol{v}_1, -2), HL(\boldsymbol{v}_2, -2)$ are *parallel* if $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly dependent. Then we have the following simple lemma.

Lemma 7.2 *Let $\boldsymbol{x} : U \longrightarrow LC^*$ be a spacelike hypersurface. Consider two points $u_1, u_2 \in U$. Then*

(1) $\boldsymbol{x}^\ell(u_1) = \boldsymbol{x}^\ell(u_2)$ *if and only if* $TPH(\boldsymbol{x}, u_1) = TPH(\boldsymbol{x}, u_2)$.

(2) $\widetilde{\boldsymbol{x}}^\ell(u_1) = \widetilde{\boldsymbol{x}}^\ell(u_2)$ *if and only if* $TPH(\boldsymbol{x}, u_1)$, $TPH(\boldsymbol{x}, u_2)$ *are parallel.*

Eventually, we have tools for the study of the contact between hypersurfaces and parabolic hyperquadric.

Let $\boldsymbol{x}_i^\ell : (U, u_i) \longrightarrow (LC^*, \boldsymbol{v}_i)$ $(i = 1, 2)$ be lightcone Gaussian image germs of spacelike hypersurface germs $\boldsymbol{x}_i : (U, u_i) \longrightarrow (LC^*, \boldsymbol{u}_i)$. We say that $\boldsymbol{x}_1^\ell$ and $\boldsymbol{x}_2^\ell$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\phi : (U, u_1) \longrightarrow (U, u_2)$ and $\Phi : (LC^*, \boldsymbol{v}_1) \longrightarrow (LC^*, \boldsymbol{v}_2)$ such that $\Phi \circ \boldsymbol{x}_1^\ell = \boldsymbol{x}_2^\ell \circ \phi$. If both the regular sets of $\boldsymbol{x}_i^\ell$ are dense in (U, u_i), it follows from Proposition A.2 that $\boldsymbol{x}_1^\ell$ and $\boldsymbol{x}_2^\ell$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian immersion germs $\mathcal{L}_4^1 : (U, u_1) \longrightarrow (\Delta_4, z_1)$ and $\mathcal{L}_4^2 : (U, u_2) \longrightarrow (\Delta_4, z_2)$ are Legendrian equivalent. This condition is also equivalent to the condition that two generating families H_1 and H_2 are P-$\mathcal{K}$-equivalent by Theorem A.3. Here, $H_i : (U \times LC^*, (u_i, \boldsymbol{v}_i)) \longrightarrow \mathbb{R}$ is the lightcone height function germ of $\boldsymbol{x}_i$.

On the other hand, we denote that $h_{i,v_i}(u) = H_i(u, \boldsymbol{v}_i)$, then we have $h_{i,v_i}(u) = \mathfrak{h}_{v_i} \circ \boldsymbol{x}_i(u)$. By Theorem 7.1, $K(\boldsymbol{x}_1(U), TPH(\boldsymbol{x}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPH(\boldsymbol{x}_2, u_2), \boldsymbol{v}_2)$ if and only if h_{1,v_1} and h_{1,v_2} are $\mathcal{K}$-equivalent. Therefore, we can apply the arguments in the appendix to our situation. We denote $Q(\boldsymbol{x}, u_0)$ the local ring of the function germ $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$, where $\boldsymbol{v}_0 = \boldsymbol{x}^\ell(u_0)$. We remark that we can explicitly write the local ring as follows:

$$Q(\boldsymbol{x}, u_0) = \frac{C^\infty_{u_0}(U)}{\langle\langle \boldsymbol{x}(u), \boldsymbol{x}^\ell(u_0)\rangle + 2\rangle_{C^\infty_{u_0}(U)}},$$

where $C^\infty_{u_0}(U)$ is the local ring of function germs at u_0 with the unique maximal ideal $\mathfrak{M}_{u_0}(U)$.

Theorem 7.3 *Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (LC^*, \boldsymbol{u}_i)$ $(i = 1, 2)$ be hypersurfaces germs such that the corresponding Legendrian map germs $\pi_{4,2} \circ \mathcal{L}^i_4 : (U, u_i) \longrightarrow (LC^*, \boldsymbol{v}_i)$ are Legendrian stable. Then the following conditions are equivalent:*

(1) *Lightcone Gauss image germs $\boldsymbol{x}^\ell_1$ and $\boldsymbol{x}^\ell_2$ are $\mathcal{A}$-equivalent.*
(2) *H_1 and H_2 are P-$\mathcal{K}$-equivalent.*
(3) *h_{1,v_1} and h_{1,v_2} are $\mathcal{K}$-equivalent.*
(4) *$K(\boldsymbol{x}_1(U), TPH(\boldsymbol{x}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPH(\boldsymbol{x}_2, u_2), \boldsymbol{v}_2)$.*
(5) *$Q(\boldsymbol{x}_1, u_1)$ and $Q(\boldsymbol{x}_2, u_2)$ are isomorphic as $\mathbb{R}$-algebras.*

Proof. By the previous arguments (mainly from Theorem 7.1), it has been already shown that conditions (3) and (4) are equivalent. Other assertions follow from Theorem 5.2 and Proposition A.4. □

In the next section, we will prove that the assumption of the theorem is generic in the case when $n \leq 6$. For general dimensions, we need the topological theory (cf., Proposition A.7).

Theorem 7.4 *Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (LC^*, \boldsymbol{x}_i(u_i))$ $(i = 1, 2)$ be spacelike hypersurface germs such that the map germ given by $\pi_{H_i} : (H_i^{-1}(\boldsymbol{v}_i), (u_i, \boldsymbol{v}_i)) \longrightarrow (LC^*, \boldsymbol{v}_i)$ at any point $u_i \in U$ is an MT-stable map germ, where H_i is the lightcone height function of $\boldsymbol{x}_i$ and $\boldsymbol{v}_i = \boldsymbol{x}^\ell_i(u_i)$. If $Q(\boldsymbol{x}_1, u_1)$ and $Q(\boldsymbol{x}_2, u_2)$ are isomorphic as $\mathbb{R}$-algebras, then $(\boldsymbol{x}^\ell_1(U), u_1)$ and $(\boldsymbol{x}^\ell_2(U), u_2)$ are stratified equivalent as set germs.*

In general we have the following proposition.

Proposition 7.5 *Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (LC^*, \boldsymbol{x}_i(u_i))$ $(i = 1, 2)$ be spacelike hypersurface germs such that their lightcone parabolic sets have no interior points as subspaces of U. If hyperbolic Gauss image germs $\boldsymbol{x}^\ell_1$, $\boldsymbol{x}^\ell_2$ are $\mathcal{A}$-equivalent, then*

$$K(\boldsymbol{x}_1(U), TPH(\boldsymbol{x}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPH(\boldsymbol{x}_2, u_2), \boldsymbol{v}_2).$$

In this case, $(\boldsymbol{x}_1^{-1}(TPH(\boldsymbol{x}_1, \boldsymbol{v}_1)), u_1)$ and $(\boldsymbol{x}_2^{-1}(TPH(\boldsymbol{x}_2, \boldsymbol{v}_2)), u_2)$ are diffeomorphic as set germs.

Proof. The lightcone parabolic set is the set of singular points of the lightcone Gauss image. So the corresponding Legendrian lifts $\mathcal{L}^i_4$ satisfy the hypothesis of Proposition A.2. If lightcone Gauss image germs $\boldsymbol{x}_1^\ell,\ \boldsymbol{x}_2^\ell$ are $\mathcal{A}$-equivalent, then $\mathcal{L}_4^1,\ \mathcal{L}_4^2$ are Legendrian equivalent, so that $H_1,\ H_2$ are P-$\mathcal{K}$-equivalent. Therefore, $h_{1,v_1},\ h_{1,v_2}$ are $\mathcal{K}$-equivalent. By Theorem 8.1, this condition is equivalent to the condition that $K(\boldsymbol{x}_1(U), TPH(\boldsymbol{x}_1, u_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPH(\boldsymbol{x}_2, u_2), \boldsymbol{v}_2)$.

On the other hand, we have $(\boldsymbol{x}_i^{-1}(TPH(\boldsymbol{x}_i, u_i)), u_i) = (h_{i,v_i}^{-1}(0), u_i)$. It follows from this fact that $(\boldsymbol{x}_1^{-1}(TPH(\boldsymbol{x}_1, u_1)), u_1)$ and $(\boldsymbol{x}_2^{-1}(TPH(\boldsymbol{x}_2, u_2)), u_2)$ are diffeomorphic as set germs because the $\mathcal{K}$-equivalence preserves the zero level sets. □

For a spacelike hypersurface germ $\boldsymbol{x} : (U, u_0) \longrightarrow (LC^*, \boldsymbol{x}(u_0))$, we call $(\boldsymbol{x}^{-1}(TPH(\boldsymbol{x}, u_0)), u_0)$ the *tangent parabolic indicatrix germ* of $\boldsymbol{x}$. By Proposition 7.5, the diffeomorphism type of the tangent parabolic indicatrix germ is an invariant of the $\mathcal{A}$-classification of the lightcone Gauss image germ of $\boldsymbol{x}$. Moreover, by the above results, we can borrow some basic invariants from the singularity theory on function germs. We need $\mathcal{K}$-invariants for function germ. The local ring of a function germ is a complete $\mathcal{K}$-invariant for generic function germs. It is, however, not a numerical invariant. The $\mathcal{K}$-codimension (or, Tyurina number) of a function germ is a numerical $\mathcal{K}$-invariant of function germs [28]. We denote that

$$\text{P-ord}(\boldsymbol{x}, u_0) = \dim \frac{C^\infty_{u_0}(U)}{\langle \langle \boldsymbol{x}(u), \boldsymbol{x}^\ell(u_0)\rangle + 2, \langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}^\ell(u_0)\rangle \rangle_{C^\infty_{u_0}}}.$$

Usually P-ord$(\boldsymbol{x}, u_0)$ is called the *$\mathcal{K}$-codimension of* h_{v_0}. However, we call it the *order of contact with the tangent parabolic hyperquadric* at $\boldsymbol{x}(u_0)$. We also have the notion of corank of function germs.

$$\text{P-corank}(\boldsymbol{x}, u_0) = (n-1) - \operatorname{rank} \operatorname{Hess}(h_{v_0}(u_0)),$$

where $\boldsymbol{v}_0 = \boldsymbol{x}^\ell(u_0)$.

By Proposition 4.2, $\boldsymbol{x}(u_0)$ is a lightcone parabolic point if and only if P-corank$(\boldsymbol{x}, u_0) \geq 1$. Moreover $\boldsymbol{x}(u_0)$ is a lightcone flat point if and only if P-corank$(\boldsymbol{x}, u_0) = n-1$.

On the other hand, a function germ $f : (\mathbb{R}^{n-1}, \boldsymbol{a}) \longrightarrow \mathbb{R}$ has the A_k-type singularity if f is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 \pm \cdots \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If P-corank$(\boldsymbol{x}, u_0) = 1$, the lightcone height function h_{v_0} has the A_k-type singularity at u_0 in generic. In this case we have P-ord$(\boldsymbol{x}, u_0) = k$. This number is equal to the order of contact in the classical sense (cf., [8]). This is the reason why we call P-ord$(\boldsymbol{x}, u_0)$ the order of contact with the tangent parabolic hyperquadric at $\boldsymbol{x}(u_0)$.

8 Generic properties

In this section we consider generic properties of spacelike hypersurfaces in LC^*. The main tool is a kind of transversality theorems. We consider the space of spacelike embeddings $\mathrm{Emb}_{\mathrm{sp}}(U, LC^*)$ with Whitney C^∞-topology. We also consider the function $\mathcal{H} : LC^* \times LC^* \longrightarrow \mathbb{R}$ which is given in §7. We claim that $\mathcal{H}_u$ is a submersion for any $\boldsymbol{u} \in LC^*$, where $\mathcal{H}_u(\boldsymbol{v}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v})$. For any $\boldsymbol{x} \in \mathrm{Emb}_{\mathrm{sp}}(U, LC^*)$, we have $H = \mathcal{H} \circ (\boldsymbol{x} \times id_{LC^*})$. We also have the k-jet extension

$$j_1^k H : U \times LC^* \longrightarrow J^k(U, \mathbb{R})$$

defined by $j_1^k H(u, \boldsymbol{v}) = j^k h_v(u)$. We consider the trivialisation $J^k(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^k(n-1, 1)$. For any submanifold $Q \subset J^k(n-1, 1)$, we denote that $\widetilde{Q} = U \times \{0\} \times Q$. Then we have the following proposition as a corollary of Lemma 6 in Wassermann [50]. (See also Montaldi [35]).

Proposition 8.1 *Let Q be a submanifold of $J^k(n-1,1)$. Then the set*

$$T_Q = \{\boldsymbol{x} \in \mathrm{Emb}_{\mathrm{sp}}(U, LC^*) \mid j_1^k H \text{ is transversal to } \widetilde{Q}\ \}$$

is a residual subset of $\mathrm{Emb}_{\mathrm{sp}}(U, LC^*)$. *If Q is a closed subset, then T_Q is open.*

On the other hand, we already have the canonical stratification $\mathcal{A}_0^\ell(U,\mathbb{R})$ of $J^k(\mathbb{R}^{n-1},\mathbb{R}) \setminus W^k(\mathbb{R}^{n-1},\mathbb{R})$ (cf., the appendix). By the above proposition and arguments in the appendix, we have the following theorem.

Theorem 8.2 *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, LC^*)$ such that for any $\boldsymbol{x} \in \mathcal{O}$, the germ of the corresponding lightcone Gauss image $\boldsymbol{x}^\ell$ at each point is the critical part of an MT-stable map germ.*

In the case when $n \leq 6$, for any $\boldsymbol{x} \in \mathcal{O}$, the germ of the Legendrian map $\pi_{4,2} \circ \mathcal{L}_4$ at each point is Legendrian stable.

We remark that we can also prove the multi-jet version of Proposition 8.1. As an application of such a multi-jet transversality theorem, we can show that the lightcone Gauss image is the critical part of an (global) MT-stable map for a generic spacelike hypersurface $\boldsymbol{x} : U \longrightarrow LC^*$ (cf., the appendix). However, the arguments are rather tedious, so that we omit it.

9 The Gauss-Bonnet type theorem

In this section we give the definition of normalized lightcone Gauss-Kronecker curvatures and a proof for the lightcone Gauss-Bonnet type theorem. Let M be a closed orientable $(n-1)$-dimensional manifold and $\mathcal{L}_4 : M \longrightarrow \Delta_4$ a Legendrian embedding such that $f = \pi_{41} \circ \mathcal{L}_4 : M \longrightarrow LC^*$ is an embedding. We can easily show that f is a spacelike embedding. We denote that $\mathbb{L} = \pi_{42} \circ \mathcal{L}_4 : M \longrightarrow LC^*$ which is called the *global lightcone Gauss image* of f.

We now consider the canonical projection $\pi : \mathbb{R}_1^{n+1} \longrightarrow \mathbb{R}^n$ defined by $\pi(x_0, x_1, \ldots, x_n) = (0, x_1, \ldots, x_n)$. Then we have an embedding $\pi|LC^*+ : LC^*_+ \longrightarrow \mathbb{R}^n$ and an orientation preserving diffeomorphism $\pi|S_+^{n-1} : S_+^{n-1} \longrightarrow S^{n-1}$.

The *global lightcone Gauss-Kronecker curvature function* $\mathcal{K}_\ell : M \longrightarrow \mathbb{R}$ is then defined in the usual way in terms of the global lightcone Gauss image $\mathbb{L}$ (cf., §3). We also define the lightcone Gauss map in the global sense

$$\widetilde{\mathbb{L}} : M \longrightarrow S_+^{n-1}$$

by

$$\widetilde{\mathbb{L}}(p) = \widetilde{\mathbb{L}(p)}.$$

By using the global lightcone Gauss map, we define a *normalized lightcone Gauss-Kronecker curvature function* $\overline{\mathcal{K}}_\ell : M \longrightarrow \mathbb{R}$ by the following relation:

$$\overline{\mathcal{K}}_\ell d\mathfrak{v}_M = \widetilde{\mathbb{L}}^* d\mathfrak{v}_{S_+^{n-1}},$$

where $d\mathfrak{v}_M$ (respectively, $d\mathfrak{v}_{S_+^{n-1}}$) is the volume form of M (respectively, S_+^{n-1}).

We now consider a geometric meaning of the normalized Gauss-Kronecker curvature function. We firstly calculate the Jacobi matrix of $\widetilde{\mathbb{L}}$.

Proposition 9.1 *There exist local coordinates* $(U,(u_1,\dots,u_{n-1}))$ *of* M *and* $(V,(v_1,\dots,v_{n-1}))$ *of* S^{n-1}_+ *such that the corresponding matrix* $\dfrac{-1}{\ell_0}\big((h^\ell)^j_i\big)$ *is the Jacobi matrix of* $\widetilde{\mathbb{L}}$.

Here $\mathbb{L}(p)=(\ell_0(p),\ell_1(p),\dots,\ell_n(p))$ *and* $\big((h^\ell)^j_i\big)$ *is the matrix given in Proposition* 3.4.

Proof. We define a projection $\Pi : LC^*_+ \longrightarrow S^{n-1}_+$ by $\Pi(\boldsymbol{v})=\widetilde{\boldsymbol{v}}$. We use the local notation in §3 here. Therefore, on a local coordinates $(U,(u_1,\dots,u_{n-1}))$ of M we denote that $f|U=\boldsymbol{x}:U\longrightarrow LC^*$ and assume that $\mathbb{L}(u)=\boldsymbol{x}^\ell(u)$. We observe that the tangent space of LC^*_+ at $\boldsymbol{v}\in LC^*_+$ is the hyperplane $HP(\boldsymbol{v},0)$. Since $\langle \mathbb{L},\boldsymbol{x}_{u_i}\rangle=0$, $\boldsymbol{x}_{u_i}(u)$ is a tangent vector of LC^*_+ at $\mathbb{L}(u)$, it follows that $\{\mathbb{L}(u),\boldsymbol{x}_{u_1}(u),\dots,\boldsymbol{x}_{u_{n-1}}(u)\}$ is a basis of the tangent space of LC^*_+ at $\mathbb{L}(u)$. The tangent direction of the fiber of Π is given by the lightlike vector $\mathbb{L}(u)$, and hence $\{d\Pi(\boldsymbol{x}_{u_1}),\dots,d\Pi(\boldsymbol{x}_{u_{n-1}})\}$ is a basis of the tangent space of S^{n-1}_+ at $d\Pi(\mathbb{L}(u))$. On the other hand, we have the lightcone Weingarten formula (cf., Proposition 3.4): $\mathbb{L}_{u_i}=-\sum_{j=1}^{n-1}(h^\ell)^j_i\boldsymbol{x}_{u_j}$. Since $\mathbb{L}=\ell_0\widetilde{\mathbb{L}}$, we calculate that $\ell_0\widetilde{\mathbb{L}}_{u_i}=\mathbb{L}_{u_i}-\ell_{0u_i}\widetilde{\mathbb{L}}$. It follows that

$$d\Pi(\widetilde{\mathbb{L}}_{u_i}(u))=-\sum_{j=1}^{n-1}\frac{1}{\ell_0}(h^\ell)^j_i d\Pi(\boldsymbol{x}_{u_j}(u)).$$

We can choose local coordinate $(V,(v_1,\dots v_{n-1}))$ of S^{n-1}_+ around $d\Pi\mathbb{L}(u)$ such that $(\partial/\partial v_i)=d\Pi(\boldsymbol{x}_{u_j}(u))$. This means that the Jacobi matrix of $\widetilde{\mathbb{L}}$ at $u\in U$ in the local coordinates $(U,(u_1,\dots,u_{n-1}))$ of M and $(V,(v_1,\dots,v_{n-1}))$ of S^{n-1}_+ is $\dfrac{-1}{\ell_0}\big((h^\ell)^j_i\big)$. $\square$

We have the following relation between $\overline{\mathcal{K}}_\ell$ and $\mathcal{K}_\ell$ as a simple corollary of the above proposition.

Corollary 9.2 *There exist local coordinates* $(U,(u_1,\dots,u_{n-1}))$ *of* M *and* $(V,(v_1,\dots,v_{n-1}))$ *of* S^{n-1}_+ *such that*

$$\overline{\mathcal{K}}_\ell(p)=\left(\frac{-1}{\ell_0(p)}\right)^{n-1}\frac{\sqrt{\det\left((g_{S^{n-1}_+})_{ij}(\widetilde{\mathbb{L}}(p))\right)}}{\sqrt{\det\left((g_M)_{ij}(p)\right)}}\mathcal{K}_\ell(p)$$

for any $p\in U$, *where* $g_M=\sum_{ij}(g_M)_{ij}du_idu_j$ *(respectively,* $g_{S^{n-1}_+}=\sum_{ij}(g_{S^{n-1}_+})_{ij}dv_idv_j$*) is the local representation of the Riemannian metric on* M *(respectively,* S^{n-1}_+*) induced from the Minkowski metric* $\langle,\rangle$ *with respect to the above local coordinates.*

Corollary 9.3 *For a point* $p\in M$, *the following conditions are equivalent:*

(1) *The point* $p\in M$ *is a lightcone parabolic point (i.e.,* $\mathcal{K}_\ell(p)=0$*).*
(2) *The point* $p\in M$ *is a singular point of the lightcone Gauss image* $\mathbb{L}$.
(3) *The point* $p\in M$ *is a singular point of the lightcone Gauss map* $\widetilde{\mathbb{L}}$.
(4) $\overline{\mathcal{K}}_\ell(p)=0$.

The lightcone Gauss-Bonnet type theorem is stated as follows.

Theorem 9.4 *Let* M *be a closed connected orientable* $(n-1)$*-dimensional manifold. Suppose that there exists a Legendrian embedding*

$$\mathcal{L}_4 : M\longrightarrow \Delta_4$$

such that $f = \pi_{41} \circ \mathcal{L}_4$ *is an embedding. If* n *is a odd number, then we have*

$$\int_M \overline{\mathcal{K}}_\ell d\mathfrak{v}_M = \frac{1}{2}\gamma_{n-1}\boldsymbol{\chi}(M)$$

where $\boldsymbol{\chi}(M)$ *is the Euler characteristic of* M, $d\mathfrak{v}_M$ *is the volume element of* M *and the constant* γ_{n-1} *is the volume of the unit* $(n-1)$*-sphere* S^{n-1}.

For the proof of Theorem 9.4, consider the (Euclidean) Gauss map

$$\mathbb{N} : M \longrightarrow S^{n-1}$$

on $\pi \circ f(M)$.

We need the following lemma.

Lemma 9.5 *Under the same assumptions as those of Theorem* 9.4*, the vector* $\pi \circ \widetilde{\mathbb{L}}(p)$ *is transverse to* $d(\pi \circ f)(T_pM)$ *at any point* $p \in M$.

Proof. Suppose that there is a point $p \in M$ such that the vector $\pi \circ \widetilde{\mathbb{L}}(p)$ is not transverse to $d(\pi \circ f)(T_pM)$ at p. Since $\pi \circ f(M)$ is a hypersurface in $\mathbb{R}^n$, we have $\pi \circ \widetilde{\mathbb{L}}(p) \in d(\pi \circ f)(T_pM)$. Therefore we have

$$\widetilde{\mathbb{L}}(p) \in df_p(T_pM) + \operatorname{Ker} d\pi_{f(p)}.$$

On the other hand, $\widetilde{\mathbb{L}}(p) \in N_{f(p)}(f(M))$ and $\widetilde{\mathbb{L}}(p) \notin \operatorname{Ker} d\pi_{f(p)}$, where $N_p(f(M))$ is the pseudo-normal space of $f(M)$ at $f(p)$. Since $\operatorname{Ker} d\pi_{f(p)}$ is a timelike one-dimensional subspace in $T_{f(p)}\mathbb{R}^{n+1}_1$, we have

$$\langle \widetilde{\mathbb{L}}(p), \operatorname{Ker} d\pi_{f(p)} \rangle_{\mathbb{R}} + df_p(T_pM) = T_{f(p)}\mathbb{R}^{n+1}_1.$$

However, by the assumption, the dimension of the vector space in the left hand side is at most n. This is a contradiction. □

Lemma 9.6 *Under the choice of a suitable direction of* $\mathbb{N}$, $\pi \circ \widetilde{\mathbb{L}}$ *and* $\mathbb{N}$ *are homotopic.*

Proof. Since $\widetilde{\mathbb{L}}$ is transverse to $\pi \circ f(M)$ in $\mathbb{R}^n$, $\langle \pi \circ \widetilde{\mathbb{L}}(p), \mathbb{N}(p) \rangle \neq 0$ at any $p \in M$. By the assumption that M is connected, we choose the direction of $\mathbb{N}$ such that makes $\langle \pi \circ \widetilde{\mathbb{L}}(p), \mathbb{N}(p) \rangle > 0$.

We now construct a homotopy between $\pi \circ \widetilde{\mathbb{L}}$ and $\mathbb{N}$. Let

$$F : M \times [0,1] \longrightarrow S^{n-1}$$

be defined by

$$F(p,t) = \frac{t\mathbb{N}(p) + (1-t)\pi \circ \widetilde{\mathbb{L}}(p)}{\|t\mathbb{N}(p) + (1-t)\pi \circ \widetilde{\mathbb{L}}(p)\|},$$

where $\|\cdot\|$ is the Euclidean norm.

If there exists $t' \in [0,1]$ and $p' \in M$ such that $t'\mathbb{N}(p') + (1-t')\pi \circ \widetilde{\mathbb{L}}(p') = \mathbf{0}$, then we have $\mathbb{N}(p') = -\pi \circ \widetilde{\mathbb{L}}(p')$. This contradicts to the assumption that $\langle \pi \circ \widetilde{\mathbb{L}}, \mathbb{N}(p) \rangle > 0$. Therefore F is a continuous mapping satisfying $F(p,0) = \pi \circ \widetilde{\mathbb{L}}(p)$ and $F(p,1) = \mathbb{N}(p)$ for any $p \in M$. □

Since the mapping degree is a homotopy invariant and a invariant under orientation preserving diffeomorphisms, we have the following corollary (cf., [14], Chapter 4, §9).

Corollary 9.7 *Under the same assumptions as those in Theorem* 9.4*, we have*

$$\deg \widetilde{\mathbb{L}} = \frac{1}{2}\boldsymbol{\chi}(M),$$

where $\deg \widetilde{\mathbb{L}}$ *is the mapping degree of* $\widetilde{\mathbb{L}}$.

By the definition of the normalized lightcone Gauss-Kronecker curvature $\overline{\mathcal{K}}_\ell$, we obtain:

$$\int_M \overline{\mathcal{K}}_\ell d\mathfrak{v}_M = \int_M \widetilde{\mathbb{L}}^* d\mathfrak{v}_{S^{n-1}_+} = \deg(\widetilde{\mathbb{L}}) \int_{S^{n-1}_+} d\mathfrak{v}_{S^{n-1}_+} = \deg(\widetilde{\mathbb{L}})\gamma_{n-1}.$$

The proof of Theorem 9.4 is now completed as a consequence of Corollary 9.7.

Remark Since we do not assume that n is odd in Lemma 9.6, we can apply the lemma for the case $n = 2$. In this case we consider a unit speed curve $\boldsymbol{\gamma} : S^1 \longrightarrow LC^*$. The lightcone Gauss image $\mathbb{L}$ is uniquely determined by relations

$$\langle \boldsymbol{\gamma}, \mathbb{L} \rangle = -2, \langle \boldsymbol{t}, \mathbb{L} \rangle = \langle \boldsymbol{\gamma}, \mathbb{L}' \rangle = 0,$$

where, $\boldsymbol{t}$ is the unit tangent vector of $\boldsymbol{\gamma}$. In this case, we have the lightcone Frenet-Serre type formula:

$$\mathbb{L}' = -\kappa_\ell(s)\boldsymbol{t}(s).$$

If we fix the following parameterization of the spacelike circle:

$$S^1_+ = \{(1, \cos\theta, \sin\theta) \mid 0 \leq \theta < 2\pi\},$$

then the normalized lightcone curvature is

$$\overline{\kappa_\ell}(s) = \frac{1}{\ell_0(s)}\kappa_\ell(s).$$

Without the loss of generality, we might assume that $\boldsymbol{\gamma}(S^1) \subset LC^*_+$. Since the projection $\pi : LC^*_+ \longrightarrow \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is an orientation preserving diffeomorphism, the winding numbers of $\boldsymbol{\gamma}$ and $\pi \circ \boldsymbol{\gamma}$ are the same. Therefore we have the following formula as a corollary of Lemma 9.6:

$$\frac{1}{2\pi}\int_{S^1} \overline{\kappa_\ell} ds = W(\boldsymbol{\gamma}),$$

where $W(\boldsymbol{\gamma})$ denotes the winding number of $\boldsymbol{\gamma}$.

10 Spacelike surfaces in the 3-dimensional lightcone

In this section we stick to the case $n = 3$. First of all we need to make some local calculations. Let $\boldsymbol{x} : U \longrightarrow LC^*$ be a spacelike surface, where $U \subset \mathbb{R}^2$ is an open region, and consider the *Riemannian curvature tensor*

$$R^\delta_{\alpha\beta\gamma} = \frac{\partial}{\partial u_\gamma}\left\{ \begin{matrix} \delta \\ \alpha\ \beta \end{matrix} \right\} - \frac{\partial}{\partial u_\beta}\left\{ \begin{matrix} \delta \\ \alpha\ \gamma \end{matrix} \right\} + \sum_\epsilon \left\{ \begin{matrix} \epsilon \\ \alpha\ \beta \end{matrix} \right\}\left\{ \begin{matrix} \delta \\ \epsilon\ \gamma \end{matrix} \right\} - \sum_\epsilon \left\{ \begin{matrix} \epsilon \\ \alpha\ \gamma \end{matrix} \right\}\left\{ \begin{matrix} \delta \\ \epsilon\ \beta \end{matrix} \right\}.$$

We also consider the tensor $R_{\alpha\beta\gamma\delta} = \sum_\epsilon g_{\alpha\epsilon} R^\epsilon_{\beta\gamma\delta}$. Standard calculations, analogous to those used in the study of the classical differential geometry on surfaces in Euclidean space (cf., [48]), lead to the following formula.

Proposition 10.1 *Under the above notations, we have*

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left\{g_{\beta\gamma}h^{\ell}_{\alpha\delta} - g_{\beta\gamma}h^{\ell}_{\alpha\gamma} + h^{\ell}_{\beta\gamma}g_{\alpha\delta} - h^{\ell}_{\beta\delta}g_{\alpha\gamma}\right\}.$$

We denote that

$$h^h_{\alpha\beta} = -\frac{1}{2}(g_{\alpha\beta} - h^{\ell}_{\alpha\beta}) \text{ and } h^d_{\alpha\beta} = -\frac{1}{2}(g_{\alpha\beta} + h^{\ell}_{\alpha\beta}).$$

It follows from Corollary 3.6 that we have

$$K_h = \frac{h^h_{11}h^h_{22} - h^h_{21}h^h_{12}}{g_{11}g_{22} - g_{12}g_{21}} \quad K_d = \frac{h^d_{11}h^d_{22} - h^d_{21}h^d_{12}}{g_{11}g_{22} - g_{12}g_{21}}.$$

Therefore we obtain the analogous result of *Theorema Egregium* of Gauss for the lightcone case:

Proposition 10.2 *Under the above notations, we have*

$$K_d - K_h = -\frac{R_{1212}}{g},$$

where $g = g_{11}g_{22} - g_{12}g_{21}$.

We remark that $-R_{1212}/g$ is the *sectional curvature* of the surface, so we denote it by K_s.

On the other hand, let κ^i_ℓ $(i = 1, 2)$ be eigenvalues of $((h^\ell)^i_j)$ (i.e., lightcone principal curvatures of the spacelike surface). We remind that $\kappa^i_\ell = \kappa^i_h - \kappa^i_d$, where κ^i_h (respectively, κ^i_d) is a hyperbolic (respectively, de Sitter) principal curvature. By direct calculations, we have the following "Theorema egregium" as a corollary of the above proposition.

Theorem 10.3 *The following relation holds:*

$$K_s = K_d - K_h = H_\ell = H_h - H_d.$$

We return to the global situation. Let M be a closed orientable 2-dimensional manifold and $f : M \longrightarrow LC^*$ an embedding induced by a Legendrian embedding $\mathcal{L}_4 : M \longrightarrow \Delta_4$. Under the same notations as in §3, we define a global mean curvature functions $\mathcal{H}_h$, $\mathcal{H}_d$ and $\mathcal{H}_\ell$, the global hyperbolic Gauss-Kronecker curvature $\mathcal{K}_h$ and the global de Sitter Gauss-Kronecker curvature $\mathcal{K}_d$ by using the lightcone Gauss image $\mathbb{L}$. Therefore we have a relation

$$\mathcal{K}_s = \mathcal{K}_d - \mathcal{K}_h = \mathcal{H}_\ell = \mathcal{H}_h - \mathcal{H}_d,$$

where $\mathcal{K}_s$ is the global sectional curvature function. Then we obtain a relation of the curvatures on M.

Theorem 10.4 *Let M be a closed connected orientable 2-dimensional manifold and $f : M \longrightarrow LC^*$ an embedding induced by the Legendrian embedding $\mathcal{L}_4$. Then we have*

$$\int_M \mathcal{H}_\ell d\mathfrak{a}_M = \int_M \mathcal{H}_h d\mathfrak{a}_M - \int_M \mathcal{H}_d d\mathfrak{a}_M = \int_M \mathcal{K}_d d\mathfrak{a}_M - \int_M \mathcal{K}_h d\mathfrak{a}_M = 2\pi\boldsymbol{\chi}(M).$$

Proof. By the Gauss-Bonnet theorem on M, considered as a Riemannian manifold, we have $\int_M \mathcal{K}_s d\mathfrak{a}_M = 2\pi\boldsymbol{\chi}(M)$. It follows from the previous relations that we finish the proof. $\square$

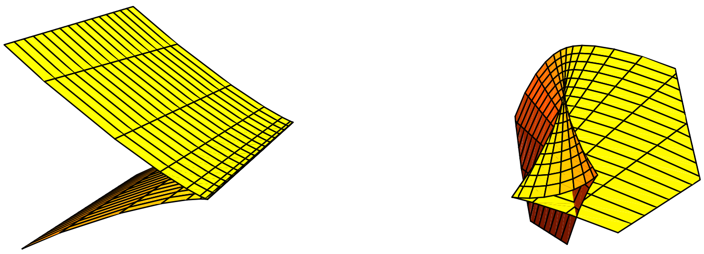
We study in the remaining of this section some generic properties of spacelike surfaces in LC^*. By Theorem 8.2 and the classification result on wave fronts (cf., [1]), we have the following local classification of singularities for the lightcone Gauss image of a generic spacelike surface in LC^*.

Theorem 10.5 *Let* $\mathrm{Emb}_{\mathrm{sp}}\,(U, LC^*)$ *be the space of embeddings from an open region* $U \subset \mathbb{R}^2$ *into* LC^* *equipped with the Whitney* C^∞*-topology. There exists an open dense subset* $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}\,(U, LC^*)$ *such that for any* $\boldsymbol{x} \in \mathcal{O}$, *the following conditions hold:*

(1) *The lightcone parabolic set* $K_\ell^{-1}(0)$ *is a regular curve. We call such a curve the lightcone parabolic curve.*

(2) *The lightcone Gauss image* $\boldsymbol{x}^\ell$ *along the lightcone parabolic curve is locally diffeomorphic to the cuspidaledge except at isolated points. At such isolated points,* $\boldsymbol{x}^\ell$ *is locally diffeomorphic to the swallowtail.*

Here, the cuspidal edge is $C = \{(x_1, x_2, x_3) | {x_1}^2 = {x_2}^3\}$ *and the swallowtail is* $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ *(cf., Fig.1).*



cuspidaledge swallowtail

Fig. 1.

Following the terminology of Whitney[51], we say that a spacelike surface $\boldsymbol{x} : U \longrightarrow LC^*$ has *the excellent lightcone Gauss image* $\boldsymbol{x}^\ell$ if $\mathcal{L}_4$ is a stable Legendrian embedding at each point. In this case, the hyperbolic Gauss image $\boldsymbol{x}^\ell$ has only cuspidaledges and swallowtails as singularities. Theorem 8.2 asserts that a spacelike surface with the excellent lightcone Gauss image is generic in the space of all spacelike surfaces in LC^*. We now consider the geometric meanings of cuspidaledges and swallowtails of the lightcone Gauss image. We have the following results analogous to the results in Banchoff et al[3].

Theorem 10.6 *Let* $\boldsymbol{x}^\ell : (U, u_0) \longrightarrow (LC^*, \boldsymbol{v}_0)$ *be the excellent lightcone Gauss image germ of a spacelike surface* $\boldsymbol{x}$ *and* $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$ *be the lightcone height function germ at* $\boldsymbol{v}_0 = \boldsymbol{x}^\ell(u_0)$. *Then we have the following:*

(1) u_0 *is a lightcone parabolic point of* $\boldsymbol{x}$ *if and only if* $\text{P-corank}(\boldsymbol{x}, u_0) = 1$ *(i.e.,*u_0 *is not a lightcone flat point of* $\boldsymbol{x}$*).*

(2) *If* u_0 *is a lightcone parabolic point of* $\boldsymbol{x}$, *then* h_{v_0} *has the* A_k*-type singularity for* $k = 2, 3$.

(3) *Suppose that* u_0 *is a lightcone parabolic point of* $\boldsymbol{x}$. *Then the following conditions are equivalent:*

(a) $\boldsymbol{x}^\ell$ *has a cuspidaledge at* u_0

(b) h_{v_0} *has the* A_2*-type singularity.*

(c) P-ord$^\pm(\boldsymbol{x}, u_0) = 2$.

(d) *The tangent parabolic indicatrix germ is a ordinary cusp, where a curve* $C \subset \mathbb{R}^2$ *is called a ordinary cusp if it is diffeomorphic to the curve given by* $\{(u_1, u_2) \mid u_1^2 - u_2^3 = 0\}$.

(e) *For each* $\varepsilon > 0$*, there exist two distinct points* $u_1, u_2 \in U$ *such that* $|u_0 - u_i| < \varepsilon$ *for* $i = 1, 2$*, both of* u_1, u_2 *are not lightcone parabolic points and the tangent parabolic quadrics to* $M = \boldsymbol{x}(U)$ *at* u_1, u_2 *are parallel.*

(4) *Suppose that* u_0 *is a lightcone parabolic point of* $\boldsymbol{x}$*. Then the following conditions are equivalent:*

(a) $\boldsymbol{x}^\ell$ *has a swallowtail at* u_0

(b) h_{v_0} *has the* A_3*-type singularity.*

(c) P-ord$^\pm(\boldsymbol{x}, u_0) = 3$.

(d) *The tangent parabolic indicatrix germ is a point or a tachnodal, where a curve* $C \subset \mathbb{R}^2$ *is called a tachnodal if it is diffeomorphic to the curve given by* $\{(u_1, u_2) \mid u_1^2 - u_2^4 = 0\}$.

(e) *For each* $\varepsilon > 0$*, there exist three distinct points* $u_1, u_2, u_3 \in U$ *such that* $|u_0 - u_i| < \varepsilon$ *for* $i = 1, 2, 3$ *and the tangent parabolic quadrics to* $M = \boldsymbol{x}(U)$ *at* u_1, u_2, u_3 *are parallel.*

(f) *For each* $\varepsilon > 0$*, there exist two distinct points* $u_1, u_2 \in U$ *such that* $|u_0 - u_i| < \varepsilon$ *for* $i = 1, 2$ *and the tangent parabolic quadrics to* $M = \boldsymbol{x}(U)$ *at* u_1, u_2 *are equal.*

Proof. We have shown that u_0 is a lightcone parabolic point if and only if P-corank$^\pm(\boldsymbol{x}, u_0) \geq 1$. Since $n = 3$, we have P-corank$^\pm(\boldsymbol{x}, u_0) \leq 2$. Since the lightcone height function germ $H : (U \times LC^*, (u_0, \boldsymbol{v}_0)) \longrightarrow \mathbb{R}$ can be considered as a generating family of the Legendrian immersion germ $\mathcal{L}_4$, h_{v_0} has only the A_k-type singularities ($k = 1, 2, 3$). This means that the corank of the Hessian matrix of h_{v_0} at a lightcone parabolic point is 1. The assertion (2) also follows. By the same reason, the conditions (3);(a),(b),(c) (respectively, (4); (a),(b),(c)) are equivalent. If the height function germ h_{v_0} has the A_2-type singularity, it is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 + u_2^3$. Since the $\mathcal{K}$-equivalence send the zero level sets, the tangent parabolic indicatrix germ is diffeomorphic to the curve given by $\pm u_1^2 + u_2^3 = 0$. This is the ordinary cusp. The normal form for the A_3-type singularity is given by $\pm u_1^2 + u_2^4$, so that the tangent parabolic indicatrix germ is diffeomorphic to the curve $\pm u_1^2 + u_2^4 = 0$. This means that the condition (3),(d) (respectively, (4),(d)) is also equivalent to the other conditions.

Suppose that u_0 is a lightcone parabolic point, by Proposition 6.1, the lightcone Gauss map has only folds or cusps. If the point u_0 is a fold point, there is a neighbourhood of u_0 on which the lightcone Gauss map is 2 to 1 except the lightcone parabolic curve (i.e, fold curve). By Lemma 7.2, the condition (3), (e) is satisfied. If the point u_0 is a cusp, the critical value set is an ordinary cusp. By the normal form, we can understand that the lightcone Gauss map is 3 to 1 inside region of the critical value. Moreover, the point u_0 is in the closure of the region. This means that the condition (4),(e) holds. We can also observe that near by a cusp point, there are 2 to 1 points which approach to u_0. However, one of those points are always lightcone parabolic points. Since other singularities do not appear for in this case, so that the condition (3),(e) (respectively, (4),(e)) characterizes a fold (respectively, a cusp).

If we consider the lightcone Gauss image instead of the lightcone Gauss map, the only singularities are cuspidaledges or swallowtails. For the swallowtail point u_0, there are self intersection curve (cf., Fig. 1) approaching to u_0. On this curve, there are two distinct point u_1, u_2 such that $\boldsymbol{x}^\ell(u_1) = \boldsymbol{x}^\ell(u_2)$. By Lemma 7.2, this means that tangent parabolic hyperquadric to

$M = \boldsymbol{x}(U)$ at u_1, u_2 are equal. Since there are no other singularities in this case, the condition (4),(f) characterize a swallowtail point of $\boldsymbol{x}^\ell$. This completes the proof. $\Box$

When considering a global embedding $f : M \longrightarrow LC^*$ induced by a Legendrian embedding $\mathcal{L}_4 : M \longrightarrow \Delta_4$, one must also pay attention to the multilocal phenomena. So we must add the double point locus, the intersection of a regular surface and the cuspidaledge and the triple point to the list of local normal forms of the singular image of lightcone Gauss images of generic embeddings. These follow from the multi-jet version of Proposition 8.1. Given a point $p_0 \in M$ and the lightlike vector $\boldsymbol{v}_0 = \mathbb{L}(p_0)$, we have the tangent parabolic quadric $TPH(f, p_0)$ of $f(M)$ at $f(p_0)$ (cf., §7). By Lemma 7.2, $\mathbb{L}(p_1) = \mathbb{L}(p_2)$ if and only if $TPH(f, p_1) = TPH(f, p_2)$.

Analogously, a triple point of the lightcone Gauss image of $f : M \longrightarrow LC^*$ corresponds to a tritangent parabolic quadric. On the other hand, we have a geometric characterizations of the swallowtail point in Theorem 10.6. Remember that a point $p \in M$ is called the lightcone parabolic point provided $\mathcal{K}_\ell(p) = 0$ which is equivalent to the condition that $\overline{\mathcal{K}}_\ell(p) = 0$ (cf., Corollary 9.3).

Denote by $T(f)$ the number of tritangent parabolic quadrics and by $SW(f)$ the number of swallowtail points of a generic embedding $f : M \longrightarrow LC^*$. By definition, the lightcone Gauss image of a hypersurface can be interpreted as a wave front in the theory of Legendrian singularities (cf., the appendix). Therefore, we have the following formula as a particular case of the relation obtained in [15] for wave fronts:

$$\boldsymbol{\chi}(\mathbb{L}(M)) = \boldsymbol{\chi}(M) + \frac{1}{2}SW(f) + T(f).$$

This together with Theorem 9.4 lead to the following:

Theorem 10.7 *Given a generic embedding $f : M \longrightarrow LC^*$, the following relation holds:*

$$\boldsymbol{\chi}(\mathbb{L}(M)) = \frac{1}{2\pi}\int_M \overline{\mathcal{K}}_\ell d\mathfrak{a}_M + \frac{1}{2}SW(f) + T(f).$$

This theorem tells us that the Euler number of the lightcone Gauss image of a generic spacelike embedding in to LC^* can be obtained in terms of the invariants of the lightcone differential geometry.

Finally, we remark that we can also apply other formulae involving the number of swallowtails and triple points on singular surfaces in a 3-manifolds (cf., [36, 38, 45]) to our situation in order to get further relations among invariants of the lightcone differential geometry.

11 Examples

In this section we give some examples. We consider a function germ $f(u_1, \dots, u_{n-1})$ around the origin with $f(0) = 1$ and $f_{u_i}(0) = 0$ $(i = 1, \dots, n-1)$. Then we have a spacelike hypersurface in LC^*_+ defined by

$$\boldsymbol{x}_f(u) = (g(u), u_1, \dots, u_{n-1}, f(u)),$$

where

$$g(u) = \sqrt{u_1^2 + \cdots + u_{n-1}^2 + f^2(u_1, \dots, u_{n-1})}$$

and $u = (u_1, \ldots, u_{n-1})$. We have $\boldsymbol{x}_f(0) = (1, 0, \ldots, 0, 1)$. We can easily calculate that $\boldsymbol{x}_{f u_i}(0) = \boldsymbol{e}_i$ $(i = 1, \ldots, n-1)$, where $\boldsymbol{e}_i$ is the canonical unit spacelike vector of $\mathbb{R}^{n+1}_1$. It follows that we have $\boldsymbol{x}^\ell_f(0) = (1, 0, \ldots, 0, -1)$. In this case, the tangent parabolic hyperquadric of $\boldsymbol{x}_f$ at $\boldsymbol{x}_f(0)$ is

$$TP_f(u) = \left(\frac{u_1^2 + \cdots + u_{n-1}^2 + 2}{2}, u_1, \ldots, u_{n-1}, 1 - \frac{u_1^2 + \cdots + u_{n-1}^2}{4} \right).$$

Therefore the tangent parabolic indicatrix germ of $\boldsymbol{x}_f$ at the origin is

$$\{ u \in (\mathbb{R}^{n-1}, 0) \mid 4f(u) + (u_1^2 + \cdots + u_{n-1}^2) - 4 = 0 \}.$$

We now give two examples in the case when $n = 3$. If we try to draw pictures of the lightcone Gauss image, it might be very hard to give a parameterization. However, the tangent parabolic indicatrix germ is very useful and easy to detect the type of singularities of the lightcone Gauss image.

Example 11.1 Consider the function given by

$$f(u_1, u_2) = 1 + \left(\frac{1}{3}u_1^3 - \frac{1}{4}u_1^2 - \frac{1}{2}u_2 \right).$$

Then

$$4f(u_1, u_2) + (u_1^2 + u_2^2) - 4 = 2\left(\frac{1}{3}u_1^3 - \frac{1}{2}u_2^2 \right),$$

so that the tangent parabolic indicatrix germ at the origin is the ordinary cusp. By Theorem 10.6, $\boldsymbol{x}_f(0)$ is a parabolic point and $\boldsymbol{x}^\ell_f(0)$ might be the cuspidaledge.

Example 11.2 Consider the function given by

$$f(u_1, u_2) = 1 + \left(\frac{1}{4}u_1^4 - \frac{1}{4}u_1^2 - \frac{1}{2}u_2 \right).$$

Then

$$4f(u_1, u_2) + (u_1^2 + u_2^2) - 4 = u_1^4 - u_2^2,$$

so that the tangent parabolic indicatrix germ at the origin is the tachnode. Therefore, $\boldsymbol{x}_f(0)$ is a parabolic point and $\boldsymbol{x}^\ell_f(0)$ might be the swallowtail.

12 Remarks on parallels and evolutes

In the last part of the paper we define the notion of parallels and evolutes of spacelike hypersurfaces in the lightcone. We do not study detailed properties here. We only describe how such notions are different from other hypersurfaces theories. Let $\boldsymbol{x} : U \longrightarrow LC^*$ be a spacelike embedding. For any fixed real number $\phi \in \mathbb{R}$, we define a Legendrian embedding $\mathcal{L}^\phi_1 : U \longrightarrow \Delta_1$ by

$$\mathcal{L}^\phi_1(u) = \left(\frac{\exp(\phi)}{2}\boldsymbol{x}(u) + \frac{\exp(-\phi)}{2}\boldsymbol{x}^\ell(u), \frac{\exp(\phi)}{2}\boldsymbol{x}(u) - \frac{\exp(-\phi)}{2}\boldsymbol{x}^\ell(u) \right).$$

We call

$$\pi_{11} \circ \mathcal{L}_1^{\phi}(u) = \frac{\exp(\phi)}{2}\boldsymbol{x}(u) + \frac{\exp(-\phi)}{2}\boldsymbol{x}^{\ell}(u)$$

the *hyperbolic parallel* of $M = \boldsymbol{x}(u)$ and

$$\pi_{12} \circ \mathcal{L}_1^{\phi}(u) = \frac{\exp(\phi)}{2}\boldsymbol{x}(u) - \frac{\exp(-\phi)}{2}\boldsymbol{x}^{\ell}(u)$$

the *de Sitter parallel* of $M = \boldsymbol{x}(U)$. Why can we call those hypersurfaces parallels? We need the notion of evolutes in order to describe the reason. What is the evolute? In the case for hypersurfaces in Euclidean space [41] (respectively, hyperbolic space [22]), it was the locus of the centers of osculating hyperspheres (respectively, hyperspheres or equidistant hypersurfaces) for the hypersurface. If the hypersurface is totally umbilic with non-zero curvature (i.e, it has the center), the evolute is just the center of the hypersurface. According to the classification of totally umbilic spacelike hypersurfaces (cf., Proposition 3.3), we give the following definition: We define the *total evolute* of $\boldsymbol{x}(U) = M$ by

$$TE_M = \Big\{ \tfrac{|\kappa_\ell(u)|}{2\sqrt{|\kappa_\ell(u)|}}\big(\boldsymbol{x}(u) + \tfrac{1}{|\kappa_\ell(u)|}\boldsymbol{x}^{\ell}(u)\big) \big| \kappa_\ell(u) \text{ is a lightcone principal curvature at } p = \boldsymbol{x}(u),\ u \in U \Big\}.$$

For a spacelike hypersurface as the above, we have the following decomposition of the total evolute:

$$TE_M(u) = HE_M \cup SE_M,$$

where

$$HE_M = \Big\{ \tfrac{\kappa_\ell(u)}{2\sqrt{\kappa_\ell(u)}}\big(\boldsymbol{x}(u) + \tfrac{1}{\kappa_\ell(u)}\boldsymbol{x}^{\ell}(u)\big) \big| \kappa_\ell(u) \text{ is a lightcone principal curvature with } \kappa_\ell(u) > 0 \text{ at } p = \boldsymbol{x}(u),\ u \in U \Big\}.$$

and

$$SE_M = \Big\{ \tfrac{-\kappa_\ell(u)}{2\sqrt{-\kappa_\ell(u)}}\big(\boldsymbol{x}(u) + \tfrac{1}{\kappa_\ell(u)}\boldsymbol{x}^{\ell}(u)\big) \big| \kappa_\ell(u) \text{ is a lightcone principal curvature with } \kappa_\ell(u) < 0 \text{ at } p = \boldsymbol{x}(u),\ u \in U \Big\}.$$

We can show that $HE_M \subset H^n(-1)$ and $SE_M \subset S^n_1$. Therefore we call HE_M (respectively, SE_M) the *hyperbolic evolute* (respectively, *de Sitter evolute*) of $\boldsymbol{x}(U) = M$.

For any fixed lightcone principal curvature κ_ℓ, we define a smooth mapping $HE_M^{\kappa_\ell} : U_+ \longrightarrow H^n(-1)$ by

$$HE_M^{\kappa_\ell}(u) = \frac{\kappa_\ell(u)}{2\sqrt{\kappa_\ell(u)}}\big(\boldsymbol{x}(u) + \frac{1}{\kappa_\ell(u)}\boldsymbol{x}^{\ell}(u)\big),$$

where $U_+ = \{u \in U \mid \kappa_\ell(u) > 0\}$. We can also define a smooth mapping $SE_{\kappa_\ell} : U_- \longrightarrow S^n_1$ by the similar way for $U_- = \{u \in U \mid \kappa_\ell(u) < 0\}$. The above mappings give local parameterizations of the evolutes. Such definitions of the evolute is reasonable compared with the definition of evolutes of hypersurfaces in Euclidean space or hyperbolic space [22, 41]. Moreover we can show that the locus of singularities of hyperbolic (respectively, de Sitter) parallels of a spacelike hypersurfaces in the lightcone is equal to the hyperbolic (respectively, de Sitter) evolute of the spacelike hypersurface (details will be described in the forthcoming paper). This fact certifies that the above definition of parallels is suitable

We remark that parallels of spacelike hypersurfaces in the lightcone are never located in the lightcone. This fact is quite different from other hypersurfaces theories. Moreover, if $\phi = 0$, we have $\mathcal{L}_1^0(u) = (\boldsymbol{x}^h(u), \boldsymbol{x}^d(u))$. Therefore $\pi_{11} \circ \mathcal{L}_1^{\phi}$ and $\pi_{12} \circ \mathcal{L}_1^{\phi}$ can be regarded as parallels of spacelike hypersurfaces $\boldsymbol{x}^h(U) \subset H^n(-1)$ and $\boldsymbol{x}^d(U) \subset S_1^n$ respectively. This means that the above notion of parallels unifies the notion of parallels of spacelike hypersurfaces in all pseudo-spheres. The detailed descriptions will be appeared in the forthcoming paper.

Appendix The theory of Legendrian singularities

In which we give a quick survey on the Legendrian singularity theory mainly due to Arnol'd-Zakalyukin [1, 52]. Almost all results have been known at least implicitly. Let $\pi : PT^*(M) \longrightarrow M$ be the projective cotangent bundle over an n-dimensional manifold M. This fibration can be considered as a Legendrian fibration with the canonical contact structure K on $PT^*(M)$. We now review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*(M) \to PT^*(M)$ and the differential map $d\pi : TPT^*(M) \to N$ of π. For any $X \in TPT^*(M)$, there exists an element $\alpha \in T^*(M)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(M)$, the property $\alpha(V) = \mathbf{0}$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(M)$ by

$$K = \{X \in TPT^*(M) | \tau(X)(d\pi(X)) = 0\}.$$

For a local coordinate neighbourhood $(U, (x_1, \dots, x_n))$ on M, we have a trivialisation $PT^*(U) \cong U \times P(\mathbb{R}^{n-1})^*$ and we call

$$((x_1, \dots, x_n), [\xi_1 : \cdots : \xi_n])$$

homogeneous coordinates, where $[\xi_1 : \cdots : \xi_n]$ are homogeneous coordinates of the dual projective space $P(\mathbb{R}^{n-1})^*$.

It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=1}^n \mu_i \xi_i = 0$, where $d\tilde{\pi}(X) = \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i}$. An immersion $i : L \to PT^*(M)$ is said to be a *Legendrian immersion* if $\dim L = n$ and $di_q(T_q L) \subset K_{i(q)}$ for any $q \in L$. We also call the map $\pi \circ i$ the *Legendrian map* and the set $W(i) = \mathrm{image}\, \pi \circ i$ the *wave front* of i. Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$.

The main tool of the theory of Legendrian singularities is the notion of generating families. Here we only consider local properites, we may assume that $M = \mathbb{R}^n$. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is a *Morse family of hypersurfaces* if the mapping

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where $(q, x) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$. In this case we have a smooth $(n-1)$-dimensional submanifold

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \mid F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ $\Phi_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\Phi_F(q,x) = \left(x, [\frac{\partial F}{\partial x_1}(q,x) : \cdots : \frac{\partial F}{\partial x_n}(q,x)] \right)$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd-Zakalyukin [1, 52].

Proposition A.1 *All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.*

We call F a *generating family* of $\Phi_F(\Sigma_*(F))$. Therefore the wave front is

$$W(\Phi_F) = \left\{ x \in \mathbb{R}^n \mid \text{there exists } q \in \mathbb{R}^k \text{ such that } F(q,x) = \frac{\partial F}{\partial q_1}(q,x) = \cdots = \frac{\partial F}{\partial q_k}(q,x) = 0 \right\}.$$

We sometime denote $\mathcal{D}_F = W(\Phi_F)$ and call it the *discriminant set* of F.

On the other hand, for any map $f : N \longrightarrow P$, we denote by $\Sigma(f)$ the set of singular points of f and $D(f) = f(\Sigma(f))$. In this case we call $f|\Sigma(f) : \Sigma(f) \longrightarrow D(f)$ the *critical part of* the mapping f. For any Morse family of hypersurfaces $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$, $(F^{-1}(0), \mathbf{0})$ is a smooth hypersurface, so we define a smooth map germ $\pi_F : (F^{-1}(0), \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ by $\pi_F(q,x) = x$. We can easily show that $\Sigma_*(F) = \Sigma(\pi_F)$. Therefore, the corresponding Legendrian map $\pi \circ \Phi_F$ is the critical part of π_F.

We now introduce an equivalence relation among Legendrian immersion germs. Let $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$ such that H preserves fibers of π and that $H(L) = L'$. A Legendrian immersion germ $i : (L.p) \subset PT^*\mathbb{R}^n$ (or, a *Legendrian map* $\pi \circ i$) at a point is said to be *Legendrian stable* if for every map with the given germ there is a neighbourhood in the space of Legendrian immersions (in the Whitney C^∞ topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has in the second neighbourhood a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs:

Proposition A.2 *Let $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs such that the representative of both of germs are proper mappings and the regular sets of the projections $\pi \circ i, \pi \circ i'$ are dense. Then i, i' are Legendrian equivalent if and only if wave front sets $W(i), W(i')$ are diffeomorphic as set germs.*

This result has been firstly pointed out by Zakalyukin [53]. The assumption in the above proposition is a generic condition for i, i'. Specially, if i, i' are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$

of the form $\Psi(x,u) = (\psi_1(q,x), \psi_2(x))$ for $(q,x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is a *$\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$* if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{\mathbf{0}\}, \dots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{\mathbf{0}\} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

(See [28].)

The main result in Arnol'd-Zakalyukin's theory [1, 52] is the following:

Theorem A.3 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then*
(1) Φ_F and Φ_G are Legendrian equivalent if and only if F, G are P-$\mathcal{K}$-equivalent.
(2) Φ_F is Legendrian stable if and only if F is a $\mathcal{K}$-versal deformation of $F \mid \mathbb{R}^k \times \{\mathbf{0}\}$.

Since F, G are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$, we do no need the notion of stably P-$\mathcal{K}$-equivalences under this situation (cf., [1]). By the uniqueness result of the $\mathcal{K}$-versal deformation of a function germ, Proposition A.2 and Theorem A.3, we have the following classification result of Legendrian stable germs (cf., [16]). For any map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define the *local ring of f* by $Q(f) = \mathcal{E}_n / f^*(\mathfrak{M}_p)\mathcal{E}_n$.

Proposition A.4 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Suppose that Φ_F, Φ_G are Legendrian stable. The the following conditions are equivalent.*
(1) $(W(\Phi_F), \mathbf{0})$ and $(W(\Phi_G), \mathbf{0})$ are diffeomorphic as germs.
(2) Φ_F and Φ_G are Legendrian equivalent.
(3) $Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$, $g = G|\mathbb{R}^k \times \{\mathbf{0}\}$.

Proof. Since Φ_F, Φ_G are Legendrian stable, these satisfy the generic condition of Proposition A.2, so that the conditions (1) and (2) are equivalent. The condition (3) implies that f, g are $\mathcal{K}$-equivalent [28, 29]. By the uniqueness of the $\mathcal{K}$-versal deformation of a function germ, F, G are P-$\mathcal{K}$-equivalent. This means that the condition (2) holds. By Theorem A.3, the condition (2) implies the condition (3). □

We now consider the following question: How does a wave front look like generically?

We have another characterization of $\mathcal{K}$-versal deformations of function germs. Let $J^\ell(\mathbb{R}^k, \mathbb{R})$ be the ℓ-jet bundle of n-variable functions which has the canonical decomposition: $J^\ell(\mathbb{R}^k, \mathbb{R}) \equiv \mathbb{R}^k \times \mathbb{R} \times J^\ell(k, 1)$. For any Morse family of hypersurfaces $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$, we define a map germ

$$j_1^\ell F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow J^\ell(\mathbb{R}^k, \mathbb{R})$$

by $j_1^\ell F(q,x) = j^\ell F_x(q)$, where $F_x(q) = F(q,x)$. We denote $\mathcal{K}^\ell(z)$ the $\mathcal{K}$-orbit through $z = j^\ell f(\mathbf{0}) \in J^\ell(k,1)$. (cf., [28]). If $f(q) = F(q, \mathbf{0})$ is ℓ-determined relative to $\mathcal{K}$, then F is a $\mathcal{K}$-versal deformation of f if and only if $j_1^\ell F$ is transversal to $\mathbb{R}^k \times \{0\} \times \mathcal{K}^\ell(z)$ (cf., [28])

We now consider the stratification of the ℓ-jet space $J^\ell(\mathbb{R}^k, \mathbb{R})$ such that $\mathcal{K}$-versal deformations are transversal to the stratification and the pull back stratification in the parameter space corresponds to the canonical stratification of the discriminant set. By Theorem A.3, such a stratification should be $\mathcal{K}$-invariant, where we have the $\mathcal{K}$-action on $J^\ell(k,1)$ (cf., [28, 29]). By this reason, we use Mather's canonical stratification here [13, 30]. Let $\mathcal{A}^\ell(k,1)$ be the canonical stratification of $J^\ell(k,1)\setminus W^\ell(k,1)$, where

$$W^\ell(k,1)=\{j^\ell f(0)\mid \dim_{\mathbb{R}}\mathcal{E}_k/((T_e\mathcal{K})(f)+\mathfrak{M}_k^\ell)\geq \ell\ \}.$$

We now define the stratification $\mathcal{A}_0^\ell(\mathbb{R}^k,\mathbb{R})$ of $J^\ell(\mathbb{R}^k,\mathbb{R})\setminus W^\ell(\mathbb{R}^k,\mathbb{R})$ by

$$\mathbb{R}^k\times(\mathbb{R}\setminus\{0\})\times(J^\ell(k,1)\setminus W^\ell(k,1)),\ \mathbb{R}^k\times\{0\}\times\mathcal{A}^\ell(k,1),$$

where $W^\ell(\mathbb{R}^k,\mathbb{R})\equiv\mathbb{R}^k\times\mathbb{R}\times W^\ell(k,1)$. In [49], Y.-H. Wan has shown that if $j_1^\ell F(0)\notin W^\ell(k,1)$ and $j_1^\ell F$ is transversal to $\mathcal{A}_0^\ell(\mathbb{R}^k,\mathbb{R})$ then $\pi_F:(F^{-1}(0),\mathbf{0})\longrightarrow(\mathbb{R}^n,\mathbf{0})$ is a MT-stable map germ. (See also [17]). Here, we call a map germ *MT-stable* if it is transversal to the canonical stratification of a jet space which is introduced in [13, 30]. The main assertion of Mather's topological stability theorem is that an MT-stable map germ is a topological stable map germ. Moreover, the critical value set of an MT-stable map germ is canonically stratified. For the classification, we refer to the following theorem of Fukuda-Fukuda [12].

Theorem A.5 *Let $f,g:(\mathbb{R}^n,\mathbf{0})\longrightarrow(\mathbb{R}^p,\mathbf{0})$ be MT-stable map germs. If $Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, then these map germs are topological equivalent.*

If we carefully read their proof, we can conclude that critical value sets (discriminant sets) of f,g are stratified equivalent. Here we say that two stratified sets are *stratified equivalent* if there exists a homeomorphism between stratified sets such that the homeomorphism maps a strata onto a strata and the restriction on each strata is smooth.

In order to apply Theorem A.5 to our situation, we need to review the theory of unfoldings of map germs. The definition of an *r-dimensional unfolding of* $f:(\mathbb{R}^n,0)\longrightarrow(\mathbb{R}^p,0)$ (originally due to Thom) is a germ $\widetilde{F}:(\mathbb{R}^n\times\mathbb{R}^r,0)\longrightarrow(\mathbb{R}^p\times\mathbb{R}^r,0)$ given by $\widetilde{F}(x,u)=(F(x,u),u)$, where $F(x,u)$ is a germ of an r dimensional parameterized family of germs with $F(x,0)=f(x)$. This definition depends on the coordinates of both of spaces $(\mathbb{R}^n\times\mathbb{R}^r,0)$ and $(\mathbb{R}^p\times\mathbb{R}^r,0)$. For our purpose, we need the coordinate free definition of unfoldings [13]. Let $f:(N,x_0)\longrightarrow(P,y_0)$ be a map germ between manifolds. An *unfolding* of f is a triple $(\widetilde{F},i,j)$ of map germs, where $i:(N,x_0)\longrightarrow(N',x_0'),\ j:(P,y_0)\longrightarrow(P',y_0')$ are immersions and j is transverse to $\widetilde{F}$, such that $\widetilde{F}\circ i=j\circ f$ and $(i,f):N\longrightarrow\{(x',y)\in N'\times P\mid\widetilde{F}(x')=j(y)\}$ is a diffeomorphism germ. The *dimension* of $(\widetilde{F},i,j)$ as an unfolding is $\dim N'-\dim N$. We can easily prove that the above two definitions are equivalent. We can show that the local ring of a map germ does not depend on the choice of the local coordinates at the points. Therefore we can define the local ring $Q(\pi_F)$ for a Morse family of hypersurfaces F. We can easily show that $Q(f)$ and $Q(\widetilde{F})$ are canonically isomorphic as $\mathbb{R}$-algebras.

We now apply the above arguments to our case. The idea used here comes from Martinet's study of stable map germs [28]. Let $F:(\mathbb{R}^k\times\mathbb{R}^n,\mathbf{0})\longrightarrow(\mathbb{R},0)$ be a Morse family of hypersurfaces. Corresponding to F, we have an unfolding of $f=F|\{\mathbf{0}\}\times\mathbb{R}^n$

$$\widetilde{F}:(\mathbb{R}^k\times\mathbb{R}^n,\mathbf{0})\longrightarrow(\mathbb{R}\times\mathbb{R}^n,\mathbf{0})$$

given by $\widetilde{F}(q,x)=(F(q,x),x)$. Then we can easily show the following lemma.

Lemma A.6 *We consider inclusions $i:(F^{-1}(0),\mathbf{0})\longrightarrow(\mathbb{R}^k\times\mathbb{R}^n,\mathbf{0})$ and $j:(\{0\}\times\mathbb{R}^n,\mathbf{0})\longrightarrow(\mathbb{R}\times\mathbb{R}^n,\mathbf{0})$. Then $(\widetilde{F},i,j)$ is an unfolding of $\pi_F:(F^{-1}(0),\mathbf{0})\longrightarrow(\mathbb{R}^n,\mathbf{0})$.*

By the previous arguments, $Q(\pi_F)$, $Q(\widetilde{F})$ and $Q(f)$ are isomorphic to each other as $\mathbb{R}$-algebras. By Theorem A.5, we have the following proposition:

Proposition A.7 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces such that π_F and π_G are MT-stable map germs. If $Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, then π_F and π_G are topological equivalent. Moreover, in this case, $\mathcal{D}_F$ and $\mathcal{D}_G$ are stratified equivalent.*

Let $F : (\mathbb{R}^n \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be a Morse family of hypersurfaces. Suppose that $j_1^\ell F(\mathbf{0}) \notin W^\ell(k,1)$ and $j_1^\ell F$ is transversal to $\mathcal{A}_0^\ell(\mathbb{R}^k, \mathbb{R})$ for sufficient large ℓ (i.e., $\operatorname{codim} W^\ell(k,1) > k+n$). By the transversality assumption, we cannot avoid strata X_j of codimension $\leq k+n$. For $n \leq 6$ and $\ell \geq 8$, by the classification of $\mathcal{K}$-simple function germs [1], $\operatorname{codim} W^\ell(k,1) > k+6$ and each strata of $\mathcal{A}^\ell(k,1)$ is a $\mathcal{K}^\ell$-orbit in $J^\ell(k,1)$. In this case, we can say that F is a $\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$ by the characterization of $\mathcal{K}$-versal deformations. Therefore Φ_F is Legendrian stable. For general $n \geq 7$, by the previous arguments, the wave front $W(\Phi_F)$ is the discriminant set of the MT-stable map germ $\pi_F : (F^{-1}(0), \mathbf{0}) \longrightarrow (\mathbb{R}^n, \mathbf{0})$.

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Shyuichi Izumiya, Department of Mathematics, Hokkaido University, Sapporo 060-0810,Japan
e-mail:izumiya@math.sci.hokudai.ac.jp