where a *discrete and faithful* $PSL_2(\mathbf{R})$-representation of Γ_g means a group homomorphism from Γ_g to $PSL_2(\mathbf{R})$ which is injective and the image of Γ_g is a discrete subgroup of $PSL_2(\mathbf{R})$. Because any Fuchsian group which is isomorphic to Γ_g can be lifted to $SL_2(\mathbf{R})$ ([Pa],[S-S]), we can start from $Hom(\Gamma_g, SL_2(\mathbf{R}))$ the set of $SL_2(\mathbf{R})$-representations of Γ_g. And T_g can be considered as the set of characters of discrete and faithful $SL_2(\mathbf{R})$-representations of Γ_g.

From this view point , we can get a real algebraic structure on T_g as follows. By using the presentation of Γ_g, $Hom(\Gamma_g, SL_2(\mathbf{R}))$ can be embeded into the product space $SL_2(\mathbf{R})^{2g}$ as the real algebraic subset $R(\Gamma)$ which is called *the space of representations* ([C-S],[Go],[M-S]) . The adjoint action of $PGL_2(\mathbf{R})$ on $R(\Gamma)$ induces the action on $\mathbf{R}[R(\Gamma)]$ the affine coordinate ring of $R(\Gamma)$ and put $\mathbf{R}[R(\Gamma)]^{PGL_2(\mathbf{R})}$ the ring of invariants under this action. Let $X(\Gamma)$ be a real algebraic set whose affine coordinate ring is isomorphic to $\mathbf{R}[R(\Gamma)]^{PGL_2(\mathbf{R})}$. Then T_g can be realized as a semialgebraic subset of $X(\Gamma)$. Hence T_g is defined by finitely many polynomial equalities and inequalities on $X(\Gamma)$. This construction is essentially due to Helling [He], and later Culler-Shalen [C-S] and Morgan-Shalen [M-S] made this process more clear and by using this procedure, Brumfiel described the real spectrum compactification of T_g [Br].

Our theme of this paper is to study the semialgebraic structure of T_g and we mainly consider the following two things. First we describe the defining equations of T_g on $X(\Gamma)$ by using 6g-6 polynomial inequalities explicitly (Theorem 3.2, 4.2). This problem is related to the construction of the global coordinates of T_g by use of small number of traces of elements of Fuchsian groups which is studied deeply by Keen ([K]) and recently by Okai and Okumura ([Ok],[O1],[O2]) by using hyperbolic geometry on $\mathbf{H}$ and the argument of the fundamental polygons of Fuchsian groups. Our treatment in this paper is rather algebraic. The second is that from a real algebraic viewpoint, we also show the well known fact that T_g is a 6g-6 dimensional cell (Theorem 3.1, 4.1.) which was proved by Teichmüller himself by use of his theory of quadratic differentials and quasi-conformal mappings.

The remainder of this paper is organized as follows. Section 2 deals with the construction of Teichmüller space T_g following Culler-Shalen [C-S] and Morgan-Shalen [M-S]. The description of defining inequalities and cell structure of T_g are shown in Section 3 and 4. In section 3 we treat the case of genus $g = 2$ and in section 4 , $g \geq 3$ cases are discussed.

2 Construction of Teichmüller space as a semialgebraic set

In this section we review the construction of Teichmüller space following [C-S],[M-S],[Sa].

2.1 The space of $SL_2(\mathbf{R})$-representations of the surface group Γ

Let $g \geq 2$ be fixed. We define the (closed) surface group of genus g by the following presentation

$$\Gamma = \Gamma_g := \langle\ \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g \mid \prod_{i=1}^{g} [\alpha_i, \beta_i] = id.\ \rangle$$

where $[\alpha_i, \beta_i] := \alpha_i \cdot \beta_i \cdot \alpha_i^{-1} \cdot \beta_i^{-1}$.

By using this presentation, we can embed $Hom(\Gamma, SL_2(\mathbf{R}))$ the set of $SL_2(\mathbf{R})$-representations of Γ into the product space $SL_2(\mathbf{R})^{2g}$ and let $R(\Gamma)$ denote the image of $Hom(\Gamma, SL_2(\mathbf{R}))$

$$\begin{array}{rcl} Hom(\Gamma, SL_2(\mathbf{R})) & \rightarrow & R(\Gamma) \subset SL_2(\mathbf{R})^{2g}. \\ \rho & \mapsto & (\rho(\alpha_1), \rho(\beta_1), \cdots, \rho(\alpha_g), \rho(\beta_g)) \end{array}$$

We identify $R(\Gamma)$ and $Hom(\Gamma, SL_2(\mathbf{R}))$. In the following we also identify a representation ρ and the image $(A_1, B_1, \cdots, A_g, B_g) \in SL_2(\mathbf{R})^{2g}$ of the system of generators $\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g\}$ of Γ under ρ. $R(\Gamma)$ is a real algebraic set and we call this *the space of* $SL_2(\mathbf{R})$*-representations of* Γ. $PGL_2(\mathbf{R})$ acts on $R(\Gamma)$ from right

$$\begin{array}{rcl} R(\Gamma) \times PGL_2(\mathbf{R}) & \longrightarrow & R(\Gamma) \\ (\rho, P) & \mapsto & P^{-1}\rho P\ . \end{array}$$

We remark that although we use the system of generators $\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g\}$ of Γ to define $R(\Gamma)$, the real algebraic structure of $R(\Gamma)$ does not depend on this system of generators. In fact if we choose another system of generators of Γ consisting of N elements and embed $Hom(\Gamma, SL_2(\mathbf{R}))$ into the product space $SL_2(\mathbf{R})^N$, we get an another real algebraic set but it is canonically isomorphic to $R(\Gamma)$.

Next we consider the following subset of $R(\Gamma)$

$$R'(\Gamma) := \{\rho \in R(\Gamma) \mid \rho \textit{ is non abelian and irreducible}\}$$

where a representation ρ is *non abelian* if $\rho(\Gamma)$ is a non abelian subgroup of $SL_2(\mathbf{R})$ and ρ is *irreducible* if $\rho(\Gamma)$ acts on $\mathbf{R}^2$ without non trivial invariant subspace. Hence if ρ is not irreducible (i.e., reducible) then there exists $P \in PGL_2(\mathbf{R})$ such that $P^{-1}\rho(\Gamma)P$ consists of upper triangular matrices, hence in particular $\rho(\Gamma)$ is solvable. We remark that the action of $PGL_2(\mathbf{R})$ on $R(\Gamma)$ preserves $R'(\Gamma)$. Next lemma is useful for the study of $R'(\Gamma)$.

Lemma 2.1 *For $\rho \in R'(\Gamma)$, there exist $g, h \in \Gamma$ such that $\rho(g)$ is a hyperbolic matrix i.e., $|tr(\rho(g))| > 2$ and $\rho(h)$ has no common fixed points of $\rho(g)$. In other words there exists $P \in PGL_2(\mathbf{R})$ such that*

$$\begin{aligned} P^{-1}\rho(g)P &= \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \quad (\lambda \neq \pm 1) \\ P^{-1}\rho(h)P &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (b \cdot c \neq 0). \ \square \end{aligned}$$

We have another characterization of $R'(\Gamma)$.

Proposition 2.1

$$\begin{aligned} R'(\Gamma) &= \{\rho \in R(\Gamma) \mid tr(\rho([a,b])) \neq 2 \ for\ some\ a, b \in \Gamma\} \\ &= R(\Gamma) - \bigcap_{a,b\in\Gamma} \{\rho \in R(\Gamma) \mid tr(\rho([a,b])) = 2\}. \end{aligned}$$

(*Proof.*)
($\Rightarrow$) Take $g, h \in \Gamma$ which satisfy the conditions of Lemma 2.1. Then $tr([\rho(g), \rho(h)]) \neq 2$.
($\Leftarrow$) If $\rho(\Gamma)$ is abelian, $[\rho(a), \rho(b)] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for any $a, b \in \Gamma$. If $\rho(\Gamma)$ has a non trivial invariant subspace, there exists $P \in PGL_2(\mathbf{R})$ such that any element of $P^{-1}\rho(\Gamma)P$ is an upper triangular matrix, hence $tr([\rho(a), \rho(b)]) = 2$ for any $a, b \in \Gamma$. $\square$

Corollary 2.1 *$R'(\Gamma)$ is open in $R(\Gamma)$.* $\square$

We can say more about $R'(\Gamma)$.

Proposition 2.2 *$R'(\Gamma)$ has the structure of a 6g-3 dimensional real analytic manifold.* $\square$

Because the action of $PGL_2(\mathbf{R})$ on $R'(\Gamma)$ is proper and without fixed points (see [Gu] Section 9) , we have the following result.

Proposition 2.3 *The quotient space $R'(\Gamma)/PGL_2(\mathbf{R})$ has the structure of a 6g-6 dimensional real analytic manifold such that the natural projection*

$$R'(\Gamma) \to R'(\Gamma)/PGL_2(\mathbf{R})$$

is a real analytic principal $PGL_2(\mathbf{R})$-bundle. □

Next we define the subset $R_0(\Gamma)$ of $R(\Gamma)$ by

$$R_0(\Gamma) := \{\rho \in R(\Gamma) \mid \rho \textit{ is discrete and faithful}\} \tag{1}$$

where a representation ρ is *discrete* if $\rho(\Gamma)$ is a discrete subgroup of $SL_2(\mathbf{R})$ and ρ is *faithful* if ρ is injective. We remark that the action of $PGL_2(\mathbf{R})$ on $R(\Gamma)$ preserves $R_0(\Gamma)$. Then another characterization of $R_0(\Gamma)$ is

Proposition 2.4

$$\begin{aligned} R_0(\Gamma) &= \{\rho \in R(\Gamma) \mid \rho \textit{ is cocompact, discrete and faithful}\} && (2)\\ &= \{\rho \in R(\Gamma) \mid \rho \textit{ is totally hyperbolic}\} && (3)\end{aligned}$$

where a representation ρ is cocompact if the quotient space $\rho(\Gamma) \setminus SL_2(\mathbf{R})$ is compact with respect to the quotient topology , and ρ is called totally hyperbolic if $\rho(h)$ is hyperbolic for any $h(\neq$ identity$) \in \Gamma$.

(*Proof.*)
(1) ⇒ (2) The fundamental group of a surface $\rho(\Gamma) \setminus \mathbf{H}$ is isomorphic to the surface group Γ, hence $\rho(\Gamma) \setminus \mathbf{H}$ is compact.
(2) ⇒ (3) Because $\rho(\Gamma)$ is discrete, any elliptic element of $\rho(\Gamma)$ is finite order. But Γ is torsion free, $\rho(\Gamma)$ has no elliptic elements. Moreover if $\rho(\Gamma)$ has a parabolic element, then $\rho(\Gamma) \setminus \mathbf{H}$ has a cusp. But $\rho(\Gamma) \setminus \mathbf{H}$ is compact, $\rho(\Gamma)$ has no parabolic elements.
(3) ⇒ (1) Faithfulness is immediate. Discreteness follows from Nielsen's theorem (see [Si] P.33 Theorem 3). □

Proposition 2.5 *$R_0(\Gamma)$ is open and closed in $R(\Gamma)$.*

(*Proof.*) We give a sketch of the proof. We recall the Jørgensen's inequalities [Jø]:

For any $\rho \in R(\Gamma)$ ρ is contained in $R_0(\Gamma)$ if and only if

$$|tr([\rho(g), \rho(h)]) - 2| + |tr(\rho(h))^2 - 4| \geq 1$$

for any pair $g, h \in \Gamma$ with $gh \neq hg$.

These inequalities are closed conditions of $R_0(\Gamma)$ in $R(\Gamma)$.

The openness of $R_0(\Gamma) \subset R(\Gamma)$ follows from the next theorem due to Weil [W]:

If G is a connected Lie group and Γ is a discrete group, then the set of cocompact, discrete and faithful representations from Γ to G is open in the set of all representations from Γ to G. □

Next we recall the notions of a semialgebraic set. Let V be a real algebraic set with its affine coordinate ring $\mathbf{R}[V]$ i.e., the ring of polynomial functions on V. A subset S of V is called *a semialgebraic subset of V* if there exist finitely many polynomial functions on V $f_i, g_{i_1}, \cdots g_{i_{m(i)}} \in \mathbf{R}[V]$ ($i = 1, \cdots, l$) such that S can be written as

$$S = \bigcup_{i=1}^{l} \{ x \in V \mid f_i(x) = 0, \; g_{i_1}(x) > 0, \; \cdots g_{i_{m(i)}}(x) > 0 \}.$$

From the above definition, any real algebraic set is a semialgebraic set. Moreover it is known that any connected component of a semialgebraic set (with respect to Euclidean topology) is also a semialgebraic set and the number of connected components of a semialgebraic set is finite (see [B-C-R] Theorem 2.4.5).

Corollary 2.2 *$R_0(\Gamma)$ consists of finitely many connected components of $R(\Gamma)$, hence $R_0(\Gamma)$ is a semialgebraic subset of $R(\Gamma)$.* □

The relation between $R'(\Gamma)$ and $R_0(\Gamma)$ is

Proposition 2.6 $R_0(\Gamma) \subset R'(\Gamma)$.

(*Proof.*) For $\rho \in R_0(\Gamma)$ because the surface group Γ is non abelian and ρ is injective, ρ is non abelian. Also because Γ is not solvable, ρ is irreducible. □

Corollary 2.3 *$R_0(\Gamma)$ has the structure of a 6g-3 dimensional real analytic manifold.* □

2.2 The space of characters of Γ

As we have seen in subsection 2.1 that $R(\Gamma)$ has the structure of a real algebraic set. Let $\mathbf{R}[R(\Gamma)]$ be its affine coordinate ring i.e., the ring of polynomial functions on $R(\Gamma)$. Then the action of $PGL_2(\mathbf{R})$ on $R(\Gamma)$ induces the action of $PGL_2(\mathbf{R})$ on $\mathbf{R}[R(\Gamma)]$

$$\begin{array}{rcl} PGL_2(\mathbf{R}) \times \mathbf{R}[R(\Gamma)] & \to & \mathbf{R}[R(\Gamma)] \\ (P, f(\rho)) & \mapsto & f(P^{-1}\rho P) \end{array}$$

and let $\mathbf{R}[R(\Gamma)]^{PGL_2(\mathbf{R})}$ be the ring of invariants of this action. For example the function $\tau_h \in \mathbf{R}[R(\Gamma)]$ $(h \in \Gamma)$ on $R(\Gamma)$ defined by

$$\tau_h(\rho) := tr(\rho(h))$$

for $\rho \in R(\Gamma)$ is an element of $\mathbf{R}[R(\Gamma)]^{PGL_2(\mathbf{R})}$. In fact $\mathbf{R}[R(\Gamma)]^{PGL_2(\mathbf{R})}$ is generated by τ_h $(h \in \Gamma)$ and is a finitely generated $\mathbf{R}$-subalgebra of $\mathbf{R}[R(\Gamma)]$ (see [He],[Ho],[Pr]).

Let $X(\Gamma)$ be a real algebraic set whose affine coordinate ring $\mathbf{R}[X(\Gamma)]$ is isomorphic to $\mathbf{R}[R(\Gamma)]^{PGL_2(\mathbf{R})}$. And let $I_h \in \mathbf{R}[X(\Gamma)]$ correspond to $\tau_h \in \mathbf{R}[R(\Gamma)]^{PGL_2(\mathbf{R})}$. Then $\mathbf{R}[X(\Gamma)]$ is generated by I_h $(h \in \Gamma)$ as $\mathbf{R}$-algebra. The injection

$$\mathbf{R}[X(\Gamma)] \cong \mathbf{R}[R(\Gamma)]^{PGL_2(\mathbf{R})} \hookrightarrow \mathbf{R}[R(\Gamma)]$$

induces the polynomial mapping

$$t : R(\Gamma) \to X(\Gamma).$$

Because $\mathbf{R}[R(\Gamma)]^{PGL_2(\mathbf{R})}$ is generated by τ_h $(h \in \Gamma)$, for a representation $\rho \in R(\Gamma)$, $t(\rho)$ can be considered as the *character* χ_ρ of ρ

$$\begin{array}{rcl} \chi_\rho : \Gamma & \to & \mathbf{R} \\ h & \mapsto & tr(\rho(h)) = \tau_h(\rho) \end{array}$$

Therefore the image $t(R(\Gamma)) \subset X(\Gamma)$ of $R(\Gamma)$ under the mapping t can be considered as the set of characters of $SL_2(\mathbf{R})$-representations of Γ. We call $X(\Gamma)$ *the space of characters of* Γ.

Moreover any element of $X(\Gamma) - t(R(\Gamma))$ can be considered as a character of $SU(2)$-representation of Γ and to explain this we need to review briefly the theory of $SL_2(\mathbf{C})$-representations of Γ following [C-S] and [M-S]. Let $R_{\mathbf{C}}(\Gamma)$ be the set of $SL_2(\mathbf{C})$-representations of Γ, then $R_{\mathbf{C}}(\Gamma)$ has the structure of a complex algebraic set and let $\mathbf{C}[R_{\mathbf{C}}(\Gamma)]$ be its affine coordinate ring. $PGL_2(\mathbf{C})$ acts on $R_{\mathbf{C}}(\Gamma)$ and also on $\mathbf{C}[R_{\mathbf{C}}(\Gamma)]$. Put $\mathbf{C}[R_{\mathbf{C}}(\Gamma)]^{PGL_2(\mathbf{C})}$ the ring of invariants of this action and let $X_{\mathbf{C}}(\Gamma)$ be a complex algebraic set whose affine coordinate ring $\mathbf{C}[X_{\mathbf{C}}(\Gamma)]$ is isomorphic to $\mathbf{C}[R_{\mathbf{C}}(\Gamma)]^{PGL_2(\mathbf{C})}$. Then the injection

$$\mathbf{C}[X_{\mathbf{C}}(\Gamma)] \cong \mathbf{C}[R_{\mathbf{C}}(\Gamma)]^{PGL_2(\mathbf{C})} \hookrightarrow \mathbf{C}[R_{\mathbf{C}}(\Gamma)]$$

induces the polynomial map

$$t_{\mathbf{C}} : R_{\mathbf{C}}(\Gamma) \to X_{\mathbf{C}}(\Gamma)$$

which is surjective. Since $R_{\mathbf{C}}(\Gamma), t_{\mathbf{C}}$ and $X_{\mathbf{C}}(\Gamma)$ are all defined over $\mathbf{Q}$, we can consider $X_{\mathbf{R}}(\Gamma)$ the set of real valued points of $X_{\mathbf{C}}(\Gamma)$. Then we can consider $X_{\mathbf{R}}(\Gamma)$ as the set of real valued characters of $SL_2(\mathbf{C})$-representations of Γ and it is known that any element of $X_{\mathbf{R}}(\Gamma)$ is either a character of $SL_2(\mathbf{R})$ or $SU(2)$-representation of Γ ([M-S] Proposition 3.1.1).

If we consider the polynomial function $tr_h \in \mathbf{C}[R_{\mathbf{C}}(\Gamma)]$ $(h \in \Gamma)$ on $R_{\mathbf{C}}(\Gamma)$ defined by

$$tr_h(\rho) := tr(\rho(h))$$

for $\rho \in R_{\mathbf{C}}(\Gamma)$, then tr_h is an element of $\mathbf{C}[R_{\mathbf{C}}(\Gamma)]^{PGL_2(\mathbf{C})}$ and write the corresponding element of $\mathbf{C}[X_{\mathbf{C}}(\Gamma)]$ also by tr_h for the sake of simplicity. Then after regarding $R(\Gamma)$ as the set of real valued points of $R_{\mathbf{C}}(\Gamma)$, there is a natural surjective homomorphism from $\mathbf{R}[X_{\mathbf{R}}(\Gamma)]$ the affine coordinate ring of $X_{\mathbf{R}}(\Gamma)$ to $\mathbf{R}[X(\Gamma)]$

$$\begin{array}{ccc} \mathbf{R}[X_{\mathbf{R}}(\Gamma)] & \to & \mathbf{R}[X(\Gamma)] \\ tr_h & \mapsto & I_h \ . \end{array}$$

Therefore there is a canonical injection from $X(\Gamma)$ to $X_{\mathbf{R}}(\Gamma)$. Hence any element of $X(\Gamma)$ is either cotained in $t(R(\Gamma))$ or can be considered as a character of $SU(2)$-representation of Γ.

We define the following subsets of $X(\Gamma)$

$$\begin{aligned} X'(\Gamma) &:= t(R'(\Gamma)) \\ U(\Gamma) &:= \{\chi \in X(\Gamma) \mid I_{[a,b]}(\chi) \neq 2 \ for\ some\ a, b \in \Gamma\} \\ &= X(\Gamma) - \bigcap_{a,b\in\Gamma} \{\chi \in X(\Gamma) \mid I_{[a,b]}(\chi) = 2\}. \end{aligned}$$

Then $U(\Gamma)$ is open in $X(\Gamma)$. By Proposition 2.1 $t^{-1}(X'(\Gamma)) = R'(\Gamma)$ and $X'(\Gamma) \subset U(\Gamma)$.

Proposition 2.7 *$X'(\Gamma)$ is open in $U(\Gamma)$. Hence $X'(\Gamma)$ is open in $X(\Gamma)$.*

(*Proof.*) Let $V(\Gamma)$ be the set of characters of $SU(2)$-representations of Γ. As $SU(2)$ is compact $V(\Gamma)$ is compact in $X_{\mathbf{R}}(\Gamma)$. Hence $U(\Gamma) = X'(\Gamma) \cup (U(\Gamma) \cap V(\Gamma))$ and $(U(\Gamma) \cap V(\Gamma))$ is compact in $U(\Gamma)$. Therefore it is enough to show that $X'(\Gamma) \cap (U(\Gamma) \cap V(\Gamma)) = \phi$. For $\rho \in R'(\Gamma)$, by lemma 2.1 there exists $g \in \Gamma$ with $|tr(\rho(g))| = |\chi_\rho(g)| > 2$. On the other hand for any $SU(2)$-representation η of Γ

$$|tr(\eta(h))| = |\chi_\eta(h)| \leq 2 \ \ for\ any\ h \in \Gamma.$$

Therefore $X'(\Gamma) \cap (U(\Gamma) \cap V(\Gamma)) = \phi$. □

Next we will show that the restriction of the mapping t to $R'(\Gamma)$

$$t : R'(\Gamma) \to X'(\Gamma)$$

is a principal $PGL_2(\mathbf{R})$-bundle. By Proposition 2.3 it is enough to show that $X'(\Gamma)$ is the $PGL_2(\mathbf{R})$ adjoint quotient of $R'(\Gamma)$. For this purpose we need to prepare two lemmas which are $SL_2(\mathbf{R})$ version of the results in [C-S] and [M-S].

Lemma 2.2 *(see [C-S] Proposition1.5.2) For $\rho_1, \rho_2 \in R'(\Gamma)$, we assume that $t(\rho_1) = t(\rho_2)$, in other words they have the same character $\chi_{\rho_1} = \chi_{\rho_2}$. Then there is $P \in PGL_2(\mathbf{R})$ such that $\rho_2 = P^{-1}\rho_1 P$.* □

Lemma 2.3 *(see [M-S] Lemma 3.1.7) For a subset U of $X'(\Gamma)$, we assume that $t^{-1}(U)$ is open in $R'(\Gamma)$ hence open in $R(\Gamma)$. Then U is open in $X'(\Gamma)$ hence in $X(\Gamma)$.* □

By the previous lemmas we conclude that

Proposition 2.8 *$t : R'(\Gamma) \to X'(\Gamma)$ can be considered as the quotient map of $R'(\Gamma)$ under the action of $PGL_2(\mathbf{R})$ i.e.,*

$$X'(\Gamma) = R'(\Gamma)/PGL_2(\mathbf{R}).$$

Therefore by Proposition 2.3 $t : R'(\Gamma) \to X'(\Gamma)$ is a principal $PGL_2(\mathbf{R})$-bundle. □

Define the closed subset $X_0(\Gamma)$ of $X(\Gamma)$ by

$$X_0(\Gamma) := \{\chi \in X(\Gamma) \mid |I_{[g,h]}(\chi) - 2| + |I_h(\chi)^2 - 4| \geq 1 \text{ for } g,h \in \Gamma \text{ with } gh \neq hg\}.$$

Then the proof of Proposition 2.5 implies $t(R_0(\Gamma)) \subset X_0(\Gamma)$.

Proposition 2.9 *1.* $X_0(\Gamma) = t(R_0(\Gamma))$.

2. $X_0(\Gamma)$ *is open in* $X'(\Gamma)$ *hence open in* $X(\Gamma)$.

3. $t^{-1}(X_0(\Gamma)) = R_0(\Gamma)$.

(*Proof.*) 1. Any representation of Γ to $SL_2(\mathbf{C})$ is discrete and faithful if and only if it satisfies Jørgensen's inequalities which we have seen in the proof of Proposition 2.5. But there are no discrete and faithful $SU(2)$-representations of Γ because $SU(2)$ is compact and Γ is an infinite group. Hence $X_0(\Gamma) \subset t(R(\Gamma))$ and it follows that $X_0(\Gamma) = t(R_0(\Gamma))$.

2. $R_0(\Gamma) \subset R'(\Gamma)$ implies $X_0(\Gamma) \subset X'(\Gamma)$. Because $R_0(\Gamma)$ is open in $R(\Gamma)$ and $t : R'(\Gamma) \to X'(\Gamma)$ is an open map by Proposition 2.3, $X_0(\Gamma)$ is open in $X'(\Gamma)$.

3. It is immediate from lemma 2.2. □

Corollary 2.4 $X_0(\Gamma)$ *is open and closed in* $X(\Gamma)$. *Therefore* $X_0(\Gamma)$ *consists of finitely many connected components of* $X(\Gamma)$ *hence it is a semialgebraic subset of* $X(\Gamma)$. □

Corollary 2.5 $t : R_0(\Gamma) \to X_0(\Gamma)$ *is also a principal* $PGL_2(\mathbf{R})$*-bundle. Hence* $X_0(\Gamma)$ *can be considered as the* $PGL_2(\mathbf{R})$ *adjoint quotient of* $R_0(\Gamma)$ *i.e.,* $X_0(\Gamma) = R_0(\Gamma)/PGL_2(\mathbf{R})$. □

We summarize the results of this subsection as the following diagram.

$$\begin{array}{ccccccc} R(\Gamma) & \supset & R'(\Gamma) & \supset & R_0(\Gamma) & & \\ t\downarrow & & \downarrow & & \downarrow & & PGL_2(\mathbf{R})\ \textit{bundle} \\ X(\Gamma) & \supset & X'(\Gamma) & \supset & X_0(\Gamma) & = & R_0(\Gamma)/PGL_2(\mathbf{R}) \end{array}$$

2.3 The relation between $SL_2(\mathbf{R})$- and $PSL_2(\mathbf{R})$-representations of Γ

Next we consider the relation between $SL_2(\mathbf{R})$- and $PSL_2(\mathbf{R})$-representations of the surface group Γ.

The group $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ ($\cong (\mathbf{Z}/2\mathbf{Z})^{2g}$) acts on $R(\Gamma)$ as follows. For any $\mu \in Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ and $\rho \in R(\Gamma)$, we define the representation $\mu \cdot \rho \in R(\Gamma)$ by

$$\mu \cdot \rho(h) := \mu(h) \cdot \rho(h) \quad (\text{ for all } h \in \Gamma).$$

Proposition 2.10 *([Pa],[S-S]) Let* $\xi : \Gamma \to PSL_2(\mathbf{R})$ *be a discrete and faithful* $PSL_2(\mathbf{R})$ *representation. Suppose* $A_i, B_i \in SL_2(\mathbf{R})$ $(i = 1, \cdots, g)$ *denote any representatives of* $\xi(\alpha_i), \xi(\beta_i) \in PSL_2(\mathbf{R})$. *Then*

$$\prod_{i=1}^{g}[A_i, B_i] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In other words, ξ *can always be lifted to a representation* $\rho \in R_0(\Gamma)$ *and the set of all liftings of* ξ *is equal to the* $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ *orbit of* ρ *in* $R_0(\Gamma)$.

$$\begin{array}{ccc} & & SL_2(\mathbf{R}) \\ & \overset{\rho}{\nearrow} & \downarrow proj. \\ \Gamma & \overset{\xi}{\to} & PSL_2(\mathbf{R}) \end{array}$$

(*Proof.*) We briefly review what Seppälä and Sorvali showed in their paper [S-S].

Let ξ be a discrete and faithful $PSL_2(\mathbf{R})$ representation. Suppose $A_i, B_i \in SL_2(\mathbf{R})$ $(i = 1, \cdots, g)$ denote any representatives of $\xi(\alpha_i), \xi(\beta_i) \in PSL_2(\mathbf{R})$. Then they showed that

$$\begin{aligned} tr([A_i, B_i]) &< -2 \quad (i = 1, \cdots, g) \\ tr([A_1, B_1] \cdots [A_j, B_j]) &< -2 \quad (j = 2, \cdots, g-1). \end{aligned}$$

In particular

$$\begin{aligned} tr([A_g, B_g]) &< -2 \\ tr([A_1, B_1] \cdots [A_{g-1}, B_{g-1}]) &< -2. \end{aligned}$$

We may suppose that $[A_1, B_1] \cdots [A_{g-1}, B_{g-1}]$ is a diagonal matrix. Then $[A_g, B_g]$ must be also diagonal, hence the above inequalities implies the conclusion. □

Corollary 2.6 *1.* $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ *acts on* $R_0(\Gamma)$ *and the quotient space* $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z}) \setminus R_0(\Gamma)$ *can be considered as the set of discrete and faithful* $PSL_2(\mathbf{R})$*-representations of* Γ.

2. *Through the mapping t ,$Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ acts also on $X_0(\Gamma)$ and the quotient space $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})\backslash X_0(\Gamma)$ can be considered as the $PGL_2(\mathbf{R})$-adjoint quotient of the set of discrete and faithful $PSL_2(\mathbf{R})$-representations of Γ.*

   We call this set Teichmüller space T_g

$$\begin{aligned} T_g &:= Hom(\Gamma, \mathbf{Z}/2\mathbf{Z}) \backslash X_0(\Gamma) \\ &= Hom(\Gamma, \mathbf{Z}/2\mathbf{Z}) \backslash R_0(\Gamma)/PGL_2(\mathbf{R}). \square \end{aligned}$$

Proposition 2.4 implies $|I_h| > 2$ (*for all* $h(\neq$ *identity*) $\in \Gamma$) on $X_0(\Gamma)$ hence the sign of I_h is constant on each connected component of $X_0(\Gamma)$. This means that $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ permutes the set of connected components of $X_0(\Gamma)$ freely. Thus

Corollary 2.7 *The quotient map $X_0(\Gamma) \to T_g$ is an unramified $(\mathbf{Z}/2\mathbf{Z})^{2g}$ -covering. Hence by taking (any) lifting of this mapping , we can consider T_g as a finite union of connected components of $X_0(\Gamma)$. Therefore T_g can be considered as a semialgebraic subset of $X_0(\Gamma)$.* $\square$

Corollary 2.8 *If $\pi_0(X_0(\Gamma))$ denotes the number of connected components of $X_0(\Gamma)$, the order of $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ divides $\pi_0(X_0(\Gamma))$. In particular*

$$2^{2g} \leq \pi_0(X_0(\Gamma)). \square$$

We summarize the result of this subsection as the following diagram.

$$\begin{array}{ccccccc} Hom(\Gamma, SL_2(\mathbf{R})) & = & R(\Gamma) & \supset & R_0(\Gamma) & & \\ & & t\downarrow & & \downarrow & & \\ & & X(\Gamma) & \supset & X_0(\Gamma) & = & R_0(\Gamma)/PGL_2(\mathbf{R}) \\ & & & & \downarrow & & \\ & & & & T_g & = & Hom(\Gamma, \mathbf{Z}/2\mathbf{Z}) \backslash X_0(\Gamma) \end{array}$$

3 Semialgebraic description of Teichmüller space T_g ($g = 2$ case)

In this section by constructing the global coordinates of $X_0(\Gamma)$, we will show the connectivity, contractibility and semialgebraic description of Teichmüller space T_2. For this purpose we need to find some semialgebraic subset of $X(\Gamma)$ containing $X_0(\Gamma)$ whose presentation as a semialgebraic set and topological structure are both simple. This is $S(\Gamma)$ stated in the following subsection.

3.1 Definition of the semialgebraic subset $S(\Gamma)$ of $X(\Gamma)$

We define the open semialgebraic subset $S(\Gamma)$ of $X(\Gamma)$ by

$$S(\Gamma) := \{\chi \in X(\Gamma) \mid I_{c_1}(\chi) < -2\}$$

where $c_1 := [\alpha_1, \beta_1] = [\alpha_2, \beta_2]^{-1} \in \Gamma$.

Proposition 3.1 $S(\Gamma) \subset X'(\Gamma)$. *Hence by Proposition 2.3* $t^{-1}(S(\Gamma)) \xrightarrow{t} S(\Gamma)$ *is a* $PGL_2(\mathbf{R})$*-bundle and we can consider* $S(\Gamma)$ *as the* $PGL_2(\mathbf{R})$*-adjoint quotient of* $t^{-1}(S(\Gamma))$ *i.e.,*

$$S(\Gamma) = t^{-1}(S(\Gamma))/PGL_2(\mathbf{R}).$$

(*Proof.*) First we show

$$S(\Gamma) \cap (X(\Gamma) - t(R(\Gamma))) = \phi.$$

As we have seen in subsection 2.2 any element of $X(\Gamma) - t(R(\Gamma))$ can be considered as a character of SU(2)-representation of Γ. Thus for $\chi \in X(\Gamma) - t(R(\Gamma))$

$$|I_h(\chi)| \leq 2 \quad for\ h \in \Gamma.$$

This shows that $S(\Gamma) \subset t(R(\Gamma))$. On the other hand Proposition 2.1 shows that $S(\Gamma) \subset X'(\Gamma)$. □

Next result is due to Seppälä and Sorvali ([S-S]).

Proposition 3.2 $X_0(\Gamma) \subset S(\Gamma)$. □

(*Proof.*) Any element $\rho = (A_1, B_1, A_2, B_2)$ of $R_0(\Gamma)$ induces a discrete and faithful $PSL_2(\mathbf{R})$-representation of Γ. Hence we have seen in the proof of Proposition 2.10 that

$$tr([A_1, B_1]) < -2.$$

This implies the conclusion. □

Corollary 3.1 *Above arguments show the following diagram.* □

$$\begin{array}{ccccccc} R(\Gamma) & \supset & R'(\Gamma) & \supset & t^{-1}(S(\Gamma)) & \supset & R_0(\Gamma) \\ t\downarrow & & \downarrow & & \downarrow & & \downarrow \\ X(\Gamma) & \supset & X'(\Gamma) & \supset & S(\Gamma) & \supset & X_0(\Gamma) \end{array}$$

3.2 Topological structure of $S(\Gamma)$

In this subsection, by constructing the global coordinates of $S(\Gamma)$, we will show that $S(\Gamma)$ consists of $2^4 \times 2$ connected components each one of which is a 6 dimensional cell. For this purpose we need some preliminaries.

We define the polynomial mapping f from $X(\Gamma)$ to $\mathbf{R}^6$. For any $\chi \in X(\Gamma)$

$$f(\chi) := (I_{\alpha_1}(\chi), I_{\beta_1}(\chi), I_{\alpha_1\beta_1}(\chi), I_{\alpha_2}(\chi), I_{\beta_2}(\chi), I_{\alpha_2\beta_2}(\chi)).$$

By the definition of I_h $(h \in \Gamma)$, for any $\rho \in R(\Gamma)$

$$f \circ t(\rho) = (tr(\rho(\alpha_1)),\ tr(\rho(\beta_1)), \cdots, tr(\rho(\alpha_2\beta_2))).$$

$$\begin{array}{ccc} R(\Gamma) & & \\ t\downarrow & \overset{f\circ t}{\searrow} & \\ X(\Gamma) & \overset{f}{\rightarrow} & \mathbf{R}^6 \end{array}$$

We write the coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ of $\mathbf{R}^6$ by $(\vec{x}, \vec{y})$ for the sake of simplicity. Next we define the polynomial function $\kappa(x, y, z)$ on $\mathbf{R}^3$ by

$$\kappa(x, y, z) := x^2 + y^2 + z^2 - xyz - 2.$$

Easy calculation shows the following lemma ([F],[G]).

Lemma 3.1 *1. For any $A, B \in SL_2(\mathbf{R})$*

$$\kappa(tr(A), tr(B), tr(AB)) = tr([A, B]).$$

2. If $(x, y, z) \in \mathbf{R}^3$ satisfies $\kappa(x, y, z) < -2$, then

$$|x| > 2,\ |y| > 2,\ |z| > 2 \text{ and } x \cdot y \cdot z > 0. \ \square$$

In particular if we put

$$V_- = \{(\vec{x}, \vec{y}) \in \mathbf{R}^6 \mid \kappa(\vec{x}) = \kappa(\vec{y}) < -2\}$$

then from the definition of $S(\Gamma)$, $f(S(\Gamma)) \subset V_-$. In fact we will see in Proposition 3.3 that $f(S(\Gamma)) = V_-$.

Lemma 3.2 *$V_- \subset \mathbf{R}^6$ consists of 2^4 connected components each one of which is a 5 dimensional cell. More precisely, put $U := V_- \cap \{(\vec{x}, \vec{y}) \in \mathbf{R}^6 \mid x_i > 0,\ y_i > 0\ (\ i = 1,2\ )\}$ and define the action of $(\mathbf{Z}/2\mathbf{Z})^4$ on $\mathbf{R}^6$ by the change of signs of the coordinates x_i and y_i ($i = 1,2$). Then U is a 5 dimensional cell and V_- can be written as*

$$V_- = \coprod_{\gamma \in (\mathbf{Z}/2\mathbf{Z})^4} \gamma(U) \quad (\ \textit{disjoint union}\ ).$$

(Proof.) For $r < -2$ put

$$W_r := \{(x,y,z) \in \mathbf{R}^3 \mid \kappa(x,y,z) = r,\ x > 0,\ y > 0,\ z > 0\}$$

and $u := x - y,\ v := x + y$ for $(x,y,z) \in W_r$. Then by Lemma 3.1.2

$$v = \sqrt{\frac{z+2}{z-2}u^2 - \frac{4}{z-2}(2 + r - z^2)} \quad > 0.$$

Hence the next mapping is homeomorphic and consequently W_r is a 2 dimensional cell.

$$\begin{array}{ccc} W_r & \simeq & \mathbf{R} \times \{z \in \mathbf{R} \mid z > 2\}. \\ (x,y,z) & \mapsto & (u,z) \end{array}$$

As $U \simeq W_r \times W_r \times \{r \in \mathbf{R} \mid r < -2\}$, U is a 5 dimensional cell and by Lemma 3.1.2

$$V_- = \coprod_{\gamma \in (\mathbf{Z}/2\mathbf{Z})^4} \gamma(U). \ \square$$

Next lemma can be shown directly by calculation but it is a key lemma for the whole story of this section.

Lemma 3.3 *Let $(A, B) \in SL_2(\mathbf{R})^2$ be a pair of hyperbolic matrices (i.e. $|tr(A)| > 2$ and $|tr(B)| > 2$) which satisfies the following condition*

$$[A,B] = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \quad (\lambda < -1). \quad \cdots 1)$$

If we put $(x,y,z) := (tr(A), tr(B), tr(AB))$, *then* $\kappa(x,y,z) < -2$ *and there exists a constant* $k \in \mathbf{R}^* := \mathbf{R} - \{0\}$ *such that* A, B *can be written as*

$$A = \begin{pmatrix} \frac{\lambda}{\lambda+1}x & \frac{1}{k}\{\frac{\lambda}{(\lambda+1)^2}x^2 - 1\} \\ k & \frac{1}{\lambda+1}x \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{\lambda+1}y & \frac{1}{k}\{\frac{1}{\lambda+1}z - \frac{\lambda}{(\lambda+1)^2}xy\} \\ k\frac{\frac{\lambda}{(\lambda+1)^2}y^2-1}{\frac{1}{\lambda+1}z-\frac{\lambda}{(\lambda+1)^2}xy} & \frac{\lambda}{\lambda+1}y \end{pmatrix} \qquad \cdots 2)$$

Conversely for any $k \in \mathbf{R}^*$ *and* $(x,y,z) \in \mathbf{R}^3$ *with* $\kappa(x,y,z) < -2$, *define* $\lambda < -1$ *by* $\lambda + \frac{1}{\lambda} = \kappa(x,y,z)$. *Then the pair of matrices* $(A,B) \in SL_2(\mathbf{R})^2$ *defined by the condition 2) satisfies 1) and* $(x,y,z) = (tr(A), tr(B), tr(AB))$.

Because the pair $(A,B) \in SL_2(\mathbf{R})^2$ *defined by the above condition 2) is uniquely determined by* $k \in \mathbf{R}^*$ *and* $(x,y,z) \in \mathbf{R}^3$ *with* $\kappa(x,y,z) < -2$, *we write it as*

$$(A,B) = (A(x,y,z,k), B(x,y,z,k)). \ \square$$

Now we can show the main result of this subsection.

Proposition 3.3 $S(\Gamma)$ *consists of* $2^4 \times 2$ *connected components each one of which is a 6 dimensional cell.*

(Proof.) First, we define the mapping Ψ

$$\Psi : t^{-1}(S(\Gamma)) \to \mathbf{R}^* \times V_- \times PGL_2(\mathbf{R}).$$

For any $\rho = (A_1, B_1, A_2, B_2) \in t^{-1}(S(\Gamma))$, we first diagonalize $[A_1, B_1]$. More presisely, by using Lemma 3.3, we can choose $P \in PGL_2(\mathbf{R})$ uniquely such that by use of the notations in Lemma 3.3, (PA_iP^{-1}, PB_iP^{-1}) $(i = 1,2)$ can be written as

$$\begin{aligned} PA_1P^{-1} &= A(tr(A_1), tr(B_1), tr(A_1B_1), 1) \\ PB_1P^{-1} &= B(tr(A_1), tr(B_1), tr(A_1B_1), 1) \\ PA_2P^{-1} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A(tr(A_2), tr(B_2), tr(A_2B_2), k) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ PB_2P^{-1} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B(tr(A_2), tr(B_2), tr(A_2B_2), k) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

where $k \in \mathbf{R}^*$ is some constant. We define the mapping Ψ by

$$\begin{array}{rcl} \Psi : t^{-1}(S(\Gamma)) & \rightarrow & \mathbf{R}^* \times V_- \times PGL_2(\mathbf{R}) \\ \rho & \mapsto & (k,\ f \circ t(\rho),\ P)\ . \end{array}$$

Lemma 3.3 tells that Ψ is bijective and also homeomorphic. From the definition, Ψ is $PGL_2(\mathbf{R})$-equivariant, hence it induces the homeomorphism Φ from $S(\Gamma)$ to $\mathbf{R}^* \times V_-$ as follows.

$$\begin{array}{ccc} t^{-1}(S(\Gamma)) & \overset{\Psi}{\simeq} & \mathbf{R}^* \times V_- \times PGL_2(\mathbf{R}) \\ t \downarrow & & \downarrow proj. \\ S(\Gamma) & \overset{\Phi}{\simeq} & \mathbf{R}^* \times V_- \end{array}$$

Moreover by Lemma 3.2, $\mathbf{R}^* \times V_-$ consists of $2^4 \times 2$ connected components each one of which is a 6 dimensional cell. □

3.3 Cell structure of Teichmüller space T_2

Next we consider the conditions which characterize the connected components of $X_0(\Gamma)$ in $S(\Gamma)$. By the definition of Φ in the proof of Proposition 3.3, the first component k of Φ can be considered as a function on $S(\Gamma)$.

Proposition 3.4 *Suppose $U \subset S(\Gamma)$ be a connected component on which the function $I_{\alpha_1} \cdot I_{\alpha_2} \cdot k$ is negative. Then there exists $\chi \in U$ such that χ is not contained in $X_0(\Gamma)$. Because $X_0(\Gamma)$ consists of finitely many connected components of $X(\Gamma)$ by Corollary 2.4 this means that $X_0(\Gamma) \cap U = \phi$.*

(Proof.) First we remark that on a connected component U of $S(\Gamma)$, the signs of the functions $I_{\alpha_1}, I_{\alpha_2}$, *and* k are constant. We consider $(\vec{x}, \vec{y}) \in V_-$ satisfying $|x_i| = |y_i| = 4$ $(i = 1, 2, 3)$. Then there are 2^4 points of V_- satisfing this condition. By use of the surjectivity of $f|_U : U \rightarrow V_-$, take $\rho = (A_1, B_1, A_2, B_2) \in t^{-1}(S(\Gamma))$ with $t(\rho) \in U$ and $f \circ t(\rho) = (\vec{x}, \vec{y})$. If $I_{\alpha_1}(t(\rho)) \cdot I_{\alpha_2}(t(\rho)) = tr(A_1) \cdot tr(A_2) = 16 > 0$, then by using the presentation of $\rho = (A_1, B_1, A_2, B_2)$ in the proof of Proposition 3.3, $tr(A_1 A_2) = -2 - k - \frac{4}{k}$ where we write $k(\rho)$ by k for the sake of simplicity. Hence if $k(\rho) = k = -2$ $(i.e., I_{\alpha_1} \cdot I_{\alpha_2} \cdot k < 0$ on U), then $tr(A_1 A_2) = 2$ and this means that $A_1 A_2 \in SL_2(\mathbf{R})$ is a parabolic matrix, thus $t(\rho)$ is not contained in $X_0(\Gamma)$. Similar argument holds for the case $I_{\alpha_1}(\rho) \cdot I_{\alpha_2}(\rho) = tr(A_1) \cdot tr(A_2) = -16 < 0$. □

From the above proof, There are 16 connected components of $S(\Gamma)$ on which the function $I_{\alpha_1} \cdot I_{\alpha_2} \cdot k$ is negative. Hence the number of connected components of $X_0(\Gamma)$, $\pi_0(X_0(\Gamma))$ is less than or equal to 16. On the other hand, as the argument in subsection 2.4 implies $\pi_0(X_0(\Gamma)) \geq 16$, we get the following result.

Theorem 3.1 $\pi_0(X_0(\Gamma)) = 16$. *Thus Teichmüller space* T_2

$$T_2 = Hom(\Gamma, \mathbf{Z}/2\mathbf{Z}) \setminus X_0(\Gamma)$$

is connected and by Proposition 3.3, it is a 6 dimensional cell in particular contractible. □

3.4 Semialgebraic structure of Teichmüller space T_2

Previous argument shows the following presentation of $X_0(\Gamma)$ as a subset of $X(\Gamma)$

$$\begin{aligned} X_0(\Gamma) &= \{\chi \in S(\Gamma) \mid I_{\alpha_1}(\chi) \cdot I_{\alpha_2}(\chi) \cdot k(\chi) > 0\ \} \\ &= \{\chi \in X(\Gamma) \mid I_{c_1} < -2 \ and \ I_{\alpha_1}(\chi) \cdot I_{\alpha_2}(\chi) \cdot k(\chi) > 0\ \} \end{aligned}$$

where $c_1 = [\alpha_1, \beta_1] \in \Gamma$. This presentation induces the following semialgebraic description of $X_0(\Gamma)$ *in* $X(\Gamma)$.

Theorem 3.2 $X_0(\Gamma)$ *can be written as a semialgebraic subset of* $X(\Gamma)$ *as follows*

$$X_0(\Gamma) = \{\chi \in X(\Gamma) \mid I_{c_1}(\chi) < -2, \frac{(I_{c_1}(\chi) + 2) \cdot I_{\alpha_1\alpha_2}(\chi)}{I_{\alpha_1}(\chi) \cdot I_{\alpha_2}(\chi)} > 2\ \}.$$

This means that for any representation $\rho = (A_1, B_1, A_2, B_2) \in R(\Gamma)$, ρ *is a discrete and faithful* $SL_2(\mathbf{R})$ *-representation of* Γ *if and only if*

$$tr([A_1, B_1]) < -2 \ and \ \frac{(tr([A_1, B_1]) + 2) \cdot tr(A_1A_2)}{tr(A_1) \cdot tr(A_2)} > 2.$$

(*Proof.*) For any $\rho = (A_1, B_1, A_2, B_2) \in t^{-1}(S(\Gamma))$, by calculating $tr(A_1A_2)$

$$\begin{aligned} &k(\rho)^2 + (tr(A_1A_2) - \frac{2tr(A_1) \cdot tr(A_2)}{tr([A_1, B_1]) + 2})k(\rho) + \\ &+(\frac{tr(A_1)^2}{tr([A_1, B_1]) + 2} - 1)(\frac{tr(A_2)^2}{tr([A_1, B_1]) + 2} - 1) = 0. \end{aligned}$$

Considering this as the quadratic equation on $k(\rho)$, the constant term is positive, hence the sign of $k(\rho)$ and the sign of the coefficient of the linear term of this equation are opposite each other. Hence for $\rho = (A_1, B_1, A_2, B_2) \in t^{-1}(S(\Gamma))$,

$$tr(A_1) \cdot tr(A_2) \cdot k(\rho) > 0 \Leftrightarrow \frac{(tr([A_1, B_1]) + 2) \cdot tr(A_1 A_2)}{tr(A_1) \cdot tr(A_2)} > 2. \square$$

Remark Because each connected component of $X_0(\Gamma)$ is separated by the action of $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ i.e., the sign conditions of the functions $I_{\alpha_1}, I_{\beta_1}, I_{\alpha_2}$ and I_{β_2}, therefore adding these 4 conditions, we can get the semialgebraic description of T_2 by use of 6 polynomial inequalities (see Corollary 2.7). □

4 Semialgebraic description of Teichmüller space T_g ($g \geq 3$ case)

In this section, we assume $g \geq 3$. We show the connectivity, contractibility and semialgebraic description of Teichmüller space T_g following the similar lines in section 3.

4.1 Definition of the semialgebraic subset $S(\Gamma)$ of $X(\Gamma)$

We define the open semialgebraic subset $S(\Gamma)$ of $X(\Gamma)$ by

$$S(\Gamma) := \{\chi \in X(\Gamma) \mid I_{c_i}(\chi) < -2 \ (i = 1, \cdots, g) \\ I_{d_j}(\chi) < -2 \ (j = 2, \cdots, g-2)\}$$

where $c_i := [\alpha_i, \beta_i] \in \Gamma$ and $d_j := c_1 c_2 \cdots c_j$.

Similar arguments of Proposition 3.1 and 3.2 show

Proposition 4.1 $S(\Gamma) \subset X'(\Gamma)$. *Hence by Proposition 2.3 ,* $t^{-1}(S(\Gamma)) \xrightarrow{t} S(\Gamma)$ *is a* $PGL_2(\mathbf{R})$*-bundle and we can consider* $S(\Gamma)$ *as the* $PGL_2(\mathbf{R})$*-adjoint quotient of* $t^{-1}(S(\Gamma))$ *i.e.,*

$$S(\Gamma) = t^{-1}(S(\Gamma))/PGL_2(\mathbf{R}). \ \square$$

Proposition 4.2 $X_0(\Gamma) \subset S(\Gamma)$. □

Moreover if a representation $\rho = (A_1, B_1, \cdots, A_g, B_g)$ is contained in $R_0(\Gamma)$, the representation $\rho_j := (A_j, B_j, A_{j+1}, B_{j+1}, \cdots, A_{j-1}, B_{j-1})$ $(j = 2, \cdots, g)$ is well defined and also an element of $R_0(\Gamma)$, hence we have

Corollary 4.1 *For $\chi \in X_0(\Gamma)$, $I_{c_i c_{i+1}}(\chi) < -2 \quad (i = 2, \cdots, g)$ where we assume that $c_{g+1} = c_1$.* □

Corollary 4.2 *Above arguments show the following diagram.* □

$$\begin{array}{ccccccc} R(\Gamma) & \supset & R'(\Gamma) & \supset & t^{-1}(S(\Gamma)) & \supset & R_0(\Gamma) \\ t\downarrow & & \downarrow & & \downarrow & & \downarrow \\ X(\Gamma) & \supset & X'(\Gamma) & \supset & S(\Gamma) & \supset & X_0(\Gamma) \end{array}$$

4.2 Topological structure of $S(\Gamma)$

In this subsection, by constructing the global coordinates of $S(\Gamma)$, we will show that $S(\Gamma)$ consists of $2^{2g} \times 2^{2g-3}$ connected components each one of which is a 6g-6 dimensional cell. For this purpose we need some preliminaries.

First we define the polynomial mapping f from $X(\Gamma)$ to $\mathbf{R}^{3g}$ by

$$f(\chi) := (I_{\alpha_1}(\chi), I_{\beta_1}(\chi), I_{\alpha_1\beta_1}(\chi), \cdots, I_{\alpha_g}(\chi), I_{\beta_g}(\chi), I_{\alpha_g\beta_g}(\chi))$$

for $\chi \in X(\Gamma)$.

$$\begin{array}{ccc} R(\Gamma) & & \\ t\downarrow & \overset{f\circ t}{\searrow} & \\ X(\Gamma) & \xrightarrow{f} & \mathbf{R}^{3g} \end{array}$$

Let $(\vec{x_1}, \cdots, \vec{x_g})$ denote the coordinates $(x_{11}, x_{12}, x_{13}, \cdots, x_{g1}, x_{g2}, x_{g3})$ of $\mathbf{R}^{3g}$. We define the semialgebraic subset V_- by

$$V_- := \{(\vec{x_1}, \cdots, \vec{x_g}) \in \mathbf{R}^{3g} \mid \kappa(\vec{x_i}) < -2 \quad (i = 1, \cdots, g)\}$$

where $\kappa(x, y, z)$ is the polynomial function on $\mathbf{R}^3$ defined in subsection 3.2. Then from the definition of $S(\Gamma)$, $f(S(\Gamma)) \subset V_-$. In fact we will see in the proof of Proposition 4.3 that $f(S(\Gamma)) = V_-$.

We can prove the next lemma by the same argument in Lemma 3.2.

Lemma 4.1 *$V_- \subset \mathbf{R}^{3g}$ consists of 2^{2g} connected components each one of which is a 3g dimensional cell. More precisely, put*

$$U := V_- \cap \{(\vec{x_1}, \cdots, \vec{x_g}) \in \mathbf{R}^{3g} \mid x_{ij} > 0 \quad (i = 1, \cdots, g \ j = 1, 2)\}$$

and define the action of $(\mathbf{Z}/2\mathbf{Z})^{2g}$ *on* $\mathbf{R}^{3g}$ *by the change of signs of* x_{ij} $(i = 1, \cdots, g\ j = 1, 2)$. *Then* U *is a* $3g$ *dimensional cell and* V_- *can be written as*

$$V_- = \coprod_{\gamma \in (\mathbf{Z}/2\mathbf{Z})^{2g}} \gamma(U) \quad (\textit{disjoint union}). \square$$

Next lemma which is shown by elementary calculation is a key lemma in this section.

Lemma 4.2 *1. For a pair of hyperbolic matrices* $(C_1, C_2) \in SL_2(\mathbf{R})^2$, *assume that* C_1 *is diagonal*

$$C_1 = \begin{pmatrix} \eta & 0 \\ 0 & \frac{1}{\eta} \end{pmatrix} \quad (\eta < -1).$$

If the traces of C_1, C_2 *and* $C_1 C_2$ *satisfy*

$$x := tr(C_1) < -2, \ y := tr(C_2) < -2 \text{ and } z := tr(C_1 C_2) < -2 \ \cdots 1)$$

then there exists $m \in \mathbf{R}^*$ *such that* C_2 *can be written as follows.*

$$C_2 = \begin{pmatrix} \frac{\eta z - y}{\eta^2 - 1} & m \\ \frac{1}{m}\left\{\frac{\eta(\eta y - z)(\eta z - y)}{(\eta^2-1)^2} - 1\right\} & \frac{\eta(\eta y - z)}{\eta^2 - 1} \end{pmatrix} \cdots 2)$$

Conversely , for any constant $m \in \mathbf{R}^*$ *and* $(x, y, z) \in \mathbf{R}^3$ *with* $x < -2, y < -2$ *and* $z < -2$, *if we put* $\eta < -1$ *with* $\eta + \frac{1}{\eta} = x$ *and define* $C_1 = \begin{pmatrix} \eta & 0 \\ 0 & \frac{1}{\eta} \end{pmatrix}$ *and* C_2 *by the condition 2), then* $(x, y, z) = (tr(C_1), tr(C_2), tr(C_1 C_2))$ *as the condition 1). We write* C_2 *defined by the condition 2) by* $C(x, y, z, m)$.

2. *Moreover for such a pair* $(C_1, C_2) \in SL_2(\mathbf{R})^2$, *we can diagonalize* $C_1 C_2$ *and* C_2 *by using the following matrices* $P, Q \in SL_2(\mathbf{R})$.

$$P := \begin{pmatrix} 1 & -\frac{m\tau\eta}{\tau^2 - 1} \\ \frac{\tau(\eta^2-1) - \eta(\eta z - y)}{m\eta(\eta^2 - 1)} & \frac{\tau\eta(\eta z - y) - (\eta^2 - 1)}{(\eta^2-1)(\tau^2-1)} \end{pmatrix}$$

where $\tau < -1$ *with* $\tau + \frac{1}{\tau} = z = tr(C_1 C_2)$ *and* $C_1 C_2 = P \begin{pmatrix} \tau & 0 \\ 0 & \frac{1}{\tau} \end{pmatrix} P^{-1}$.

$$Q := \begin{pmatrix} 1 & -\frac{m\xi}{\xi^2 - 1} \\ \frac{\xi(\eta^2-1) - (\eta z - y)}{m(\eta^2 - 1)} & \frac{\xi(\eta z - y) - (\eta^2 - 1)}{(\eta^2-1)(\xi^2-1)} \end{pmatrix}$$

where $\xi < -1$ *with* $\xi + \frac{1}{\xi} = y = tr(C_2)$ *and* $C_2 = Q \begin{pmatrix} \xi & 0 \\ 0 & \frac{1}{\xi} \end{pmatrix} Q^{-1}$.

In the following we write these P and Q by P(x,y,z,m) and Q(x,y,z,m). □

Proposition 4.3 $S(\Gamma)$ *consists of* $2^{2g} \times 2^{2g-3}$ *connected components each one of which is a 6g-6 dimensional cell.*

(Proof.) We construct the mapping Ψ

$$\Psi : t^{-1}(S(\Gamma)) \rightarrow V_{-} \times \{w \in \mathbf{R} \mid w < -2\}^{g-3} \times (\mathbf{R}^*)^{g-3} \times (\mathbf{R}^*)^g \times PGL_2(\mathbf{R})$$

as follows.

For $\rho = (A_1, B_1, \cdots, A_g, B_g) \in t^{-1}(S(\Gamma))$, put

$$\begin{aligned} (\vec{x_1}, \cdots, \vec{x_g}) &:= f \circ t(\rho) \in V_{-} \ (\ where\ \vec{x_i} := (x_{i1}, x_{i2}, x_{i3})) \\ C_i &:= [A_i, B_i] \ (i = 1, \cdots, g) \\ u_i &:= tr(C_i) = \kappa(\vec{x_i}) \ (i = 1, \cdots, g) \\ D_k &:= C_1 \cdots C_k \ (k = 1, \cdots, g-1) \\ w_k &:= tr(D_k) \ (k = 1, \cdots, g-1). \end{aligned}$$

We remark that

$$\begin{aligned} D_1 &= C_1 \\ w_1 &= u_1 \\ w_{g-1} &= u_g. \end{aligned}$$

Because of the definition of $S(\Gamma)$

$$w_1 < -2,\ u_2 < -2,\ and\ w_2 < -2.$$

Lemma 4.2.1 shows that there exists $R \in PGL_2(\mathbf{R})$ uniquely such that

$$\begin{aligned} RC_1R^{-1} &= \begin{pmatrix} \eta_1 & 0 \\ 0 & \frac{1}{\eta_1} \end{pmatrix} \ (\eta_1 < -1\ with\ \eta_1 + \frac{1}{\eta_1} = w_1) \\ RC_2R^{-1} &= C(w_1, u_2, w_2, 1). \end{aligned}$$

Then by Lemma 4.2.2 there exists $P_1 = P(w_1, u_2, w_2, 1)$ such that

$$RD_2R^{-1} = P_1 \begin{pmatrix} \eta_2 & 0 \\ 0 & \frac{1}{\eta_2} \end{pmatrix} P_1^{-1} \ (\eta_2 < -1\ with\ \eta_2 + \frac{1}{\eta_2} = w_2).$$

Similarly because

$$w_2 < -2,\ u_3 < -2,\ and\ w_3 < -2$$

Lemma 4.2.1 shows that there exists a constant $m_1 \in \mathbf{R}^*$ such that

$$RC_3R^{-1} = P_1C(w_2, u_3, w_3, m_1)P_1^{-1}$$

and by Lemma 4.2.2 there exists $P_2 = P(w_2, u_3, w_3, m_1)$ such that

$$RD_3R^{-1} = P_1P_2\begin{pmatrix} \eta_3 & 0 \\ 0 & \frac{1}{\eta_3} \end{pmatrix} P_2^{-1}P_1^{-1}\ (\eta_3 < -1\ with\ \eta_3 + \frac{1}{\eta_3} = w_3).$$

Inductively, for $j = 2, \cdots, g-1$, because

$$w_{j-1} < -2,\ u_j < -2,\ and\ w_j < -2$$

Lemma 4.2 shows

$$\begin{aligned} RC_jR^{-1} &= P_1 \cdots P_{j-2}C(w_{j-1}, u_j, w_j, m_{j-2})P_{j-2}^{-1} \cdots P_1^{-1} \\ RD_jR^{-1} &= P_1 \cdots P_{j-1}\begin{pmatrix} \eta_j & 0 \\ 0 & \frac{1}{\eta_j} \end{pmatrix} P_{j-1}^{-1} \cdots P_1^{-1} \end{aligned}$$

where $m_{j-2} \in \mathbf{R}^*$ with $m_0 = 1$, $P_{j-1} = P(w_{j-1}, u_j, w_j, m_{j-2})$ with $P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\eta_j < -1$ with $\eta_j + \frac{1}{\eta_j} = w_j$.

Moreover RC_gR^{-1} can be written as

$$RC_gR^{-1} = P_1 \cdots P_{g-2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \eta_{g-1} & 0 \\ 0 & \frac{1}{\eta_{g-1}} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P_{g-2}^{-1} \cdots P_1^{-1}.$$

On the other hand by Lemma 3.3

$$\begin{aligned} RA_1R^{-1} &= A(\vec{x_1}, k_1) \\ RB_1R^{-1} &= B(\vec{x_1}, k_1) \end{aligned}$$

for some $k_1 \in \mathbf{R}^*$ where we write $A(x_{11}, x_{12}, x_{13}, k_1)$ by $A(\vec{x_1}, k_1)$. By Lemma 4.2.2 there exist $Q_2 = Q(w_1, u_2, w_2, 1)$ and $k_2 \in \mathbf{R}^*$ such that

$$\begin{aligned} RA_2R^{-1} &= Q_2A(\vec{x_2}, k_2)Q_2^{-1} \\ RB_2R^{-1} &= Q_2B(\vec{x_2}, k_2)Q_2^{-1}. \end{aligned}$$

We put

$$S'(\Gamma) := \{\chi \in S(\Gamma) \mid m_j(\chi) > 0 \; (j = 1, \cdots, g-3)\}.$$

Proposition 4.9 *For $\rho = (A_1, B_1, \cdots, A_g, B_g) \in t^{-1}(S'(\Gamma))$ we write $x_{i1}(\rho) \cdot k_i(\rho)$ $(i = 1, \cdots, g)$ by $x_{i1} \cdot k_i$ for the sake of simplicity. Then*

$$x_{i1} \cdot k_i > 0 \quad (i = 1, \cdots, g)$$

if and only if

$$\frac{tr([A_i, B_i][A_{i+1}, B_{i+1}]) + tr[A_{i+1}, B_{i+1}]}{tr[A_i, B_i] + 2} < \frac{tr(A_i[A_{i+1}, B_{i+1}])}{tr A_i}. \quad \square$$

We omit the proof of the above propositions.

Above consideration shows the semialgebraic presentation of $X_0(\Gamma)$.

Theorem 4.2 *For $\alpha_i, \beta_i \in \Gamma$, put $c_i := [\alpha_i, \beta_i]$ $(i = 1, \cdots, g)$, and $d_j := c_1 \cdots c_j$ $(j = 1, \cdots, g-1)$. Then $\chi \in X(\Gamma)$ is contained in $X_0(\Gamma)$ if and only if χ satisfies the following 4g-6 inequalities on I_h $(\in \Gamma)$.*

$$\begin{aligned}
&I_{c_i}(\chi) < -2 \quad (i = 1, \cdots, g), \\
&I_{d_j}(\chi) < -2 \quad (j = 2, \cdots, g-2), \\
&\frac{I_{c_k c_{k+1}}(\chi) + I_{c_{k+1}}(\chi)}{I_{c_k}(\chi) + 2} < \frac{I_{\alpha_k c_{k+1}}(\chi)}{I_{\alpha_k}(\chi)} \quad (k = 1, \cdots, g), \\
&I_{d_{l+1}}(\chi)(I_{d_l}(\chi) I_{d_{l+2}}(\chi) + I_{c_{l+1}}(\chi) I_{c_{l+2}}(\chi)) \\
&\quad > 2(I_{d_l}(\chi) I_{c_{l+2}}(\chi) + I_{c_{l+1}}(\chi) I_{d_{l+2}}(\chi)) + (I_{d_{l+1}}(\chi)^2 - 4) I_{d_l c_{l+2}}(\chi) \\
&\quad (l = 1, \cdots, g-3)
\end{aligned}$$

where we assume that $c_{g+1} = c_1$.

By adding 2g inequalities which consist of the sign conditions of $I_{\alpha_i}, I_{\beta_i}$ $(i = 1, \cdots, g)$ (see Corollary 2.7), we can also describe T_g by 6g-6 polynomial inequalities in $X(\Gamma)$. $\square$

References

[B-C-R] J.Bochnak, M.Coste and M-F.Roy. Géométrie algébrique réelle , Springer Verlag (1986).

[Br] G.W.Brumfiel. The real spectrum compactification of Teichmüller space, Contem. Math. Vol.74 (1988),51-75.

[C-S] M.Culler and P.B.Shalen. Varieties of group representations and splitting of 3 manifolds, Ann. of Math. 117 (1983),109-146.

[F] R.Fricke-F.Klein. Vorlesungen uber der Theorie der Automorphen Funktionen, Vol.1, (1897) Teubner,Leipzig.

[Go] W.Goldman. Topological components of spaces of representations, Invent. Math. 93 (1988),557-607.

[Gu] R.C.Gunning. Lectures on vector bundles over Riemann surfaces, (1967) Princeton.

[He] H.Helling. Diskrete Untergruppen von $SL_2(\mathbf{R})$, Invent. Math. 17 (1972),217-229.

[Ho] R.Horowitz. Characters of free groups represented in the two dimensional linear groups, Comm.Pure Appl.Math. 25 (1972),635-649.

[Jø] T.Jørgensen. On discrete groups of Möbius transformations, Amer.J.Math. 98 (1976),739-749.

[K] L.Keen. On Fricke Moduli, Ann. of Math. Studies 66 (1971),205-224.

[M-S] J.W.Morgan and P.B.Shalen. Valuations,trees and degenerations of hyperbolic structures , Ann. of Math. 120 (1984),401-476.

[Ok] T.Okai. An explicit description of the Teichmüller space as holonomy representations and its applications , Hiroshima Math. J. 22 (1992), 259-271.

[O1] Y.Okumura. On the global real analytic coordinates for Teichmüller spaces, J.Math.Soc.Japan Vol.42,No.1.(1990),91-101.

[O2] Y.Okumura. a note of his talk at KANSUU-RON summer seminar 92' (in Japanese).

[Pa] S.J.Patterson. On the cohomology of Fuchsian groups, Glasgow Math. 16 (1975),123-140.

[Pr] C.Procesi. The Invariant Theory of $n \times n$ Matrices, Adv. in Math. 19 (1976),306-381.

Semialgebraic description of Teichmüller space

Yohei Komori
Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606, Japan

Abstract

We give a concrete semialgebraic description of Teichmüller space T_g of the closed surface group Γ_g of genus $g(\geq 2)$. We also show the connectivity and contractibility of T_g from a view point of $SL_2(\mathbf{R})$-representations of Γ_g.

1 Introduction

Teichmüller space T_g of compact Riemann surfaces of genus $g(\geq 2)$ is the moduli space of *marked* Riemann surfaces of genus g. Thanks to the uniformization theorem due to Klein, Koebe and Poincaré, any compact Riemann surface of genus $g(\geq 2)$ can be obtained as the quotient space $G \setminus \mathbf{H}$ where $\mathbf{H}$ is the upper half plane and G is a cocompact Fuchsian group i.e., a cocompact discrete subgroup of $PSL_2(\mathbf{R})$. And as an abstract group , G is isomorphic to the surface group Γ_g which has the following presentation

$$\Gamma_g := \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g | \prod_{i=1}^{g} (\alpha_i \cdot \beta_i \cdot \alpha_i^{-1} \cdot \beta_i^{-1}) = id. \rangle \ .$$

From this view point, T_g can be considered as the deformation space of a Fuchsian group which is isomorphic to Γ_g and this is called *Fricke moduli* studied by Fricke himself and more precisely by Keen ([F],[K]).

In this article, we consider this Fricke moduli from a view point of $SL_2(\mathbf{R})$-representations of the surface group Γ_g. We treat T_g as the $PGL_2(\mathbf{R})$-adjoint quotient of the set of discrete and faithful $PSL_2(\mathbf{R})$-representations of Γ_g

$$T_g = \{\Gamma_g \to PSL_2(\mathbf{R}) \ : \ discrete\, and\, faithful\}/PGL_2(\mathbf{R})$$

[Sa] K.Saito. The Teichmüller Space and a certain modular function from a view point of group representations, Algebraic Geometry and Related Topics- Proceedings of the International Symposium, Inchoen, Republic of Korea (1992),41-89.

[S-S] M.Seppälä and T.Sorvali. Traces of commutators of Möbius Transformations, Math. Scand. 68 (1991),53-58.

[Si] C.L.Siegel. Topics in complex function theory, Vol.2 (1971) John Wiley and Sons,Inc.

[W] A.Weil. On discrete subgroups of Lie groups, Ann. of Math. 72 (1960),369-384, Ann. of Math. 75 (1962),578-602.