BINARY MARKET MODELS WITH MEMORY

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ABSTRACT. We construct a binary market model with memory that approximates a continuous-time market model driven by a Gaussian process equivalent to Brownian motion. We give a sufficient conditions for the binary market to be arbitrage-free. In a case when arbitrage opportunities exist, we present the rate at which the arbitrage probability tends to zero as the number of periods goes to infinity.

1. INTRODUCTION

Let $T \in (0, \infty)$. We consider the stock price process $(S_t)_{0 \le t \le T}$ that is governed by the stochastic differential equation

$$dS_t = S_t(bdt + \sigma dY_t) \qquad (0 \le t \le T), \tag{1.1}$$

where σ and the initial value S_0 are positive constants, and $b \in \mathbf{R}$. In the classical Black-Scholes model, Brownian motion is used as the driving noise process Y, and the resulting price process S becomes Markovian. In [1, 2], the following Gaussian process $(Y_t)_{0 \le t \le T}$ with stationary increments is used instead as the driving noise process Y in (1.1):

$$Y_t = B_t - \int_0^t \left\{ \int_{-\infty}^s pe^{-(q+p)(s-u)} dB_u \right\} ds \qquad (0 \le t \le T), \tag{1.2}$$

where p and q are real constants such that

$$0 < q < \infty, \quad -q < p < \infty,$$

and $(B_t)_{t \in \mathbf{R}}$ is a one-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ satisfying $B_0 = 0$. The parameters p and q describe the memory of Y, and the resulting stock price process S becomes non-Markovian. An empirical study on S&P 500 data in [3] shows that the model captures very well the memory effect when the market is stable.

It should be noticed that (1.2) is not a semimartingale representation of Y with respect to the P-augmentation $(\mathcal{F}_t)_{0 \le t \le T}$ of the filtration generated by $(Y_t)_{0 \le t \le T}$ since (B_t) is not $(\mathcal{F}_t)$-adapted. However, by innovation theory as described in Liptser and Shiryayev [11], we can show that Y is actually an $(\mathcal{F}_t)$-semimartingale ([1, Theorem 3.1]). In fact, using the prediction theory for Y which is developed in [2], we see ([9, Theorem 2.1]) that there exists a one-dimensional Brownian motion $(W_t)_{0 \le t \le T}$, called the *innovation process*, satisfying

$$\sigma(W_s : 0 \le s \le t) = \sigma(Y_s : 0 \le s \le t) \qquad (0 \le t \le T),$$

1991 *Mathematics Subject Classification.* Primary 91B28; secondary 60F17.
Key words and phrases. Financial market with memory, binary market, arbitrage.
This work is partially supported by the Australian Research Council grant DP0345577.

and

$$Y_t = W_t - \int_0^t \left\{ \int_0^s l(s,u) dW_u \right\} ds \qquad (t \in [0,T]), \tag{1.3}$$

where $l(t,s)$ is a Volterra kernel given explicitly by

$$l(t,s) = pe^{-(p+q)(t-s)} \left\{ 1 - \frac{2pq}{(2q+p)^2 e^{2qs} - p^2} \right\} \quad (0 \le s \le t \le T). \tag{1.4}$$

Thus the process Y has the virtue that it possesses the property of a stationary increments process with memory and the simple semimartingale representation (1.3) with (1.4) simultaneously. We know of no other process with this kind of properties. The two properties of Y become a great advantage, for example, in its parameter estimation (see [9, Section 5]).

Several authors use fractional Brownian motion as the driving noise process (see, e.g., Comte and Renault [5], Rogers [12], and Willinger et al. [15]). However this approach is not entirely satisfactory since fractional Brownian motion is not a semimartingale (Lin [10] and Rogers [12]), whence there exists no equivalent martingale measure in the corresponding market. On the other hand, the market defined by (1.1) with (1.2) or (1.3) and (1.4) is arbitrage-free and complete since the process Y becomes a Brownian motion under a suitable probability measure (see [1, Section 3]). Moreover, for this model, we can obtain explicit results such as the solution to the expected logarithmic utility maximization from terminal wealth (see [2]).

As is well known, binary approximation of the Black-Scholes model plays a very important role for the model in many ways. Sottinen [13] constructed a binary market model that approximates the market driven by fractional Brownian motion, and investigated the arbitrage opportunities in the binary model.

In this paper, we construct a binary market model with memory that approximates the continuous-time market model driven by Y in (1.3). However, rather than considering the special kernel $l(t,s)$ in (1.4), we take a general bounded measurable Volterra kernel $l(t,s)$. Since $l(t,s)$ given by (1.4) is bounded, the results thus obtained apply to the special case (1.4). We remark that any centered Gaussian process $Y = (Y_t)_{0 \le t \le T}$ that is equivalent to a Brownian motion has a canonical representation of the form (1.3) with $l(t,s)$ satisfying square integrability (see Hida and Hitsuda [8, Chapter VI]). Thus, in this paper, we consider a subclass consisting of Y for which $l(t,s)$ is bounded. As in [13], the key feature to the construction of the approximating binary market is to prove a Donsker-type theorem for the process Y (Theorem 2.1).

Unlike the market driven by fractional Brownian motion, the market driven by Y in (1.3) is arbitrage-free (see, e.g., the proof of [1, Theorem 3.3]). However, the approximating binary market may admit arbitrage opportunities. We consider conditions for their existence or non-existence. We also study the rate at which the aribtrage probability tends to zero as the number of periods goes to infinity.

This paper is organized as follows. In Section 2, we prove a Donsker-type theorem for the driving process Y in (1.3) with bounded kernel $l(t,s)$. In Section 3, we consider a discrete-time approximation of the stock price process S in (1.1). As a special case, we obtain the desired approximating binary model. In Section 4, we study arbitrage opportunities in the binary model.

2. A Donsker-type theorem

Let $T \in (0, \infty)$. In what follows, we write $C = C_T$ for positive constants, depending on T, which may not be necessarily equal to each other. Let n be a positive integer. In Sections 2 and 3, we write

$$\sum_{s \le t} X_s = \sum\nolimits_{i=1}^{\lfloor nt \rfloor} X_{\frac{i}{n}}, \qquad \prod_{s \le t} X_s = \prod\nolimits_{i=1}^{\lfloor nt \rfloor} X_{\frac{i}{n}}.$$

Let $l(t, s)$ be a bounded measurable function on $[0, T] \times [0, T]$ that vanishes whenever $s > t$. Let $W = (W_t)_{0 \le t \le T}$ be a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. We define the process $Y = (Y_t)_{0 \le t \le T}$ by (1.3).

We put, for $t, u \in [0, T]$,

$$z(t, u) := \int_u^t l(s, u) ds, \qquad y(t, u) := 1 - z(t, u).$$

Then both $z(t, u)$ and $y(t, u)$ are bounded and continuous on $[0, T] \times [0, T]$, and it holds that $Y_t = \int_0^t y(t, u) dW_u$ for $0 \le t \le T$. Let C be a positive constant satisfying, for $(t_1, u), (t_2, u) \in [0, T] \times [0, T]$,

$$|z(t_1, u) - z(t_2, u)| = |y(t_1, u) - y(t_2, u)| \le C|t_1 - t_2|. \tag{2.1}$$

Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $E[\xi_1] = 0$ and $E[(\xi_1)^2] = 1$. We also assume that

$$E[(\xi_1)^4] < \infty. \tag{2.2}$$

We define the process $W^{(n)} = (W_t^{(n)})_{0 \le t \le T}$ by

$$W_t^{(n)} := \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i \qquad (0 \le t \le T),$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. The process $W^{(n)}$ converges weakly to W in the Skorohod space by Donsker's theorem (see, e.g., Billingsley [4, Theorem 16.1]). We define the process $Y^{(n)} = (Y_t^{(n)})_{0 \le t \le T}$ by

$$Y_t^{(n)} := \int_0^t y(\tfrac{\lfloor nt \rfloor}{n}, s) dW_s^{(n)} \qquad (0 \le t \le T).$$

Then it follows that

$$Y_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{i}{n}) \xi_i \qquad (0 \le t \le T).$$

Here is the Donsker-type theorem for Y.

Theorem 2.1. *The process $Y^{(n)}$ converges weakly to Y as $n \to \infty$.*

Proof. We first show that the finite-dimensional distributions of $Y^{(n)}$ converge to those of Y as $n \to \infty$. Thus, for $a_1, \dots, a_d \in \mathbf{R}$ and $t_1, \dots, t_d \in [0, T]$, we show that $X^{(n)}$ converges to a normal distribution with variance $\mathrm{Var}(X)$, where $X^{(n)} :=$

$\sum_{k=1}^{d} a_k Y_{t_k}^{(n)}$ and $X := \sum_{k=1}^{d} a_k Y_{t_k}$. We have

$$\begin{aligned}\mathrm{Var}(X^{(n)}) &= \sum_{k,l=1}^{d} a_k a_l \frac{1}{n} \sum_{i=1}^{\lfloor n(t_k \wedge t_l)\rfloor} y(\tfrac{\lfloor nt_k\rfloor}{n}, \tfrac{i}{n}) y(\tfrac{\lfloor nt_l\rfloor}{n}, \tfrac{i}{n}) \\ &= \sum_{k,l=1}^{d} a_k a_l \int_0^{\frac{\lfloor n(t_k\wedge t_l)\rfloor}{n}} y(\tfrac{\lfloor nt_k\rfloor}{n}, \tfrac{\lfloor ns\rfloor+1}{n}) y(\tfrac{\lfloor nt_l\rfloor}{n}, \tfrac{\lfloor ns\rfloor+1}{n}) ds,\end{aligned}$$

where $t \wedge s := \min(t, s)$. The function $(t_1, t_2, u) \mapsto y(t_1, u)y(t_2, u)$ is continuous, whence uniformly continuous, on the compact set $[0, T]^3$. From this and the fact that $0 \le t - (\lfloor nt\rfloor/n) < 1/n$, we see that

$$\lim_{n\to\infty} \mathrm{Var}(X^{(n)}) = \sum_{k,l=1}^{d} a_k a_l \int_0^{t_k\wedge t_l} y(t_k, s)y(t_l, s)ds = \mathrm{Var}(X). \tag{2.3}$$

We may assume $\mathrm{Var}(X) > 0$. For, otherwise, (2.3) implies that $X^{(n)}$ converges to $X = 0$ in law. We put $b_i^{(n)} := \sum_{k=1}^{d} a_k y(\frac{\lfloor nt_k\rfloor}{n}, \frac{i}{n})$ and $X_i^{(n)} := n^{-1/2} b_i^{(n)} \xi_i$ for $n, i = 1, 2, \dots$. Then we have $X^{(n)} = \sum_{i=1}^{\lfloor nT\rfloor} X_i^{(n)}$ for $n = 1, 2, \dots$. We need to show the following Lindeberg's condition: for every $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{i=1}^{\lfloor nT\rfloor} E\left[(X_i^{(n)})^2 \mathbf{1}_{\{|X_i^{(n)}|>\epsilon\sigma^{(n)}\}}\right] = 0, \tag{2.4}$$

where $\sigma^{(n)} := \sqrt{\mathrm{Var}(X^{(n)})}$. Choose a positive constant M satisfying $|b_i^{(n)}| \le M$ for $n, i = 1, 2, \dots$. Then since $|X_i^{(n)}| \le Mn^{-1/2}|\xi_i|$, we have

$$\begin{aligned}&\sum_{i=1}^{\lfloor nT\rfloor} E\left[(X_i^{(n)})^2 \mathbf{1}_{\{|X_i^{(n)}|>\epsilon\sigma^{(n)}\}}\right] \le \sum_{i=1}^{\lfloor nT\rfloor} E\left[(Mn^{-1/2}\xi_i)^2 \mathbf{1}_{\{|Mn^{-1/2}\xi_i|>\epsilon\sigma^{(n)}\}}\right] \\ &= \sum_{i=1}^{\lfloor nT\rfloor} M^2 n^{-1} E\left[(\xi_1)^2 \mathbf{1}_{\{|\xi_1|\ge M^{-1}\sigma^{(n)}\sqrt{n}\}}\right] \le M^2 T E\left[(\xi_1)^2 \mathbf{1}_{\{|\xi_1|\ge M^{-1}\sigma^{(n)}\sqrt{n}\}}\right].\end{aligned}$$

We obtain (2.4) from this. By (2.4) and (2.3), we can apply the central limit theorem (cf. [4, Theorem 7.2]), so that $X^{(n)}$ converges to X in law, as desired.

Next we show that, for $0 \le t_1 \le t \le t_2 \le T$ and $n = 1, 2, \dots$,

$$E\left[|Y_t^{(n)} - Y_{t_1}^{(n)}|^2 |Y_{t_2}^{(n)} - Y_t^{(n)}|^2\right] \le C|t_2 - t_1|^2. \tag{2.5}$$

The theorem follows from this and [4, Theorem 15.6]. However, if $t_2 - t_1 < 1/n$, then either t_1 and t or t and t_2 lie in the same subinterval $[\frac{m}{n}, \frac{m+1}{n})$ for some m, whence the left hand side of (2.5) is zero. Therefore we may assume that $t_2 - t_1 \ge 1/n$.

We show that

$$E\left[|Y_t^{(n)} - Y_s^{(n)}|^4\right] \le C|t - s|^2 \tag{2.6}$$

for t, s and n satisfying

$$0 \le s < t \le T, \qquad t - s \ge \tfrac{1}{n}. \tag{2.7}$$

This implies (2.5) under the condition $t_2 - t_1 \geq 1/n$ since

$$E\left[|Y_t^{(n)} - Y_{t_1}^{(n)}|^2 |Y_{t_2}^{(n)} - Y_t^{(n)}|^2\right] \leq E\left[|Y_t^{(n)} - Y_{t_1}^{(n)}|^4\right]^{1/2} E\left[|Y_{t_2}^{(n)} - Y_t^{(n)}|^4\right]^{1/2}$$
$$\leq C|t - t_1||t_2 - t| \leq C|t_2 - t_1|^2.$$

For distinct i, j, k and l, we have

$$E[(\xi_i)^3 \xi_j] = E[(\xi_i)^2 \xi_j \xi_k] = E[\xi_i \xi_j \xi_k \xi_l] = 0.$$

Hence, for t, s and n satisfying (2.7), $E[|Y_t^{(n)} - Y_s^{(n)}|^4]$ is equal to

$$\begin{aligned}
&n^{-2} E\left[\left\{\sum\nolimits_{i=1}^{\lfloor nt \rfloor} (y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{i}{n}) - y(\tfrac{\lfloor ns \rfloor}{n}, \tfrac{i}{n}))\xi_i\right\}^4\right] \\
&= E[(\xi_1)^4] n^{-2} \sum_{i=1}^{\lfloor nt \rfloor} \{y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{i}{n}) - y(\tfrac{\lfloor ns \rfloor}{n}, \tfrac{i}{n})\}^4 \\
&+ \frac{6}{n^2} E[(\xi_1)^2]^2 \sum_{1 \leq i < j \leq \lfloor nt \rfloor} \{y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{i}{n}) - y(\tfrac{\lfloor ns \rfloor}{n}, \tfrac{i}{n})\}^2 \{y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{j}{n}) - y(\tfrac{\lfloor ns \rfloor}{n}, \tfrac{j}{n})\}^2 \\
&= (I_1 + I_2) E[(\xi_1)^4] + 6(J_1 + J_2 + J_3) E[(\xi_1)^2]^2,
\end{aligned}$$

where

$$I_1 := n^{-2} \sum_{i=1}^{\lfloor ns \rfloor} \{y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{i}{n}) - y(\tfrac{\lfloor ns \rfloor}{n}, \tfrac{i}{n})\}^4, \qquad I_2 := n^{-2} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{i}{n})^4$$

and

$$\begin{aligned}
J_1 &:= n^{-2} \sum_{(i,j) \in \Lambda_1} \{y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{i}{n}) - y(\tfrac{\lfloor ns \rfloor}{n}, \tfrac{i}{n})\}^2 \{y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{j}{n}) - y(\tfrac{\lfloor ns \rfloor}{n}, \tfrac{j}{n})\}^2, \\
J_2 &:= n^{-2} \sum_{(i,j) \in \Lambda_2} \{y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{i}{n}) - y(\tfrac{\lfloor ns \rfloor}{n}, \tfrac{i}{n})\}^2 y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{j}{n})^2, \\
J_3 &:= n^{-2} \sum_{(i,j) \in \Lambda_3} y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{i}{n})^2 y(\tfrac{\lfloor nt \rfloor}{n}, \tfrac{j}{n})^2
\end{aligned}$$

with

$$\begin{aligned}
\Lambda_1 &:= \{(i,j) : 1 \leq i < j \leq \lfloor ns \rfloor\}, \\
\Lambda_2 &:= \{(i,j) : 1 \leq i \leq \lfloor ns \rfloor,\ \lfloor ns \rfloor < j \leq \lfloor nt \rfloor\}, \\
\Lambda_2 &:= \{(i,j) : \lfloor ns \rfloor < i < j \leq \lfloor nt \rfloor\}.
\end{aligned}$$

By (2.7), we have

$$\lfloor nt \rfloor - \lfloor ns \rfloor \leq nt - ns + 1 = n(t - s + \tfrac{1}{n}) \leq 2n(t - s),$$

so that

$$\#\Lambda_1 \leq Cn^2, \quad \#\Lambda_2 \leq Cn^2(t - s), \quad \#\Lambda_3 \leq Cn^2(t - s)^2.$$

Therefore, using (2.1), we obtain, for t, s and n satisfying (2.7),

$$\begin{aligned}
|I_1| &\le Cn^{-2}\cdot n\cdot(t-s)^4 = Cn^{-1}(t-s)^4 \le C(t-s)^5 \le C(t-s)^2,\\
|I_2| &\le Cn^{-2}\cdot n(t-s) = Cn^{-1}(t-s) \le C(t-s)^2,\\
|J_1| &\le Cn^{-2}\cdot n^2\cdot(t-s)^4 = C(t-s)^4 \le C(t-s)^2,\\
|J_2| &\le Cn^{-2}\cdot n^2(t-s)\cdot(t-s)^2 = C(t-s)^3 \le C(t-s)^2,\\
|J_3| &\le Cn^{-2}\cdot n^2(t-s)^2 = C(t-s)^2.
\end{aligned}$$

Thus (2.6) follows. □

Denote by ΔX and $[X]$ the jump and quadratic variation processes of a process X, respectively, i.e.,

$$\Delta X_t := X_t - \lim_{s\uparrow t} X_s, \qquad [X]_t := \sum_{s\le t} (\Delta X_s)^2 .$$

Theorem 2.2. *The process $\Delta Y^{(n)}$ converges to zero in probability, while $[Y^{(n)}]$ converges to the deterministic process $(t)_{0\le t\le T}$ in probability.*

Proof. From (2.6) with (2.7), we have

$$E\left[(\Delta Y_t^{(n)})^4\right] \le E\left[(Y_t^{(n)} - Y_{t-\frac{1}{n}}^{(n)})^4\right] \le Cn^{-2},$$

so that, as $n\to\infty$,

$$E\left[\sup_{0\le t\le T}(\Delta Y_t^{(n)})^4\right] \le E\left[\sum_{t\le T}(\Delta Y_t^{(n)})^4\right] = \sum_{t\le T} E\left[(\Delta Y_t^{(n)})^4\right] \le C\frac{nT}{n^2} \to 0.$$

Thus $\Delta Y^{(n)}$ converges to zero in probability.

We put $Z_t^{(n)} := \int_0^t z(\frac{\lfloor nt\rfloor}{n}, s)dW_s^{(n)}$ for $0\le t\le T$. Then we have $Y_t^{(n)} = W_t^{(n)} - Z_t^{(n)}$, whence

$$[Y^{(n)}]_t = [W^{(n)}]_t - 2\sum_{s\le t}(\Delta W_s^{(n)})(\Delta Z_s^{(n)}) + [Z^{(n)}]_t.$$

Since $z(u,u)=0$, we have

$$Z_t^{(n)} - Z_{t-\frac{1}{n}}^{(n)} = \frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor-1}\{z(\tfrac{\lfloor nt\rfloor}{n},\tfrac{i}{n}) - z(\tfrac{\lfloor nt\rfloor-1}{n},\tfrac{i}{n})\}\xi_i \quad (=0 \quad \text{if } \lfloor nt\rfloor = 1).$$

From this and (2.1), $E[(\Delta Z_t^{(n)})^2]$ is at most

$$E\left[(Z_t^{(n)} - Z_{t-\frac{1}{n}}^{(n)})^2\right] = \frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor-1}\{z(\tfrac{\lfloor nt\rfloor}{n},\tfrac{i}{n}) - z(\tfrac{\lfloor nt\rfloor-1}{n},\tfrac{i}{n})\}^2 \le \frac{nT}{n}\cdot\frac{C^2}{n^2} = \frac{C}{n^2}.$$

Since $[Z^{(n)}]_t$ is increasing, we see that

$$E\left[\sup_{0\le t\le T}[Z^{(n)}]_t\right] = E\left[[Z^{(n)}]_T\right] = \sum_{t\le T} E\left[(\Delta Z_t^{(n)})^2\right] \le nT\frac{C}{n^2} = \frac{C}{n}. \tag{2.8}$$

Thus $[Z^{(n)}]$ converges to zero in probability.

We have

$$[W^{(n)}]_t - t = \frac{\lfloor nt \rfloor}{n} - t + \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor}\{(\xi_i)^2 - 1\}.$$

Let $\epsilon > 0$. Then, from (2.2) and Kolmogorov's inequality (see, e.g, Williams [14, Section 14.6]), we see that

$$P\left(\sup_{0\le t\le T}\frac{1}{n}\left|\sum\nolimits_{i=1}^{\lfloor nt \rfloor}\{(\xi_i)^2 - 1\}\right| \ge \epsilon\right) = P\left(\sup_{0\le t\le T}\left|\sum\nolimits_{i=1}^{\lfloor nt \rfloor}\{(\xi_i)^2 - 1\}\right| \ge n\epsilon\right)$$
$$\le \frac{1}{\epsilon^2 n^2}\sum_{i=1}^{\lfloor nT \rfloor} E\left[(\xi_i^2-1)^2\right] \le \frac{nT}{\epsilon^2 n^2} E\left[(\xi_1^2-1)^2\right] \to 0 \quad (n\to\infty).$$

From this and the fact that $0 \le t - (\lfloor nt \rfloor/n) < 1/n$, we see that $[W^{(n)}]$ converges to the deterministic process (t) in probability.

By Schwarz's inequality, we have

$$\left|\sum\nolimits_{s\le t}(\Delta W_s^{(n)})(\Delta Z_s^{(n)})\right| \le [W^{(n)}]_t^{1/2}[Z^{(n)}]_t^{1/2} \le [W^{(n)}]_T^{1/2}[Z^{(n)}]_T^{1/2},$$

whence, by (2.8),

$$E\left[\sup_{0\le t\le T}\left|\sum\nolimits_{s\le t}(\Delta W_s^{(n)})(\Delta Z_s^{(n)})\right|\right] \le E\left[[W^{(n)}]_T^{1/2}[Z^{(n)}]_T^{1/2}\right]$$
$$\le E\left[[W^{(n)}]_T\right]^{1/2} E\left[[Z^{(n)}]_T\right]^{1/2} \le T^{1/2}\cdot(Cn^{-1})^{1/2} = Cn^{-1/2}.$$

Thus the process $(\sum_{s\le t}(\Delta W_s^{(n)})(\Delta Z_s^{(n)}))$ also converges to zero in probability. Combining, we see that $[Y^{(n)}]$ converges to (t) in probability. □

3. Approximating binary market

Let $T \in (0,\infty)$ and let Y be as defined in Section 2. We consider the stock price process S that is governed by the following more general stochastic differential equaltion than (1.1):

$$dS_t = S_t\{b(t)dt + \sigma dY_t\} \qquad (0 \le t \le T),$$

where σ and the initial value S_0 are positive constants, and $b(\cdot)$ is a deterministic continuous function on $[0,T]$. The solution S is given by

$$S_t := S_0 \exp\left\{\sigma Y_t + \int_0^t b(s)ds - \tfrac{1}{2}\sigma^2 t\right\} \qquad (0 \le t \le T).$$

For $n = 1, 2, \dots$, we consider the process $S^{(n)} = (S_t^{(n)})_{0\le t\le T}$ defined by

$$S_t^{(n)} := \prod_{s\le t}\left\{1 + \sigma\Delta Y_s^{(n)} + \frac{1}{n}b(\tfrac{\lfloor ns \rfloor}{n})\right\} \qquad (0 \le t \le T),$$

where $Y^{(n)}$ is as in Section 2. The aim of this section is to prove that $S^{(n)}$ converges weakly to the process S.

As in [13, (10) and (11)], we put

$$Y_t^{(1,n)} := \sum_{s\le t}\Delta Y_s^{(n)}\mathbf{1}_{\{|\Delta Y_s^{(n)}|<\frac{1}{2}\sigma^{-1}\}}, \quad Y_t^{(2,n)} := \sum_{s\le t}\Delta Y_s^{(n)}\mathbf{1}_{\{|\Delta Y_s^{(n)}|\ge\frac{1}{2}\sigma^{-1}\}}.$$

Then we have

$$Y_t^{(n)} = Y_t^{(1,n)} + Y_t^{(2,n)}, \tag{3.1}$$

$$[Y^{(1,n)}]_t = \sum_{s\le t} (\Delta Y_s^{(n)})^2 \mathbf{1}_{\{|\Delta Y_s^{(n)}|<\frac{1}{2}\sigma^{-1}\}}, \tag{3.2}$$

$$[Y^{(2,n)}]_t = \sum_{s\le t} (\Delta Y_s^{(n)})^2 \mathbf{1}_{\{|\Delta Y_s^{(n)}|\ge\frac{1}{2}\sigma^{-1}\}}, \tag{3.3}$$

$$[Y^{(n)}]_t = [Y^{(1,n)}]_t + [Y^{(2,n)}]_t. \tag{3.4}$$

Lemma 3.1. *The process $[Y^{(2,n)}]$ converges to zero in probability, whence $[Y^{(1,n)}]$ converges to the deterministic process (t) in probability. The process $Y^{(2,n)}$ converges to zero in probability, whence $Y^{(1,n)}$ converges weakly to Y.*

Proof. Let $\epsilon > 0$. Then, by (3.3), we have

$$P\left(\sup_{0\le t\le T} [Y^{(2,n)}]_t \ge \epsilon\right) \le P\left(\sup_{0\le t\le T} [Y^{(2,n)}]_t > 0\right) = P\left(\sup_{0\le t\le T} |\Delta Y_t^{(n)}| \ge \tfrac{1}{2}\sigma^{-1}\right).$$

Since the process $\Delta Y^{(n)}$ converges to zero in probability by Theorem 2.2, $[Y^{(2,n)}]$ converges to zero in probability. Therefore, by Theorem 2.2 and (3.4), $[Y^{(1,n)}]$ converges to zero in probability.

In the same way, since

$$P\left(\sup_{0\le t\le T} |Y_t^{(2,n)}| \ge \epsilon\right) \le P\left(\sup_{0\le t\le T} |\Delta Y_t^{(n)}| \ge \tfrac{1}{2}\sigma^{-1}\right),$$

it follows from Theorem 2.2 that $Y^{(2,n)}$ converges to zero in probability. Therefore, by Theorem 2.1, (3.1) and [4, Theorem 4.1], $Y^{(1,n)}$ converges weakly to Y. □

Theorem 3.2. *The process $S^{(n)}$ converges weakly to S.*

Proof. Write $S_t^{(n)} = S_t^{(1,n)} S_t^{(2,n)}$, where

$$S_t^{(1,n)} := \prod_{s\le t} \left\{1 + \sigma\Delta Y_s^{(1,n)} + \frac{1}{n} b(\tfrac{\lfloor ns\rfloor}{n})\right\}$$
$$S_t^{(2,n)} := \prod_{s\le t} \left\{1 + \sigma\Delta Y_s^{(2,n)}\right\},$$

and the processes $Y^{(i,n)}$ are as above. We claim the following: (i) $S^{(1,n)}$ *converges weakly to* S; (ii) $S^{(2,n)}$ *converges to one in probability.*

By [4, Problem 1, Page 28], the claim (ii) implies that $S^{(1,n)}(S^{(2,n)} - 1)$ converges to zero in probability. Since

$$S_t^{(n)} = S_t^{(1,n)}(S_t^{(2,n)} - 1) + S_t^{(1,n)},$$

we see from (i) and [4, Theorem 4.1] that $S^{(n)}$ converges weakly to S, as desired.

We first prove (ii). Let $\epsilon > 0$. Then

$$P\left(\sup_{0\le t\le T} |S_t^{(2,n)} - 1| \ge \epsilon\right) \le P\left(\sup_{0\le t\le T} |\Delta Y_t^{(n)}| > \tfrac{1}{2}\sigma^{-1}\right).$$

Since the process $\Delta Y^{(n)}$ converges to zero in probability by Theorem 2.2, $S^{(2,n)}$ converges to one in probability. Thus (ii) follows. Next we prove (i). Since the exponential is a continuous functional in the Skorohod topology, it is enough to prove that $\log S^{(1,n)}$ converges weakly to the process $(\sigma Y_t + \int_0^t b(s)ds - \frac{1}{2}\sigma^2 t)$. Notice that $|\sigma \Delta Y_t^{(1,n)}| + \frac{1}{n}|b(\frac{\lfloor nt \rfloor}{n})| < \frac{3}{4}$ for sufficiently large n and $t \in [0,T]$, whence the logarithm $\log S^{(1,n)}$ is well defined for such n.

We have

$$\log(1+x) = x - \frac{1}{2}x^2 + r(x)x^3 \qquad (|x| < 1),$$

where $r(x)$ is a bounded function on $|x| \le \frac{3}{4}$. Hence

$$\begin{aligned}\log S_t^{(1,n)} = \sum_{s \le t} \Bigg\{ & \sigma \Delta Y_s^{(1,n)} + \frac{1}{n} b(\tfrac{\lfloor ns \rfloor}{n}) - \frac{1}{2}\left(\sigma \Delta Y_s^{(1,n)} + \frac{1}{n} b(\tfrac{\lfloor ns \rfloor}{n})\right)^2 \\ & + r\left(\sigma \Delta Y_s^{(1,n)} + \frac{1}{n} b(\tfrac{\lfloor ns \rfloor}{n})\right) \cdot \left(\sigma \Delta Y_s^{(1,n)} + \frac{1}{n} b(\tfrac{\lfloor ns \rfloor}{n})\right)^3 \Bigg\} \\ = \sigma Y_t^{(1,n)} + \sum_{s \le t} \frac{1}{n} b(\tfrac{\lfloor ns \rfloor}{n}) - \frac{1}{2}\Phi_t^{(n)} + \Psi_t^{(n)},\end{aligned}$$

where

$$\begin{aligned}\Phi_t^{(n)} &:= \sum_{s \le t} \left(\frac{1}{n} b(\tfrac{\lfloor ns \rfloor}{n}) + \sigma \Delta Y_s^{(1,n)}\right)^2, \\ \Psi_t^{(n)} &:= \sum_{s \le t} r\left(\sigma \Delta Y_s^{(1,n)} + \frac{1}{n} b(\tfrac{\lfloor ns \rfloor}{n})\right) \cdot \left(\sigma \Delta Y_s^{(1,n)} + \frac{1}{n} b(\tfrac{\lfloor ns \rfloor}{n})\right)^3.\end{aligned}$$

We have $\Phi_t^{(n)} = n^{-2}\sum_{s \le t} b(\frac{\lfloor ns \rfloor}{n})^2 + 2\sigma \Gamma_t^{(N)} + \sigma^2 [Y^{(1,n)}]_t$, where

$$\Gamma_t^{(n)} := \sum\nolimits_{s \le t} \frac{1}{n} b(\tfrac{\lfloor ns \rfloor}{n}) \Delta Y_s^{(1,n)}.$$

Since $b(\cdot)$ is bounded, the first term $n^{-2}\sum_{s \le t} b(\frac{\lfloor ns \rfloor}{n})^2$ goes to 0 as $n \to \infty$. By Lemma 3.1, the third term $\sigma^2 [Y^{(1,n)}]$ converges to $(\sigma^2 t)$ in probability. As for the second term, it holds that

$$\sup_{0 \le t \le T} \left|\Gamma_t^{(n)}\right| \le C \sup_{s \le T} |\Delta Y_s^{(1,n)}| \le C \le |\Delta Y_t^{(n)}|.$$

Since $\Delta Y^{(n)}$ converges to zero in probability by Theorem 2.2, so does $\Gamma^{(n)}$. Thus the process (Φ_t) converges to $(\sigma^2 t)$. Since

$$\sup_{0 \le t \le T} \Psi_t \le C \left(\frac{1}{n} + \sup_{s \le T} |\Delta Y_s^{(1,n)}|\right) \Phi_T,$$

we see that the process (Ψ_t) converges to zero in probability. Using these fact as well as Lemma 3.1 and [4, Theorem 4.1], we see that $\log S^{(1,n)}$ converges weakly to $(\sigma Y_t + \int_0^t b(s)ds - \frac{1}{2}\sigma^2 t)$. □

If we take the i.i.d. random variables $\{\xi_i\}$ so that

(3.5) $$P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2,$$

then we obtain the desired approximating binary market model.

4. Arbitrage opportunities in the binary market

In this section, we study the arbitrage opportunities in the approximating binary market model with memory constructed in Section 3. For simplicity, we assume that the function $b(\cdot)$ is a real constant as in (1.1).

Let $N \in \mathbf{N}$, $r, b \in \mathbf{R}$, and $\sigma \in (0, \infty)$. The number N corresponds to n in Sections 2 and 3. Let the function $y(t, u)$ be as in Section 2. We define

$$r^{(N)} := \frac{r}{N}, \qquad b^{(N)} := \frac{b}{N}.$$

The $\lfloor NT \rfloor$-period market $\mathcal{M}^{(N)}$ consists of a share of the money market with price process $(B_n^{(N)})_{n=0,1,\dots,\lfloor NT \rfloor}$ and a stock with price process $(S_n^{(N)})_{n=0,1,\dots,\lfloor NT \rfloor}$. The prices are governed respectively by

$$B_n^{(N)} = B_{n-1}^{(N)}(1 + r^{(N)}) \quad (n = 1, \dots, \lfloor NT \rfloor), \quad B_0^{(N)} = 1,$$
$$S_n^{(N)} = S_{n-1}^{(N)}(1 + b^{(N)} + X_n^{(N)}) \quad (n = 1, \dots, \lfloor NT \rfloor), \quad S_0^{(N)} = s_0,$$

where s_0 is a positive constant,

$$X_n^{(N)} := \sigma \Delta Y_{\frac{n}{N}}^{(N)} = \frac{\sigma}{\sqrt{N}} \sum_{i=1}^{n} \left\{ y(\tfrac{n}{N}, \tfrac{i}{N}) - y(\tfrac{n-1}{N}, \tfrac{i}{N}) \right\} \xi_i$$

and $\{\xi_i\}$ are i.i.d. random variables satisfying (3.5). Theorem 3.2 implies that the binary market model $\mathcal{M}^{(N)}$ approximates the continuous-time market model with bond price process (e^{rt}) and stock price process S in (1.1).

Given the values of $\xi_1, \dots, \xi_{n-1}$, the random variable $X_n^{(N)}$ takes the following two possible values u_n and d_n: $d_1 = -\sigma/\sqrt{N}$, $u_1 = \sigma/\sqrt{N}$, and for $n = 2, \dots, N$,

$$d_n \equiv d_n(\xi_1, \dots, \xi_{n-1}) = \frac{\sigma}{\sqrt{N}} \sum_{i=1}^{n-1} \left\{ y(\tfrac{n}{N}, \tfrac{i}{N}) - y(\tfrac{n-1}{N}, \tfrac{i}{N}) \right\} \xi_i - \frac{\sigma}{\sqrt{N}},$$
$$u_n \equiv u_n(\xi_1, \dots, \xi_{n-1}) = \frac{\sigma}{\sqrt{N}} \sum_{i=1}^{n-1} \left\{ y(\tfrac{n}{N}, \tfrac{i}{N}) - y(\tfrac{n-1}{N}, \tfrac{i}{N}) \right\} \xi_i + \frac{\sigma}{\sqrt{N}}.$$

We investigate the arbitrage opportunities in $\mathcal{M}^{(N)}$. Let C be a positive constant satisfying

$$|y(t, u) - y(s, u)| \le C|t - s| \qquad (0 \le t, s, u \le T). \tag{4.1}$$

Theorem 4.1. *Suppose that $T < 1/C$. Then there exists an integer N_0 such that for each $N \ge N_0$, the market $\mathcal{M}^{(N)}$ is arbitrage-free.*

Proof. From the condition $TC < 1$, we have an integer N_0 satisfying

$$\frac{b}{N} - \frac{\sigma}{\sqrt{N}}(TC + 1) > -1, \quad |r - b| < \sqrt{N}(1 - TC)\sigma \quad (N \ge N_0). \tag{4.2}$$

By (4.1), we have, for $n = 1, \dots, \lfloor NT \rfloor$,

$$\begin{aligned} \min_{\xi \in \{-1,1\}^{n-1}} d_n(\xi) &= -\frac{\sigma}{\sqrt{N}} \sum_{i=1}^{n-1} |y(\tfrac{n}{N}, \tfrac{i}{N}) - y(\tfrac{n-1}{N}, \tfrac{i}{N})| - \frac{\sigma}{\sqrt{N}} \\ &\ge -\frac{\sigma}{\sqrt{N}} \left(\frac{n-1}{N} C + 1 \right) \ge -\frac{\sigma}{\sqrt{N}} (TC + 1). \end{aligned}$$

This and (4.2) yield, for $N \geq N_0$ and $n = 1, \ldots, \lfloor NT \rfloor$,

$$b^{(N)} + X_n^{(N)} \geq \frac{b}{N} + \min_{\xi \in \{-1,1\}^{n-1}} d_n(\xi) > -1,$$

whence $S_n > 0$.

We show that $\mathcal{M}^{(N)}$ is arbitrage-free for $N \geq N_0$. By Dzhaparidze [6, Proposition 6.1.2], $\mathcal{M}^{(N)}$ is free from arbitrage opportunities if and only if

$$d_n < r^{(N)} - b^{(N)} < u_n \qquad (n = 1, \ldots, \lfloor NT \rfloor). \tag{4.3}$$

However, we have

$$\begin{aligned}\max_{\xi \in \{-1,1\}^{n-1}} d_n(\xi) &= \frac{\sigma}{\sqrt{N}} \sum_{i=1}^{n-1} |y(\tfrac{n}{N}, \tfrac{i}{N}) - y(\tfrac{n-1}{N}, \tfrac{i}{N})| - \frac{\sigma}{\sqrt{N}} \\ &\leq -\frac{\sigma}{\sqrt{N}} \left(1 - \frac{n-1}{N} C\right) \leq -\frac{\sigma}{\sqrt{N}} (1 - TC),\end{aligned}$$

and

$$\begin{aligned}\min_{\xi \in \{-1,1\}^{n-1}} u_n(\xi) &= -\frac{\sigma}{\sqrt{N}} \sum_{i=1}^{n-1} |y(\tfrac{n}{N}, \tfrac{i}{N}) - y(\tfrac{n-1}{N}, \tfrac{i}{N})| + \frac{\sigma}{\sqrt{N}} \\ &\geq \frac{\sigma}{\sqrt{N}} \left(1 - \frac{n-1}{N} C\right) \geq \frac{\sigma}{\sqrt{N}} (1 - TC).\end{aligned}$$

Thus, by (4.2), (4.3) holds for $N \geq N_0$. □

By Theorem 4.1, the market $\mathcal{M}^{(N)}$ is arbitrage-free for T small enough and N large enough. However, in general, the market $\mathcal{M}^{(N)}$ may admit arbitrage opportunities, as we see below.

Suppose that there exists a positive constant C such that $l(s,u) \geq C$ for $0 \leq u < s \leq T$. Let $T > 1/C$. We assume that $r \leq b$. Then, $d_{\lfloor NT \rfloor}(-1, \ldots, -1)$ is

$$\frac{\sigma}{\sqrt{N}} \sum_{i=1}^{\lfloor NT \rfloor - 1} \int_{\frac{\lfloor NT \rfloor - 1}{N}}^{\frac{\lfloor NT \rfloor}{N}} l(s, \tfrac{i}{N}) ds - \frac{\sigma}{\sqrt{N}} > \frac{\sigma}{\sqrt{N}} \left(\frac{C(\lfloor NT \rfloor - 1)}{N} - 1 \right).$$

Since $TC > 1$, it follows that $d_{\lfloor NT \rfloor}(-1, \ldots, -1) > r_N - b_N$ or

$$S_{\lfloor NT \rfloor} > (1 + r_N) S_{\lfloor NT \rfloor - 1}$$

for N large enough. Therefore, if the value of $(\xi_1, \ldots, \xi_{\lfloor NT \rfloor - 1})$ turns out to be $(-1, \ldots, -1)$, then we have an arbitrage opportunity: we may buy stocks at time $\lfloor NT \rfloor - 1$ using money obtained by shortselling bonds. In a similar fashion, we can show that if $T > 1/C$, $r < b$ and N is large enough, then the value $(1, \ldots, 1)$ of $(\xi_1, \ldots, \xi_{\lfloor NT \rfloor - 1})$ gives an arbitrage opportunity.

Put

$$P_N = P\left(\bigcup_{n=1}^{\lfloor NT \rfloor} \left\{ d_n < r^{(N)} - b^{(N)} < u_n \right\}^c \right).$$

As we see in the proof of Theorem 4.1, the binary market $\mathcal{M}^{(N)}$ is arbitrage-free if and only if $P_N = 0$. The next theorem gives the rate at which the arbitrage probability P_N tends to zero as $N \to \infty$.

Theorem 4.2. *There exists a positive constant $C' = C'_T$ such that, for each $\alpha \in (0,1)$, we have $N(\alpha) \in \mathbf{N}$ satisfying*

$$P_N \le \frac{C'}{N^\alpha} \qquad (N \ge N(\alpha)).$$

Proof. Set $\beta := (\alpha+1)/2$, and choose $N(\alpha) \in \mathbf{N}$ so large that

$$N^{\beta/2} C\sqrt{T} < \sqrt{N} - |(r-b)/\sigma|, \qquad N^{\beta/2} > 4 \qquad (N \ge N(\alpha)). \tag{4.4}$$

Then we have $d_1 < r^{(N)} - b^{(N)} < u_1$. For $N \ge N(\alpha)$ and $n = 2, \dots, \lfloor NT \rfloor$, we put $\lambda := N^{\beta/2}$ and

$$s_{n-1} := \left[N \sum_{i=1}^{n-1} \left\{ y(\tfrac{n}{N}, \tfrac{i}{N}) - y(\tfrac{n-1}{N}, \tfrac{i}{N}) \right\}^2 \right]^{1/2}, \qquad M_{n-1} := \max_{1 \le m \le n-1} \left| \sum\nolimits_{i=1}^{m} \eta_i \right|,$$

where $\eta_i := \sqrt{N} \left\{ y(\tfrac{n}{N}, \tfrac{i}{N}) - y(\tfrac{n-1}{N}, \tfrac{i}{N}) \right\} \xi_i$ for $i = 1, 2, \dots$. By (4.1), we have $s_{n-1} \le C\sqrt{T}$. This and (4.4) imply that

$$\begin{aligned} P\left(\frac{r-b}{N} \le d_n \right) &\le P\left(\frac{r-b}{\sigma} + \sqrt{N} \le M_{n-1} \right) \le P(M_{n-1} \ge \lambda C\sqrt{T}) \\ &\le P(M_{n-1} \ge \lambda s_{n-1}). \end{aligned}$$

Similarly we have

$$P\left(u_n \le \frac{r-b}{N} \right) \le P(M_{n-1} \ge \lambda s_{n-1}).$$

Since $\frac{1}{4}\lambda > 1$ and

$$\max_{1 \le i \le n-1} |\eta_i| = \max_{1 \le i \le n-1} |\sqrt{N}\{ y(\tfrac{n}{N}, \tfrac{i}{N}) - y(\tfrac{n-1}{N}, \tfrac{i}{N}) \}| \le s_{n-1},$$

it follows from [4, (12.16), Page 89] that

$$P(M_{n-1} \ge \lambda s_{n-1}) \le \frac{C_0}{\lambda^4}$$

for some constant $C_0 > 0$ independent of N and n (notice that η_i here corresponds to ξ_i in [4, (12.16), Page 89]). Hence, P_N is at most

$$\sum_{n=2}^{\lfloor NT \rfloor} \left\{ P\left(\frac{r-b}{N} \le d_n \right) + P\left(u_n \le \frac{r-b}{N} \right) \right\} \le \frac{2\lfloor NT \rfloor C_0}{N^{2\beta}} \le \frac{2TC_0}{N^\alpha}.$$

Thus the theorem follows. □

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