The curved traveling front of the Allen-Cahn equation

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In this talk we consider the Allen-Cahn equation:

$$u_t = \triangle u + f(u) \qquad (x, y, t) \in \mathbf{R}^2 \times \mathbf{R}_+ \tag{1}$$

where f is of "bistable type". The typical example of the nonlinear term f is

$$f(u) = u(1-u)(u-a), \qquad 0 < a < \frac{1}{2}. \tag{2}$$

The constant states 0 and 1 are stable under the diffusion-free system. By the assumption of a, the region of the state 1 is getting larger and larger and finally it covers the whole space. When the state 1 propagates, we can observe the characteristic profiles. In the one-dimensional space, one of the typical solutions is a *traveling wave* solution which never changes its shape without translation. Substituting $u(x,t) = \Phi(x - ct)$, we have

$$\Phi_{\xi\xi} - c\Phi_\xi + f(\Phi) = 0.$$

Actually for the nonlinearity (2), we have

$$\Phi(\xi) = \frac{1}{2}\left(1 - \tanh\frac{\xi}{2\sqrt{2}}\right), \quad c = \sqrt{2}\left(\frac{1}{2} - a\right).$$

As the appropriate singular limit, the interface between two states 1 and 0 becomes sharp and we can get the interface equation (see e.g. [4]):

$$V = H + k \tag{3}$$

where V is a normal velocity, H is the curvature, and k is a given constant. This equation is also observed in the filamentary vortex of the Ginzburg–Landau equation confined in a plane [3] and the BZ reaction [6].

The typical solutions of (3) are the circles and lines. In the case $k \neq 0$, unfortunately, some interfaces may possess some self–intersection points eventually, even if the initial interface has none. If the interface is represented by the graph $y = v(x,t)$, the equation (3) is reduced to

$$v_t = \frac{v_{xx}}{1+v_x^2} + k\sqrt{1+v_x^2} \qquad x \in \mathbf{R},\, t > 0, \tag{4}$$

Deckelnick et al in [3] proved the existence of the traveling curved front and studied the stability of the front under some restricted assumptions for u_0. The authors relaxed the assumption for the initial data and classified all the traveling fronts in [7, 8]. They proved the following (see [7, Proposition 1.1, Theorem 1.2]).

Theorem 1 *Any traveling front of* (3) *with velocity* ${}^t(0,c)$ *is one of the three, after appropriate translations,*

(i) *lines* $y = m_* x$, *and* $y = -m_* x$

(ii) *a traveling curved front* $\Gamma_c(t)$ *which possesses two asymptotes* $y = \pm m_* x$,

(iii) *stationary circles with radius* $1/|k|$ *only in the case* $c = 0$,

where $m_* := \sqrt{c^2 - k^2}/k$. *Moreover the explicit form of the traveling curved front* $\Gamma_c(t) = \{y = \varphi(x) + ct\}$ *with speed* $c(\geq k)$ *is given in*

$$\begin{aligned} x(\theta;c) &:= \frac{\theta}{c} + \frac{k}{c\sqrt{c^2-k^2}} \log \left| \frac{1 + \sqrt{\dfrac{c+k}{c-k}} \tan\dfrac{\theta}{2}}{1 - \sqrt{\dfrac{c+k}{c-k}} \tan\dfrac{\theta}{2}} \right|, \\ y(\theta;c) &:= -\frac{1}{c} \log\left(\frac{c\cos\theta - k}{c-k} \right), \end{aligned}$$

for $\theta \in (-\arctan m_*, \arctan m_*)$.

The traveling curved front $\Gamma_c(t)$ is "V-shaped", which connects two asymptotes. The existence of this traveling front is also reported in [2, 3] and in a liquid BZ reaction [6].

The asymptotic stability of the curved traveling front in (4) is discussed in [3, 8]. It is proved that the traveling curved front is asymptotically stable, if the initial perturbation is restricted to

$$BC_0^1 := \{v \in C^1(\mathbf{R}) \mid \sup_{-\infty<x<\infty} (|v(x)| + |v_x(x)|) < \infty,\ \lim_{|x|\to\infty} v(x) = 0\}.$$

and that if you take the perturbation space

$$BC^1 := \{v \in C^1(\mathbf{R}) \mid \sup_{-\infty<x<\infty}(|v(x)| + |v_x(x)|) < \infty\},$$

instead of BC_0^1, the traveling curved front is not asymptotically stable (see [8, Theorem 1.1 and Theorem 4.1]).

By the above observation, we can expect that a “V-shaped” traveling wave solution of (1) exists. Actually we have the following theorem.

Theorem 2 *There exists a traveling wave solution* $u(x, y, t) = U(x, y - ct)$ *of* (1) *such that*

$$\lim_{R\to\infty} \sup_{(x,y)\in D_R} \left| U(x, y) - \Phi\left(\frac{k}{c}(y - m_*|x|)\right)\right| = 0$$

where

$$D_R := \{(x, y) \mid x^2 + y^2 \geq R^2\}.$$

Bonnet and Hamel [1] showed the existence of the “V-shaped” traveling wave solutions, if f is of the “ignition temperature” type (mono-stable type) instead of (2). Hamel and Monneau [5] shows the uniqueness of the traveling front of the corresponding singular limit problem.

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