Navier-Stokes Equations in a Rotating Frame in $\mathbb{R}^3$ with Initial Data Nondecreasing at Infinity

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Abstract

Three-dimensional rotating Navier-Stokes equations are considered with a constant Coriolis parameter Ω and initial data nondecreasing at infinity. In contrast to the non-rotating case ($\Omega = 0$), it is shown for the problem with rotation ($\Omega \neq 0$) that Green's function corresponding to the linear problem (Stokes + Coriolis combined operator) does not belong to $L^1(\mathbb{R}^3)$. Moreover, the corresponding integral operator is unbounded in the space $L^\infty_\sigma(\mathbb{R}^3)$ of solenoidal vector fields in $\mathbb{R}^3$ and the linear (Stokes+Coriolis) combined operator does not generate a semigroup in $L^\infty_\sigma(\mathbb{R}^3)$. Local in time, uniform in Ω unique solvability of the rotating Navier-Stokes equations is proven for initial velocity fields in the space $L^\infty_{\sigma,a}(\mathbb{R}^3)$ which consists of L^∞ solenoidal vector fields satisfying vertical averaging property such that their baroclinic component belongs to a homogeneous Besov space $\dot{B}^0_{\infty,1}$ which is smaller than L^∞ but still contains various periodic and almost periodic functions. This restriction of initial data to $L^\infty_{\sigma,a}(\mathbb{R}^3)$

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which is a subspace of $L^\infty_\sigma(\mathbb{R}^3)$ is essential for the combined linear operator (Stokes + Coriolis) to generate a semigroup. The proof of uniform in Ω local in time unique solvability requires detailed study of the symbol of this semigroup and obtaining uniform in Ω estimates of the corresponding operator norms in Banach spaces. Using the rotation transformation, we also obtain local in time, uniform in Ω solvability of the classical 3D Navier-Stokes equations in $\mathbb{R}^3$ with initial velocity and vorticity of the form $\mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + \frac{\Omega}{2} e_3 \times y$, $\mathrm{curl}\mathbf{V}(0) = \mathrm{curl}\tilde{\mathbf{V}}_0(y) + \Omega e_3$ where $\tilde{\mathbf{V}}_0(y) \in L^\infty_{\sigma,a}(\mathbb{R}^3)$.

1 Introduction

In this paper we study initial value problem for the three-dimensional rotating Navier-Stokes equations in $\mathbb{R}^3$ with initial data nondecreasing at infinity:

$$\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U} + \Omega e_3 \times \mathbf{U} + \nu \mathrm{curl}^2 \mathbf{U} = -\nabla p, \;\; \nabla \cdot \mathbf{U} = 0, \tag{1.1}$$

$$\mathbf{U}(t,x)|_{t=0} \;\; = \;\; \mathbf{U}_0(x) \tag{1.2}$$

where $x = (x_1, x_2, x_3)$, $\mathbf{U}(t,x) = (U_1, U_2, U_3)$ is the velocity field and p is the pressure. In Eqs. (1.1) e_3 denotes the vertical unit vector and Ω is a constant Coriolis parameter; the term $\Omega e_3 \times \mathbf{U}$ restricted to divergence free vector fields is called the Coriolis operator. The initial velocity field $\mathbf{U}_0(x)$ depends on three variables x_1, x_2 and x_3. We consider initial data in spaces of solenoidal vector fields $L^\infty_\sigma(\mathbb{R}^3)$ nondecreasing at infinity ($L^\infty(\mathbb{R}^3)$ restricted to the divergence free subspace). The consideration of solutions not decaying at infinity is essential in the development of rigorous mathematical theory of 3D rotating turbulence (homogeneous statistical solutions [10]). In this paper we prove local (in time), *uniform in* Ω unique solvability of the rotating Navier-Stokes equations in $\mathbb{R}^3$ under the condition that the initial velocity $\mathbf{U}_0 \in L^\infty_{\sigma,a}(\mathbb{R}^3)$, which is a subspace of $L^\infty_\sigma(\mathbb{R}^3)$ having vertical averaging property. We take initial data in the space $L^\infty_{\sigma,a}(\mathbb{R}^3) = \{u \in L^\infty(\mathbb{R}^3) : u - \overline{u} \in \dot{B}^0_{\infty,1}\}$ where $\dot{B}^0_{\infty,1}$ is a Besov space which contains various periodic and almost periodic functions (see Appendix B). Here $\overline{u}$ denotes the vertical average of u. We use $\dot{B}^0_{\infty,1}$ since the Riesz operator is bounded in $\dot{B}^0_{\infty,1}$ but not in L^∞. The space $L^\infty_{\sigma,a}(\mathbb{R}^3)$ is a subspace of $L^\infty_\sigma(\mathbb{R}^3)$ which consists of bounded vector fields satisfying *vertical averaging property*. It is shown that the linear combined operator (Stokes + Coriolis) generates a uniformly bounded semigroup on $L^\infty_{\sigma,a}(\mathbb{R}^3)$.

The above initial value problem (1.1)-(1.2) for the 3D rotating Navier-Stokes Equations is equivalent, via rotation transformation with respect to the vertical axis e_3, to the initial value problem for the classical (non-rotating) 3D Navier-Stokes Equations with initial data of the type $\mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + \frac{\Omega}{2} e_3 \times y$:

$$\partial_t \mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{V} + \nu \mathrm{curl}^2 \mathbf{V} = -\nabla q, \;\; \nabla \cdot \mathbf{V} = 0, \tag{1.3}$$

$$\mathbf{V}(t,y)|_{t=0} \quad = \quad \mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + \frac{\Omega}{2} e_3 \times y \tag{1.4}$$

where $y = (y_1, y_2, y_3)$, $\mathbf{V}(t,y) = (V_1, V_2, V_3)$ is the velocity field and q is the pressure. Since $\mathrm{curl}(\frac{\Omega}{2} e_3 \times y) = \Omega e_3$, the vorticity vector at initial time $t = 0$ is $\mathrm{curl}\mathbf{V}(0,y) = \mathrm{curl}\tilde{\mathbf{V}}_0(y) + \Omega e_3$. This connection between initial value problems for the 3D Navier-Stokes Equations is made precise in the last section of the paper. Using the rotation transformation, our results for initial value problem (1.1)-(1.2) imply local (in time), uniform in Ω solvability of the Navier-Stokes equations (1.3)-(1.4) in $\mathbb{R}^3$ under the condition that the initial velocity is of the form $\mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + \frac{\Omega}{2} e_3 \times y$ with $\tilde{\mathbf{V}}_0(y) \in L^\infty_{\sigma,a}(\mathbb{R}^3)$.

Let $\mathbf{J}$ be the matrix such that $\mathbf{J}\mathbf{a} = e_3 \times \mathbf{a}$ for any vector field $\mathbf{a}$. Then

$$\mathbf{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.5}$$

We define the Stokes operator $\mathbf{A}$:

$$\mathbf{A}\mathbf{U} = \nu \mathrm{curl}^2 \mathbf{U} = -\nu \Delta \mathbf{U} \tag{1.6}$$

on divergence free vector fields. Let $\mathbf{P}$ be the projection operator on divergence free fields. We recall that the operator $\mathbf{P}$ is related to the Riesz operators:

$$\mathbf{P} = \{P_{ij}\}_{i,j=1,2,3}, \;\; P_{ij} = \delta_{ij} + R_i R_j; \tag{1.7}$$

where $\delta_{i,j}$ is Kronecker's delta and R_j are the *scalar Riesz operators* defined by

$$R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-1/2} = \sigma(\frac{i\xi_j}{|\xi|}) \qquad \text{for } j = 1,2,3 \tag{1.8}$$

where $i = \sqrt{-1}$ (see e.g. [27]).

We transform (1.1)-(1.2) into the abstract operator differential equation for $\mathbf{U}$

$$\mathbf{U}_t + \mathbf{A}(\Omega)\mathbf{U} + \mathbf{P}(\mathbf{U} \cdot \nabla)\mathbf{U} = 0, \tag{1.9}$$

where

$$\mathbf{A}(\Omega)\mathbf{U} = \mathbf{A}\mathbf{U} + \Omega \mathbf{S}\mathbf{U}, \;\; \mathbf{S} = \mathbf{P}\mathbf{J}\mathbf{P} \tag{1.10}$$

and we have used $\mathbf{PJU} = \mathbf{PJPU}$ on solenoidal vector fields. The main difficulty that we face in our studies of local uniform in Ω solvability for Eqs. (1.1)-(1.2), (1.3)-(1.4) is that the Coriolis term is an unbounded operator in $L^\infty_\sigma(\mathbb{R}^3)$. We find that it is necessary to restrict initial data on a subspace of $L^\infty_\sigma(\mathbb{R}^3)$ on which the combined operator (Stokes + Coriolis) generates a semigroup. Then uniform in Ω time-local solvability of the full nonlinear problem is proven with detailed study of the symbol of this semigroup and obtaining uniform in Ω estimates of the corresponding operator norms in Banach spaces.

It is important to note that mathematical techniques for Eqs. (1.1)-(1.2) with initial data on compact manifolds (bounded domains and periodic lattices in $\mathbb{R}^3$) and for initial data in $L^p(\mathbb{R}^3)$, $1 < p < +\infty$ spaces of functions that decay at infinity are very different from those for initial data non-decaying at infinity in $\mathbb{R}^3$. In the former case, the Coriolis operator is a bounded zero order pseudo-differential operator with a *skew-symmetric* matrix symbol. Then local in time solvability for fixed Ω immediately follows by repeating classical arguments on local solvability of the 3D Navier-Stokes equations. Uniform in Ω solvability does not always hold for bounded domains and it requires careful consideration in each case. We note that for initial data on periodic lattices and in bounded cylindrical domains in $\mathbb{R}^3$ the time interval $[0, T]$ for existence of strong solutions is uniform in Ω. Moreover, regularization of solutions occurs for large Ω. Global regularity for large Ω of solutions of the three-dimensional Navier-Stokes equations (1.1)-(1.2), (1.3)-(1.4) with initial data $\mathbf{U}_0(x)$ on arbitrary periodic lattices and in bounded cylindrical domains in $\mathbb{R}^3$ was proven in [2], [3] and [20] without any conditional assumptions on the properties of solutions at later times. The method of proving global regularity for large fixed Ω is based on the analysis of fast singular oscillating limits (singular limit $\Omega \to +\infty$), nonlinear averaging and cancellation of oscillations in the nonlinear interactions for the vorticity field. It uses harmonic analysis tools of lemmas on restricted convolutions and Littlewood-Paley dyadic decomposition to prove global regularity of the limit resonant three-dimensional Navier-Stokes equations which holds without any restriction on the size of initial data and strong convergence theorems for large Ω.

The mathematical theory of the Navier-Stokes equations in $\mathbb{R}^n$ $(n = 2, 3)$ with initial data in spaces of functions non-decaying at infinity is more difficult than those on bounded domains or with periodic boundary conditions and it was developed only recently although there are earlier works to construct mild solutions for L^∞ initial data [6],[8]. Since energy is infinite for the corresponding solutions, classical energy methods for estimating norms of solutions or Galerkin approximation procedures cannot be used and new techniques are required. For example, Giga, Inui and Matsui [12] showed the time-local existence of strong solutions to the Navier-Stokes equations with non-decaying initial data in $L^\infty_\sigma(\mathbb{R}^n)$, $n = 2, 3$. Moreover, they proved the uniqueness under the same conditions. There are several related works for L^∞ initial data [7],[19]. We do not intend to exhaust references on this topic. Giga, Matsui and Sawada [13] proved the global in time solvability of the 2D Navier-Stokes equations with initial velocity in $L^\infty_\sigma(\mathbb{R}^2)$ without smallness nor integrability condition on initial velocity.

Although there are several earlier works on the solvability of the Navier-Stokes equations with initial data in Besov type spaces, it requires decay at space infinity. The space $\dot{B}^0_{\infty,1}$ was first used to solve the Boussinesq equations by Sawada and Taniuchi [25] (see Taniuchi[28] for recent improvement). Recently, Hieber-Sawada [16] and Sawada [24] constructed a unique local solution for the Navier-Stokes equations (1.3) with initial data $Mx + v_0$ where M is a trace free matrix and $v_0 \in \dot{B}^0_{\infty,1}$. This includes (1.4). However, their existence time estimate depends on Ω, since the term $\Omega \mathbf{e}_3 \times \mathbf{\Phi}$ is regarded as a perturbation. This is a major difference between our and their approaches. Although we restrict initial data v_0 in $L^\infty_{\sigma,a}$, as noticed in Remark 4.1 (iii) we may take an arbitrary element of $\dot{B}^0_{\infty,1}$ provided that it is divergence free. The reason we use smaller space is to give a framework to study the limit $\Omega \to \infty$ in the future.

2 Linear problem and calculation of symbols of pseudo-differential operators

In this section we solve linear problem using Fourier transform and calculate symbols of the corresponding pseudo-differential operators in $\mathbb{R}^3$. We consider the linear problem (Stokes+Coriolis):

$$\begin{aligned} \partial_t \mathbf{\Phi} - \nu \Delta \mathbf{\Phi} + \Omega e_3 \times \mathbf{\Phi} = -\nabla \pi,\ \nabla \cdot \mathbf{\Phi} = 0, \\ \mathbf{\Phi}(t,x)|_{t=0} = \mathbf{\Phi}_0(x). \end{aligned} \tag{2.1}$$

After applying projection $\mathbf{P}$ on divergence free vector fields, the above equation (2.1) can be written in operator form as follows

$$\mathbf{\Phi}_t + \mathbf{A}\mathbf{\Phi} + \Omega \mathbf{S} \mathbf{\Phi} = 0,\ \mathbf{\Phi}(t)|_{t=0} = \mathbf{\Phi}_0. \tag{2.2}$$

We introduce Fourier integrals:

$$\begin{aligned} F\mathbf{u}(\xi) = \hat{\mathbf{u}}(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\xi \cdot x} \mathbf{u}(x) dx, \\ F^{-1}\mathbf{v}(x) = \check{\mathbf{v}}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix\cdot \xi} \mathbf{v}(\xi) d\xi. \end{aligned} \tag{2.3}$$

Clearly, $\xi \cdot \hat{\mathbf{u}}(\xi) = 0$ if $\mathbf{u}$ is divergence free. Recall that the operators $\mathbf{P}$ and curl in Fourier representation have symbols $\sigma(\mathbf{P})$ and $\sigma(\text{curl})$:

$$\sigma(\mathbf{P}) = \mathbf{I} - \frac{1}{|\xi|^2} \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_2\xi_1 & \xi_2^2 & \xi_2\xi_3 \\ \xi_3\xi_1 & \xi_3\xi_2 & \xi_3^2 \end{pmatrix},\ \sigma(\text{curl}) = i \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}. \tag{2.4}$$

Here $\mathbf{I}$ is the 3×3 identity matrix. In what follows, we shall freely denote singular integral operator, say R_j in (1.8), by its symbol, say $i\xi_j/|\xi|$ for simplicity.

We also define the *vector Riesz operator* $\mathbf{R}$ by introducing its symbol:

$$\sigma(\mathbf{R}) \equiv \mathbf{R}(\xi) = \begin{pmatrix} 0 & -\frac{\xi_3}{|\xi|} & \frac{\xi_2}{|\xi|} \\ \frac{\xi_3}{|\xi|} & 0 & -\frac{\xi_1}{|\xi|} \\ -\frac{\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & 0 \end{pmatrix}. \tag{2.5}$$

We note that the symbol $\mathbf{R}(\xi)$ is a 3×3 skew-symmetric matrix. The vector Riesz operator $\mathbf{R}$ acting in the space of divergence free vector fields has the property:

$$\mathbf{R}^2 = -\mathbf{I}. \tag{2.6}$$

In fact, since $\mathbf{R}(\xi)\mathbf{v} = \frac{1}{|\xi|}\xi \times \mathbf{v}$, we calculate for any solenoidal vector field $\mathbf{v}$

$$\begin{aligned}\mathbf{R}(\xi)^2\mathbf{v} &= \mathbf{R}(\xi)(\frac{1}{|\xi|}\xi \times \mathbf{v}) = \frac{1}{|\xi|^2}\xi \times (\xi \times \mathbf{v}) \\ &= \frac{1}{|\xi|^2}((\xi \cdot \mathbf{v})\xi - (\xi \cdot \xi)\mathbf{v}) = -\frac{1}{|\xi|^2}|\xi|^2\mathbf{v} = -\mathbf{v}.\end{aligned}$$

Here, we used divergence free condition $(\xi \cdot v) = 0$. Because the scalar Riesz operators R_j satisfy $\sum_{j=1}^3 R_j^2 = -1$, it seems natural to call the operator $\mathbf{R}$ the vector Riesz operator. We now calculate 3×3 matrix symbol $\mathbf{S}(\xi)$ of the zero order pseudo-differential operator $\mathbf{S}$:

$$\sigma(\mathbf{S}) \equiv \mathbf{S}(\xi) = \mathbf{P}(\xi)\mathbf{J}\mathbf{P}(\xi). \tag{2.7}$$

We make an important observation that the operator $\mathbf{S} = \mathbf{PJP}$ is related to the Riesz operators and the curl operator. One can easily show by direct matrix multiplication that

$$\mathbf{S}(\xi) \equiv \mathbf{P}(\xi)\mathbf{J}\mathbf{P}(\xi) = \left(\frac{\xi_3}{|\xi|}\right)\mathbf{R}(\xi). \tag{2.8}$$

It implies that the symbol of the operator $\mathbf{S}$ commutes with the symbols of the operator *curl* and the Stokes operator $\mathbf{A}$. The symbol $\mathbf{S}(\xi)$ of the operator $\mathbf{S}$ is a homogeneous funstion of degree zero and it is expressed in terms of the scalar Riesz operators R_j for $j = 1, 2, 3$ (cf. (1.8)). Eqs. (2.5) and (2.8) imply

$$\mathbf{S} = R_3 \begin{pmatrix} 0 & R_3 & -R_2 \\ -R_3 & 0 & R_1 \\ R_2 & -R_1 & 0 \end{pmatrix}. \tag{2.9}$$

We recall that the Riesz operators R_j are bounded operators in $L^p(\mathbb{R}^3)$ for $1 < p < \infty$ and $BMO(\mathbb{R}^3)$. Here, BMO is the space of functions of bounded mean oscillations (e.g. [27]). However, the Riesz operators are not bounded in $L^\infty(\mathbb{R}^3)$. We also note that the Riesz operators R_j are bounded from $L^\infty(\mathbb{R}^3)$ to $BMO(\mathbb{R}^3)$.

Since Riesz operators are bounded in $BMO(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$ $(1 < p < +\infty)$, we have

Proposition 2.1. (1) $\mathbf{S} : BMO(\mathbb{R}^3) \to BMO(\mathbb{R}^3)$ *is a bounded operator.*
(2) $\mathbf{S} : L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$, $1 < p < +\infty$, *is a bounded operator.*
(3) *The symbol* $\mathbf{S}(\xi) : \mathbb{R}^3 \to \mathbb{R}^3$ *of the operator* $\mathbf{S}$ *is a* 3×3 *matrix with the following properties:*

$$\begin{aligned}&\text{(a)} \quad (\mathbf{S}(\xi))^* = -\mathbf{S}(\xi) \quad \textit{(skew-symmetric matrix)}, \\ &\text{(b)} \quad (\mathbf{S}(\xi))^2 = -\frac{\xi_3^2}{|\xi|^2}\mathbf{I} = \left(\frac{i\xi_3}{|\xi|}\right)\left(\frac{i\xi_3}{|\xi|}\right)\mathbf{I} \ \textit{i.e.}\ \mathbf{S}^2 = R_3^2\,\mathbf{I}\end{aligned} \tag{2.10}$$

where $\frac{i\xi_3}{|\xi|}$ *is the symbol of the Riesz operator* R_3.
(4) $|\mathbf{S}(\xi)\mathbf{v}| = |\mathbf{v}|$ *on the linear subspace of* $\mathbb{R}^3$ *with the property* $\xi \cdot \mathbf{v} = 0$ *(subspace of solenoidal vector fields). Here* $|\mathbf{v}|$ *denotes length of the vector* $\mathbf{v} \in \mathbb{R}^3$.

Remark 2.1. The operator $\mathbf{S}$ is *not* a bounded operator in $L^\infty_\sigma(\mathbb{R}^3)$, however, $\mathbf{S} : L^\infty_\sigma(\mathbb{R}^3) \to BMO(\mathbb{R}^3)$.

Eq. (2.10) is useful in calculating the operator $\exp(\mathbf{S})$ directly using infinite series:

$$\exp(\mathbf{S}) = \sum_{j=0}^{+\infty} \frac{1}{j!}\mathbf{S}^j. \tag{2.11}$$

Then we can solve linear Stokes+Coriolis problem (2.1), (2.2) in $BMO(\mathbb{R}^3)$ and in $L^p(\mathbb{R}^3)$, $1 < p < +\infty$. Since the operators commute, the solution of (2.2) is given by

$$\mathbf{\Phi}(t) = \exp((-\mathbf{A} - \Omega\mathbf{S})t)\mathbf{\Phi}_0 = \exp(\nu t\Delta)E(-\Omega t)\mathbf{\Phi}_0, \tag{2.12}$$

where $E(-\Omega t) = \exp(-\Omega t\mathbf{S})$. Of course, in Eqs. (2.12), $\exp(\nu t\Delta)$ is the usual semigroup generated by the heat kernel. Since $\mathbf{S}$ is a bounded operator in $BMO(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$, $1 < p < +\infty$, the operator $\exp(\Omega\mathbf{S}t)$ is also a bounded operator in these spaces. It is defined by convergent series:

$$\exp(\Omega\mathbf{S}t) = \sum_{j=0}^{+\infty} \frac{1}{j!}(\Omega t)^j\mathbf{S}^j. \tag{2.13}$$

We can solve linear Stokes+Coriolis problem (2.1) using Fourier transform in $\mathbb{R}^3$. After applying Fourier transform and projecting on divergence free subspace, we obtain

$$\begin{aligned} \partial_t\mathbf{\Phi}(t,\xi) + \nu|\xi|^2\mathbf{\Phi}(t,\xi) + \Omega\mathbf{S}(\xi)\mathbf{\Phi}(t,\xi) = 0,& \\ \mathbf{\Phi}(t,\xi)|_{t=0} = \mathbf{\Phi}_0(\xi).& \end{aligned} \tag{2.14}$$

Direct calculation using infinite series (2.13) and the property (2.10) of $\mathbf{S}$ implies that

$$\exp(\Omega\mathbf{S}(\xi)t) = \cos(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{I} + \sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}(\xi), \tag{2.15}$$

where $\mathbf{R}(\xi)$ is defined in (2.5).

Then the solution of (2.14) is given by

$$\mathbf{\Phi}(t,\xi) = e^{-\nu|\xi|^2 t}\left(\cos(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{I} - \sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}(\xi)\right)\mathbf{\Phi}_0(\xi). \tag{2.16}$$

In physical space the solution is given by convolution of inverse Fourier transform of $e^{-\nu|\xi|^2 t}\cos(\frac{\xi_3}{|\xi|}\Omega t)$ and $e^{-\nu|\xi|^2 t}\sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}(\xi)$ with $\mathbf{\Phi}_0(x)$.

Thus, the symbol of the vector pseudo-differential operator $\exp(-\mathbf{A}(\Omega)t)$ corresponding to the linear problem (Stokes Operator + $\Omega\mathbf{S}$) is given by

$$\sigma(\exp(-\mathbf{A}(\Omega)t)) = e^{-\nu|\xi|^2 t}\cos(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{I} - e^{-\nu|\xi|^2 t}\sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}(\xi), \tag{2.17}$$

where $\mathbf{R}$ is the vector Riesz operator with the 3×3 matrix symbol $\mathbf{R}(\xi)$ defined above; $\mathbf{I}$ is the 3×3 identity matrix. From the calculations outlined in Appendix A it follows that

$$F^{-1}\left(e^{-\nu|\xi|^2 t}\cos(\frac{\xi_3}{|\xi|}\Omega t)\right), F^{-1}\left(e^{-\nu|\xi|^2 t}\sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}(\xi)\right) \in L^q(\mathbb{R}^3), 1 < q < +\infty. \tag{2.18}$$

The symbol $\sigma(\exp(-\mathbf{A}(\Omega)t))$ is discontinuous at $\xi = 0$ since the functions $e^{-\nu|\xi|^2 t}\sin(\frac{\xi_3}{|\xi|}\Omega t)\frac{\xi_j}{|\xi|}$, $j = 1, 2$ are discontinuous at $\xi = 0$. Therefore, the integral kernel given by Fourier transform of the symbol cannot belong to $L^1(\mathbb{R}^3)$. More detailed consideration of the Fourier transform given in the Appendix A shows that it behaves as $|x|^{-3}$ for large $|x|$ and that it is not a bounded operator in $L^\infty_\sigma(\mathbb{R}^3)$.

We state a uniform boundedness of $\exp(-\mathbf{A}(\Omega)t)$ in $BMO(\mathbb{R}^3)$ which will be needed in Section 4:

Proposition 2.2. $\exp(-\mathbf{A}(\Omega)t) : BMO(\mathbb{R}^3) \to BMO(\mathbb{R}^3)$ *is a bounded operator and*

$$|| \exp(-\mathbf{A}(\Omega)t)||_{BMO\to BMO} \le C, \tag{2.19}$$

where C *is independent of* Ω *and* $t > 0$*.*

Proof: The fact that $\exp(-\mathbf{A}(\Omega)t)$ is a bounded operator in $BMO(\mathbb{R}^3)$ follows from the formula for its symbol (2.17) together with Lemma B.2 and the fact that the Riesz operators are bounded in $BMO(\mathbb{R}^3)$. We note that dependence on Ω appears only in $\cos(\frac{\xi_3}{|\xi|}\Omega t)$ and $\sin(\frac{\xi_3}{|\xi|}\Omega t)$ which are functions of the Riesz operator R_3 (e.g. $\cos(\frac{\xi_3}{|\xi|}\Omega t) = (e^{i\frac{\xi_3}{|\xi|}\Omega t} + e^{-i\frac{\xi_3}{|\xi|}\Omega t})/2$). Since the spectrum of the Riesz operator R_3 is included in the pure imaginary axis (Appendix B; Lemma B.5), the operator norm of $\exp(\alpha R_3) : BMO \to BMO$ is bounded by 1 independent of $\alpha \in \mathbb{R}$. Since $\exp(-\mathbf{A}(\Omega)t) = e^{\nu\Delta t}E(-\Omega t)$ and $||e^{\nu\Delta t}||_{BMO\to BMO} \le C_0$ with $C_0 > 0$ independent of t and ν, the uniform bound for $\exp(\alpha R_3)$ now yields (2.19).

Remark 2.2. (i) $\exp(-\mathbf{A}(\Omega)t)$ is a bounded operator from $L^\infty_\sigma(\mathbb{R}^3)$ to $BMO(\mathbb{R}^3)$; however, it is not a bounded operator from $L^\infty_\sigma(\mathbb{R}^3)$ to itself.
(ii) $\exp(-\mathbf{A}(\Omega)t) : L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$, $1 < p < +\infty$ is a bounded operator.

3 Stokes-Coriolis semigroup and splitting of initial data having vertical averaging property

It was shown in the previous section that $\exp(-\mathbf{A}(\Omega)t)\mathbf{U}$ does not belong to $L^\infty_\sigma(\mathbb{R}^3)$ for general $\mathbf{U} \in L^\infty_\sigma(\mathbb{R}^3)$. However, $\exp(-\mathbf{A}(\Omega)t)\mathbf{U} \in L^\infty_\sigma(\mathbb{R}^3)$ if $\mathbf{U}$ belongs to a subspace $L^\infty_{\sigma,a}(\mathbb{R}^3)$ of $L^\infty_\sigma(\mathbb{R}^3)$, which we now define.

First, we introduce *vertical averaging property*.

Definition 3.1. (vertical averaging)
Let $\mathbf{U} \in L^\infty_\sigma(\mathbb{R}^3)$. We say that $\mathbf{U}$ admits vertical averaging if

$$\lim_{L\to+\infty} \frac{1}{2L}\int_{-L}^{L} \mathbf{U}(x_1,x_2,x_3)dx_3 \equiv \overline{\mathbf{U}}(x_1,x_2)$$

exists almost everywhere. The vector field $\overline{\mathbf{U}}(x_1,x_2)$ is called *vertical average* of $\mathbf{U}(x_1,x_2,x_3)$.

Remark 3.1. (i) Clearly, all periodic and almost periodic functions (or vector fields) admit vertical averaging.
(ii) The vector field $\overline{\mathbf{U}}(x_1,x_2) = (\overline{U}_1(x_1,x_2), \overline{U}_2(x_1,x_2), \overline{U}_3(x_1,x_2))$ has zero horizontal divergence:

$$\nabla\cdot\overline{\mathbf{U}} = \partial_{x_1}\overline{U}_1 + \partial_{x_2}\overline{U}_2 = 0. \tag{3.1}$$

(iii)Supposing $\mathbf{U} \in L^p_\sigma(\mathbb{R}^3)$ for $1 < p < \infty$, the vertical average always exists; moreover, $\overline{\mathbf{U}} \equiv 0$.
(iv)If $\mathbf{U} \in L^\infty(\mathbb{R}^3)$ admits vertical averaging (at (x_1,x_2)), then we have uniform convergence property, i.e.,

$$\lim_{L\to\infty}\sup_{|r|\le M}\frac{1}{2L}\int_{-L}^{L}\mathbf{U}(x_1,x_2,x_3+r)dx_3 = \overline{\mathbf{U}}(x_1,x_2)$$

for each $M > 0$. Indeed, we may assume that $\overline{\mathbf{U}}(x_1,x_2) = 0$ by considering $\mathbf{U} - \overline{\mathbf{U}}$ instead of $\mathbf{U}$. We suppress the dependence of (x_1,x_2). Since

$$\int_{-L}^{L}\mathbf{U}(x_3+r)dx_3 = \left(\int_{-L-r}^{L+r} - \int_{-L-r}^{-L+r}\right)\mathbf{U}(x_3)dx_3,$$

we observe that

$$\left|\frac{1}{2L}\int_{-L}^{L}\mathbf{U}(x_3+r)dx_3\right| \le \frac{L+r}{L}\frac{1}{2(L+r)}\left|\int_{-L-r}^{L+r}\mathbf{U}(x_3)dx_3\right| + ||\mathbf{U}||_\infty\frac{2r}{2L}.$$

We take supremum in $r \in [-M, M]$ and send L to ∞ to get the desired result.

Eq. (3.1) follows if we apply vertical averaging operation to the 3D divergence free equation $\nabla\cdot\mathbf{U} = \partial_{x_1}U_1 + \partial_{x_2}U_2 + \partial_{x_3}U_3 = 0$ and notice that

$$\lim_{L\to+\infty}\frac{1}{2L}\int_{-L}^{L}\frac{\partial U_3}{\partial x_3}dx_3 = \lim_{L\to+\infty}\frac{1}{2L}(U_3(x_1,x_2,L) - U_3(x_1,x_2,-L)) = 0, \tag{3.2}$$

since $U_3 \in L^\infty(\mathbb{R}^3)$.

The operation of vertical averaging defined above is called 'barotropic projection' and the vector field $\overline{\mathbf{U}}(x_1, x_2)$ is called 'barotropic component' of $\mathbf{U}(x_1, x_2, x_3)$. Then the 'baroclinic component' $\mathbf{U}^\perp(x_1, x_2, x_3)$ is defined as

$$\mathbf{U}^\perp(x_1, x_2, x_3) = \mathbf{U}(x_1, x_2, x_3) - \overline{\mathbf{U}}(x_1, x_2). \tag{3.3}$$

Now we define the space $L^\infty_{\sigma,a}(\mathbb{R}^3)$.

Definition 3.2. (Space for initial data) We define a subspace of L^∞_σ of the form

$$L^\infty_{\sigma,a}(\mathbb{R}^3) = \{\mathbf{U} \in L^\infty_\sigma(\mathbb{R}^3);\ \mathbf{U} \text{ admits vertical averaging and } \mathbf{U}^\perp \in \dot{B}^0_{\infty,1}\}.$$

Here $\dot{B}^0_{\infty,1}$ is the homogeneous Besov space (see Appendix B on details of its definition and properties). The space $L^\infty_{\sigma,a}(\mathbb{R}^3)$ is a Banach space with the norm

$$||\mathbf{U}||_{L^\infty_{\sigma,a}} = ||\overline{\mathbf{U}}||_{L^\infty(\mathbb{R}^2;\mathbb{R}^3)} + ||\mathbf{U}^\perp||_{\dot{B}^0_{\infty,1}}.$$

Indeed, let $\{\mathbf{U}_j\}$ be a Cauchy sequence of $L^\infty_{\sigma,a}$. Since $||f||_{\dot{B}^0_{\infty,1}} \le C||f||_\infty$, $\mathbf{U}_j$ converges to some $\mathbf{U} \in L^\infty_\sigma$ uniformly in $\mathbb{R}^3$. Since $\overline{\mathbf{U}_j}$ exists, so does $\overline{\mathbf{U}}$. Since $||\overline{f}||_\infty \le C||f||_\infty$, we conclude that $\overline{\mathbf{U}_j} \to \overline{\mathbf{U}}$ uniformly in $\mathbb{R}^2$. Since $\{\mathbf{U}_j^\perp\}$ is a Cauchy sequence in $\dot{B}^0_{\infty,1}$, there is a limit $\mathbf{v} \in \dot{B}^0_{\infty,1}$. However, $\mathbf{U}_j \to \mathbf{U}$, $\overline{\mathbf{U}_j} \to \overline{\mathbf{U}}$, so $\mathbf{v}$ must be equal to $\mathbf{U}^\perp$.

Remark 3.2. The space $L^\infty_{\sigma,a}$ has a topological direct sum decomposition of the form

$$L^\infty_{\sigma,a} = \mathcal{W} \oplus \mathcal{B}^0$$

with

$$\mathcal{W} = \{\mathbf{U} \in L^\infty_\sigma;\ \partial U_i/\partial x_3 \equiv 0 \quad \text{in distributional sense } \mathbb{R}^3 \text{ for } i = 1,2,3\},$$
$$\mathcal{B}^0 = \{\mathbf{U} \in \dot{B}^0_{\infty,1} \cap L^\infty_\sigma;\ \overline{\mathbf{U}}(x_1, x_2) \equiv 0 \quad \text{a.e. } (x_1, x_2) \in \mathbb{R}^2\}.$$

Indeed, for $\mathbf{U} \in L^\infty_{\sigma,a}$ we observe that $\overline{\mathbf{U}} \in \mathcal{W}$ and $\mathbf{U}^\perp \in \mathcal{B}^0$. Moreover, $\mathcal{W} \cap \mathcal{B}^0 = \{0\}$. The closedness of $\mathcal{W}$ and $\mathcal{B}^0$ can be proved using Definition 3.2.

The advantage of the Besov space $\dot{B}^0_{\infty,1}$ is that the Riesz operators and, consequently, the operator $\exp(-\mathbf{A}(\Omega)t)$ are bounded operators in this space. Also, this space contains all locally Lipschitz periodic functions with zero mean value and all almost periodic functions of the form

$$\sum_{j=1}^\infty \alpha_j e^{\sqrt{-1}\lambda_j \cdot x} \qquad \text{with } \{\alpha_j\}_{j=1}^\infty \in l^1,\ \{\lambda_j\} \subset \mathbb{R}^3 \setminus \{0\}.$$

Let $\mathbf{U} \in L^\infty_{\sigma,a}(\mathbb{R}^3)$. Then $\mathbf{U}$ admits vertical averaging and we have the following representation (splitting)

$$\mathbf{U} = \overline{\mathbf{U}} + \mathbf{U}^\perp, \tag{3.4}$$

where $\overline{\mathbf{U}}(x_1, x_2)$ is a 2D-3C vector field (vector field with three components where each component $\{\overline{U}_j(x_1, x_2)\}_{j=1}^3$ depends only on two variables x_1 and x_2); $\overline{U}_j(x_1, x_2) \in L^\infty(\mathbb{R}^2)$.

We have

$$\exp(-\mathbf{A}(\Omega)t)\mathbf{U} = \exp(\nu t\Delta)\overline{\mathbf{U}} + \exp(-\mathbf{A}(\Omega)t)\mathbf{U}^\perp, \tag{3.5}$$

where we used

$$\exp(-\Omega t\mathbf{S})\overline{\mathbf{U}} = \overline{\mathbf{U}}. \tag{3.6}$$

Then the first term in (3.5) is the classical heat kernel. In order to estimate the second term in (3.5), we need to show that the norm of the operator $\exp(-\mathbf{A}(\Omega)t)$ in the Besov space $\dot{B}^0_{\infty,1}$ is independent of Ω.

Proposition 3.1. *The operator* $\exp(-\mathbf{A}(\Omega)t) : \dot{B}^0_{\infty,1} \to \dot{B}^0_{\infty,1}$ *is a bounded operator and*

$$|| \exp(-\mathbf{A}(\Omega)t)||_{\dot{B}^0_{\infty,1}\to\dot{B}^0_{\infty,1}} \leq C, \tag{3.7}$$

where C *is independent of* Ω *and* $t > 0$.

Proof: A direct calculation using the Gauss kernel yields

$$||e^{\nu t\Delta}||_{\dot{B}^0_{\infty,1}\to\dot{B}^0_{\infty,1}} \leq 1.$$

It suffices to prove the uniform boundedness of $E(-\Omega t) : \dot{B}^0_{\infty,1} \to \dot{B}^0_{\infty,1}$. Since (2.15) implies that

$$E(-\Omega t) = \cos(-iR_3\Omega t)\mathbf{I} + \mathbf{R}\sin(-iR_3\Omega t)$$

and the Riesz operator $\mathbf{R}$ is bounded in $\dot{B}^0_{\infty,1}$, it suffices to prove a uniform bound for $\cos(-iR_3\Omega t)$ and $\sin(-iR_3\Omega t)$. The dependence on Ω appears only in cos and sin functions. Note that the operator norm of $\exp(\alpha R_3) : \dot{B}^0_{\infty,1} \to \dot{B}^0_{\infty,1}$ is bounded by 1 independent of $\alpha \in \mathbb{R}$ since the spectrum of the Riesz operator R_3 is included in the pure imaginary axis (Appendix B; Lemma B.4). Since cos and sin can be expressed by exponential functions, e.g. $\cos(-iR_3\Omega t) = (\exp(R_3\Omega t) + \exp(-R_3\Omega t))/2$, we have a uniform bound independent of Ω and t for $E(-\Omega t)$. This yields (3.7).

In the remainder of this section we shall prove that $\exp(-\mathbf{A}(\Omega)t)$ is a uniformly bounded semigroup in $L^\infty_{\sigma,a}$. Since we have Proposition 3.1 together with (3.5) and (3.6), it suffices to prove

Proposition 3.2. *The operator* $\exp(-\mathbf{A}(\Omega)t)$ *maps from* $L^\infty_{\sigma,a}$ *to itself for all* $t > 0$.

Proof: It suffices to show that $\exp(-\mathbf{A}(\Omega)t)\mathbf{U} = \exp(-\Omega t\mathbf{S})\exp(\nu t\Delta)\mathbf{U} \in \mathcal{B}^0$ if $\mathbf{U} \in \mathcal{B}^0$. We first prove that $\exp(\nu t\Delta)\mathbf{U} \in \mathcal{B}^0$ if $\mathbf{U} \in \mathcal{B}^0$. Since

$$(\exp(\nu t\Delta)\mathbf{U})(x) = \int_{\mathbb{R}^3}\left(\int_{-\infty}^{\infty}\mathbf{U}(x_1-y_1,x_2-y_2,x_3-y_3)g_{\nu t}(y_3)dy_3\right)g_{\nu t}(y_1,y_2)dy_1dy_2$$

with the Gauss kernel $g_{\nu t}$, it suffices to prove that

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}\int_{-\infty}^{\infty}\mathbf{U}(x_1-y_1,x_2-y_2,x_3-y_3)g_{\nu t}(y_3)dy_3dx_3 = 0$$

for a.e. (x_1-y_1,x_2-y_2). This follows from the uniform convergence property Remark 3.1(iv), since $g_{\nu t}(y_3)$ is integrable for large y_3. We thus proved that $\overline{\exp(\nu t\Delta)\mathbf{U}} = 0$. The divergence free property is clear, so we conclude that $\exp(\nu t\Delta)\mathbf{U} \in \mathcal{B}^0$ if $\mathbf{U} \in \mathcal{B}^0$. The proof will be complete if we prove

$$E(-\Omega t)\mathbf{U} \in \mathcal{B}^0 \quad \text{if } \mathbf{U} \in \mathcal{B}^0.$$

We give the proof of this fact in the Appendix (Lemma B.6).

4 Local solvability independent of the speed of rotation

In this section we prove time-local existence and uniqueness for (1.1)-(1.2) on some time $[0,T_0]$ with T_0 independent of $\Omega \in \mathbb{R}$. The differential equations are formally transformed into the integral equation of the form:

$$\text{(I)} \qquad \mathbf{U}(t) = \exp(-\mathbf{A}(\Omega)t)\mathbf{U}_0 - N(\mathbf{U},t;\Omega) \qquad \text{for } t>0.$$

Here the nonlinear term $N(\mathbf{U},t;\Omega) = N(\mathbf{U},\mathbf{U},t;\Omega)$ is defined by

$$N(\mathbf{U},\mathbf{V},t;\Omega) = \int_0^t \exp(-\mathbf{A}(\Omega)(t-s))\mathbf{P}\mathrm{div}(\mathbf{U}\otimes\mathbf{V})(s)\;ds.$$

We call a solution of the integral equation (I) a mild solution of the rotational Navier-Stokes equations. Since $\mathbf{PU} = \mathbf{U}$ for divergence free vector field and $\mathbf{P}\Delta = \Delta\mathbf{P}$, we have

$$\mathbf{A}(\Omega) = -\mathbf{P}\Delta + \Omega\mathbf{PJ} = -\Delta + \Omega\mathbf{PJP}.$$

Note that

$$\exp(-\mathbf{A}(\Omega)t) = e^{t\Delta}E(-\Omega t),$$

where $e^{t\Delta}$ is the solution operator of the heat equation (in this section we put $\nu = 1$ for simplicity of notations). For an interval $I \subset [-\infty,\infty]$ and a Banach space X let $C(I;X)$ denote the space of all continuous functions with valued in X. The space $C_w(I;X)$ denotes the space of all X-valued star weakly continuous functions.

The goal of this section is to prove the following theorems.

Theorem 4.1. (Existence and uniqueness of mild solution U)
Suppose that $\mathbf{U}_0 \in L^\infty_{\sigma,a}(\mathbb{R}^3)$. *Then*
(1) *There exist* $T_0 > 0$ *independent of* Ω *and a unique solution* $\mathbf{U} = \mathbf{U}(t)$ *of* (I) *such that*

$$\mathbf{U} \in C([\delta, T_0]; L^\infty_\sigma) \cap C_w([0, T_0]; L^\infty_\sigma) \tag{4.1}$$

for any $\delta > 0$.
(2) *The solution* $\mathbf{U}$ *satisfies*

$$\sup_{t \in (0,T_0)} ||t^{1/2} \nabla \mathbf{U}||_{L^\infty_\sigma} < \infty \qquad \text{and} \qquad \nabla \mathbf{U} \in C([\delta, T_0]; L^\infty_\sigma) \tag{4.2}$$

for any $\delta > 0$.

Theorem 4.2. (Existence of classical solution U)
Suppose that $\mathbf{U}_0 \in L^\infty_{\sigma,a}(\mathbb{R}^3)$. *Let* $\mathbf{U} = \mathbf{U}(t)$ *be a solution of* (I) *satisfying (4.1) and (4.2). If we set*

$$\nabla p(t) = \nabla \sum_{j,k=1}^{3} R_j R_k \mathbf{U}^j \mathbf{U}^k(t) - \Omega \begin{pmatrix} R_1 \ (R_2\mathbf{U}^1 - R_1\mathbf{U}^2) \\ R_2 \ (R_2\mathbf{U}^1 - R_1\mathbf{U}^2) \\ R_3 \ (R_2\mathbf{U}^1 - R_1\mathbf{U}^2) \end{pmatrix} \qquad \text{for } t > 0, \tag{4.3}$$

then the pair $(\mathbf{U}, \nabla p)$ *is a classical solution of (1.1)-(1.2).*

Such a solution (satisfying (4.1)-(4.3)) is unique. In fact a stronger version is available.

Theorem 4.3. (Uniqueness of classical solution U)
Suppose that $\mathbf{U}_0 \in L^\infty_{\sigma,a}(\mathbb{R}^3)$. *Let*

$$\mathbf{U} \in L^\infty((0,T) \times \mathbb{R}^3), \quad p \in L^1_{loc}([0,T); BMO)$$

be a solution of (1.1)-(1.2) in a distributional sense for some $T > 0$. *Then the pair* $(\mathbf{U}, \nabla p)$ *is unique. Furthermore, the relation (4.3) holds.*

Remark 4.1. (i)For a lower estimate for $T_0 > 0$ we get

$$T_0 \geq C / ||\mathbf{U}_0||^2_{L^\infty_{\sigma,a}}$$

with C independent of Ω.
(ii)For regularity we can get the same results as in [12]. The remark except (i) after Theorem 1 in [12] holds for our equation (I).
(iii) From the proof given below it is rather clear that one can take initial data in $\mathcal{W} + \dot{B}^0_{\infty,1}$, which is larger than $L^\infty_{\sigma,a}$. In particular, this class includes $\dot{B}^0_{\infty,1} \cap L^\infty_\sigma$ for which local existence is discussed in [24].
(iv) If in addition we assume that $\overline{\mathbf{U}_0} \in BUC$ so that $\mathbf{U}_0 \in BUC$, then by construction

our solution $\mathbf{U} \in C([0,T_0]; BUC)$; here, BUC denotes the space of all bounded uniformly continuous functions in $\mathbb{R}^3$. Indeed, since $\dot{B}^0_{\infty,1} \subset BUC$ (see e.g. Example 2.3(iv) in [24]), $\mathbf{U}_0 \in BUC$. Since $e^{t\Delta}\overline{\mathbf{U}_0} \in C([0,\infty); BUC)$ (see Proposition A.1.1 in [12]) and $E(-\Omega t)\mathbf{U}_0^\perp \in C([0,\infty); \dot{B}^0_{\infty,1})$, it is easy to see that $\mathbf{U}_j \in C([0,\infty); BUC)$. Thus its uniform limit $\mathbf{U}$ belongs to $C([0,T_0]; BUC)$.

We note that Theorem 4.2 follows from Theorem 4.1 as observed in [12], where the case $\Omega = 0$ is discussed. We also note that the uniqueness (Theorem 4.3) can be proved along the line of [14],[18], where the case $\Omega = 0$ is discussed. We won't repeat the proofs. The proof of Theorem 4.1 is based on a standard iteration method, and is similar to that of [12]. We have already prepared two estimates for $\exp(-\mathbf{A}(\Omega)t)$ in BMO and Besov spaces (Proposition 2.2 and Proposition 3.1). We further estimate its spatial derivatives.

Lemma 4.1. *There exists a constant $C > 0$ (depending only on space dimensions) that satisfies*

$$||\nabla e^{t\Delta} f||_{L^\infty} \leq C t^{-1/2} ||f||_{BMO}, \quad t > 0$$

for $f \in BMO$.

Proof In [[9],Lemma 2.1] Carpio obtained for the Gauss kernel $g_t = g_t(x)$ that

$$||\nabla g_t||_{\mathcal{H}^1} \leq C t^{-1/2}, \quad t > 0.$$

Here, $\mathcal{H}^1$ denotes the Hardy space. Since the dual space of the space $\mathcal{H}^1$ is BMO, we have

$$||\nabla e^{t\Delta} f||_{L^\infty} \leq ||\nabla g_t||_{\mathcal{H}^1} ||f||_{BMO} \leq C t^{-1/2} ||f||_{BMO}.$$

Lemma 4.1 was proved.

Using the above lemma, Proposition 2.2 and Proposition 3.1, the linear term is estimated as follows.

Lemma 4.2. *There exists a constant C (independent of Ω, t, f) that satisfies*

$$||\exp(-\mathbf{A}(\Omega)t) f||_{L^\infty} \leq C ||f||_{L^\infty_{\sigma,a}}, \quad t > 0, \quad \text{and}$$
$$||\nabla \exp(-\mathbf{A}(\Omega)t) f||_{L^\infty} \leq C t^{-1/2} ||f||_{L^\infty_{\sigma,a}}, \quad t > 0$$

for all $f = (f_i)_{1\leq i \leq 3} \in L^\infty_{\sigma,a}$.

Proof By (3.5), Proposition 3.1 and $||\cdot||_{L^\infty} \leq ||\cdot||_{\dot{B}^0_{\infty,1}}$ we get

$$\begin{aligned}
||\exp(-\mathbf{A}(\Omega)t) f||_{L^\infty} &= ||e^{t\Delta}\overline{f} + e^{t\Delta}\exp(-t\Omega\mathbf{PJP}) f^\perp||_{L^\infty} \\
&\leq ||e^{t\Delta}\overline{f}||_{L^\infty} + ||e^{t\Delta}\exp(-t\Omega\mathbf{PJP}) f^\perp||_{L^\infty} \\
&\leq ||\overline{f}||_{L^\infty} + ||\exp(-t\Omega\mathbf{PJP}) f^\perp||_{L^\infty} \\
&\leq ||\overline{f}||_{L^\infty} + ||\exp(-t\Omega\mathbf{PJP}) f^\perp||_{\dot{B}^0_{\infty,1}} \\
&\leq ||\overline{f}||_{L^\infty} + C||f^\perp||_{\dot{B}^0_{\infty,1}} \\
&\leq C||f||_{L^\infty_{\sigma,a}}.
\end{aligned}$$

Similarly Lemma 4.1, Proposition 2.2 and $||\cdot||_{BMO} \le ||\cdot||_{L^\infty}$ imply that

$$
\begin{aligned}
||\nabla \exp(-\mathbf{A}(\Omega)t)f||_{L^\infty} &= ||\nabla e^{t\Delta}\overline{f} + \nabla e^{t\Delta}\exp(-t\Omega\mathbf{PJP})f^\perp||_{L^\infty} \\
&\le ||\nabla e^{t\Delta}\overline{f}||_{L^\infty} + ||\nabla e^{t\Delta}\exp(-t\Omega\mathbf{PJP})f^\perp||_{L^\infty} \\
&\le Ct^{-1/2}||\overline{f}||_{BMO} + Ct^{-1/2}||\exp(-t\Omega\mathbf{PJP})f^\perp||_{BMO} \\
&\le Ct^{-1/2}(||\overline{f}||_{BMO} + ||f^\perp||_{BMO}) \\
&\le Ct^{-1/2}(||\overline{f}||_{L^\infty} + ||f^\perp||_{L^\infty}) \\
&\le Ct^{-1/2}(||\overline{f}||_{L^\infty} + ||f^\perp||_{\dot{B}^0_{\infty,1}}) \\
&\le Ct^{-1/2}||f||_{L^\infty_{\sigma,a}}.
\end{aligned}
$$

We have proved Lemma 4.2.

Next we prepare estimates for the nonlinear term.

Lemma 4.3. (Derivative estimate)
There exists a constant C (independent of Ω, t, F and f) that satisfies

$$
\begin{aligned}
&||\exp(-\mathbf{A}(\Omega)t)\mathbf{P}\mathrm{div}F||_{L^\infty} \le Ct^{-1/2}||F||_{BMO}, \quad t>0, \quad \textit{and} \\
&||\nabla\exp(-\mathbf{A}(\Omega)t)\mathbf{P}f||_{L^\infty} \le Ct^{-1/2}||f||_{BMO} \quad t>0
\end{aligned}
$$

for all $F = (F_{i,j})_{1\le i,j\le 3} \in BMO$, with $\mathrm{div}F \in BMO$ and for all $f = (f_i)_{1\le i\le 3} \in BMO$.

Proof It is easy to see that

$$\mathbf{P}\mathrm{div}F = \mathrm{div}F + \mathrm{div}(\mathbf{P}-I)F^t,$$

where F^t is transposed matrix of F. We rewrite

$$\exp(-\mathbf{A}(\Omega)t)\mathbf{P}\mathrm{div}F = e^{t\Delta}E(-\Omega t)\mathrm{div}\{F + (\mathbf{P}-I)F^t\}.$$

Since the symbol of the operator $e^{t\Delta}E(-\Omega t)\mathrm{div}$ is represented by

$$
\begin{aligned}
&\exp(-t|\xi|^2)\{\cos(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{I} - \mathbf{R}(\xi)\sin(\frac{\xi_3}{|\xi|}\Omega t)\}i\xi_k \\
&= i\xi_k\exp(-t|\xi|^2)\cos(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{I} - i\xi_k\mathbf{R}(\xi)\exp(-t|\xi|^2)\sin(\frac{\xi_3}{|\xi|}\Omega t),
\end{aligned}
$$

one sees that

$$
\begin{aligned}
||e^{t\Delta}E(-\Omega t)\mathrm{div}||_{BMO\to L^\infty} &\le ||\nabla e^{t\Delta}||_{BMO\to L^\infty}||\cos(-iR_3\Omega t)||_{BMO\to BMO} \\
&\quad + ||\mathrm{curl}e^{t\Delta}||_{BMO\to L^\infty}||\sin(-iR_3\Omega t)||_{BMO\to BMO}|||\mathbf{R}||_{BMO\to BMO} \\
&\le Ct^{-1/2}, \qquad (4.4)
\end{aligned}
$$

where $C > 0$ is independent of Ω and t. Thus, by Lemma 4.1, Proposition 2.2 and boundedness of the operator $\mathbf{P}$ in BMO we have

$$\begin{aligned}||\exp(-\mathbf{A}(\Omega)t)\mathbf{P}\mathrm{div}F||_{L^\infty} &\le Ct^{-1/2}||F+(\mathbf{P}-I)F^t||_{BMO}\\ &\le Ct^{-1/2}(||F||_{BMO}+||(\mathbf{P}-I)F^t||_{BMO}) \le Ct^{-1/2}||F||_{BMO}.\end{aligned}$$

Similarly, we get by (4.4)

$$||\nabla\exp(-\mathbf{A}(\Omega)t)\mathbf{P}f||_{L^\infty} \le Ct^{-1/2}||f||_{BMO}$$

because the symbol of the operator $\nabla\exp(-\mathbf{A}(\Omega)t)$ is the essentially same as that of $e^{t\Delta}E(-\Omega t)\mathrm{div}$. We have proved Lemma 4.3.

Proof of Theorem 4.1. We use the following successive iteration:

$$\mathbf{U}_1(t) = \exp(-\mathbf{A}(\Omega)t)\mathbf{U}_0, \qquad \mathbf{U}_{j+1}(t) = \exp(-\mathbf{A}(\Omega)t)\mathbf{U}_0 - N(\mathbf{U}_j,t;\Omega) \qquad \text{for } j\ge 1.$$

For $j \ge 1$ and $T > 0$ we set

$$K_j = K_j(T) = \sup_{0<s<T}||\mathbf{U}_j(s)||_{L^\infty} \quad \text{and} \quad K_j' = K_j'(T) = \sup_{0<s<T}(s^{1/2}||\nabla\mathbf{U}_j(s)||_{L^\infty}).$$

Put $K_0 = ||\mathbf{U}_0||_{L^\infty_{\sigma,a}}$ and note that K_0 is independent of $T > 0$. It follows from Lemma 4.3 and $||\cdot||_{BMO} \le ||\cdot||_{L^\infty}$ that

$$\begin{aligned}||N(\mathbf{U}_j,t;\Omega)||_{L^\infty} &\le \int_0^t ||\exp(-\mathbf{A}(\Omega)(t-s))\mathbf{P}\mathrm{div}(\mathbf{U}_j\otimes\mathbf{U}_j)(s)||_{L^\infty}\,ds\\ &\le \int_0^t ||\exp(-\mathbf{A}(\Omega)(t-s))\mathbf{P}\mathrm{div}||_{BMO\to L^\infty}||(\mathbf{U}_j\otimes\mathbf{U}_j)(s)||_{BMO}\,ds\\ &\le \int_0^t C(t-s)^{-1/2}||(\mathbf{U}_j\otimes\mathbf{U}_j)(s)||_{BMO}\,ds\\ &\le Ct^{1/2}\sup_{0<s<t}||(\mathbf{U}_j\otimes\mathbf{U}_j)(s)||_{BMO} \le Ct^{1/2}\sup_{0<s<t}||(\mathbf{U}_j\otimes\mathbf{U}_j)(s)||_{L^\infty}\\ &\le Ct^{1/2}\sup_{0<s<t}(||\mathbf{U}_j(s)||^2_{L^\infty}) \le Ct^{1/2}(\sup_{0<s<t}||\mathbf{U}_j(s)||_{L^\infty})^2. \qquad (4.5)\end{aligned}$$

Similarly we have from Lemma 4.3

$$\begin{aligned}||\nabla N(\mathbf{U}_j,t;\Omega)||_{L^\infty} &\le \int_0^t ||\nabla\exp(-\mathbf{A}(\Omega)(t-s))\mathbf{P}\mathrm{div}(\mathbf{U}_j\otimes\mathbf{U}_j)(s)||_{L^\infty}\,ds\\ &\le C\int_0^t (t-s)^{-1/2}||\mathrm{div}(\mathbf{U}_j\otimes\mathbf{U}_j)(s)||_{BMO}\,ds\\ &\le C\int_0^t (t-s)^{-1/2}||\mathrm{div}(\mathbf{U}_j\otimes\mathbf{U}_j)(s)||_{L^\infty}\,ds\\ &\le C\int_0^t (t-s)^{-1/2}s^{-1/2}s^{1/2}||\nabla\mathbf{U}_j(s)||_{L^\infty}||\mathbf{U}_j(s)||_{L^\infty}\,ds\\ &\le C\sup_{0<s<t}(s^{1/2}||\nabla\mathbf{U}_j(s)||_{L^\infty})\sup_{0<s<t}||\mathbf{U}_j(s)||_{L^\infty}. \qquad (4.6)\end{aligned}$$

By the above estimates and Lemma 4.2 there exist constants C_0, C_1, C_2 and C_3 independent of Ω and T such that

$$K_{j+1}(T) \leq C_0 K_0 + C_1 T^{1/2}(K_j(T))^2,$$
$$K'_{j+1}(T) \leq C_2 K_0 + C_3 T^{1/2} K_j(T) K'_j(T)$$

for $j \geq 1$. Taking T_0 small so that $T_0 < \inf(1/(4C_0C_1K_0)^2, 1/(4C_0C_3K_0)^2)$, we get

$$\sup_{j\geq 1} K_j(T) \leq 2C_0K_0 \quad \text{and} \quad \sup_{j\geq 1} K'_j(T) \leq 2C_2K_0 \qquad \text{if} \quad T \leq T_0.$$

Next we shall prove the convergence. For $j \geq 1$ and $T > 0$ put

$$L_j = L_j(T) = \sup_{0<s<T} ||\mathbf{U}_j(s) - \mathbf{U}_{j-1}(s)||_{L^\infty},$$
$$L'_j = L'_j(T) = \sup_{0<s<T} (s^{1/2}||\nabla \mathbf{U}_j(s) - \nabla \mathbf{U}_{j-1}(s)||_{L^\infty}).$$

Since

$$\begin{aligned} \mathbf{U}_{j+1}(t) - \mathbf{U}_j(t) = & N(\mathbf{U}_j, \mathbf{U}_j, t; \Omega) - N(\mathbf{U}_j, \mathbf{U}_{j-1}, t; \Omega) \\ & + N(\mathbf{U}_j, \mathbf{U}_{j-1}, t; \Omega) - N(\mathbf{U}_{j-1}, \mathbf{U}_{j-1}, t; \Omega), \end{aligned} \tag{4.7}$$

similarly as in (4.5) and (4.6) we get

$$\begin{aligned} & ||\mathbf{U}_{j+1}(t) - \mathbf{U}_j(t)||_{L^\infty} \\ & \leq \int_0^t C(t-s)^{-1/2}(||\mathbf{U}_j(s)||_{L^\infty} + ||\mathbf{U}_{j-1}(s)||_{L^\infty})||(\mathbf{U}_j - \mathbf{U}_{j-1})(s)||_{L^\infty}\, ds \\ & \leq Ct^{1/2} \sup_{0<s<T}(||\mathbf{U}_j(s)||_{L^\infty} + ||\mathbf{U}_{j-1}(s)||_{L^\infty}) \sup_{0<s<T} ||(\mathbf{U}_j - \mathbf{U}_{j-1})(s)||_{L^\infty} \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} ||\nabla \mathbf{U}_{j+1}(t) - \nabla \mathbf{U}_j(t)||_{L^\infty} \quad & \leq C \int_0^t (t-s)^{-1/2} s^{-1/2} \{ s^{1/2} ||\nabla \mathbf{U}_j(s)||_{L^\infty} ||(\mathbf{U}_j - \mathbf{U}_{j-1})(s)||_{L^\infty} \\ & + s^{1/2} ||\mathbf{U}_j(s)||_{L^\infty} ||\nabla(\mathbf{U}_j - \mathbf{U}_{j-1})(s)||_{L^\infty} \}\, ds \\ & + C \int_0^t (t-s)^{-1/2} s^{-1/2} \{ s^{1/2} ||\nabla \mathbf{U}_{j-1}(s)||_{L^\infty} ||(\mathbf{U}_j - \mathbf{U}_{j-1})(s)||_{L^\infty} \\ & + s^{1/2} ||\mathbf{U}_{j-1}(s)||_{L^\infty} ||\nabla(\mathbf{U}_j - \mathbf{U}_{j-1})(s)||_{L^\infty} \}\, ds. \end{aligned} \tag{4.9}$$

Hence there exist $C_4, C_5 > 0$ independent of Ω and T such that

$$L_{j+1}(T) \leq C_4 K_0 T^{1/2} L_j(T),$$
$$L'_{j+1}(T) \leq C_5 K_0 T^{1/2} (L_j(T) + L'_j(T))$$

for $j \geq 1$. Taking T_1 small so that $T_1 < 1/(2(C_4 + C_5)K_0)^2$, it is easy to see that

$$\sup_{j\geq 1} \frac{L_{j+1}(T)}{L_j(T)} < \frac{1}{2} \qquad \text{and} \qquad \sup_{j\geq 1} \frac{L_{j+1}(T) + L'_{j+1}(T)}{L_j(T) + L'_j(T)} < \frac{1}{2} \qquad \text{if} \quad T \leq T_1.$$

Thus, choosing $T < \min(T_0, T_1)$, the approximations $\{\mathbf{U}_j(t)\}_{j\geq 1}$ and $\{t^{1/2}\nabla \mathbf{U}_j(t)\}_{j\geq 1}$ are Cauchy sequences in $L^\infty((0,T) \times \mathbb{R}^3)$. Denote its limits by $\mathbf{U}(t)$ and $\mathbf{V}(t)$, respectively. Since $\mathbf{U}_j$ satisfies (4.1), so does $\mathbf{U}$. Similar calculation as in (4.8) and (4.9) yields that

$$\begin{aligned} N(\mathbf{U}_j, t; \Omega) &\rightarrow N(\mathbf{U}, t; \Omega) \quad \text{in } L^\infty((0,T) \times \mathbb{R}^3) \quad \text{as } j \to \infty, \\ \nabla N(\mathbf{U}_j, t; \Omega) &\rightarrow \nabla N(\mathbf{U}, t; \Omega) \quad \text{in } L^\infty((0,T) \times \mathbb{R}^3) \quad \text{as } j \to \infty, \end{aligned}$$

which guarantees that $t^{1/2}\nabla \mathbf{U} = \mathbf{V}$ and that the limit $\mathbf{U}$ solves the integral equation (I). The properties (4.2) for $\mathbf{U}$ are also inherited from $\mathbf{U}_j$'s.

It remains to prove the uniqueness. We set $\mathbf{W} = \mathbf{U}_1 - \mathbf{U}_2$ and observe that

$$\mathbf{W}(t) = N(\mathbf{U}_1, \mathbf{U}_1, t; \Omega) - N(\mathbf{U}_2, \mathbf{U}_2, t; \Omega).$$

Then the same calculation as (4.7) and (4.8) gives us $\mathbf{W} \equiv 0$.

5 Concluding remarks

The above results for the 3D rotating Navier-Stokes Equations can be formulated for solutions of the three-dimensional Navier-Stokes Equations with initial data of the form $\mathbf{V}(t,y)|_{t=0} = \mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + \frac{\Omega}{2} e_3 \times y$:

$$\partial_t \mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{V} + \nu \text{curl}^2 \mathbf{V} = -\nabla q, \;\; \nabla \cdot \mathbf{V} = 0, \tag{5.1}$$

$$\mathbf{V}(t,y)|_{t=0} \;\; = \;\; \mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + \frac{\Omega}{2} e_3 \times y \tag{5.2}$$

where $y = (y_1, y_2, y_3)$, $\mathbf{V}(t,y) = (V_1, V_2, V_3)$ is the velocity field and q is the pressure. In Eqs. (5.1) e_3 denotes the vertical unit vector and Ω is a constant parameter. The field $\tilde{\mathbf{V}}_0(y)$ depends on three variables y_1, y_2 and y_3. Since $\text{curl}(\frac{\Omega}{2} e_3 \times y) = \Omega e_3$, the vorticity vector at initial time $t = 0$ is

$$\text{curl}\mathbf{V}(0,y) = \text{curl}\tilde{\mathbf{V}}_0(y) + \Omega e_3. \tag{5.3}$$

In (5.2) we take $\tilde{\mathbf{V}}_0(y) \in L^\infty_{\sigma,a}(\mathbb{R}^3)$.

We now detail the canonical rotation transformation between the original vector field $\mathbf{V}(t,y)$ and the vector field $\mathbf{U}(t,x)$. Let $\mathbf{J}$ be the matrix such that $\mathbf{J}\mathbf{a} = e_3 \times \mathbf{a}$ for any vector field $\mathbf{a}$. Then

$$\mathbf{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \;\; \mathbf{\Upsilon}(t) \equiv e^{\Omega \mathbf{J} t/2} = \begin{pmatrix} \cos(\frac{\Omega t}{2}) & -\sin(\frac{\Omega t}{2}) & 0 \\ \sin(\frac{\Omega t}{2}) & \cos(\frac{\Omega t}{2}) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.4}$$

For any fixed parameter Ω we introduce the following fundamental rotation transformation:

$$\mathbf{V}(t,y) = e^{+\Omega\mathbf{J}t/2}\mathbf{U}(t, e^{-\Omega\mathbf{J}t/2}y) + \frac{\Omega}{2}\mathbf{J}y,\ x = e^{-\Omega\mathbf{J}t/2}y. \tag{5.5}$$

The transformation (5.5) is invertible:

$$\mathbf{U}(t,x) = e^{-\Omega\mathbf{J}t/2}\mathbf{V}(t, e^{+\Omega\mathbf{J}t/2}x) - \frac{\Omega}{2}\mathbf{J}x,\ y = e^{+\Omega\mathbf{J}t/2}x. \tag{5.6}$$

The transformations (5.5)-(5.6) establish one-to-one correspondence between solenoidal vector fields $\mathbf{V}(t,y)$ and $\mathbf{U}(t,x)$. We note that $x = y$ for $t = 0$ and therefore $\tilde{\mathbf{V}}_0(y) = \tilde{\mathbf{V}}_0(x)$. Let $x = (x_h, x_3)$ where $x_h = (x_1, x_2, 0)$, $|x_h|^2 = x_1^2 + x_2^2$ and similarly for y.

The following identities hold for the vector fields $\mathbf{V}(t,y)$ and $\mathbf{U}(t,x)$ and pressure π:
1. $\nabla_y \cdot \mathbf{V}(t,y) = \nabla_x \cdot \mathbf{U}(t,x)$.
2. $\nabla_y \pi = \mathbf{\Upsilon}(t)\nabla_x \pi$.
3. $\mathrm{curl}_y \mathbf{V}(t,y) = \mathbf{\Upsilon}(t)\mathrm{curl}_x \mathbf{U}(t,x) + \Omega e_3$, $\quad \mathrm{curl}_y^2 \mathbf{V}(t,y) = \mathbf{\Upsilon}(t)\mathrm{curl}_x^2 \mathbf{U}(t,x)$.
4. $\frac{D}{Dt}\mathbf{V}(t,y) = \mathbf{\Upsilon}(t)\left(\frac{D}{Dt}\mathbf{U}(t,x) + \Omega\mathbf{J}\mathbf{U} - \frac{\Omega^2}{2}x_h\right)$ where $\frac{D}{Dt}$ are the corresponding Lagrangian derivatives, $\mathbf{J}\mathbf{U} = e_3 \times \mathbf{U}$.

The above identities 1-4 imply that the transformation (5.5)-(5.6) is canonical for Eqs. (5.1)-(5.2). From the property 1 it follows that $\nabla_x \cdot \mathbf{U}(t,x) = 0$ since $\nabla_y \cdot \mathbf{V}(t,y) = 0$. Now using 2-4 and the fact that $\mathbf{\Upsilon}(t)$ is unitary, we can express each term in (5.1) in x and t variables to obtain the equations for $\mathbf{U}(t,x)$. Under the canonical rotation transformation (5.5)-(5.6) Eqs. (5.1)-(5.2) turn into Navier-Stokes system (5.7)-(5.8) with an additional Coriolis term $\Omega e_3 \times \mathbf{U}$ and modified initial data and pressure:

$$\partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla_x)\mathbf{U} + \nu\mathrm{curl}_x^2\mathbf{U} + \Omega\mathbf{e}_3 \times \mathbf{U} = -\nabla_x p,\ \nabla_x \cdot \mathbf{U} = 0, \tag{5.7}$$

$$\mathbf{U}(t,x)|_{t=0} = \mathbf{U}(0,x) = \tilde{\mathbf{V}}_0(x), \tag{5.8}$$

where $x = y$ at $t = 0$ and $x_h = (x_1, x_2)$. The systems Eqs. (5.1)-(5.2) and (5.7)-(5.8) are equivalent for every Ω and the pair of transformations (5.5)-(5.6) establishes one-to-one correspondence between their fully three-dimensional solutions.

We now state our theorem for the initial value problem (5.1)-(5.2).

Theorem 5.1. (Existence of classical solution V)
Suppose $\tilde{\mathbf{V}}_0 \in L^\infty_{\sigma,a}(\mathbb{R}^3)$. Then there exists a classical solution $(\mathbf{V}, \nabla q)$ of (5.1)-(5.2) satisfying

$$\nabla q(t) = \nabla \sum_{j,k=1}^{3} R_j R_k \mathbf{V}^j \mathbf{V}^k(t) - \Omega \begin{pmatrix} R_1\ (R_2\mathbf{V}^1 - R_1\mathbf{V}^2) \\ R_2\ (R_2\mathbf{V}^1 - R_1\mathbf{V}^2) \\ R_3\ (R_2\mathbf{V}^1 - R_1\mathbf{V}^2) \end{pmatrix} + \nabla\frac{\Omega^2|y_h|^2}{4} \qquad \text{for } t > 0.$$

Such a solution is unique provided that $\mathbf{V} - \frac{\Omega}{2}\mathbf{J}y \in L^\infty(\mathbb{R}^n \times (0,T))$.

This follows from Theorem 4.2 and 4.3.

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A Appendix: Calculation of integral kernels

In this section we analyze inverse Fourier transform of $e^{-\nu|\xi|^2 t}\cos(\frac{\xi_3}{|\xi|}\Omega t)$, which gives the integral kernel in the convolution operator with $\mathbf{\Phi}_0(x)$ (diagonal terms in (2.16)-(2.17)). The calculation of the inverse Fourier transform of $e^{-\nu|\xi|^2 t}\sin(\frac{\xi_3}{|\xi|}\Omega t)\frac{\xi_j}{|\xi|}$ (off-diagonal terms in (2.16)-(2.17)) is similar. The integral kernel is obtained in the form $(2\pi)^{-3/2}F^{-1}(e^{-\nu|\xi|^2 t}\cos(\frac{\xi_3}{|\xi|}\Omega t) - e^{-\nu|\xi|^2 t}\sin(\frac{\xi_3}{|\xi|}\Omega t)\frac{\xi_j}{|\xi|})$ since $F^{-1}mFf = (2\pi)^{-3/2}(F^{-1}m) * f$ for a symbol m and a function f.

We have

$$F^{-1}\left(e^{-\nu|\xi|^2 t}\cos(\frac{\xi_3}{|\xi|}\Omega t)\right) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\cos(\frac{\xi_3}{|\xi|}\Omega t)e^{-\nu|\xi|^2 t}e^{ix\cdot\xi}d\xi, \tag{A.1}$$

where $x = (x_1, x_2, x_3)$ and we denote $|x|^2 = x_1^2 + x_2^2 + x_3^2$, $|x'|^2 = x_1^2 + x_2^2$. Using spherical coordinates with center at 0 and azimuthal angle θ measured from the axis determined by the vector x, one has $(0 \le \theta \le \pi,\ 0 \le \psi \le 2\pi,\ \rho = |\xi|)$

$$\frac{\xi_3}{|\xi|} = -\frac{|x'|}{|x|}\sin\theta\sin\psi + \frac{x_3}{|x|}\cos\theta. \tag{A.2}$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^3}\cos(\frac{\xi_3}{|\xi|}\Omega t)e^{-\nu|\xi|^2 t}e^{ix\cdot\xi}d\xi \\
&= \int_0^{+\infty}\int_0^{2\pi}\int_0^{\pi}\cos\left(\frac{\Omega t|x'|}{|x|}\sin\theta\sin\psi\right)\cos\left(\frac{\Omega t x_3}{|x|}\cos\theta\right)e^{-\nu\rho^2 t}e^{i|x|\rho\cos\theta}\rho^2\sin\theta d\rho d\psi d\theta \\
&= 2\pi\int_0^{+\infty}\int_0^{\pi}J_0\left(\frac{\Omega t|x'|}{|x|}\sin\theta\right)\cos\left(\frac{\Omega t x_3}{|x|}\cos\theta\right)e^{-\nu\rho^2 t}e^{i|x|\rho\cos\theta}\rho^2\sin\theta d\rho d\theta,
\end{aligned} \tag{A.3}$$

where we have used the identity

$$\int_0^{2\pi}\cos\left(\frac{\Omega t|x'|}{|x|}\sin\theta\sin\psi\right)d\psi = 2\pi J_0\left(\frac{\Omega t|x'|}{|x|}\sin\theta\right). \tag{A.4}$$

Let $\mu = \cos\theta$. Then we have from (A.3)

$$\int_{\mathbb{R}^3} \cos(\frac{\xi_3}{|\xi|}\Omega t) e^{-\nu|\xi|^2 t} e^{ix\cdot\xi} d\xi$$
$$= 2\pi \int_0^{+\infty} \int_{-1}^{1} J_0\left(\frac{\Omega t|x'|}{|x|}\sqrt{1-\mu^2}\right) \cos\left(\frac{\Omega t x_3}{|x|}\mu\right) e^{-\nu\rho^2 t} \cos(|x|\rho\mu)\rho^2 d\rho d\mu \quad \text{(A.5)}$$

since the function $\sin(|x|\rho\mu)$ is odd in μ and other functions are even in μ.

Now we calculate the integral in (A.5) involving integration with respect to ρ. We have after somewhat lengthy but elementary calculations (which also involves shifting contour of integration in complex plane) or from the Table of Integrals in [[15],page 529, 3.952]:

$$\int_0^{+\infty} e^{-\nu\rho^2 t} \cos(|x|\rho\mu)\rho^2 d\rho = \frac{\sqrt{\pi}}{4(\sqrt{\nu t})^3}\left(1 - \frac{|x|^2\nu^2}{2\nu t}\right) e^{-\frac{|x|^2\mu^2}{4\nu t}}. \quad \text{(A.6)}$$

Substituting (A.6) into (A.5), we obtain

$$\int_{\mathbb{R}^3} \cos(\frac{\xi_3}{|\xi|}\Omega t) e^{-\nu|\xi|^2 t} e^{ix\cdot\xi} d\xi$$
$$= 2\pi \frac{\sqrt{\pi}}{4(\sqrt{\nu t})^3} \int_{-1}^{1} J_0\left(\frac{\Omega t|x'|}{|x|}\sqrt{1-\mu^2}\right) \cos\left(\frac{\Omega t x_3}{|x|}\mu\right)\left(1 - \frac{|x|^2\nu^2}{2\nu t}\right) e^{-\frac{|x|^2\mu^2}{4\nu t}} d\mu. \text{(A.7)}$$

For $\Omega = 0$ the above expression reduces to the heat kernel $G_{\nu t}(x) = \frac{1}{(4\pi\nu t)^{3/2}} e^{-\frac{|x|^2}{4\nu t}}$. In fact, since $J_0\left(\frac{\Omega t|x'|}{|x|}\sqrt{1-\mu^2}\right)|_{\Omega=0} = 1$, $\cos\left(\frac{\Omega t x_3}{|x|}\mu\right)|_{\Omega=0} = 1$ and

$$\int_{-1}^{1}\left(1 - \frac{|x|^2\mu^2}{2\nu t}\right) e^{-\frac{|x|^2\mu^2}{4\nu t}} d\mu = 2e^{-\frac{|x|^2}{4\nu t}},$$

we get

$$2\pi \frac{\sqrt{\pi}}{4(\sqrt{\nu t})^3} \int_{-1}^{1}\left(1 - \frac{|x|^2\nu^2}{2\nu t}\right) e^{-\frac{|x|^2\mu^2}{4\nu t}} d\mu$$
$$= 2\pi \frac{\sqrt{\pi}}{4(\sqrt{\nu t})^3} 2(4\pi\nu t)^{3/2} \frac{1}{(4\pi\nu t)^{3/2}} e^{-\frac{|x|^2}{4\nu t}} = (2\pi)^{\frac{3}{2}\cdot 2} G_{\nu t}(x).$$

Hence the kernel is given by $(2\pi)^{-3/2} F^{-1}(e^{-\nu|\xi|^2 t}\cos(\frac{\xi_3}{|\xi|}\Omega t)) = G_{\nu t}(x)$ if $\Omega = 0$.

Let Ω and t be fixed. The asymptotics of the integral kernel in $|x|$ can be analyzed using (A.7). Clearly, it is bounded for $|x| \to 0$. Now we deduce the behaviour for large $|x|$. The main obstacle to a rapid decay of the kernel for large $|x|$ is that the term $e^{-\frac{|x|^2\mu^2}{4\nu t}}$ appears in combination with $|x|^2\mu^2$ and $e^{-\frac{|x|^2\mu^2}{4\nu t}}|_{\mu=0} = 1$. The main contribution to the kernel asymptotics for large $|x|$ is given in the integral (A.7) by a small interval containing $\mu = 0$. If we expand the expression $J_0\left(\frac{\Omega t|x'|}{|x|}\sqrt{1-\mu^2}\right)\cos\left(\frac{\Omega t x_3}{|x|}\mu\right)$ under integral in powers of μ (valid uniformly in $|x|$ since $\frac{|x_3|}{|x|}, \frac{|x'|}{|x|} \le 1$), then first we recover the term (heat kernel) $\times$ $J_0(\frac{\Omega t|x'|}{|x|})$ which clearly rapidly decays as $|x| \to +\infty$. Since the function under integral is even in μ, the next term will be of the form (function independent of μ) $\times$ $\mu^2(1-\frac{|x|^2\mu^2}{2\nu t})e^{-\frac{|x|^2\mu^2}{4\nu t}}$. Its asymptotic behaviour for large $|x|$ is given by the integral:

$$\int_{-1}^{1}\mu^2(1-\frac{|x|^2\mu^2}{2\nu t})e^{-\frac{|x|^2\mu^2}{4\nu t}}d\mu = \frac{1}{|x|^3}\int_{-|x|}^{|x|}\eta^2(1-\frac{\eta^2}{2\nu t})e^{-\frac{\eta^2}{4\nu t}}d\eta \sim \frac{C(\nu t)}{|x|^3} \text{ for large } |x|.$$

Therefore, the integral kernel behaves as $\frac{1}{|x|^3}$ for large $|x|$. In particular, the integral kernel does not belong to $L^1(\mathbb{R}^3)$. The corresponding integral operator cannot be viewed as a bounded operator in $L^\infty(\mathbb{R}^3)$ since a characteristic function of the outside of a large ball is always mapped to ∞ by this operator.

The above analysis and similar considerations for $F^{-1}\left(e^{-\nu|\xi|^2 t}\sin(\frac{\xi_3}{|\xi|}\Omega t)\frac{\xi_j}{|\xi|}\right)$ show that

$$F^{-1}\left(e^{-\nu|\xi|^2 t}\cos(\frac{\xi_3}{|\xi|}\Omega t)\right), F^{-1}\left(e^{-\nu|\xi|^2 t}\sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}(\xi)\right) \in L^q(\mathbb{R}^3), 1 < q < +\infty.$$

It is clear without any calculations that $F^{-1}\left(e^{-\nu|\xi|^2 t}\sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}(\xi)\right)$ does not belong to $L^1(\mathbb{R}^3)$ since $e^{-\nu|\xi|^2 t}\sin(\frac{\xi_3}{|\xi|}\Omega t)\frac{\xi_j}{|\xi|}$ $(j = 1, 2)$ are discontinuous at $\xi = 0$.

B Appendix: Boundedness of the operator $E(-\Omega t)$ in the homogeneous Besov space $\dot{B}^0_{\infty,1}$ uniformly in Ω and t

In this section we introduce the homogeneous Besov spaces $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$ and show boundedness of the operator $E(-\Omega t)$ uniformly in the Coriolis parameter Ω and time t. The boundedness in the Besov space is indispensable to estimate linear term in L^∞ to obtain local existence and uniqueness theorems. The dependence of the operator $E(-\Omega t)$ on the parameter Ω appears in the form $\exp(\omega R_3)$ with $\omega \in \mathbb{R}$. Except Lemma B.6 all statements in this section hold for general dimension $n = 1, 2, 3, \ldots$.

Before introducing the homogeneous Besov spaces, we prepare some notations. By $\mathcal{S}$ we denote the class of rapidly decreasing functions. The dual of $\mathcal{S}$, the space of tempered distributions is denoted by $\mathcal{S}'$. By $\mathcal{H}^1$ we denote the Hardy space. It is well known that the dual space of the Hardy space $\mathcal{H}^1$ is BMO, the space of functions of bounded mean oscillations. Let $\{\phi_j\}_{j=-\infty}^{\infty}$ be the Littlewood-Paley dyadic decomposition satisfying

$$\widehat{\phi}_j(\xi) = \widehat{\phi}_0(2^{-j}\xi) \in C_c^\infty(\mathbb{R}^n), \quad \mathrm{supp}\widehat{\phi}_0 \subset \{1/2 < |\xi| < 2\}, \quad \sum_{j=-\infty}^{\infty} \widehat{\phi}_j(\xi) = 1 \quad (\xi \neq 0). \tag{B.1}$$

Definition B.1. (See, e.g. [5] page 146)
The homogeneous Besov space $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$ is defined by

$$\dot{B}^s_{p,q} \equiv \left\{ f \in \mathcal{Z}'; ||f; \dot{B}^s_{p,q}|| < \infty \right\}$$

for $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, where

$$||f; \dot{B}^s_{p,q}|| \equiv \begin{cases} \left[\sum_{j=-\infty}^{\infty} 2^{jsq} ||\phi_j * f; L^p||^q \right]^{1/q} & \text{if} \quad q < \infty, \\ \sup_{-\infty \le j \le \infty} 2^{js} ||\phi_j * f; L^p|| & \text{if} \quad q = \infty. \end{cases}$$

Here $\mathcal{Z}'$ is the topological dual space of the space $\mathcal{Z}$, which is defined by $\mathcal{Z} \equiv \big\{ f \in \mathcal{S};\ D^\alpha \hat{f}(0) = 0$ for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \big\}$.

The above definition yields that all polynomials vanish in $\dot{B}^s_{p,q}$, however, it is well known that

$$\dot{B}^s_{p,q} \cong \{ f \in \mathcal{S}';\ ||f; \dot{B}^s_{p,q}|| < \infty \text{ and } f = \sum_{j=\infty}^{\infty} \phi_j * f \text{ in } \mathcal{S}' \} \tag{B.2}$$

if

$$s < n/p \quad \text{or} \quad (s = n/p \text{ and } q = 1). \tag{B.3}$$

Since indices of our target space $\dot{B}^0_{\infty,1}(\mathbb{R}^n)$ satisfy (B.3), the space $\dot{B}^s_{p,q}$ can be regarded as (B.2). For the details and examples one can consult e.g. [24],[25],[28].

The key lemma of this section is as follows.

Lemma B.1. (Boundedness of convolution-type operator)
For $h \in \mathcal{S}'$ let $T = h*$ be a convolution-type operator defined on $\mathcal{S}$. Assume that T is regarded as a bounded operator $\mathcal{H}^1 \to \mathcal{H}^1$. Then, the operator T is bounded from $\dot{B}^0_{\infty,1}$ to itself. Its norm $||T||_{\dot{B}^0_{\infty,1} \to \dot{B}^0_{\infty,1}}$ is bounded by $C||T||_{\mathcal{H}^1 \to \mathcal{H}^1}$ with C depending only on n.

Proof: By the definition of the Besov norm we have

$$||Tf; \dot{B}^0_{\infty,1}|| = \sum_{j=-\infty}^{\infty} ||\phi_j * Tf||_\infty = \sum_{j=-\infty}^{\infty} ||\phi_j * h * f||_\infty.$$

Since only three terms in the family $\{\mathrm{supp}\widehat{\phi_j}\}_j$ are nonzero for any fixed point $\xi \in \mathbb{R}^n$, we derive

$$\begin{aligned}||Tf; \dot{B}^0_{\infty,1}|| &= \sum_{j,k\in\mathbb{Z},\ |j-k|\le 2} ||\phi_j * h * \phi_k * f||_\infty \\ &\le \sum_{j,k\in\mathbb{Z},\ |j-k|\le 2} ||\phi_j * h||_1 ||\phi_k * f||_\infty.\end{aligned}$$

The fact $||\cdot||_{L^1} \le ||\cdot||_{\mathcal{H}^1}$ and the assumption yield

$$\begin{aligned}||Tf; \dot{B}^0_{\infty,1}|| &\le \sum_{j,k\in\mathbb{Z},\ |j-k|\le 2} ||\phi_j * h||_{\mathcal{H}^1} ||\phi_k * f||_\infty \\ &\le C \sum_{j,k\in\mathbb{Z},\ |j-k|\le 2} ||\phi_j||_{\mathcal{H}^1} ||\phi_k * f||_\infty.\end{aligned}$$

Here, $||\phi_j||_{\mathcal{H}^1} = ||\phi_j||_{L^1} + \sum_{k=1}^n ||iR_k\phi_j||_{L^1}$ is a constant independent of j since $||\phi_j||_{\mathcal{H}^1} = ||\phi_0||_{\mathcal{H}^1}$. Indeed, we obtain that

$$\begin{aligned}||\phi_j||_1 &= \int_{\mathbb{R}^n} |\phi_j(x)|dx = \int_{\mathbb{R}^n} |(F^{-1}(F\phi_j)(\xi))(x)|dx \\ &= \int_{\mathbb{R}^n} |(F^{-1}(F\phi_0)(2^{-j}\xi))(x)|dx = 2^{jn}\int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} e^{i2^jx\xi}(F\phi_0(\xi))(x)d\xi|dx \\ &= 2^{jn}\int_{\mathbb{R}^n} |\phi_0(2^jx)|dx = \int_{\mathbb{R}^n} |\phi_0(x)|dx = ||\phi_0||_1\end{aligned}$$

and similarly

$$\begin{aligned}||iR_k\phi_j||_1 &= \int_{\mathbb{R}^n} |iR_k\phi_j(x)|dx = \int_{\mathbb{R}^n} |(F^{-1}\frac{i\xi_k}{|\xi|}(F\phi_j)(\xi))(x)|dx \\ &= \int_{\mathbb{R}^n} |(F^{-1}\frac{i\xi_k}{|\xi|}(F\phi_0)(2^{-j}\xi))(x)|dx = 2^{jn}\int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} e^{i2^jx\xi}\frac{i\xi_k}{|\xi|}(F\phi_0(\xi))(x)d\xi|dx \\ &= 2^{jn}\int_{\mathbb{R}^n} |iR_k\phi_0(2^jx)|dx = \int_{\mathbb{R}^n} |iR_k\phi_0(x)|dx = ||iR_k\phi_0||_1\end{aligned}$$

for all k with $1 \le k \le n$. Thus we conclude

$$||Tf; \dot{B}^0_{\infty,1}|| \le C||\phi_0||_{\mathcal{H}^1} \sum_{j,k\in\mathbb{Z},\ |j-k|\le 2} ||\phi_k * f||_\infty \le 3C||\phi_0||_{\mathcal{H}^1} \sum_{j=-\infty}^{\infty} ||\phi_j * f||_\infty = C||f; \dot{B}^0_{\infty,1}||.$$

This establishes the result.

Remark B.1. The proof can be easily modified to obtain a bound for $||T||_{\dot{B}^0_{\infty,q}\to\dot{B}^0_{\infty,q}}$ for arbitrary $q \in [1,\infty]$ including $q = \infty$.

Lemma B.2. (Theorem 7.30 in [11], [17], Mikhlin-type theorem in the Hardy space and BMO)
Suppose $k > n/2$. Let $m(\xi) \in C^k(\mathbb{R}^n \setminus \{0\})$ satisfy

$$|D^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \quad (\xi \ne 0) \qquad \text{for all } \ |\alpha| = \alpha_1 + \cdots + \alpha_n \le k. \tag{B.4}$$

Then the operator defined by $T_m = F^{-1} m F$ is bounded from $\mathcal{H}^1$ to itself and from BMO to itself.

Lemma B.3. (Boundedness of Resolvent operator)
Consider the operator $\lambda - iR_j : \dot{B}^0_{\infty,1} \to \dot{B}^0_{\infty,1}$ for $j = 1, 2, 3$. Then, Spec$(iR_j) \subset \mathbb{R}$. Here Spec$(K)$ denotes the spectrum set of an operator K.

Proof: Assume $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since it is easy to see that $m(\xi) = 1/(\lambda + \frac{\xi_j}{|\xi|})$ satisfies (B.4), Lemma B.2 guarantees that $(\lambda - iR_j)^{-1}$ exists and bounded from $\mathcal{H}^1$ to itself. So, it follows from Lemma B.1 that $(\lambda - iR_j)^{-1}$ exists and bounded from $\dot{B}^0_{\infty,1}$ to itself. Thus $\lambda \in \mathbb{C} \setminus \mathbb{R}$ belong to the resolvent set.

Finally we will show uniform boundedness for $\exp(\omega R_j)$ independent of $\omega \in \mathbb{R}$.

Lemma B.4. (Uniform boundedness of the operator $\exp(\omega R_j)$-Besov-case)
For $f \in \dot{B}^0_{\infty,1}$ and $\omega \in \mathbb{R}$ we have

$$|| \exp(\omega R_j) f; \dot{B}^0_{\infty,1}|| \le ||f; \dot{B}^0_{\infty,1}||.$$

Proof: By spectrum mapping theorem we have

$$\begin{aligned} || \exp(\omega R_j); \dot{B}^0_{\infty,1} \to \dot{B}^0_{\infty,1}|| &= \sup\{|z|; z \in \text{Spec}(\exp(\omega R_j))\} \\ &= \sup\{|z|; z \in \exp(-i\omega \ \text{Spec}(iR_j))\} \\ &= \sup\{|\exp(-i\omega z)|; z \in \text{Spec}(iR_j)\}. \end{aligned}$$

It follows from Lemma B.3 that

$$|| \exp(\omega R_j); \dot{B}^0_{\infty,1} \to \dot{B}^0_{\infty,1}|| \ \le \ \sup\{|\exp(-i\omega z)|; z \in \mathbb{R}\}.$$

Since $|\exp(-i\omega z)| = 1$ when $z \in \mathbb{R}$, we obtain the desired result.

Lemma B.5. (Uniform boundedness of the operator $\exp(\omega R_j)$-BMO-case)
Let $\lambda - iR_j$ be the operator in BMO. Then Spec$(iR_j) \subset \mathbb{R}$ and $|| \exp(\omega R_j) f; BMO|| \le ||f; BMO||$.

The proof parallels that of Lemma B.3 and Lemma B.4; we need not use Lemma B.1.

Lemma B.6. (Persistency of vertical averaging property)
Assume that $n = 3$. If $\mathbf{U} \in \mathcal{B}^0$, then $E(-\Omega t)\mathbf{U} \in \mathcal{B}^0$, where $E(-\Omega t) = \exp(-t\Omega \mathbf{S})$.

Proof: It suffices to prove that $\overline{R_j f} = 0$ if $\overline{f} = 0$ for $f \in \dot{B}^0_{\infty,1}$, where R_j is a scalar Riesz operator and f is a scalar function. We approximate f by a finite sum $\sum \phi_j * f$.

We set $f_l = \sum_{|k| \le l} \phi_k * f$ for $l > 0$. By a similar argument to prove that $\overline{\exp(\nu t \Delta)\mathbf{U}} = 0$ for $\overline{\mathbf{U}} = 0$ in the proof of Proposition 3.2 we obtain that $\overline{R_j \phi_k * f} = 0$ if $\overline{f} = 0$, since $R_j \phi_k$ is a rapidly decreasing function. This implies that $\overline{R_j f_l} = 0$. Since $f_l \to f$ in $\dot{B}^0_{\infty,1}$ as $l \to \infty$, the Riesz operator R_j is bounded and the subspace of the zero vertical average is closed in $\dot{B}^0_{\infty,1}$, we conclude that $\overline{R_j f} = 0$.

Remark B.2. The fact that Mikhlin's condition (B.4) implies that a bound for the operator $T_m = F^{-1} m F$ in $\dot{B}^0_{\infty,1}$ can be proved directly without using Lemma B.1; see e.g. Amann [1]. However, Lemma B.1 is not included in [1] and seems to be new.

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