

# **NORMAL INTERMEDIATE SUBFACTORS**

**Tamotsu Teruya**

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# NORMAL INTERMEDIATE SUBFACTORS

TAMOTSU TERUYA

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ABSTRACT. Let  $N \subset M$  be an irreducible inclusion of type II<sub>1</sub> factors with finite Jones index. We shall introduce the notion of normality for intermediate subfactors of the inclusion  $N \subset M$ . If the depth of  $N \subset M$  is 2, then an intermediate subfactor  $K$  for  $N \subset M$  is normal in  $N \subset M$  if and only if the depths of  $N \subset K$  and  $K \subset M$  are both 2. In particular, if  $M$  is the crossed product  $N \rtimes G$  of a finite group  $G$ , then  $K = N \rtimes H$  is normal in  $N \subset M$  if and only if  $H$  is a normal subgroup of  $G$ .

## 1. INTRODUCTION

The index theory for type II<sub>1</sub> factors initiated by V. Jones [9] and the classification of subfactors has been studied by many people ([4], [6], [7], [8], [10], [11], [12], [13], [14, 15], [21], ...). A. Ocneanu [19] introduced the concept of a paragroup to classify subfactors. By using the so-called standard invariant equivalent to the paragroups, S. Popa [24],[23] classified subfactors under more general conditions. Paragroup or the standard invariant for an inclusion of type II<sub>1</sub> factors with finite Jones index is a

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*Key words and phrases.* subfactor, Jones index, normal, Hopf algebra, Kac algebra, strongly outer action, modular lattice.

group like object which contains finite groups. So the theory of finite groups may be considered as part of the index theory for an inclusion of type  $\text{II}_1$  factors with finite Jones index. It is well known that if  $\alpha : G \rightarrow \text{Aut}(N)$  is an outer action of a finite group  $G$  on a type  $\text{II}_1$  factor  $N$  and  $K$  is an intermediate subfactor for  $N \subset N \rtimes_{\alpha} G$ , then there is a subgroup  $H$  of  $G$  such that  $K = N \rtimes_{\alpha} H$  (see for instance [18]). On the other hand, Y. Watatani [30] showed that there exist only finitely many intermediate subfactors for an irreducible inclusion with finite index. So it is natural to consider intermediate subfactors as “quantized subgroups” in the index theory for an inclusion of type  $\text{II}_1$  factors. The notion of normality for subgroups plays important role in the theory of finite groups. In this note we introduce the notion of normality for intermediate subfactors of irreducible inclusions.

D. Bisch [1] and A. Ocneanu [20] gave a nice characterization of intermediate subfactors of a given irreducible inclusion  $N \subset M$  in terms of Jones projections and Ocneanu’s Fourier transform  $\mathcal{F} : N' \cap M_1 \rightarrow M' \cap M_2$ . We define normal intermediate subfactors as follows:

*Definition.* Let  $N \subset M$  be an irreducible inclusion of type  $\text{II}_1$  factors with finite index and  $K$  an intermediate subfactor of the inclusion  $N \subset M$ . Then  $K$  is a *normal intermediate subfactor* of the inclusion  $N \subset M$  if  $e_K \in \mathcal{Z}(N' \cap M_1)$  and  $\mathcal{F}(e_K) \in \mathcal{Z}(M' \cap M_2)$ , where  $e_K$  is the Jones projection for the inclusion  $K \subset M$ .

Every finite dimensional Hopf  $C^*$ -algebra (Kac algebra) gives rise to an irreducible

inclusion of AFD  $II_1$  factors, which are characterized by depth 2 (see for example [20], [27], [28], [33]). Let  $M$  be the crossed product algebra  $N \rtimes \mathbf{H}$  of  $N$  by an outer action of a finite dimensional Hopf  $C^*$ -algebra  $\mathbf{H}$ . Unfortunately, there is no one-to-one correspondence between the intermediate subfactors of  $N \subset M$  and the subHopf  $C^*$ -algebras of  $\mathbf{H}$  in general. But we get the next result:

*Theorem.* Let  $N \subset M$  be an irreducible, depth 2 inclusion of type  $II_1$  factors with finite index, i.e.,  $M$  is described as the crossed product algebra  $N \rtimes \mathbf{H}$  of  $N$  by an outer action of a finite dimensional Hopf  $C^*$ -algebra  $\mathbf{H}$ . Let  $K$  be an intermediate subfactor of  $N \subset M$  and  $e_K$  is the Jones projection for  $K \subset M$ . Then  $K$  is described as the crossed product algebra  $N \rtimes \mathbf{K}$  of  $N$  by an outer action of a subHopf  $C^*$  algebra  $\mathbf{K}$  of  $\mathbf{H}$  if and only if  $e_K$  is an element of the center of the relative commutant algebra  $N' \cap M_1$ , where  $M_1$  is the basic extension for  $N \subset M$ .

Let  $N \subset M$  be an irreducible inclusion of type  $II_1$  factors with finite index and  $M_1$  the basic extension for  $N \subset M$ . Let  $K$  be an intermediate subfactor of  $N \subset M$  and  $K_1$  the basic extension for  $K \subset M$ . Then  $K_1$  is an intermediate subfactor of  $M \subset M_1$ . For the Jones projections  $e_K$  and  $e_{K_1}$  for the inclusions  $K \subset M$  and  $K_1 \subset M_1$ , respectively, since  $\mathcal{F}(e_K) = \lambda e_{K_1}$  for some scalar  $\lambda$ , we get the next theorem:

*Theorem.* If the depth of a given irreducible inclusion  $N \subset M$  is 2, then an intermediate subfactor  $K$  of  $N \subset M$  is normal in  $N \subset M$  if and only if the depths of  $N \subset K$  and  $K \subset M$  are both 2.

The author [29] showed that if  $M$  is the crossed product  $N \rtimes G$  of finite group  $G$  and  $K = N \rtimes H$ , then  $H$  is a normal subgroup of  $G$  if and only if  $K \subset M \simeq K \subset K \rtimes F$  for some finite group  $F$ , i.e., the depth of  $K \subset M$  is 2. Hence we have  $H$  is a normal subgroup of  $G$  if and only if  $K$  is a normal intermediate subfactor of  $N \subset M$  by the previous theorem. Therefore our notion of normality for intermediate subfactors is an extension of that in the theory of finite groups.

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## 2. PRELIMINARIES

**2.1. intermediate subfactors.** We recall here some results for intermediate subfactors. Let  $N \subset M$  be a pair of type  $II_1$  factors. We denote by  $\mathcal{L}(N \subset M)$  the set of all intermediate von Neumann subalgebras of  $N \subset M$ . The set  $\mathcal{L}(N \subset M)$  forms a lattice under the two operations  $\vee$  and  $\wedge$  defined by

$$K_1 \vee K_2 = (K_1 \cup K_2)'' \text{ and } K_1 \wedge K_2 = K_1 \cap K_2.$$

If the relative commutant algebra  $N' \cap M$  is trivial, then  $\mathcal{L}(N \subset M)$  is exactly the lattice of intermediate subfactors for  $N \subset M$ . In fact for any  $K \in \mathcal{L}(N \subset M)$ ,  $\mathcal{Z}(K) = K' \cap K \subset N' \cap M = \mathbb{C}$ . If  $M$  is the crossed product  $N \rtimes_{\alpha} G$  for an outer action  $\alpha$  of a finite group  $G$ , then it is well known that the intermediate subfactor lattice  $\mathcal{L}(N \subset M)$  is isomorphic to the subgroup lattice  $\mathcal{L}(G)$  (see [17], [18]). In [30] Y. Watatani proved the next theorem.

*Theorem.* Let  $N \subset M$  be a pair of type II<sub>1</sub> factors. If  $[M : N] < \infty$  and  $N' \cap M = \mathbb{C}$ , then  $\mathcal{L}(N \subset M)$  is a finite lattice.

Later we were noted that this theorem was shown by S. Popa implicitly [22].

From now on we assume that  $[M : N] < \infty$  and  $N' \cap M = \mathbb{C}$ . Let

$$N \subset M \subset M_1 \subset M_2$$

be the Jones tower of  $N \subset M$  obtained by iterating the basic extension. Let  $e_N \in M_1$  and  $e_M \in M_1$  be the Jones projections for  $N \subset M$  and  $M \subset M_1$ , respectively. We denote by  $\mathcal{F}$ , Ocneanu's Fourier transform from  $N' \cap M_1$  onto  $M' \cap M_2$  i.e.,

$$\mathcal{F}(x) = [M : N]^{-\frac{3}{2}} E_{M'}^{N'}(x e_N e_M), \quad x \in N' \cap M_1,$$

where  $E_{M'}^{N'}$  is the conditional expectation from  $N'$  onto  $M'$ . For  $K \in \mathcal{L}(N \subset M)$ , if  $e_K$  is the Jones projection for  $K \subset M$ , then  $e_K$  is an element of  $N' \cap M_1$ . In fact  $K_1 = \langle M, e_K \rangle = J_M K' J_M \subset J_M N' J_M = M_1$  and hence  $e_K \in K' \cap K_1 \subset N' \cap M_1$ .

D. Bisch [1] and A. Ocneanu [20] gave the next characterization of intermediate subfactors in terms of Jones projections in  $N' \cap M_1$ .

*Theorem.* Let  $p$  be a projection in  $N' \cap M_1$ . There exists an intermediate subfactor  $K \in \mathcal{L}(N \subset M)$  such that  $p = e_K$  if and only if

- (1)  $p \geq e_N$ ,
- (2)  $\mathcal{F}(p) = \lambda q$  for some  $\lambda \in \mathbb{C}$  and some projection  $q \in M' \cap M_2$ .

In this case,  $q$  is the Jones projection  $e_{K_1}$  for  $K_1 \subset M_1$ .

For the convenience, we prove the next lemmas (see for example [1], [26]).

**Lemma 2.1.** *With the above notations, we have*

$$e_K = [K : N][M : N]E_{M_1}^{M_2}(e_M e_N e_{K_1}),$$

where  $E_{M_1}^{M_2}$  is the trace preserving conditional expectation form  $M_2$  onto  $M_1$ .

*Proof.* Since  $e_M \leq e_{K_1}$ , we have

$$e_M e_N e_{K_1} = e_M e_{K_1} e_N e_{K_1} = e_M E_{K_1}^{M_1}(e_N).$$

Since  $E_{K_1}^{M_1}(e_N)e_K = E_{K_1}^{M_1}(e_N e_K) = E_{K_1}^{M_1}(e_N)$ , by [21], we have

$$\begin{aligned} E_{K_1}^{M_1}(e_N) &= [M : K]E_M^{K_1}(E_{K_1}^{M_1}(e_N)e_K)e_K \\ &= [M : K]E_M^{M_1}(e_N)e_K \\ &= \frac{[M : K]}{[M : N]}e_K \\ &= \frac{1}{[K : N]}e_K. \end{aligned}$$

Therefore we have

$$e_M e_N e_{K_1} = \frac{1}{[K : N]}e_M e_K.$$



And hence we have

$$E_{M_1}^{M_2}(e_M e_N e_{K_1}) = \frac{1}{[K : N]} E_{M_1}^{M_2}(e_M) e_K = \frac{1}{[K : N][M : N]} e_K.$$

We get the result.  $\square$

**Lemma 2.2.** *Let  $K$  be an intermediate subfactor for  $N \subset M$ . Let  $K \subset M \subset K_1 \subset K_2$  and  $N \subset M \subset M_1 \subset M_2$  be the Jones towers for  $K \subset M$  and  $N \subset M$ , respectively. If  $e_{K_1}$  is the Jones projection for  $K_1 \subset M_1$ , then there exists a  $*$ -isomorphism  $\varphi$  of  $K_2$  onto  $e_{K_1} M_2 e_{K_1}$  such that  $\varphi(x) = x e_{K_1}$  for  $x \in K_1$  and  $\varphi(e_M^{K_1}) = e_M$ , where  $e_M^{K_1}$  and  $e_M$  are the Jones projections for  $M \subset K_1$  and  $M \subset M_1$ , respectively.*

*Proof.* Since  $e_{K_1} \in K_1' \subset M'$ , it is obvious that  $(M \subset K_1) \simeq (M e_{K_1} \subset K_1 e_{K_1})$ . Therefore it is enough to show that  $e_{K_1} M_2 e_{K_1}$  is the basic extension for  $M e_{K_1} \subset K_1 e_{K_1}$  with the Jones projection  $e_M$ . By the fact that  $e_M = e_{K_1} e_M e_{K_1}$ ,  $e_M$  is an element of  $e_{K_1} M_2 e_{K_1}$ . Let  $\widetilde{K_2}$  be the basic extension for  $K_1 \subset M_1$ . Since  $e_{K_1} \widetilde{K_2} e_{K_1} = K_1 e_{K_1}$ , we get by Lemma 2.1,

$$\begin{aligned} E_{K_1 e_{K_1}}^{e_{K_1} M_2 e_{K_1}}(e_M) &= E_{e_{K_1} \widetilde{K_2} e_{K_1}}^{e_{K_1} M_2 e_{K_1}}(e_M) \\ &= e_{K_1} E_{\widetilde{K_2}}^{M_2}(e_M) e_{K_1} \\ &= E_{\widetilde{K_2}}^{M_2}(e_M) \\ &= \frac{1}{[M : K]} e_{K_1}. \end{aligned}$$

We can see that

$$Me_{K_1} = (K_1 \cap \{e_M\}')e_{K_1} = K_1e_{K_1} \cap \{e_M\}'.$$

Therefore  $e_{K_1}M_2e_{K_1}$  is the basic extension for  $Me_{K_1} \subset K_1e_{K_1}$  by [21].  $\square$

**2.2. Finite dimensional Hopf  $C^*$ -algebras.** In this subsection we recall some facts about finite dimensional Hopf  $C^*$ -algebras.

Let  $\mathbf{H}$  be a finite dimensional Hopf  $C^*$ -algebra with a comultiplication  $\Delta_{\mathbf{H}}$  and an anti-pode  $S_{\mathbf{H}}$ . Let  $\mathbf{K}$  be a subHopf  $C^*$ -algebra of  $\mathbf{H}$ , i.e.,  $\mathbf{K}$  is a  $*$ -subalgebra of  $\mathbf{H}$ ,  $S_{\mathbf{H}}(\mathbf{K}) \subset \mathbf{K}$  and  $\Delta_{\mathbf{H}}(\mathbf{K}) \subset \mathbf{K} \otimes \mathbf{K}$ .

**Lemma 2.3.** *Define the subset  $\mathbf{K}^\perp$  of  $\mathbf{H}^*$  by*

$$\mathbf{K}^\perp = \{ f \in \mathbf{H}^* \mid (f, k) = 0, \forall k \in \mathbf{K} \},$$

where  $(, ) : \mathbf{H}^* \times \mathbf{H} \rightarrow \mathbb{C}$  is the dual pairing defined by  $(f, h) = f(h)$ ,  $f \in \mathbf{H}^*$ ,  $h \in \mathbf{H}$ .

Then  $\mathbf{K}^\perp$  is an ideal of  $\mathbf{H}^*$ .

*Proof.* Let  $g$  be an element of  $\mathbf{K}^\perp$  and  $f$  an element of  $\mathbf{H}^*$ . Then the element  $gf$  of  $\mathbf{H}^*$  is determined by the equation

$$(gf, h) = (g \otimes f, \Delta_{\mathbf{H}}(h)), \forall h \in \mathbf{H}.$$

By virtue of  $\Delta_{\mathbf{H}}(\mathbf{K}) \subset \mathbf{K} \otimes \mathbf{K}$ , we get

$$(gf, k) = (g \otimes f, \Delta_{\mathbf{H}}(k)) = 0, \forall k \in \mathbf{K}.$$

Therefore  $gf$  is an element of  $\mathbf{K}^\perp$ . Similarly,  $fg \in \mathbf{K}^\perp$ .  $\square$

By the above lemma, there exists the central projection  $p \in \mathbf{H}^*$  such that  $\mathbf{K}^\perp = p\mathbf{H}^*$ .

We put  $e_{\mathbf{K}} = 1 - p$ .

**Proposition 2.4.** *With the above notation, the reduced algebra  $e_{\mathbf{K}}\mathbf{H}^*$  is the dual Hopf  $C^*$ -algebra of  $\mathbf{K}$ .*

*Proof.* Suppose that  $k \in \mathbf{K}$  and  $(y, k) = 0, \forall y \in e_{\mathbf{K}}\mathbf{H}^*$ . Then

$$(f, k) = (e_{\mathbf{K}}f, k) + (pf, k) = (e_{\mathbf{K}}f, k) = 0, \forall f \in \mathbf{H}^*.$$

Therefore  $k = 0$ . Conversely, suppose that  $y \in e_{\mathbf{K}}\mathbf{H}^*$  and  $(y, k) = 0, \forall k \in \mathbf{K}$ . Then  $y \in \mathbf{K}^\perp \cap e_{\mathbf{K}}\mathbf{H}^* = \{0\}$ . Hence the form  $(, )|_{e_{\mathbf{K}}\mathbf{H}^* \times \mathbf{K}}$  establishes a duality between  $\mathbf{K}$  and  $e_{\mathbf{K}}\mathbf{H}^*$ . So we can identify  $e_{\mathbf{K}}\mathbf{H}^*$  with  $\mathbf{K}^*$ . Then for  $y \in \mathbf{K}^*$  and  $k_1, k_2 \in \mathbf{K}$ , we have

$$\begin{aligned} (y, k_1 k_2) &= (\Delta_{\mathbf{H}^*}(y), k_1 \otimes k_2) \\ &= (\Delta_{\mathbf{H}^*}(y)(e_{\mathbf{K}} \otimes e_{\mathbf{K}}), k_1 \otimes k_2). \end{aligned}$$

Hence  $\Delta_{\mathbf{K}^*}(y) = \Delta_{\mathbf{H}^*}(y)(e_{\mathbf{K}} \otimes e_{\mathbf{K}})$ . Similarly, we have  $S_{\mathbf{K}^*} = S_{\mathbf{H}^*}|_{\mathbf{K}^*}$  by the fact that

$$\overline{(y^*, k^*)} = (S_{\mathbf{H}^*}(y), k), \forall y \in \mathbf{K}^*, \forall k \in \mathbf{K}.$$

Therefore  $e_{\mathbf{K}}\mathbf{H}^*$  is again a Hopf  $C^*$ -algebra with the dual algebra  $\mathbf{K}$ .  $\square$

**Theorem 2.5.** *Let  $\mathbf{H}$  be a finite dimensional Hopf  $C^*$ -algebra. The number of sub-Hopf  $C^*$ -algebras of  $\mathbf{H}$  is finite.*

*Proof.* By the above proposition, the map  $\mathbf{K} \mapsto e_{\mathbf{K}}$  from the set of subHopf  $C^*$ -algebras of  $\mathbf{H}$  to central projections of  $\mathbf{H}^*$  is injection. Since the number of central projections of  $\mathbf{H}^*$  is finite, so is that of subHopf  $C^*$  algebras of  $\mathbf{H}$ .  $\square$

*Remark.* Since every finite dimensional Hopf  $C^*$ -algebra (Kac algebra) admits an “outer” action on the AFD  $\text{II}_1$  factor [33], the above theorem immediately follows from [30, Theorem 2.2].

*Definition.* Let  $\mathbf{H}$  be any Hopf algebra.

- (1) The left adjoint action of  $\mathbf{H}$  on itself is given by

$$(ad_l h)(k) = \sum h_1 k (S_{\mathbf{H}}(h_2)),$$

for all  $h, k \in \mathbf{H}$ .

- (2) The right adjoint action of  $\mathbf{H}$  on itself is given by

$$(ad_r h)(k) = \sum (S_{\mathbf{H}}(h_1)) k h_2,$$

for all  $h, k \in \mathbf{H}$ .

- (3) A subHopf algebra  $\mathbf{K}$  of  $\mathbf{H}$  is called *normal* if both

$$(ad_l \mathbf{H})(\mathbf{K}) \subset \mathbf{K} \text{ and } (ad_r \mathbf{H})(\mathbf{K}) \subset \mathbf{K}.$$

See [16, pp. 33].

The next proposition is useful later.

**Proposition 2.6.** *Let  $H$  be a finite dimensional Hopf algebra with a counit  $\varepsilon_H$  and  $K$  a subHopf algebra of  $H$ . Then  $K$  is normal if and only if  $HK^+ = K^+H$ , where  $K^+ = K \cap \ker \varepsilon_H$ .*

See for a proof [16, pp. 35].

**2.3. Bimodules.** In this subsection we recall some facts about the bimodule calculus associated with an inclusion of type  $II_1$  factors (see for example [20],[31]).

Let  $A, B, C$  be type  $II_1$  factors and let  $\alpha = {}_A H_B, \beta = {}_A K_B, \gamma = {}_B L_C$  be  $A$ - $B$ ,  $A$ - $B$  and  $B$ - $C$  Hilbert bimodules, respectively. We write  $\alpha\gamma$  for the  $A$ - $C$  Hilbert bimodule  ${}_A H_B \otimes_B {}_B L_C$ . We denote by  $\langle \alpha, \beta \rangle$  the dimension of the space of  $A$ - $B$  intertwiners from  ${}_A H_B$  to  ${}_A K_B$ . The conjugate Hilbert space  $H^*$  of  ${}_A H_B$  is naturally a  $B$ - $A$  bimodule with  $B$ - $A$  actions defined by

$$b \cdot \xi^* \cdot a = (a^* \xi b^*)^* \quad \text{for } a \in A \text{ and } b \in B,$$

where  $\xi^* = \langle \cdot, \xi \rangle_H \in H^*$  for  $\xi \in {}_A H_B$ . We denote by  $\bar{\alpha}$  the conjugate  $B$ - $A$  Hilbert bimodule associated with  $\alpha$ .

**Proposition 2.7 (Frobenius reciprocity).** *Let  $A, B, C$  be type  $II_1$  factors, and  $\alpha = {}_A H_B, \beta = {}_B K_C$  and  $\gamma = {}_A L_C$  be Hilbert bimodules. Then*

$$\langle \alpha\beta, \gamma \rangle = \langle \alpha, \gamma\bar{\beta} \rangle = \langle \beta, \bar{\alpha}\gamma \rangle.$$

See for a proof [20], [31].

**Example 2.8.** Let  $M$  be a type  $\text{II}_1$  factor with the normalized trace  $\tau_M$ . As usual we let  $L^2(M)$  be the Hilbert space obtained by completing  $M$  in the norm  $\|x\|_2 = \sqrt{\tau_M(x^*x)}$ ,  $x \in M$ . Let  $\eta : M \rightarrow L^2(M)$  be the canonical implementation. Let  $J : L^2(M) \rightarrow L^2(M)$  be the modular conjugation defined by  $J\eta(x) = x^*$ ,  $x \in M$ . For  $\theta \in \text{Aut}(M)$ , we define  ${}_M L^2(\theta)_M$ , the  $M$ - $M$  Hilbert bimodule, by

- (1)  ${}_M L^2(\theta)_M = L^2(M)$  as a Hilbert space,
- (2)  $x \cdot \xi \cdot y = xJ\theta(y)^*J\xi$ ,  $x, y \in M$ ,  $\xi \in L^2(M)$ .

Then for  $\theta, \theta_1, \theta_2 \in \text{Aut}(M)$  we have

$$\overline{{}_M L^2(\theta)_M} \simeq {}_M L^2(\theta^{-1})_M$$

$${}_M L^2(\theta_1)_M \otimes_M {}_M L^2(\theta_2)_M \simeq {}_M L^2(\theta_1\theta_2)_M.$$

A bimodule  $\alpha = {}_A H_B$  is called irreducible if  $\langle \alpha, \alpha \rangle = 1$ , i.e.,  $\text{End}_{A-B}({}_A H_B) \simeq \mathbb{C}$ . If  $\langle \alpha, \alpha \rangle < \infty$ ,  $\alpha = {}_A H_B$ , then we can get an  $A$ - $B$  irreducible bimodule by cutting  ${}_A H_B$  by a minimal projection in  $\text{End}_{A-B}({}_A H_B)$ .

**Example 2.9.** Let  $N \subset M$  be an inclusion of type  $\text{II}_1$  factors. We define the  $N$ - $M$  bimodule  ${}_N L^2(M)_M$  by actions

$$x \cdot \xi \cdot y = xJy^*J\xi, \quad \xi \in L^2(M), x \in N, y \in M.$$

Then we can see that

$$\text{End}({}_N L^2(M)_M) \simeq N' \cap M.$$

In particular, if  $N' \cap M = \mathbb{C}$ , then  ${}_N L^2(M)_M$  is an irreducible  $N$ - $M$  bimodule.

The next lemma is well known.

**Lemma 2.10.** *Let  $N \subset M$  be a pair of type  $II_1$  factors with finite index and  $M_1$  the basic extension for the inclusion  $N \subset M$ . For  $\theta \in \text{Aut}(N)$ ,  ${}_N L^2(M)_{\theta(N)} \simeq {}_N L^2(M)_N$  if and only if there exists a unitary  $u \in M_1$  such that  $uxu^* = \theta(x)$ , for all  $x \in N$ , where  ${}_N L^2(M)_{\theta(N)}$  is defined as in Example 2.9.*

**Example 2.11.** Let  $\gamma : G \rightarrow \text{Aut}(N)$  be an outer action of a finite group  $G$  on a type  $II_1$  factor  $N$ . Let  $M = N \rtimes_\gamma G$  be the crossed product and  $\rho$  the  $N$ - $M$  bimodule  ${}_N L^2(M)_M$  defined as in Example 2.9. If  $\{\lambda_g \mid g \in G\}$  is a unitary implementation for the crossed product, then each element  $x \in M$  is written in the form  $x = \sum_{g \in G} x_g \lambda_g$ ,  $x_g \in N$ . This implies that the irreducible decomposition of  $\rho \bar{\rho} = {}_N L^2(M)_N$  is

$$\bigoplus_{g \in G} {}_N (\overline{N \lambda_g} \|\cdot\|^2)_N \simeq \bigoplus_{g \in G} {}_N L^2(\gamma_g)_N,$$

where  ${}_N L^2(\gamma_g)_N$  is the  $N$ - $N$  bimodule as in Example 2.8.

### 3. DEFINITION OF NORMAL INTERMEDIATE SUBFACTORS

In this section, we shall introduce the notion of normality for intermediate subfactors and study its properties.

Let  $N \subset M$  be a pair of type  $II_1$  factors with  $[M : N] < \infty$ . Let  $N \subset M \subset M_1 \subset M_2$  be the Jones tower of  $N \subset M$ , obtained by iterating the basic extensions. We

denote by  $\mathcal{F}$ , Ocneanu's Fourier transform from  $N' \cap M_1$  onto  $M' \cap M_2$  i.e.,

$$\mathcal{F}(x) = [M : N]^{-\frac{3}{2}} E_{M'}^{N'}(x e_N e_M), \quad x \in N' \cap M_1,$$

where  $E_{M'}^{N'}$  is the conditional expectation from  $N'$  onto  $M'$ .

**Definition 3.1.** Let  $K$  be an intermediate subfactor of  $N \subset M$  and  $e_K$  the Jones projection for the inclusion  $K \subset M$ . Then we call that  $K$  is *normal* in  $N \subset M$  if  $e_K$  and  $\mathcal{F}(e_K)$  are elements of the centers of  $N' \cap M_1$  and  $M' \cap M_2$ , respectively.

**Lemma 3.2.** *Let  $K$  be an intermediate subfactor for an irreducible inclusion  $N \subset M$  of type  $II_1$  factors with finite index. Let  $K_1$  and  $M_1$  be the basic extensions for  $K \subset M$  and  $N \subset M$ , respectively. Then  $K$  is normal in  $N \subset M$  if and only if  $K_1$  is normal in  $M \subset M_1$ .*

*Proof.* Since  $\mathcal{F}(e_K) = \lambda e_{K_1}$  for some  $\lambda \in \mathbb{C}$ , it is obvious by the definition.  $\square$

**Proposition 3.3.** *Let  $N$  be the fixed point algebra  $M^{(G, \alpha)}$  of a type  $II_1$  factor  $M$  by an outer action  $\alpha$  of a finite group  $G$ . If  $K = M^{(H, \alpha)}$  is an intermediate subfactor associated with a subgroup  $H$  of  $G$ , then  $K$  is normal in  $N \subset M$  if and only if  $H$  is a normal subgroup of  $G$ .*

*Proof.* Let  $\{u_g \mid g \in G\}$  be unitary operators on  $L^2(M)$  defined by  $u_g \eta(x) = \eta(\alpha_g(x))$ ,  $x \in M$ , where  $L^2(M)$  and  $\eta$  are defined as in Example 2.8. Then  $N = M \cap \{u_g \mid g \in G\}'$ ,  $M_1 = (M \cup \{u_g \mid g \in G\})''$  and  $N' \cap M_1 = \{u_g \mid g \in G\}'' \simeq \mathbb{C}G$ . Since  $K = M^H = M \cap \{u_h \mid h \in H\}'$ , the Jones projection  $e_K$  for  $K \subset M$  is  $\frac{1}{\#H} \sum_{h \in H} u_h$ .



Since

$$u_g e_K u_g^* = \frac{1}{\#H} \sum_{h \in H} u_{ghg^{-1}} \quad \text{for any } g \in G,$$

$H$  is a normal subgroup if and only if  $e_K$  is an element of the center of  $N' \cap M_1$ . Since  $M' \cap M_2$  is a commutative algebra,  $\mathcal{F}(e_K)$  is always an element of the center of  $M' \cap M_2$ . So we get the result.  $\square$

**Proposition 3.4.** *Let  $M$  be the crossed product  $N \rtimes_{\alpha} G$  of a  $II_1$  factor  $N$  by an outer action  $\alpha$  of a finite group  $G$ . If  $K = N \rtimes_{\alpha} H$  is an intermediate subfactor associated with a subgroup  $H$  of  $G$ , then  $K$  is normal in  $N \subset M$  if and only if  $H$  is a normal subgroup of  $G$ .*

*Proof.* This immediately follows from Lemma 3.2 and Proposition 3.3.  $\square$

**Proposition 3.5.** *Let  $\alpha : G \rightarrow \text{Aut}(P)$  be an outer action of a finite group  $G$  on a type  $II_1$  factor  $P$  and  $H$  a subgroup of  $G$ . Let  $M$  be the fixed point algebra  $P^{(H, \alpha)}$  and  $N$  the fixed point algebra  $P^{(G, \alpha)}$ . For  $K \in \mathcal{L}(N \subset M)$ , there is a subgroup  $A$  of  $G$  such that  $H \subset A \subset G$  and  $K = P^{(A, \alpha)}$ . Then  $K$  is a normal intermediate subfactor of  $N \subset M$  if and only if  $AgH = HgA$ , for  $\forall g \in G$ .*

*Proof.* Let  $\{u_g \mid g \in G\}$  be unitary operators on  $L^2(P)$  defined by  $u_g \eta(x) = \eta(\alpha_g(x))$ ,  $x \in P$ , where  $L^2(P)$  and  $\eta$  are defined as in Example 2.8. Let  $P_1$  be the basic extension for  $N \subset P$ . Then

$$N' \cap P_1 = \left\{ \sum_{g \in G} x_g u_g \mid x_g \in \mathbb{C} \right\} \simeq \mathbb{C}G.$$

Let  $e_M^P$  be the Jones projection for  $M \subset P$ . Then

$$e_M^P = \frac{1}{\#H} \sum_{h \in H} u_h.$$

Let  $M_1$  be the basic extension for  $N \subset M$ . Then by Lemma 2.2,

$$\begin{aligned} N' \cap M_1 &\simeq e_M^P (N' \cap P_1) e_M^P \\ &= \left\{ \sum_{g \in G} \sum_{h, k \in H} x_g u_{h g k} \mid x_g \in \mathbb{C} \right\}. \end{aligned}$$

Therefore

$$\begin{aligned} e_K^M \in \mathcal{Z}(N' \cap M_1) &\Leftrightarrow e_M^P e_K^P e_M^P (= e_K^P = \frac{1}{\#A} \sum_{a \in A} u_a) \in \mathcal{Z}(e_M^P (N' \cap P_1) e_M^P) \\ &\Leftrightarrow \sum_{a \in A} \sum_{h, k \in H} u_{a h g k} = \sum_{a \in A} \sum_{h, k \in H} u_{h g k a} \text{ for } \forall g \in G \\ &\Leftrightarrow A g H = H g A \text{ for } \forall g \in G. \end{aligned}$$

Since  $M' \cap M_2$  is a commutative algebra, we get the the result.  $\square$

**Proposition 3.6.** *Let  $\alpha : G \rightarrow \text{Aut}(P)$  be an outer action of a finite group  $G$  on a type  $II_1$  factor  $P$  and  $H$  a subgroup of  $G$ . Let  $M$  be the crossed product  $P \rtimes_{\alpha} G$  and  $N$  the crossed product  $P \rtimes_{\alpha} H$ . For  $K \in \mathcal{L}(N \subset M)$ , there is a subgroup  $A$  of  $G$  such that  $H \subset A \subset G$  and  $K = P \rtimes_{\alpha} A$ . Then  $K$  is a normal intermediate subfactor of  $N \subset M$  if and only if  $A g H = H g A$ , for  $\forall g \in G$ .*

*Proof.* This immediately follows from Lemma 3.2 and the above proposition.  $\square$

**Proposition 3.7.** *Let  $N \subset M$  and  $Q \subset P$  be irreducible inclusions of type  $II_1$  factors with finite indices. Then both of  $N \otimes P$  and  $M \otimes Q$  are normal intermediate subfactors*

of  $N \otimes Q \subset M \otimes P$ .

*Proof.* Let  $M_1 = \langle M, e_N \rangle$  and  $P_1 = \langle P, e_Q \rangle$  be the basic extension for  $N \subset M$  and  $Q \subset P$  with the Jones projections  $e_N$  and  $e_Q$ , respectively. Then  $M_1 \otimes P_1$  is the basic extension for  $N \otimes Q \subset M \otimes P$  with the Jones projection  $e_N \otimes e_Q$ . Moreover,  $e_N \otimes 1$  and  $1 \otimes e_Q$  are the Jones projections for  $N \otimes P \subset M \otimes P$  and  $M \otimes Q \subset M \otimes P$ , respectively. Since  $N \subset M$  and  $Q \subset P$  are irreducible,  $e_N$  and  $e_Q$  are elements of the centers of  $N' \cap M_1$  and  $Q' \cap P_1$ , respectively by [21, Proposition 1.9]. And hence  $e_N \otimes 1$  and  $1 \otimes e_Q$  are elements of the center of  $(N \otimes Q)' \cap (M_1 \otimes P_1) = (N' \cap M_1) \otimes (Q' \cap P_1)$ . Similarly, we can observe that  $\mathcal{F}(e_N \otimes 1)$  and  $\mathcal{F}(1 \otimes e_Q)$  are elements of the center of  $(M \otimes P)' \cap (M_2 \otimes P_2)$ , where  $M_2$  and  $P_2$  are the basic extension for  $M \subset M_1$  and  $P \subset P_1$ , respectively. We have thus proved the proposition  $\square$

In [30] Y. Watatani introduced the notion of *quasi-normal intermediate subfactors* to study the modular identity for intermediate subfactor lattices.

*Definition.* Let  $N \subset M$  be an inclusion of type II<sub>1</sub> factors with finite index and  $K$  an intermediate subfactor of  $N \subset M$ . Then  $K$  is *quasi-normal* (or *doubly commuting*) if for any  $L \in \mathcal{L}(N \subset M)$ ,

$$\begin{array}{ccc} K & \subset & K \vee L \\ \cup & & \cup \\ K \wedge L & \subset & L \end{array}$$

and

$$\begin{array}{ccc} K_1 & \subset & K_1 \vee L_1 \\ \cup & & \cup \\ K_1 \wedge L_1 & \subset & L_1 \end{array}$$

are commuting squares (see for example [5]), where  $K_1$  and  $L_1$  are the basic extension for  $K \subset M$  and  $L \subset M$ , respectively.

**Proposition 3.8.** *Let  $N \subset M$  be an irreducible inclusion of type  $II_1$  factors with finite index. If  $K$  is a normal intermediate subfactor of  $N \subset M$  then  $K$  is quasi-normal in  $N \subset M$*

*Proof.* Suppose that the Jones projection  $e_K$  for  $K \subset M$  is an element of the center of  $N' \cap M_1$ . Then since for any intermediate subfactor  $L$  of  $N \subset M$ , the Jones projection  $e_K^{K \vee L}$  for  $K \subset (K \vee L)$  is also a central projection in  $K' \cap (K \vee L)_1$ , we have

$$\begin{array}{ccc} K & \subset & K \vee L \\ \cup & & \cup \\ K \wedge L & \subset & L \end{array}$$

is a commuting square. Similarly, if  $\mathcal{F}(e_K)$  is an element of the center of  $M' \cap M_2$ ,

then

$$\begin{array}{ccc} K_1 & \subset & K_1 \vee L_1 \\ \cup & & \cup \\ K_1 \wedge L_1 & \subset & L_1 \end{array}$$

is a commuting square. Therefore if  $K$  is normal in  $N \subset M$ , then  $K$  is quasi-normal.  $\square$

We have a characterization of normal intermediate subfactors in terms of bimodules. Let  $K$  be an intermediate subfactor of an irreducible inclusion  $N \subset M$  of type  $II_1$  factors with finite index. We note that  $e_K$  is in the center of  $N' \cap M_1$  if and only if for any  $T \in \text{End}({}_N L^2(M)_N)$ ,  $T L^2(K) \subset L^2(K)$ .

**Proposition 3.9.** *Let  $K$  be an intermediate subfactor for an irreducible inclusion  $N \subset M$  of type  $II_1$  factors with finite index. Let  $\alpha$  be the  $N$ - $K$  bimodule  ${}_N L^2(K)_K$  and  $\beta$  the  $K$ - $M$  bimodule  ${}_K L^2(M)_M$ . If  $\rho$  is the  $N$ - $M$  bimodule  $\alpha\beta = {}_N L^2(M)_M$ , then  $K$  is normal in  $N \subset M$  if and only if*

$$(1) \langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle,$$

$$(2) \langle \bar{\beta}\beta, \bar{\rho}\rho \rangle = \langle \bar{\beta}\beta, \bar{\beta}\beta \rangle.$$

*Proof.* Since  $\text{End}({}_N L^2(K)_K) = N' \cap \langle N, e_N^K \rangle \simeq e_K(N' \cap M_1)e_K$  by Lemma 2.2, if  $e_K$  is an element of the center of  $N' \cap M_1$ , then for any irreducible  $N$ - $N$  bimodule  $\sigma$  contained in  $\alpha\bar{\alpha}$ , the multiplicity of  $\sigma$  in  $\alpha\bar{\alpha}$  is equal to the multiplicity of  $\sigma$  in  $\rho\bar{\rho}$ .

Therefore we have

$$\langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle.$$

Conversely, suppose that  $e_K$  is not an element of the center of  $N' \cap M_1$ . Then there exist minimal projections  $p, q \in e_K(N' \cap M_1)e_K$  such that

$$p \sim q \text{ in } (N' \cap M_1) \quad \text{and} \quad p \not\sim q \text{ in } e_K(N' \cap M_1)e_K.$$

Therefore we have

$$\langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle \neq \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle.$$

And hence  $e_K$  is an element of the center of  $(N' \cap M_1)$  if and only if

$$\langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle.$$

Similarly, we can see that  $e_{K_1}$  is an element of the center of  $(M' \cap M_2)$  if and only if

$$\langle \bar{\beta}\beta, \bar{\rho}\rho \rangle = \langle \bar{\beta}\beta, \bar{\beta}\beta \rangle.$$

Since  $\mathcal{F}(e_K) = \lambda e_{K_1}$  for some  $\lambda \in \mathbb{C}$ , we get the result.  $\square$

**Theorem 3.10.** *Let  $K$  be an intermediate subfactor for an irreducible inclusion  $N \subset M$  of type  $II_1$  factors with finite index. If the depths of  $N \subset K$  and  $K \subset M$  are both 2, then  $K$  is normal in  $N \subset M$ .*

*Proof.* Let  $\alpha$  be the  $N$ - $K$  bimodule  ${}_N L^2(K)_K$  and  $\beta$  the  $K$ - $M$  bimodule  ${}_K L^2(M)_M$ .

By the assumption, we have

$$\alpha\bar{\alpha}\alpha \simeq \underbrace{\alpha \oplus \alpha \oplus \cdots \oplus \alpha}_{[K:N]\text{times}}$$

and

$$\overline{\beta\beta\beta} \simeq \underbrace{\overline{\beta \oplus \beta \oplus \dots \oplus \beta}}_{[M:K]\text{times}}.$$

And hence

$$\langle \alpha\overline{\alpha}, \alpha\overline{\alpha} \rangle = \langle \alpha\overline{\alpha}\alpha, \alpha \rangle = [K : N]$$

and

$$\langle \overline{\beta\beta}, \overline{\beta\beta} \rangle = \langle \overline{\beta\beta\beta}, \overline{\beta} \rangle = [M : K]$$

by Frobenius reciprocity. Since  $N \subset M$  is irreducible, if  $\rho$  is the  $N$ - $M$  bimodule  ${}_N L^2(M)_M (= \alpha\beta)$ , then

$$1 = \langle \rho, \rho \rangle = \langle \alpha\beta, \alpha\beta \rangle = \langle \overline{\alpha\alpha}, \overline{\beta\beta} \rangle.$$

And hence we have

$$\begin{aligned} \langle \alpha\overline{\alpha}, \rho\overline{\rho} \rangle &= \langle \alpha\overline{\alpha}, \alpha\beta\overline{\beta\alpha} \rangle \\ &= \langle \alpha\overline{\alpha}\alpha, \alpha\beta\overline{\beta} \rangle \\ &= [K : N] \langle \alpha, \alpha\beta\overline{\beta} \rangle \\ &= [K : N] \langle \overline{\alpha\alpha}, \overline{\beta\beta} \rangle = [K : N], \end{aligned}$$

i.e.,

$$\langle \alpha\overline{\alpha}, \rho\overline{\rho} \rangle = \langle \alpha\overline{\alpha}, \alpha\overline{\alpha} \rangle.$$

Similarly, we have

$$\langle \overline{\beta\beta}, \overline{\rho\rho} \rangle = \langle \overline{\beta\beta}, \overline{\beta\beta} \rangle.$$

So we get the result by Lemma 3.9.  $\square$

**Proposition 3.11.** *Let  $M_0, N_0, K$  be intermediate subfactors for an irreducible inclusion  $N \subset M$  of type  $II_1$  factors with finite index such that*

$$N \subset N_0 \subset K \subset M_0 \subset M.$$

*If  $K$  is normal in  $N \subset M$ , then  $K$  is also normal in  $N_0 \subset M_0$ .*

*Proof.* Let  $\alpha = {}_N L^2(K)_K$ ,  $\alpha_0 = {}_{N_0} L^2(K)_K$ ,  $\beta = {}_K L^2(M)_M$  and  $\beta_0 = {}_K L^2(M_0)_{M_0}$ .

Since

$$\alpha\bar{\alpha} = {}_N L^2(K)_K \otimes_K {}_K L^2(K)_N = {}_N L^2(K)_K \otimes_K {}_K L^2(K)_K \otimes_K {}_K L^2(K)_N,$$

we have

$$\langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle = \langle \bar{\alpha}\alpha\bar{\alpha}\alpha, {}_K L^2(K)_K \rangle$$

by Frobenius reciprocity. Since  $\langle \alpha\bar{\alpha}, \alpha\beta\bar{\beta}\bar{\alpha} \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle$  by the assumption, we have

$$\langle \bar{\alpha}\alpha\bar{\alpha}\alpha, \beta\bar{\beta} \rangle = \langle \bar{\alpha}\alpha\bar{\alpha}\alpha, {}_K L^2(K)_K \rangle,$$

i.e., the irreducible  $K$ - $K$  sub-bimodules of  $\bar{\alpha}\alpha\bar{\alpha}\alpha$  contained in  $\beta\bar{\beta}$  is only  ${}_K L^2(K)_K$ .

Since  $\bar{\alpha}_0\alpha_0$  is contained in  $\bar{\alpha}\alpha$  and  $\beta_0\bar{\beta}_0$  is contained in  $\beta\bar{\beta}$ , we have

$$\langle \bar{\alpha}_0\alpha_0\bar{\alpha}_0\alpha_0, \beta_0\bar{\beta}_0 \rangle = \langle \bar{\alpha}_0\alpha_0\bar{\alpha}_0\alpha_0, {}_K L^2(K)_K \rangle,$$

i.e.,

$$\langle \alpha_0\bar{\alpha}_0, \alpha_0\beta_0\bar{\beta}_0\bar{\alpha}_0 \rangle = \langle \alpha_0\bar{\alpha}_0, \alpha_0\bar{\alpha}_0 \rangle.$$

By the same argument, we have

$$\langle \bar{\beta}_0\beta_0, \bar{\beta}_0\bar{\alpha}_0\alpha_0\beta_0 \rangle = \langle \bar{\beta}_0\beta_0, \bar{\beta}_0\beta_0 \rangle.$$



We have thus proved the proposition.  $\square$

#### 4. NORMAL INTERMEDIATE SUBFACTORS FOR DEPTH 2 INCLUSIONS

It is well-known that the crossed product of a finite dimensional Hopf  $C^*$  algebra (Kac algebra) is characterized by the depth 2 condition. In this section we study normal intermediate subfactors for depth 2 inclusions.

**4.1. The action of  $K' \cap K_1$  on  $M$ .** Let  $N \subset M$  be an irreducible, depth 2 inclusion of type  $II_1$  factors with finite index. Let  $N \subset M \subset M_1 \subset M_2$  be the Jones tower for  $N \subset M$ . We put  $A = N' \cap M_1$  and  $B = M' \cap M_2$ . Then  $A$  and  $B$  are dual pair of Hopf  $C^*$ -algebras with a pairing

$$(a, b) = [M : N]^2 \tau(ae_M e_N b), \text{ for } a \in A \text{ and } b \in B,$$

where  $e_N$  and  $e_M$  are the Jones projections for  $N \subset M$  and  $M \subset M_1$ , respectively.

Define a bilinear map  $A \times M \rightarrow M$  (denoted by  $a \odot x$ ) by setting

$$a \odot x = [M : N] E_M^{M_1}(ax e_N),$$

for  $x \in M$  and  $a \in A$ . This map is a left action of Hopf  $C^*$  algebra  $A$  and

$$N = M^A = \{ x \in M \mid a \odot x = \varepsilon(a)x, \forall a \in A \},$$

where  $\varepsilon : A \rightarrow \mathbb{C}$  is the counit determined by  $ae_N = \varepsilon(a)e_N$  (see [28]).

**Proposition 4.1.** *Let  $K$  be an intermediate subfactor of  $N \subset M$  and  $K_1$  the basic extension for  $K \subset M$ . We put  $H = K' \cap K_1$ . If  $a$  is an element of  $H$ , then*

$$[M : K]E_M^{K_1}(axe_K) = [M : N]E_M^{M_1}(axe_N), \quad \forall x \in M.$$

*This implies*

$$K = M^H = \{ x \in M \mid a \odot x = \varepsilon(a)x, \forall a \in H \}.$$

*Proof.* Since  $e_K = \frac{[M:N]}{[M:K]}E_{K_1}^{M_1}(e_N)$  by [26], we have

$$\begin{aligned} [M : K]E_M^{K_1}(axe_K) &= [M : K]E_M^{K_1}\left(ax \frac{[M : N]}{[M : K]}E_{K_1}^{M_1}(e_N)\right) \\ &= [M : N]E_M^{K_1}(E_{K_1}^{M_1}(axe_N)) \\ &= [M : N]E_M^{M_1}(axe_N) \end{aligned}$$

for  $\forall a \in H$  and  $\forall x \in M$ .  $\square$

**4.2. Hopf algebra structures on  $K' \cap K_1$ .** Let  $N \subset M$  be an irreducible, depth 2 inclusion of type  $\text{II}_1$  factors with finite index and  $K$  an intermediate subfactor of  $N \subset M$ . Then the depth of  $K \subset M$  is not 2 in general. In this subsection we shall prove that if the depth of  $K \subset M$  is 2, then  $H = K' \cap K_1$  is a subHopf  $C^*$  algebra of  $A = N' \cap M_1$ .

By Lemma 2.2, there exists an isomorphism  $\varphi$  of  $K_2$  onto  $e_{K_1}M_2e_{K_1}$  such that  $\varphi(x) = xe_{K_1}$  for  $x \in K_1$  and  $\varphi(e_M^{K_1}) = e_M$ , where  $K \subset M \subset K_1 \subset K_2$  is the Jones tower for the inclusion  $K \subset M$  and  $e_M^{K_1}$  is the Jones projection for  $M \subset K_1$ .

**Lemma 4.2.** *With the above notation, we have*

$$[M : K]^2 \tau(he_M^{K_1} e_K k) = [M : N]^2 \tau(he_{M \cap N} \varphi(k))$$

for  $\forall h \in H = K' \cap K_1$  and  $\forall k \in M' \cap K_2$ .

*Proof.* By the fact that  $e_{K_1} e_N e_{K_1} = E_{K_1}^{M_1}(e_N) e_{K_1} = \frac{[M:K]}{[M:N]} e_K e_{K_1}$ , we have  $\varphi(e_K) = e_K e_{K_1} = \frac{[M:N]}{[M:K]} e_{K_1} e_N e_{K_1}$ . Therefore

$$\begin{aligned} [M : K]^2 \tau(he_M^{K_1} e_K k) &= [M : K]^2 [K : N] \tau(\varphi(he_M^{K_1} e_K k)) \\ &= [M : K]^2 [K : N] \frac{[M : N]}{[M : K]} \tau(\varphi(h) e_{M \cap K_1} e_N e_{K_1} \varphi(k)) \\ &= [M : N]^2 \tau(he_{M \cap N} \varphi(k)). \end{aligned}$$

□

**Lemma 4.3.** *Let  $N \subset M$  be an irreducible, depth 2 inclusion of type  $II_1$  factors with finite index and  $K$  an intermediate subfactor for  $N \subset M$ . Let  $N \subset M \subset M_1 \subset M_2$  and  $K \subset M \subset K_1 \subset K_2$  be the Jones towers for  $N \subset M$  and  $K \subset M$ , respectively. If the depth of  $K \subset M$  is 2, then for any  $b \in M' \cap M_2$ , there exist elements  $\{x_i\}, \{y_i\}$  of  $N' \cap M_1$  such that*

$$b = \sum_i x_i e_M y_i$$

and

$$\sum_i E_{K_1}^{M_1}(x_i) e_M E_{K_1}^{M_1}(y_i) \in (K' \cap K_1) e_M (K' \cap K_1),$$

where  $E_{K_1}^{M_1}$  is the trace preserving conditional expectation from  $M_1$  onto  $K_1$ .

*Proof.* Since the depth of  $N \subset M$  is 2,

$$(N' \cap M_1)e_M(N' \cap M_1) = N' \cap M_2.$$

And hence any element  $b \in M' \cap M_2$  is written in the form

$$b = \sum_i x_i e_M y_i, \quad x_i, y_i \in N' \cap M_1.$$

Since the depth of  $K \subset M$  is 2,

$$(K' \cap K_1)e_M^{K_1}(K' \cap K_1) = K' \cap K_2,$$

where  $e_M^{K_1}$  is the Jones projection for  $M \subset K_1$ . By Lemma 2.2, we have

$$(K' \cap K_1)e_M(K' \cap K_1) = e_{K_1}(K' \cap M_2)e_{K_1}.$$

Therefore we have

$$\begin{aligned} e_{K_1} b e_{K_1} &= e_{K_1} \left( \sum_i x_i e_M y_i \right) e_{K_1} \\ &= \sum_i E_{K_1}^{M_1}(x_i) e_M E_{K_1}^{M_1}(y_i) \in (K' \cap K_1) e_M (K' \cap K_1). \end{aligned}$$

we have thus proved the lemma.  $\square$

**Proposition 4.4.** *Suppose that the depth of  $N \subset M$  is 2. Let  $K$  be an intermediate subfactor for  $N \subset M$  and  $K_1$  the basic extension for  $K \subset M$ . If the depth of  $K \subset M$  is 2, then  $H = K' \cap K_1$  is a subHopf algebra of  $A = N' \cap M_1$ .*

*Proof.* Let  $S_A$  be an antipode of  $A$ , i.e.,  $S_A : A \rightarrow A$  is an anti-algebra morphism determined by

$$(S_A(a), b) = \overline{(a^*, b^*)} \quad \text{for } \forall a \in A \text{ and } \forall b \in B = M' \cap M_2.$$

Since  $Be_NB = N' \cap M_2$  by the assumption, for any  $a \in A$ , there exist  $x_i, y_i \in B$  such that  $a = \sum_i x_i e_N y_i$ . Then  $S_A(a) = \sum_i y_i e_N x_i$  (see for example [28]). By the assumption and Lemma 4.2,  $H$  and  $B_{e_{K_1}} = e_{K_1} B e_{K_1}$  are the dual pair of Hopf algebras with a pairing

$$(h, k) = [M : N]^2 \tau(h e_M e_N k) \text{ for } \forall h \in H \text{ and } \forall k \in B_{e_{K_1}}.$$

By the fact that  $\varphi(e_K) = \frac{[M:N]}{[M:K]} e_{K_1} e_N e_{K_1}$ , for  $h \in H$ , there exist  $s_n, t_n \in B_{e_{K_1}}$  such that  $h e_{K_1} = \varphi(h) = \sum_n s_n e_N t_n$ , where  $\varphi$  is defined in Lemma 2.2. Then for  $\forall b \in B$ , we have

$$\begin{aligned} (S_A(h), b) &= \overline{(h^*, b^*)} \\ &= [M : N]^2 \tau(b e_N e_M h) \\ &= [M : N]^2 \sum_n \tau(b e_N e_M s_n e_N t_n) \\ &= [M : N]^2 \sum_n \tau(b E_{M_1}^{M'}(e_M s_n) e_N t_n) \\ &= [M : N]^2 \sum_n \tau(e_M s_n) \tau(b e_N t_n) \\ &= [M : N] \sum_n \tau(e_M s_n) \tau(b t_n). \end{aligned}$$

Since  $S_H(h) e_{K_1} = S_{H_{e_{K_1}}}(h e_{K_1}) = \sum_n t_n e_N s_n$  by the fact that  $e_{K_1} \in H'$ , we have, for

$\forall b \in B,$

$$\begin{aligned}
(S_H(h), b) &= [M : N]^2 \tau(S_{H e_{K_1}}(h e_{K_1}) e_M e_N b) \\
&= [M : N]^2 \sum_n \tau(t_n e_N s_n e_M e_N b) \\
&= [M : N] \sum_n \tau(s_n e_M) \tau(t_n b).
\end{aligned}$$

Therefore we have  $S_A(h) = S_H(h) \in H$ , i.e.,  $S_A(H) \subset H$ .

Let  $\Delta_A$  be a comultiplication of  $A$ , i.e.,  $\Delta_A : A \rightarrow A \otimes A$  is determined by

$$(a, b_1 b_2) = (\Delta_A(a), b_1 \otimes b_2) \text{ for } \forall b_1, b_2 \in B.$$

For  $h \in H$ , we denote  $\Delta_A(h)$  by  $\sum_{(h)} h_{(1)} \otimes h_{(2)}$ . Since  $e_M = e_{K_1} e_M$  and  $e_{K_1} h = h e_{K_1}$ , we have

$$\begin{aligned}
(h, b) &= [M : N]^2 \tau(h e_{K_1} e_M e_N b) \\
(4.1) \quad &= [M : N]^2 \tau(h e_M e_N b e_{K_1}) \\
&= (h, b e_{K_1}) \text{ for } \forall h \in H \text{ and } \forall b \in B.
\end{aligned}$$

Since  $e_{K_1}$  is an element of the center of  $B$  by the proof of Theorem 3.10, we have

$$\begin{aligned}
(h, b_1 b_2) &= (h, b_1 e_{K_1} b_2 e_{K_1}) \\
&= (\Delta_A(h), b_1 e_{K_1} \otimes b_2 e_{K_1}) \\
&= \sum_{(h)} (h_{(1)}, b_1 e_{K_1}) (h_{(2)}, b_2 e_{K_1}) \\
&= \sum_{(h)} [M : N]^2 \tau(e_{K_1} h_{(1)} e_M e_N b_1) [M : N]^2 \tau(e_{K_1} h_{(2)} e_M e_N b_2) \\
&= \sum_{(h)} (E_{K_1}^{M_1}(h_{(1)}), b_1) (E_{K_1}^{M_1}(h_{(2)}), b_2), \text{ for } \forall b_1, b_2 \in B.
\end{aligned}$$

Since  $\sum_{(h)} S_A(h_{(1)})e_M h_{(2)} \in B$  by [28], we have

$$\Delta_A(H) \subset H \otimes H$$

by Lemma 4.3. We have thus proved the theorem.  $\square$

**Theorem 4.5.** *Let  $N \subset M$  be an irreducible, depth 2 inclusion of type  $II_1$  factors with finite index and  $K$  an intermediate subfactor for  $N \subset M$ . Let  $N \subset M \subset M_1 \subset M_2$  and  $K \subset M \subset K_1 \subset K_2$  be the Jones towers for  $N \subset M$  and  $K \subset M$ , respectively. Then the depth of  $K \subset M$  is 2 if and only if  $e_{K_1}$  is an element of the center of  $M' \cap M_2$ , where  $e_{K_1}$  is the Jones projection for  $K_1 \subset M_1$ .*

*Proof.* Suppose that the depth of  $K \subset M$  is 2. Then by the proof of Theorem 3.10,  $e_{K_1}$  is an element of the center of  $M' \cap M_2$ .

Conversely, suppose that  $e_{K_1}$  is an element of the center of  $M' \cap M_2$ . Then for any  $h \in H = K' \cap K_1$ , we have

$$\begin{aligned} (S_A(h), b) &= \overline{(h^*, b^*)} \\ &= [M : N]^2 \tau(b e_N e_M h) \\ &= [M : N]^2 \tau(e_{K_1} b e_{K_1} e_N e_M h) \\ &= (S_A(h), e_{K_1} b e_{K_1}) \quad \text{for } \forall b \in B = M' \cap M_2 \end{aligned}$$

and hence  $S_A(H) \subset H$ . Similarly, for any  $h \in H$ , we have

$$\begin{aligned} (\Delta_A(h), x \otimes y) &= (h, xy) \\ &= (h, e_{K_1} x e_{K_1} y e_{K_1}) \\ &= (\Delta_A(h), e_{K_1} x e_{K_1} \otimes e_{K_1} y e_{K_1}) \quad \text{for } \forall x, y \in M' \cap M_2, \end{aligned}$$

and hence  $\Delta_A(H) \subset H \otimes H$ . Therefore  $H$  is a subHopf algebra of  $N' \cap M_1$ . By Proposition 4.1, we have  $K = M^H$ . So the depth of  $K \subset M$  is 2.  $\square$

**Corollary 4.6.** *Let  $N \subset M$  be an irreducible, depth 2 inclusion of type  $II_1$  factors with finite index and  $K$  an intermediate subfactor for  $N \subset M$ . Let  $N \subset M \subset M_1 \subset M_2$  and  $K \subset M \subset K_1 \subset K_2$  be the Jones towers for  $N \subset M$  and  $K \subset M$ , respectively. The depth of  $N \subset K$  is 2 if and only if  $e_K$  is an element of the center of  $N' \cap M_1$ , where  $e_K$  is the Jones projection for  $K \subset M$ .*

*Proof.* Let  $K_{-1}$  and  $N_{-1}$  be the tunnel constructions for  $N \subset K$  and  $N \subset M$ , respectively. Then the depth of  $N_{-1} \subset N$  is 2 and, the depth of  $N \subset K$  is 2 if and only if the depth of  $K_{-1} \subset N$  is 2. And hence, by Theorem 4.5, we get the corollary.  $\square$

**Theorem 4.7.** *Let  $N \subset M$  be an irreducible, depth 2 inclusion of type  $II_1$  factors with finite index and  $K$  an intermediate subfactor for  $N \subset M$ . Then  $K$  is a normal intermediate subfactor of  $N \subset M$  if and only if the depths of  $N \subset K$  and  $K \subset M$  are both 2.*



*Proof.* This immediately follows from Theorem 4.5 and Corollary 4.6.  $\square$

**Theorem 4.8.** *Let  $N \subset M$  be an irreducible, depth 2 inclusion of type  $II_1$  factors with finite index and  $K$  an intermediate subfactor for  $N \subset M$ . Then  $K$  is a normal intermediate subfactor of  $N \subset M$  if and only if  $K' \cap K_1$  is a normal subHopf algebra of  $N' \cap M_1$ , where  $K_1$  and  $M_1$  are the basic extensions for  $N \subset M$  and  $K \subset M$ , respectively.*

*Proof.* Suppose that  $K$  is a normal intermediate subfactor of  $N \subset M$ . Then  $H = K' \cap K_1$  is a subHopf algebra of  $A = N' \cap M_1$  by Proposition 4.4. Let  $\varepsilon_H$  is a counit of  $H$ . Then

$$xe_K = \varepsilon_H(x)e_K \quad \text{for } x \in H.$$

Therefore  $H^+ = H \cap \ker \varepsilon_H = H(1 - e_K)$ . Since  $(1 - e_K)$  is an element of the center of  $A$  by the assumption, we have

$$H^+A = AH^+.$$

Hence  $H$  is a normal subHopf algebra of  $A$  by Proposition 2.6. Conversely, we suppose that  $H$  is a normal subHopf algebra of  $A$ . Then by Proposition 4.4 and Theorem 4.5,  $e_{K_1}$  is element of the center of  $M' \cap M_2$ . Since  $H^+ = H(1 - e_K)$ , we have  $(1 - e_K)A = A(1 - e_K)$  by Proposition 2.6. This implies  $e_K$  is an element of the center of  $N' \cap M_1$  and hence  $K$  is a normal intermediate subfactor of  $N \subset M$ .  $\square$

**4.3. Lattices of normal intermediate subfactors.** Let  $N \subset M$  be an irreducible, depth 2 inclusion of type  $II_1$  factors with finite index. In this subsection we shall prove that the set of all normal intermediate subfactors of the inclusion  $N \subset M$ , denoted by  $\mathcal{N}(N \subset M)$ , is a sublattice of  $\mathcal{L}(N \subset M)$ . Moreover,  $\mathcal{N}(N \subset M)$  is a modular lattice.

**Lemma 4.9.** *Let  $L$  and  $K$  be intermediate subfactors of  $N \subset M$  and  $L_1$  and  $K_1$  the basic extensions for  $L \subset M$  and  $K \subset M$ , respectively. Then the basic extension  $(L \wedge K)_1$  for  $(L \wedge K) \subset M$  is  $L_1 \vee K_1$  and the basic extension  $(L \vee K)_1$  for  $(L \vee K) \subset M$  is  $L_1 \wedge K_1$ .*

*Proof.* By the fact that  $(L \cap K)' = (L' \cup K')''$ , we have

$$(L \wedge K)_1 = J(L \wedge K)'J = L_1 \vee K_1.$$

Similarly, by the fact that  $(L \cup K)' = L' \cap K'$ , we have

$$(L \vee K)_1 = J(L \cup K)'J = L_1 \wedge K_1.$$

□

We note that if we denote by  $e_A$  the Jones projection for  $A \subset M$ , then for  $L, K \in \mathcal{L}(N \subset M)$ , we have  $e_{L \wedge K} = e_L \wedge e_K$ . But  $e_{L \vee K} \neq e_L \vee e_K$  in general (see [26]).

**Theorem 4.10.** *Let  $N \subset M$  be an irreducible, depth 2 inclusion of type  $II_1$  factors with finite index. Then the set of all normal intermediate subfactors  $\mathcal{N}(N \subset M)$  is a sublattice of  $\mathcal{L}(N \subset M)$ .*

*Proof.* Let  $L$  and  $K$  be normal intermediate subfactors of  $N \subset M$ . Since  $e_L$  and  $e_K$  are elements of the center of  $N' \cap M_1$  by the assumption, we have  $e_{L \wedge K} = e_L \wedge e_K \in \mathcal{Z}(N' \cap M_1)$  by the above argument. Observe that

$$(L \vee K)' \cap (L \vee K)_1 = (L' \cap L_1) \cap (K' \cap K_1).$$

Since  $L' \cap L_1$  and  $K' \cap K_1$  are invariants under the left and right adjoint action of  $N' \cap M_1$  (see Definition 2.2), so is  $(L \vee K)' \cap (L \vee K)_1$ . Therefore we can see that  $(L \vee K)' \cap (L \vee K)_1$  is a normal subHopf algebra  $N' \cap M_1$  by the definition. Since  $L \vee K$  is a normal intermediate subfactor of  $N \subset M$  by Theorem 4.8, we have  $e_{L \vee K} \in \mathcal{Z}(N' \cap M_1)$ . Applying the same argument for the dual inclusion  $M \subset M_1$ , we conclude that  $L \wedge K$  and  $L \vee K$  are normal intermediate subfactors of  $N \subset M$ .  $\square$

**Corollary 4.11.** *Let  $N \subset M$  be an irreducible, depth 2 inclusion of type  $II_1$  factors. Then  $\mathcal{N}(N \subset M)$  is a modular lattice.*

*Proof.* This immediately follows from Proposition 3.8, Theorem 4.10 and [30, Theorem 3.9].  $\square$

**Theorem 4.12.** *Let  $N \subset M$  be an irreducible, depth 2 inclusion of type  $II_1$  factors with finite index. Then every maximal chain from  $M$  to  $N$  in  $\mathcal{N}(N \subset M)$  has the same length, i.e., for  $A_i (i = 1, 2, \dots, m)$ ,  $B_j (j = 1, 2, \dots, n) \in \mathcal{N}(N \subset M)$ , if*

$$M = A_0 > A_1 > \dots > A_m = N$$

and

$$M = B_0 > B_1 > \cdots > B_n = N,$$

then  $m = n$ , where  $X > Y$  means  $X \supset Y$  and  $X \supseteq K \supseteq Y$ , implies  $K = X$  or  $K = Y$  for  $X, Y, K \in \mathcal{N}(N \subset M)$ .

*Proof.* Since we have the Jordan-Dedekind chain condition holding in modular lattices, this immediately follows from the previous corollary.  $\square$

**Example 4.13.** We denote by  $S_n$  the symmetric group on  $n$  letters,  $x_1, x_2, \dots, x_n$  and  $\sigma = (1, 2, 3, \dots, n)$  the element of  $S_n$  with order  $n$  and  $\langle \sigma \rangle$  the cyclic group generated by  $\sigma$ . Let  $\gamma : S_n \rightarrow \text{Aut}(P)$  be an outer action of  $S_n$  on a type  $\text{II}_1$  factor  $P$  and let

$$N = P^{\gamma\sigma} \subset M = P \rtimes_{\gamma} S_{n-1}.$$

Then we can see that  $S_n = S_{n-1}\langle \sigma \rangle = \langle \sigma \rangle S_{n-1}$  and  $S_{n-1} \cap \langle \sigma \rangle = \{e\}$ . Therefore the depth of  $N \subset M$  is 2 (see [25, 32]). We put  $K = P \rtimes_{\gamma} A_{n-1}$ , where  $A_{n-1}$  is the alternating group consists of the even permutations on  $x_1, x_2, \dots, x_{n-1}$ . If  $n$  is odd, then the length of  $\mathcal{N}(N \subset M)$  is 3 and if  $n$  is even, then that is 2 (we shall show this fact later in Example 5.4).

## 5. SOME EXAMPLES

In this section we shall give some examples of normal intermediate subfactors and non normal ones.

**5.1. Group type inclusions.** Let  $\gamma : G \rightarrow \text{Aut}(P)$  be an outer action of a discrete group  $G$  on a type  $\text{II}_1$  factor. Let  $A$  and  $B$  be finite subgroups of  $G$  such that  $A \cap B = \{e\}$ . Let  $N$  be the fixed point algebra  $P^{(A,\gamma)}$  and  $M$  the crossed product  $P \rtimes_{\gamma} B$ . Then  $N \subset M$  is an irreducible inclusion by [2] and  $P$  is normal in  $N \subset M$  by Theorem 3.10. In this subsection we consider inclusions of this type.

**Proposition 5.1.** *With the above notation, let  $H$  be a subgroup of  $B$  and  $K$  the crossed product  $P \rtimes H$ . Then  $K$  is normal in  $N \subset M$  if and only if  $H$  is a normal subgroup of  $B$  and  $AH \cap BA = AH \cap HA$ .*

*Proof.* Let  $\alpha = {}_N L^2(P)_P$  and  $\beta = {}_P L^2(M)_M$ . Let  $\beta_1 = {}_P L^2(K)_K$  and  $\beta_2 = {}_K L^2(M)_M$ . Then we have

$$\bar{\alpha}\alpha = \bigoplus_{a \in A} {}_P L^2(\gamma_a)_P$$

$$\beta_1 \bar{\beta}_1 = \bigoplus_{h \in H} {}_P L^2(\gamma_h)_P$$

$$\beta \bar{\beta} = \bigoplus_{b \in B} {}_P L^2(\gamma_b)_P,$$

as in Example 2.11. Since  $A \cap B = \{e\}$ , we have

$$(ab = a'b', a, a' \in A, b, b' \in B) \iff (a = a' \text{ and } b = b').$$

Therefore if  $\rho = {}_N L^2(M)_M (= \alpha\beta)$ , then

$$\begin{aligned} \langle \alpha\beta_1 \overline{(\alpha\beta_1)}, \rho\bar{\rho} \rangle &= \langle \alpha\beta_1 \bar{\beta}_1 \bar{\alpha}, \alpha\beta \bar{\beta} \bar{\alpha} \rangle \\ &= \langle \bar{\alpha} \alpha \beta_1 \bar{\beta}_1, \beta \bar{\beta} \bar{\alpha} \alpha \rangle \\ &= \#(AH \cap BA) \end{aligned}$$

and

$$\begin{aligned} \langle \alpha\beta_1 \overline{(\alpha\beta_1)}, \alpha\beta_1 \overline{(\alpha\beta_1)} \rangle &= \langle \bar{\alpha} \alpha \beta_1 \bar{\beta}_1, \beta_1 \bar{\beta}_1 \bar{\alpha} \alpha \rangle \\ &= \#(AH \cap HA). \end{aligned}$$

Hence  $e_K \in \mathcal{Z}(N' \cap M_1)$  if and only if  $(AH \cap BA) = (AH \cap HA)$  by Proposition 3.9.

Suppose  $K$  is normal in  $N \subset M$ . Then  $K$  is also normal in  $P \subset M$  by Proposition 3.11. Therefore  $H$  is a normal subgroup of  $B$  by Proposition 3.4. Conversely, if  $H$  is a normal subgroup of  $B$ , i.e., the depth of  $K \subset M$  is 2, then we have

$$\begin{aligned} \langle \bar{\beta}_2 \beta_2, \bar{\rho} \rho \rangle &= \langle \bar{\beta}_2 \beta_2, \bar{\beta}_2 \bar{\beta}_1 \bar{\alpha} \alpha \beta_1 \beta_2 \rangle \\ &= \langle \beta_2 \bar{\beta}_2 \beta_2 \bar{\beta}_2, \bar{\beta}_1 \bar{\alpha} \alpha \beta_1 \rangle \\ &= [B : H] \langle \beta_2 \bar{\beta}_2, \bar{\beta}_1 \bar{\alpha} \alpha \beta_1 \rangle \\ &= [B : H] = \langle \bar{\beta}_2 \beta_2, \bar{\beta}_2 \beta_2 \rangle. \end{aligned}$$

This proves the proposition.  $\square$

Let  $G$  be a finite group with two subgroups  $A, B$  satisfying  $G = AB$  and  $A \cap B = \{e\}$ . By the uniqueness of the decomposition of an element in  $G = AB = BA$ , we

can represent  $ab$  for  $a \in A, b \in B$  as

$$ab = \alpha_a(b)\beta_{b^{-1}}(a^{-1})^{-1} \in BA.$$

Then the matched pair  $(A, B, \alpha, \beta)$  appears (see for example [25]).

**Proposition 5.2.** *Let  $(A, B, \alpha, \beta)$  be the matched pair defined as above and let*

$$M = P \rtimes_{\gamma} B \supset N = P^{(A, \gamma)} = \{x \in P \mid \gamma_a(x) = x, \forall a \in A\},$$

where  $\gamma$  is an outer action of  $G$  on  $II_1$  factor  $P$ . Then the depth of  $N \subset M$  is 2.

See for a proof [25, 32].

**Theorem 5.3.** *Let  $G$  be a finite group with two subgroups  $A, B$  satisfying  $G = AB$  and  $A \cap B = \{e\}$  and  $(A, B, \alpha, \beta)$  the associated matched pair. Let  $\gamma : G \rightarrow \text{Aut}(P)$  be an outer action of  $G$  on a type  $II_1$  factor  $P$  and let*

$$M = P \rtimes_{\gamma} B \supset N = P^{(A, \gamma)} = \{x \in P \mid \gamma_a(x) = x, \forall a \in A\}.$$

*If  $H$  is a subgroup of  $B$  and  $K = P \rtimes_{\gamma} H \in \mathcal{L}(N \subset M)$ , then  $K$  is a normal intermediate subfactor for  $N \subset M$  if and only if*

- (1)  $H$  is a normal subgroup of  $B$ ,
- (2)  $\alpha_a(H) = H, \forall a \in A$ , i.e.,  $AH = HA$

*In particular, if  $G$  is a semi direct product  $B \rtimes A$ , then  $K$  is a normal in  $N \subset M$  if and only if  $H$  is a normal subgroup of  $G$ .*

*Proof.* Since  $BA = AB = G$ , we have  $(AH \cap BA) = AH$ . By Proposition 5.1, we have  $K$  is normal intermediate subfactor in  $N \subset M$  if and only if  $H$  is a normal subgroup of  $B$  and  $(AH \cap HA) = AH$ , i.e.,  $AH = HA$  since  $\#HA = \#AH$ .  $\square$

**Example 5.4.** Let  $N = P^\sigma \subset M = P \rtimes_\gamma S_{n-1}$  be the irreducible inclusion defined as in Example 4.13. The depth of  $N \subset M$  is 2 by Proposition 5.2. We put  $K = P \rtimes_\gamma A_{n-1}$ . If  $n$  is odd, then  $\sigma$  is an even permutation and we can see that  $A_n = A_{n-1}\langle\sigma\rangle = \langle\sigma\rangle A_{n-1}$ . Therefore  $K$  is normal in  $N \subset M$  by Theorem 5.3. If  $n$  is even, then  $\sigma$  is an odd permutation. Since the product of an even and odd permutation in either order is odd, and the product of two odd permutation is even,  $A_{n-1}\langle\sigma\rangle$  is not subgroup of  $S_n$  and hence  $A_{n-1}\langle\sigma\rangle \neq \langle\sigma\rangle A_{n-1}$ . Therefore  $K$  is not normal in  $N \subset M$  by Theorem 5.3.

Since  $S_{n-1}$  is a maximal subgroup of  $S_n$ , we have if  $\langle\sigma^k\rangle S_{n-1} = S_{n-1}\langle\sigma^k\rangle$ , then  $\langle\sigma^k\rangle = \langle\sigma\rangle$  or  $k = 0 \pmod{n}$ , i.e., there is no normal intermediate subfactor  $K$  of  $N \subset M$  such that  $N \subsetneq K \subsetneq P$  by Theorem 5.2.

*Remark.* By Example 5.4, we have completed the proof of Example 4.13.

**5.2. Strongly outer actions and intermediate subfactors.** In this subsection we shall study relations between strongly outer actions introduced by Choda and Kosaki [3] and normal intermediate subfactors.

Let  $N \subset M$  be a pair of type  $\text{II}_1$  factors, and we set

$$\text{Aut}(M, N) = \{ \theta \in \text{Aut}(M) \mid \theta(N) = N \}.$$



Let

$$N(= M_{-1}) \subset M(= M_0) \subset M_1 \subset M_2 \subset \dots$$

be the Jones tower of the pair  $N \subset M$ , and  $e_k (\in M_k)$  the Jones projection for the pair  $M_{k-2} \subset M_{k-1}$ . Then each automorphism  $\theta \in \text{Aut}(M, N)$  is extended to all  $M_n$  subject to the condition  $\theta(e_i) = e_i$ .

*Definition.* An automorphism  $\theta \in \text{Aut}(M, N)$  is said to be strongly outer if the following condition is satisfied for all  $k \geq -1$ :

$$a \in M_k \text{ satisfies } ax = \theta(x)a \text{ for all } x \in N \Rightarrow a = 0.$$

An action  $\alpha$  of a group  $G$  into  $\text{Aut}(M, N)$  is said to be strongly outer if  $\alpha_g$  is strongly outer for all  $g \in G$  except for the identity  $e$ .

For  $\theta \in \text{Aut}(M, N)$ , let  ${}_N L^2(\theta)_N$  be the  $N$ - $N$  bimodule as in Example 2.8. M. Choda and H. Kosaki [3] gave the next characterization of strongly outer automorphisms.

*Theorem.* For  $\theta \in \text{Aut}(M, N)$ , if  ${}_N L^2(\theta)_N$  does not appear in the irreducible decomposition of  $(\rho\bar{\rho})^k$ ,  $k = 1, 2, \dots$ , then  $\theta$  is strongly outer, where  $\rho$  is the  $N$ - $M$  bimodule  ${}_N L^2(M)_M$ .

**Lemma 5.5.** *Let  $B \subset A$  be an irreducible pair of type  $II_1$  factors with finite index.*

*Let  $\gamma : G \rightarrow \text{Aut}(A, B)$  be an outer action of a finite group  $G$  and  $\alpha = {}_B L^2(A)_A$ . If  $\bar{\alpha} \neq {}_A L^2(\gamma_g)_A$  for all  $g \in G$  except for the identity  $e$ , then  $B' \cap (A \rtimes_\gamma G) = \mathbb{C}$ . In particular, if  $\gamma$  is strongly outer, then  $B \subset A \rtimes_\gamma G$  is irreducible.*

*Proof.* Let  $\beta = {}_A L^2(A \rtimes_\gamma G)_{A \rtimes_\gamma G}$  and  $\rho = {}_B L^2(A \rtimes_\gamma G)_{A \rtimes_\gamma G} (= \alpha\beta)$ . Then we have

$$\beta\bar{\beta} = {}_A L^2(A \rtimes_\gamma G)_A \simeq \bigoplus_{g \in G} {}_A L^2(\gamma_g)_A.$$

Therefore

$$\begin{aligned} \langle \rho, \rho \rangle &= \langle \alpha\beta, \alpha\beta \rangle \\ &= \langle \bar{\alpha}\alpha, \beta\bar{\beta} \rangle = 1. \end{aligned}$$

This implies that  $B' \cap (A \rtimes_\gamma G) = \mathbb{C}$ .  $\square$

**Proposition 5.6.** *Let  $B \subset A$  be an irreducible pair of type  $II_1$  factors with finite index. Let  $\gamma : G \rightarrow \text{Aut}(A, B)$  be an outer action of a finite group  $G$  and  $\alpha = {}_B L^2(A)_A$ . Then  $A$  is normal in  $B \subset A \rtimes_\gamma G$  if and only if  $\bar{\alpha}\alpha\bar{\alpha}\alpha \neq {}_A L^2(\gamma_g)_A$  for all  $g \in G$  except for the identity  $e$ .*

*Proof.* Suppose that  $\bar{\alpha}\alpha\bar{\alpha}\alpha \neq {}_A L^2(\gamma_g)_A$  for all  $g \in G$  except for the identity  $e$ . Let  $\beta = {}_A L^2(A \rtimes_\gamma G)_{A \rtimes_\gamma G}$  and  $\rho = {}_B L^2(A \rtimes_\gamma G)_{A \rtimes_\gamma G} (= \alpha\beta)$ . Since  $\beta\bar{\beta} \simeq \bigoplus_{g \in G} {}_A L^2(\gamma_g)_A$ , we have

$$\begin{aligned} \langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle &= \langle \alpha\bar{\alpha}, \alpha\beta\bar{\beta}\bar{\alpha} \rangle \\ &= \langle \bar{\alpha}\alpha\bar{\alpha}\alpha, \beta\bar{\beta} \rangle \\ &= \langle \bar{\alpha}\alpha\bar{\alpha}\alpha, {}_A L^2(A)_A \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle. \end{aligned}$$

Since  $\langle \beta\bar{\beta}, \bar{\alpha}\alpha \rangle = 1$  by Lemma 5.5, we have

$$\begin{aligned} \langle \bar{\beta}\beta, \bar{\rho}\rho \rangle &= \langle \bar{\beta}\beta, \bar{\beta}\bar{\alpha}\alpha\beta \rangle \\ &= \langle \beta\bar{\beta}\beta, \bar{\alpha}\alpha\beta \rangle \\ &= \#G \langle \bar{\beta}\beta, \bar{\alpha}\alpha \rangle \\ &= \#G = \langle \bar{\beta}\beta, \bar{\beta}\beta \rangle. \end{aligned}$$

Therefore  $A$  is normal in  $B \subset A \rtimes_{\gamma} G$  by Lemma 3.9.

Conversely, suppose that  $\bar{\alpha}\alpha\bar{\alpha}\alpha \succ_A L^2(\gamma_g)_A$  for some  $g(\neq e) \in G$ . Then we have

$$\begin{aligned} \langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle &= \langle \bar{\alpha}\alpha\bar{\alpha}\alpha, \beta\bar{\beta} \rangle \\ &\cong \langle \bar{\alpha}\alpha\bar{\alpha}\alpha, {}_A L^2(A)_A \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle. \end{aligned}$$

And hence  $A$  is not normal in  $B \subset A \rtimes_{\gamma} G$ .  $\square$

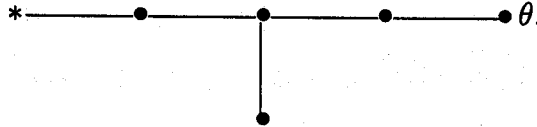
**Theorem 5.7.** *Let  $B \subset A$  be an irreducible pair of type  $II_1$  factors with finite index.*

*If  $\gamma : G \rightarrow \text{Aut}(M, N)$  is a strongly outer action of a finite group  $G$ , then  $A$  is normal intermediate subfactor for the inclusion  $B \subset A \rtimes_{\gamma} G$ .*

*Proof.* This immediately follows from the previous proposition.  $\square$

**Example 5.8.** Let  $B \subset A$  be an inclusion of type  $II_1$  factors with the principal graph

$E_6$ ,



We put  $\alpha = {}_B L^2(A)_A$ . Then we have

$$\bar{\alpha}\alpha\bar{\alpha}\alpha \succ {}_A L^2(\theta)_A.$$

By Proposition 5.6,  $A$  is not normal in  $B \subset A \rtimes_{\theta} \mathbb{Z}/2\mathbb{Z}$ .

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DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, SAPPORO 060 JAPAN

*E-mail address:* t-teruya@math.hokudai.ac.jp